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Generalized Hamiltonian systems on subvarieties: constant rank case

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Abstract

For the constraint variety in symplectic manifold, the solvable Hamiltonian vector fields on the constraint are investigated. According to P.A.M. Dirac [3], the space of solvable Hamiltonian systems is determined by the geometric restriction of the symplectic form to the constraint. Solvability condition of the generalized Hamiltonian systems is extended to singular varieties and applied under some assumption on singularities. The constraint being a smooth submanifold in a symplectic space was considered in [6]. In this paper, we investigate the solvability of generalized Hamiltonian systems and the constraint invariants on singular constraints in the constant rank case.

Keywords: Symplectic manifold, Hamiltonian systems, Symplectic constraints

Mathematics Subject Classification: 53D05, 51N10, 53D22, 70H05, 15A04

1 Introduction

Let K be a submanifold of $(\mathbb{R}^{2n}, \omega)$ with the symplectic structure in Darboux form $\omega = \sum_{i=1}^{n} dy_i \wedge dx_i$. The generalized Hamiltonian system on K (generalized Hamiltonian dynamics [3,11,12]) is defined as a sub-bundle L of $T\mathbb{R}^{2n}$, $\tau : T\mathbb{R}^{2n} \to \mathbb{R}^{2n}$, over K, which is a Lagrangian submanifold of $(T\mathbb{R}^{2n}, \dot{\omega})$, $\dot{\omega}\mid_L = 0$ with the associated symplectic form $\dot{\omega} = \sum_{i=1}^{n} d\dot{y}_i \wedge dx_i - d\dot{x}_i \wedge dy_i$. Then, locally L is expressed as

$$L = \{v \in T\mathbb{R}^{2n} : \tau(v) \in K, \text{ and, for any } u \in T_{\tau(v)}K, \omega(u, v) = -dh(u)\}, \tag{1.1}$$

by function h which is locally defined on K. In the coordinates we use, the generalized Hamiltonian system (1.1) is generated by the Morse family $F : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$, which works only locally on K (cf. [1,7,13]),

$$F(x, y, \lambda) = \sum_{i=1}^{k} a_i(x, y)\lambda_i + b(x, y), \tag{1.2}$$

where K is defined by smooth functions $a_i : \mathbb{R}^{2n} \to \mathbb{R}$, $1 \le i \le k$, with the maximal rank condition $rank(\frac{\partial a_i}{\partial x_l}(x, y), \frac{\partial a_i}{\partial y_l}(x, y)) = k$, $K = \{(x, y) \in \mathbb{R}^{2n} : a_i(x, y) = 0,\ i = 1, \ldots, k\}$ and $b : \mathbb{R}^{2n} \to \mathbb{R}$ is an arbitrary smooth extension of $h : K \to \mathbb{R}$. In what follows we consider mainly germs of functions and mappings at zero or representants of this germs.

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Fundamental property of such systems is their local smooth solvability, i.e. existence, for each $v \in L$, of a smooth family $\alpha : U \times (-\epsilon, \epsilon) \ni (\bar{v}, t) \mapsto \mathbb{R}^{2n}$ of smooth solutions of L in the neighbourhood U of v in L such that $\dot{\alpha}_{\bar{v}}(0) = \bar{v}$. Condition for smooth solvability of generalized Hamiltonian systems (cf. [4,5,9])

$$\{\frac{\partial F}{\partial \lambda_i}, F\}(x, y, \lambda) = 0, i = 1, \ldots, k \text{ for } (\mathrm{x, y}, \lambda) \in \mathrm{K} \times \mathbb{R}^{\mathrm{k}}$$

reads as a linear equation for $\lambda_j,\ j = 1, \ldots, k$ (cf. [6,10]) with canonical matrix $A(x, y) = (\{a_i, a_j\})$;

$$\sum_{j=1}^{k} \{a_i, a_j\}(x, y)\lambda_j = \{b, a_i\}(x, y),\ i = 1, \ldots, k, \tag{1.3}$$

where $\{\bullet, \bullet\}$ is a Poisson bracket induced by ω.

If k is even and $detA(x, y) \neq 0$ on K, then the only smoothly solvable sections of L define the solvable Hamiltonian systems $X_b \in \Gamma(TK)$,

$$X_b = \frac{\partial \hat{F}}{\partial y}(x, y)\frac{\partial}{\partial x} - \frac{\partial \hat{F}}{\partial x}(x, y)\frac{\partial}{\partial y}\ |_K,\ \text{where } \hat{F}(x, y) = b(x, y) + \sum_{i=1}^{k} \lambda_i(x, y)a_i(x, y)$$

and $\lambda(x, y)$ is a unique smooth solution of (1.3). If K is coisotropic then $\{a_i, a_j\} = 0$ and $\{b, a_j\} = 0, 1 \leq j \leq k$. In this case, the generalized Hamiltonian system L is smoothly solvable for b fulfilling the above equations, and after reduction defines smoothly solvable Hamiltonian systems on the reduced space. The constant rank condition of $A(x, y)$ at all points of K is related to the special cases of submanifolds of $(\mathbb{R}^{2n}, \omega)$ like in the coisotropic case. We denote $V_q = T_qK \cap (T_qK)^{\omega},\ q = (x, y)$, as a kernel of $A(q)$ at each $q \in K$, $dimV_q = l$. The two form induced on the quotient space $(T_qK)^{\omega}/V_q$ is nondegenerated for $k = 1, \ldots, 2n - 1$ and $dim(T_qK)^{\omega}/V_q = k - l$ is an even number $l \leq max\{k, 2n - k\}$. The constant rank condition of $A(q)$ along K implies that $V = \bigcup_{q \in K} V_q$ is an integrable, characteristic distribution of $\omega\ |_K$ and it is a smoothly solvable submanifold of L with $b \equiv 0$. It can be written in the form

$$V_q = \{\sum_{i=1}^{k} \lambda_i(\frac{\partial a_i}{\partial y}(x, y)\frac{\partial}{\partial x} - \frac{\partial a_i}{\partial x}(x, y)\frac{\partial}{\partial y}\}_{\lambda \in \mathbb{R}^k},$$

with its sections given by smooth solutions of the equation $\lambda(x, y) \in KerA(x, y),\ (x, y) \in K$. All smoothly solvable Hamiltonian systems on K are given as solutions of Eq. (1.3) with b fulfilling conditions of pointwise solvability of (1.3).

In this paper, we first show the properties of generalized Hamiltonian systems as a constrained Lagrangian varieties in symplectic tangent bundle to symplectic manifold $(\mathbb{R}^{2n}, \omega)$ equipped with the canonical symplectic structure induced by ω. The basic symplectic invariant, which is the kernel of ω restricted to the constraint K, is directly related to the question of solvability of Hamiltonian vector fields over K. In fact these vector fields are constructed by smooth solutions of so-called tangential solvability condition (1.3). The properties of solutions of this condition in the singular case of K are investigated in Sect. 3. In Sect. 4, we present a simple proof of the Main Theorem based on series of partial results concerning solvability domains in generalized Hamiltonian systems.

2 Basic notations and main results

Let K (or the representative of its germ at $(0,0)$) be a subvariety of $(\mathbb{R}^{2n},\omega)$ defined by smooth functions $(a_i(x,y))_{i=1,\dots,k}$. A singular point of K is a point where K is not a smooth submanifold of $\mathbb{R}^{2n}$ locally around that point. By Σ_K, we denote the singular point set of K. Suppose that

$$rank\left(\frac{\partial a_i}{\partial x_j}(0,0), \frac{\partial a_i}{\partial y_j}(0,0)\right)_{1\le i\le k, 1\le j\le n} = r.$$

Changing the order of $a_1(x,y),\cdots,a_k(x,y)$, we may assume that $da_1(0,0),\cdots,da_r(0,0)$ are linearly independent. We set

$$K_r = \{(x,y)\in\mathbb{R}^{2n} \ | \ a_1(x,y)=\cdots=a_r(x,y)=0\}. \tag{2.1}$$

Note that if K is smooth, then $r=k$ and $K_r=K$. Let us introduce the Hamiltonian immersion mapping $\phi:\mathbb{R}^{2n}\times\mathbb{R}^k\to T\mathbb{R}^{2n}$,

$$\phi(x,y,\lambda) = (x,y,\frac{\partial F}{\partial y}(x,y,\lambda), -\frac{\partial F}{\partial x}(x,y,\lambda)). \tag{2.2}$$

Then, the generalized Hamiltonian system on K is given in the form

$$L_F = \phi(K\times\mathbb{R}^k) = \{(x,y,\frac{\partial F}{\partial y}(x,y,\lambda), -\frac{\partial F}{\partial x}(x,y,\lambda)) \mid (x,y,\lambda)\in K\times\mathbb{R}^k)\} \tag{2.3}$$

with the tangential solvability condition

$$A(x,y)\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_k\end{pmatrix} = \begin{pmatrix}\{b,a_1\}(x,y)\\ \vdots\\ \{b,a_k\}(x,y)\end{pmatrix}, \quad (x,y)\in K\subset\mathbb{R}^{2n}. \tag{2.4}$$

By $\tilde{S}_F$, we denote the tangential stationary set,

$$\tilde{S}_F = \{(x,y,\lambda)\in K\times\mathbb{R}^k : \sum_{j=1}^{k}\{a_i,a_j\}(x,y)\lambda_j = \{b,a_i\}(x,y), 1\le i\le k\} \tag{2.5}$$

and its image S_F in the tangent bundle $T\mathbb{R}^{2n}$, $\tau: T\mathbb{R}^{2n}\to\mathbb{R}^{2n}$; $S_F=\phi(\tilde{S}_F)$ $(\subset L_F)$. For a smooth solution $\lambda(x,y)$ of Eq. (2.4), we set

$$Q_{\lambda(x,y)} = \{\phi(x,y,\lambda(x,y)) : (x,y)\in K\} \tag{2.6}$$

Lemma 1 *(1) $L_F\cap\tau^{-1}(K-\Sigma_K)$ is a Lagrangian submanifold of $(T\mathbb{R}^{2n},\dot{\omega})$.*
(2) The restriction of ϕ to the regular point set of K

$$\phi\mid_{(K-\Sigma_K)\times\mathbb{R}^k} : (K-\Sigma_K)\times\mathbb{R}^k\to L_F\cap\tau^{-1}(K-\Sigma_K)$$

is a diffeomorphism, and for a point $(x,y)\in K-\Sigma_K$, the restricted map

$$\phi\mid_{\{(x,y)\}\times\mathbb{R}^k} : \{(x,y)\}\times\mathbb{R}^k\to L_F\cap\tau^{-1}((x,y))$$

is an affine isomorphism with $\phi(x,y,0) = (x,y,\frac{\partial b}{\partial x}(x,y), -\frac{\partial b}{\partial y}(x,y))$.

(3) Let $(x, y) \in \Sigma_K$ *and let* $\operatorname{rank}\left(\frac{\partial a_i}{\partial x_j}(x, y),\ \frac{\partial a_i}{\partial y_j}(x, y)\right) = r < k$. *Then,*

$$\phi\mid_{\{(x,y)\}\times\mathbb{R}^k}: \{(x, y)\} \times \mathbb{R}^k \to L_F \cap \tau^{-1}((x, y)) \subset T_{(x,y)}\mathbb{R}^{2n}$$

is an affine map of rank r with $\phi(x, y, 0) = (x, y, \frac{\partial b}{\partial x}(x, y), -\frac{\partial b}{\partial y}(x, y))$.

Definition 1 We say that a set of smooth functions $(\lambda_1(x, y), \cdots, \lambda_k(x, y))$ in the neighbourhood of $(0, 0)$ is a smooth solution of (2.4) defined on K if it satisfies Eq. (2.4) at every point (x, y) of K.

Definition 2 Let Q be a subset of L_F. A solution of Q is a C^1-curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^{2n}$ such that $(\gamma(t), \frac{d\gamma}{dt}(t)) \in Q,\ -\epsilon < t < \epsilon$.

In what follows we often abbreviate $(x, y, \dot{x}, \dot{y}) \in T\mathbb{R}^{2n}$ to $(q, \dot{q})$.

Definition 3 Let Q be a subset of L_F. A point $(q, \dot{q}) \in Q$ is a solvable point of Q if there exists a solution γ of Q such that

$$\gamma(0) = q, \quad \frac{d\gamma}{dt}(0) = \dot{q}.$$

Q is solvable if Q consists only of solvable points.

Next we define the notion of smooth solvability of subsets of L_F. First we give a definition for the case where K is a smooth submanifold of $\mathbb{R}^{2n}$ and Q is a smooth submanifold of L_F. Note that in this case, by Lemma 1, L_F is a submanifold of $T\mathbb{R}^{2n}$ and $\phi : K \times \mathbb{R}^k \to L_F$ is a diffeomorphism.

Definition 4 Suppose that K is a smooth submanifold of $\mathbb{R}^{2n}$ and Q is a smooth submanifold of L_F. Let

$$\tilde{Q} = \phi^{-1}(Q) \subset K \times \mathbb{R}^k.$$

Note that in this case ϕ is a diffeomorphism and $\tilde{Q}$ is a smooth manifold. We say that a point $(q_0, \dot{q}_0) \in Q$ with $(q_0, \dot{q}_0) = \phi(q_0, \lambda_0), \quad (q_0, \lambda_0) \in \tilde{Q}$ is a smoothly solvable point of Q if there exists
a small neighbourhood $\tilde{W}$ of (q_0, λ_0) in $\tilde{Q}$, a positive number $\epsilon > 0$ and a smooth map

$$\tilde{\gamma} : \tilde{W} \times (-\epsilon, \epsilon) \to \mathbb{R}^{2n}$$

such that for every $(q, \dot{q}) \in \phi(\tilde{W})\ (\subset Q)$ with $(q, \dot{q}) = \phi(q, \lambda),\ (q, \lambda) \in \tilde{W}$, the curve $\gamma_{(q,\lambda)} : (-\epsilon, \epsilon) \to \mathbb{R}^{2n}$ defined by

$$\gamma_{(q,\lambda)}(t) := \tilde{\gamma}(q, \lambda, t)$$

is a smooth solution of Q with initial condition

$$(\gamma_{(q,\lambda)}(0), \frac{d\gamma_{(q,\lambda)}}{dt}(0)) = (q, \dot{q}) = \phi(q, \lambda).$$

We say that Q is smoothly solvable if Q consists only of its smoothly solvable points.

If K is singular, for a definition of smooth solvability of subsets of L_F, we consider subsets Q of L_F such that

$$Q \cap \tau^{-1}(K - \Sigma_K) \text{is dense in } Q. \tag{2.7}$$

For such a subset Q, let $\tilde{Q}$ be a subset of $\phi^{-1}(Q)$ $(\subset K \times \mathbb{R}^k)$ such that

$$\phi(\tilde{Q}) = Q \quad \text{and} \quad \tilde{Q} = \overline{\tilde{Q} \cap ((K - \Sigma_K) \times \mathbb{R}^k)} \tag{2.8}$$

where for a subset B of $\mathbb{R}^{2n} \times \mathbb{R}^k$, $\overline{B}$ denotes the topological closure of B in $\mathbb{R}^{2n} \times \mathbb{R}^k$. Since $K - \Sigma_K$ is dense in K, such $\tilde{Q}$ is uniquely determined.

Definition 5 (General case) Let Q be a subset of L_F satisfying (2.7) and let $\tilde{Q}$ be a subset of $\phi^{-1}(Q)$ satisfying (2.8). We say that a point $(q_0, \dot{q}_0) \in Q$ with $(q_0, \dot{q}_0) = \phi(q_0, \lambda_0)$, $(q_0, \lambda_0) \in \tilde{Q}$ is a *smoothly solvable point* of Q if there exist a submanifold $\tilde{N}$ of $\mathbb{R}^{2n} \times \mathbb{R}^k$ and a small neighbourhood $\tilde{W}$ of (q_0, λ_0) in $\mathbb{R}^{2n} \times \mathbb{R}^k$ with $\tilde{Q} \cap \tilde{W} \subset \tilde{N}$, a positive number $\epsilon > 0$ and a smooth map

$$\tilde{\gamma} : (\tilde{N} \cap \tilde{W}) \times (-\epsilon, \epsilon) \to \mathbb{R}^{2n},$$

such that for every $(q, \dot{q}) \in Q \cap \phi(\tilde{W})$ with $(q, \dot{q}) = \phi(q, \lambda)$, $(q, \lambda) \in \tilde{Q} \cap \tilde{W}$, the curve $\gamma_{(q,\lambda)} : (-\epsilon, \epsilon) \to \mathbb{R}^{2n}$ defined by $\gamma_{(q,\lambda)}(t) = \tilde{\gamma}(q, \lambda, t)$ is a solution of Q satisfying the initial condition

$$(\gamma_{(q,\lambda)}(0), \frac{d\gamma_{(q,\lambda)}}{dt}(0)) = (q, \dot{q}) = \phi(q, \lambda).$$

We say that Q is smoothly solvable if Q consists only of its smoothly solvable points.

Let $L_F(\subset T\mathbb{R}^{2n})$ be a generalized Hamiltonian system on K generated by a Morse family

$$F(x, y, \lambda) = \sum_{i=1}^{k} a_i(x, y)\lambda_i + b(x, y), \qquad F : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R} \tag{2.9}$$

with $\{a_i(x, y), 1 \leq i \leq k\}$ functions defining K no longer submanifold of $\mathbb{R}^{2n}$ (as it was in (1.2) and a smooth function $b : \mathbb{R}^{2n} \to \mathbb{R}$.

Main Theorem.

Suppose that

(1) the rank of the matrix $(\{a_i(x, y), a_j(x, y)\})$ is constant on K_r,
(2) the set of regular points $K - \Sigma_K$ is dense in K and
(3) for every point $(x, y) \in K_r$

$$\begin{pmatrix} \{b, a_1\}(x, y) \\ \vdots \\ \{b, a_k\}(x, y) \end{pmatrix} \in (\{a_i, a_j\}(x, y)) \left(\mathbb{R}^k\right). \tag{2.10}$$

Then, S_F is smoothly solvable. Moreover, any smoothly solvable subset of L_F is a subset of S_F.

Let us notice that the second assertion of Main Theorem is a consequence of the following fact,

Proposition 2 *Any solvable subset of L_F is a subset of S_F without any assumptions made on K and the matrix $(\{a_i(x, y), a_j(x, y)\})$.*

Proof Let Q be solvable subset of L_F. Suppose that $(x, y, \dot{x}, \dot{y}) = \phi(x, y, \lambda)$ is a solvable point of Q. Then, there exists a solution $\gamma : (-\epsilon, \epsilon) \to K$ $(\subset \mathbb{R}^{2n})$ of Q such that $(x, y, \dot{x}, \dot{y}) = (\gamma(0), d\gamma/dt(0))$. Since $\gamma(t) \in K$ for all $t \in (-\epsilon, \epsilon)$, we have $a_\ell(\gamma(t)) = 0, \quad \ell = 1, \cdots, k$. Therefore

$$d(a_\ell(\gamma(t))/dt = \sum_{i=1}^{n} \frac{\partial a_\ell}{\partial x_i}(\gamma(t))\frac{d\gamma_i}{dt}(t) + \frac{\partial a_\ell}{\partial y_i}(\gamma(t))\frac{d\gamma_{n+i}}{dt}(t) = 0.$$

Thus,

$$\sum_{i=1}^{n} \frac{\partial a_\ell}{\partial x_i}(x, y)\dot{x}_i + \frac{\partial a_\ell}{\partial y_i}(x, y)\dot{y}_i = 0.$$

On the other hand, since $(x, y, \dot{x}, \dot{y}) = \phi(x, y, \lambda)$, we can write

$$\dot{x}_i = \frac{\partial F}{\partial y_i}(x, y, \lambda) = \sum_{m=1}^{k} \frac{\partial a_m}{\partial y_i}(x, y)\lambda_m + \frac{\partial b}{\partial y_i}(x, y)$$

$$\dot{y}_i = -\frac{\partial F}{\partial x_i}(x, y, \lambda) = -\sum_{m=1}^{k} \frac{\partial a_m}{\partial x_i}(x, y)\lambda_m - \frac{\partial b}{\partial x_i}(x, y).$$

Then using these equations, we have

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \frac{\partial a_\ell}{\partial x_i}(x, y)\dot{x}_i + \frac{\partial a_\ell}{\partial y_i}(x, y)\dot{y}_i \\ &= \sum_{i=1}^{n} \frac{\partial a_\ell}{\partial x_i}(x, y)\left(\sum_{m=1}^{k} \frac{\partial a_m}{\partial y_i}(x,y)\lambda_m + \frac{\partial b}{\partial y_i}(x, y)\right) \\ &\quad - \sum_{i=1}^{n} \frac{\partial a_\ell}{\partial y_i}(x, y)\left(\sum_{m=1}^{k} \frac{\partial a_m}{\partial x_i}(x, y)\lambda_m + \frac{\partial b}{\partial x_i}(x, y)\right) \\ &= -\sum_{m=1}^{k} \{a_\ell, a_m\}(x, y)\lambda_m - \{a_\ell, b\}(x, y) \end{aligned}$$

Thus, $(\lambda_1, \cdots, \lambda_k)$ is a solution of (2.4) and $(x, y, \dot{x}, \dot{y}) = \phi(x, y, \lambda) \in S_F$. □

Example 1 Let us consider the cuspidal surface,

$$K = \{(x, y) \in \mathbb{R}^4 \mid a_1(x, y) = y_2, a_2(x, y) = y_1^3 - x_2^2\}$$
$$K_1 = \{(x, y) \in \mathbb{R}^4 \mid a_1(x, y) = y_2\}.$$

We have

$$\{a_1, a_2\} = -2x_2,$$
$$\{b, a_1\} = -\frac{\partial b}{\partial x_2},$$
$$\{b, a_2\} = -2x_2\frac{\partial b}{\partial y_2} - 3y_1^2\frac{\partial b}{\partial x_1}$$

with the structure Eq. (2.4),

$$-2x_2\lambda_2 = -\frac{\partial b}{\partial x_2}, \tag{2.11}$$

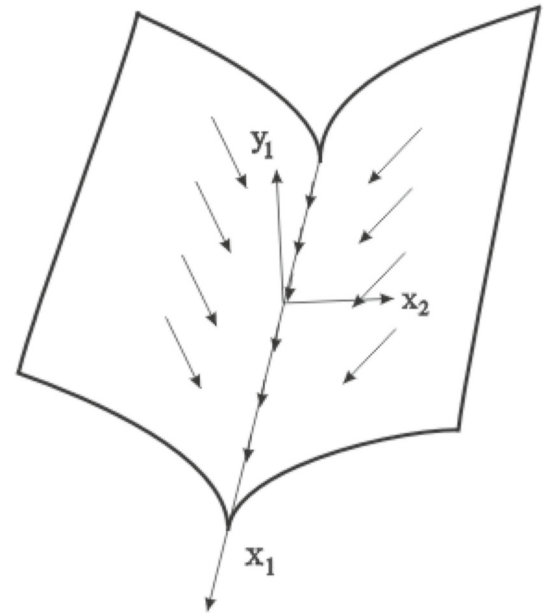

Fig. 1 $K \subset K_1$

$$2x_2\lambda_1 = -2x_2\frac{\partial b}{\partial y_2} - 3y_1^2\frac{\partial b}{\partial x_1}. \tag{2.12}$$

Let $L_F \subset T\mathbb{R}^4$ be generated by a family

$$F(x, y, \lambda) = y_2\lambda_1 + (y_1^3 - x_2^2)\lambda_2 + x_2^2x_1 + y_1,$$
$$L_F = \{(3y_1^2\lambda_2 + 1)\frac{\partial}{\partial x_1} + \lambda_1\frac{\partial}{\partial x_2} - x_2^2\frac{\partial}{\partial y_1} + (2x_2\lambda_2 - 2x_2x_1)\frac{\partial}{\partial y_2} \mid (\lambda_1, \lambda_2) \in \mathbb{R}^2\}.$$

We get

$$\lambda_1 = -\frac{3}{2}y_1^2x_2, \quad \lambda_2 = x_1$$

as a unique smooth solution of Eq. (2.4). Thus, the Hamiltonian vector field

$$S_F = (1 + 3y_1^2x_1)\frac{\partial}{\partial x_1} - \frac{3}{2}y_1^2x_2\frac{\partial}{\partial x_2} - x_2^2\frac{\partial}{\partial y_1}$$

is tangent to K (see figure 1.).

As the special cases of Main Theorem, we have the following two theorems

Theorem 1 *Let $k \leq n$. Suppose that the set of regular points is dense in K. Then, L_F is smoothly solvable if and only if*

$$\{a_i, a_j\} = 0 \quad \text{and} \quad \{b, a_i\} = 0 \quad \text{on } K, \quad 1 \leq i, j \leq k. \tag{2.13}$$

Proof If (2.13) holds, then all $\lambda \in \mathbb{R}^k$ are solutions of (2.4) and we have $L_F = S_F$. Then from Main Theorem, L_F is smoothly solvable. Conversely if L_F is smoothly solvable, then by Proposition 2, $L_F = S_F$. Hence, all $\lambda \in \mathbb{R}^k$ are solutions of (2.4), which happens only if $\{a_i, a_j\} = 0$ on K, what defines the coisotropic leafs (Lagrangian in the case if $k = n$) and the condition $\{b, a_i\} = 0$ on $K \quad 1 \leq i, j \leq k$ gives the constancy of b on leafs. ◻

Theorem 2 *Suppose that* $\det(\{a_\ell, a_m\}(x, y)) \neq 0$ *on K. Then,*

(1) k is even and K is a smooth submanifold of $\mathbb{R}^{2n}$ *of codimension k,*
(2) there exists a unique smooth solution $\lambda(x, y)$ *of (2.4) defined on K and* $S_F = Q_{\lambda(x,y)}$*. Moreover,* $S_F = Q_{\lambda(x,y)}$ *is a unique smoothly solvable subset of L such that* $\tau(Q) = K$*.*

Proof Since $(\{a_\ell, a_m\}(x, y))$ is a skew-symmetric matrix, if $\det(\{a_\ell, a_m\}(x, y)) \neq 0$, then the rank of $(\{a_\ell, a_m\}(x, y))$ is the size k of $(\{a_\ell, a_m\}(x, y))$ and it is even. Moreover, since

$$(\{a_\ell, a_m\}(x, y)) = \begin{pmatrix} \frac{\partial a_i}{\partial x_j} & \frac{\partial a_i}{\partial y_j} \end{pmatrix} \begin{pmatrix} O & I_n \\ -I_n & O \end{pmatrix}^t \begin{pmatrix} \frac{\partial a_i}{\partial x_j} & \frac{\partial a_i}{\partial y_j} \end{pmatrix}$$

and since the rank of $(\{a_\ell, a_m\}(x, y))$ is k, rank $\begin{pmatrix} \frac{\partial a_i}{\partial x_j} & \frac{\partial a_i}{\partial y_j} \end{pmatrix} = k$. Therefore, K is a smooth submanifold of $\mathbb{R}^{2n}$ of codimension k. Next, since $(\{a_\ell, a_m\}(x, y)$ has the maximal rank k, there exists a unique smooth solution $\lambda(x, y)$ of (2.4) defined on K and $S_F = Q_{\lambda(x,y)}$. Since $\tau \mid_{Q_{\lambda(x,y)}} : Q_{\lambda(x,y)} \to K$ is a diffeomorphism, by Proposition 1, $S_F = Q_{\lambda(x,y)}$ is a unique smoothly solvable subset of L_F such that $\tau(Q) = K$. □

3 Solutions of solvability equations

In this section, we check the existence of smooth solutions of Eq. (2.4) and get results which we use in the proof of the Main Theorem.

From now on throughout this section, we suppose that $(0, 0) \in K$ and the rank of the matrix $(\partial a_i/\partial x_j(0, 0)\ \partial a_i/\partial y_j(0, 0))$ is r. First we investigate the bundle structure of the kernels and images of $A(x, y) = (\{a_i, a_j\}(x, y))$.

For a small neighbourhood U of $(0, 0)$, $K_r \cap U$ is a codimension r smooth submanifold of $\mathbb{R}^{2n}$ and $K \cap U \subset K_r \cap U$. Considering $A(x, y)$ as a linear mapping, we set

$$\begin{aligned} A_{K_r\cap U} &= \bigcup_{(x,y)\in K_r\cap U} \{(x, y)\} \times A(x, y)(\mathbb{R}^k) \\ N_{(x,y)} &= \text{ the kernel of } A(x, y) = \{\lambda \in \mathbb{R}^k \mid A(x, y)\lambda = 0\} \\ N_{K_r\cap U} &= \bigcup_{(x,y)\in K_r\cap U} \{(x, y)\} \times N_{(x,y)} \\ N_{K\cap U} &= \bigcup_{(x,y)\in K\cap U} \{(x, y)\} \times N_{(x,y)} \end{aligned}$$

Proposition 3 *Suppose that*

rank of $(\{a_i, a_j\}(x, y))$ *is constant and equal s on* $K_r \cap U$*.*

Then,

(1) $A_{K_r\cap U}$ *is a smooth vector bundle over* $K_r \cap U$ *of rank s.*
(2) $A_{K\cap U}$ *is a topological vector bundle over* $K \cap U$ *of rank s.*
(3) $N_{K_r\cap U}$ *is a smooth vector bundle over* $K_r \cap U$ *of rank* $k - s$*.*
(4) $N_{K\cap U}$ *is a topological vector bundle over* $K \cap U$ *of rank* $k - s$*.*

Proof (1) Since the rank of $(\{a_i, a_j\}(x, y))$ is constant and equal s on $K_r \cap U$, $A_{K_r\cap U}$ is a vector bundle over $K_r \cap U$ of rank s. It may be smooth or not. Choosing U small enough, we may assume that there exist vectors $e_1, \cdots, e_s \in \mathbb{R}^k$ such that $\{A(x, y)e_1, \cdots, A(x, y)e_s\}$ is a basis of $A(x, y)(\mathbb{R}^k)$ and they depend smoothly on $(x, y) \in K_r \cap U$. Thus, $A_{K_r\cap U}$ is a smooth vector bundle over $K_r \cap U$ of rank r.

(2) This case is a direct corollary of 1). The smoothness is spoiled on Σ_K.

(3) Let $e_1, \cdots, e_s \in \mathbb{R}^k$ be the same vectors as in the proof of 1). Let $e_{s+1}, \cdots, e_k \in \mathbb{R}^k$ be vectors such that $\{e_1, \cdots, e_s, e_{s+1}, \cdots, e_k\}$ is a basis of $\mathbb{R}_k$. Since $\{A(x, y)e_1, \cdots, A(x, y)e_s\}$ is a basis of $A(x, y)(\mathbb{R}^k)$, then $Ae_{s+j}(x, y)$ is a linear combination of $\{A(x, y)e_1, \cdots, A(x, y)e_s\}$:

$$A(x, y)e_{s+i}(x, y) = \sum_{j=1}^{s} \alpha_{s+i,j}(x, y)A(x, y)e_j.$$

Then, $\{e_{s+1} - \sum_{j=1}^{s} \alpha_{s+1,j}(x, y)e_j, \cdots, e_k - \sum_{j=1}^{s} \alpha_{k,j}(x, y)e_j\}$ is a basis of $N_{(x,y)}$ and they depend smoothly on $(x, y) \in K_r \cap U$. Thus, $N_{K_r \cap U}$ is a smooth vector bundle over $K_r \cap U$ of rank $k - s$.

(4) This case is a direct corollary of 3). The smoothness is spoiled on Σ_K.

□

Let us recall the definitions of $\tilde{S}_F$ and S_F (see (2.5)) and set

$$\begin{aligned}
\tilde{S}_F\ |_{K\cap U} &= \{(x, y, \lambda) \in (K \cap U) \times \mathbb{R}^k : (x, y, \lambda) \in \tilde{S}_F\}\\
S_F\ |_{K\cap U} &= \phi(\tilde{S}_F\ |_{K\cap U})\ (\subset L)\\
\tilde{S}_F\ |_{K_r\cap U} &= \{(x, y, \lambda) \in (K_r \cap U) \times \mathbb{R}^k : (x, y, \lambda) \in \tilde{S}_F\}\\
S_F\ |_{K_r\cap U} &= \phi(\tilde{S}_F\ |_{K_r\cap U})\ (\subset L)
\end{aligned}$$

As a corollary of Proposition 3 we have,

Proposition 4 *(Relation of $\tilde{S}_F$ and $N_{K\cap U}$)*
Suppose that

$$\text{rank}\ (\{a_i, a_j\}(x, y))\ \text{ is constant and equal to } s \text{ on } K_r \cap U.$$

Let $\lambda(x, y)$ be a smooth solution of (2.4) defined on $K_r \cap U$. Then

$$\tilde{S}_F\ |_{K\cap U} = \lambda + N_{K\cap U} := \bigcup_{(x,y)\in K\cap U} \{(x, y)\} \times (\lambda(x, y) + N(x, y)),$$

$$\tilde{S}_F\ |_{K_r\cap U} = \lambda + N_{K_r\cap U} := \bigcup_{(x,y)\in K_r\cap U} \{(x, y)\} \times (\lambda(x, y) + N(x, y)).$$

Proof Let $(x, y, \mu) \in \tilde{S}_F\ |_{K\cap U}$, then by definition

$$(\{a_i(x, y), a_j(x, y)\})\mu = \{b, a\}(x, y) :=^t (\{b, a_1\}(x, y), \ldots, \{b, a_k\}(x, y)).$$

Since $\lambda(x, y)$ is a solution of (2.4) defined on $K_r \cap U$, then

$$(\{a_i(x, y), a_j(x, y)\})\lambda(x, y) = \{b, a\}(x, y).$$

Therefore, $(\{a_i(x, y), a_j(x, y)\})(\lambda(x, y) - \mu) = \{b, a\}(x, y) = 0$ and $(x, y, \lambda(x, y) - \mu) \in \lambda + N_{K\cap U}$. This proves $\tilde{S}_F\ |_{K\cap U} \subset \lambda + N_{K\cap U}$. The opposite inclusion can be proved similarly. The proof of $\tilde{S}_F\ |_{K_r\cap U} = \lambda + N_{K_r\cap U}$ is the same. □

Lemma 5 *Let $a_1(x), \cdots, a_k(x)$ be smooth functions on $\mathbb{R}^m$ such that the variety $K = \{(x \in \mathbb{R}^m \mid a_\ell(x) = 0,\ \ell = 1, \ldots, k\}$ is smooth or the set of regular points of K is dense in K. Let X be a smooth vector field on $\mathbb{R}^m$. If*

$$Xa_i(x) = 0,\ i = 1, \ldots, k,\ \text{ on } K,$$

Then, the integral curves of X preserve K.

Proof In the case if K is a smooth submanifold of $\mathbb{R}^m$ the condition $Xa_i(x) = 0, i = 1, \ldots, k,$ on K implies that X is tangent to K and we may regard X as a smooth tangent vector field on K. Therefore, integral curves of X preserve K.

To prove the Lemma in singular case, it suffices to prove that for any point $x \in K$ there exists a small number $\epsilon > 0$ such that for an integral curve $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^m$ of X with $\gamma(0) = x$, we have $\gamma(-\epsilon, \epsilon) \subset K$.

Let Σ_K denote the singular point set of K. For a point $x \in \mathbb{R}^m$, let $\gamma_x(t)$ denote an integral curve of X with $\gamma_x(0) = x$. Now we have two cases

(1) $x_0 \in K - \Sigma_K$. In this case, from the argument workout in smooth case, there exists a small number $\epsilon > 0$ such that for an integral curve $\gamma_{x_0} : (-\epsilon, \epsilon) \to \mathbb{R}^m$ of X with $\gamma_{x_0}(0)$, we have $\gamma(-\epsilon, \epsilon) \subset K$.

(2) $x_0 \in \Sigma_K$. In this case let $\gamma_{x_0} : (-\epsilon_0, -\epsilon_0) \to \mathbb{R}^m$ be an integral curve of X such that $\gamma_{x_0}(0) = x_0$. Suppose that there no exist small numbers $\epsilon > 0$ with $0 < \epsilon < \epsilon_0$ such that $\gamma(-\epsilon, \epsilon) \subset K$. Then, there exists a series of real numbers $\{t_\ell \mid \ell \in \mathbb{N}\}$ such that

$$\lim_{\ell \to \infty} t_\ell = 0 \quad \text{and} \quad \gamma_{x_0}(t_\ell) \notin K.$$

We take $\ell_0 \in \mathbb{N}$ large enough. Then, $\gamma_{x_0}(t_{\ell_0}) \notin K$. Let $B(x, \delta)$ denote a δ-neighbourhood of x in $\mathbb{R}^m$:

$$B(x, \delta) := \{y \in \mathbb{R}^m \mid \|y - x\| < \delta\}.$$

Since $\mathbb{R}^m - K$ is open in $\mathbb{R}^m$, there exists $\delta_1 > 0$ such that

$$B(\gamma_{x_0}(t_{\ell_0}), \delta_1) \subset \mathbb{R}^m - K.$$

Then, by the fundamental theorem of ordinary differential equations, there exists $\delta_2 > 0$ such that

$$\text{if} \quad x \in B(x_0, \delta_2) \quad \text{then} \quad \gamma_x(t_{\ell_0}) \in B(\gamma_{x_0}(t_{\ell_0}), \delta_1).$$

Since $K - \Sigma_K$ is dense in K, there exists a point $x \in B(x_0, \delta_2) \cap (K - \Sigma_K) \neq \emptyset$. Then

$$\gamma_x(t_{\ell_0}) \in B(\gamma_{x_0}(t_{\ell_0}), \delta_1) \subset \mathbb{R}^m - K.$$

Since $x \in K - \Sigma_K$, this contradicts the assertion of the first case, for which we can take t_{ℓ_0} arbitrarily small. Thus, there exists a small number ϵ with $0 < \epsilon < \epsilon_0$ such that $\gamma_{x_0}(-\epsilon, -\epsilon) \subset K$.

□

Proposition 6 *Suppose that the set of regular points of K is dense in K. Let $(x_0, y_0) \in K$ and U be a neighbourhood of (x_0, y_0) in $\mathbb{R}^{2n}$. If the linear Eq. (2.4) has a smooth solution $\lambda(x, y) = (\lambda_1(x, y), \ldots, \lambda_k(x, y))$ defined on $K_r \cap U$, then the set*

$$Q_{\lambda(x,y)} = \{(x, y, \frac{\partial F}{\partial y}(x, y, \lambda(x, y)), -\frac{\partial F}{\partial x}(x, y, \lambda(x, y)) \mid (x, y) \in K \cap U\} \; (\subset L_F)$$

is smoothly solvable.

Proof Let $\lambda(x, y) = (\lambda_1(x, y), \ldots, \lambda_k(x, y))$ be a smooth solution of (2.4) defined on $K_r \cap U$. Consider the following vector field

$$\begin{aligned} X_{\lambda(x,y)} &= \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(x, y, \lambda(x, y)) \frac{\partial}{\partial x_i} - \frac{\partial F}{\partial x_i}(x, y, \lambda(x, y)) \frac{\partial}{\partial y_i} \\ &= \sum_{i=1}^{n} (\sum_{\ell=1}^{k} \frac{\partial a_\ell}{\partial y_i}(x, y)\lambda_\ell(x, y) + \frac{\partial b}{\partial y_i}(x, y)) \frac{\partial}{\partial x_i} \\ &\quad - \sum_{i=1}^{n} (\sum_{\ell=1}^{k} \frac{\partial a_\ell}{\partial x_i}(x, y)\lambda_\ell(x, y) + \frac{\partial b}{\partial x_i}(x, y)) \frac{\partial}{\partial y_i} \end{aligned}$$

on $K_r \cap U$. Since $\lambda(x, y) = (\lambda_1(x, y), \ldots, \lambda_k(x, y))$ is a solution of (2.4) defined on $K_r \cap U$, we see that $X_{\lambda(x,y)}(a_\ell) = 0$ on $K_r \cap U$ for $\ell = 1, \cdots, k$. Therefore, $X_{\lambda(x,y)}$ is a smooth tangent vector field on $K_r \cap U$. Now choosing U small, we may assume that $K_r \cap U$ is diffeomorphic to a Euclidean space. Therefore, we can apply Lemma 5 to this situation and we see that the integral curves of $X_{\lambda(x,y)}$ preserve K. Let us define

$$\tilde{N} = \{(x, y, \lambda(x, y)) \mid (x, y) \in K_r \cap U\}.$$

Then, $\tilde{N}$ is a $2n - r$ dimensional submanifold of $\mathbb{R}^{2n} \times \mathbb{R}^k$. Taking U small enough, if necessary, there exists a positive number $\epsilon > 0$ such that for every $(x, y) \in K_r \cap U$, there exists an integral curve

$$\gamma_{(x,y)} : (-\epsilon, \epsilon) \to \mathbb{R}^{2n} \text{of } X_{\lambda(x,y)} \text{such that } \gamma_{(x,y)}(0) = (x, y).$$

Consider the map

$$\tilde{\gamma} : \tilde{N} \times (-\epsilon, \epsilon) \to \mathbb{R}^{2n} \quad \text{defined by} \quad \tilde{\gamma}((x, y, \lambda(x, y), t) = \gamma_{(x,y)}(t).$$

Since $X_{\lambda(x,y)}$ is a smooth vector field on $K_r \cap U$, $\tilde{\gamma} : \tilde{N} \times (-\epsilon, \epsilon) \to \mathbb{R}^{2n}$ is a smooth map. Thus, for a point $(q, \dot{q}) = \phi(q, \lambda(q)) \in Q_{\lambda(x,y)}$, the curve

$$\gamma_{(q,\lambda(q))} : (-\epsilon, \epsilon) \to \mathbb{R}^{2n} \quad \text{defined by} \quad \gamma_{(q,\lambda(q))}(t) = \tilde{\gamma}(q, \lambda, t) = \gamma_q(t)$$

is a solution of L_F with initial condition

$$(\gamma_{(q,\lambda(q))}(0), \frac{d\gamma_{(q,\lambda(q))}}{dt}(0)) = (q, \dot{q}) = \phi(q, \lambda(q)).$$

From the definition of $Q_{\lambda(x,y)}$ and the fact that $\gamma_{(x,y)}(t)$ is an integral curve of $X_{\lambda(x,y)}$, we see that $(\gamma_{(x,y)}(t), d\gamma_{(x,y)}/dt(t)) \in Q_{\lambda(x,y)}$. Hence, $\gamma_{(q,\lambda(q))}$ is a solution of $Q_{\lambda(x,y)}$. Now by Definition 5, $(q, \dot{q}) = \phi(q, \lambda(q))$ $(\in Q_{\lambda(x,y)})$ is a smoothly solvable point of $Q_{\lambda(x,y)}$. Thus, $Q_{\lambda(x,y)}$ is smoothly solvable. □

Proposition 7 *Let U be a small neighbourhood of* $(0, 0)$ *in* $\mathbb{R}^{2n}$. *Suppose that*

(1) $(\{b, a_1\}(x, y), \cdots, \{b, a_k\}(x, y)) \in A(x, y)(\mathbb{R}^k)$ for every $(x, y) \in K_r \cap U$ and
(2) the rank of $(\{a_i, a_j\}(x, y))$ is constant and equal s on $K_r \cap U$.

Then, Eq. (2.4) has a smooth solution defined on $K_r \cap U$.

Proof Suppose that the rank of $(\{a_i, a_j\}(x, y))$ is constant and equal s on $K_r \cap U$. Then by Proposition 3, $A_{K_r \cap U}$ is a smooth vector bundle of rank s on $K_r \cap U$. Choosing U small enough and changing the order of the standard normal vectors $e_1 = (1, 0, \cdots, 0), \cdots, e_k =$

$(0, \cdots, 0, 1)$, we may assume that the image of the first s vectors $\{A(x, y)e_1, \cdots, A(x, y)e_s\}$ is a basis of $A(x, y)(\mathbb{R}^k)$.

Since $(\{b, a_1\}(x, y), \cdots, \{b, a_k\}(x, y)) \in A(x, y)(\mathbb{R}^k)$ for every $(x, y) \in K_r \cap U$, it is a linear combination of $A(x, y)e_1, \cdots, A(x, y)e_s$;

$$(\{b, a_1\}(x, y), \cdots, \{b, a_k\}(x, y\}) = \sum_{i=1}^{s} \lambda_i(x, y)A(x, y)(e_i).$$

Since $\{b, a_1\}(x, y), \cdots, \{b, a_k\}(x, y)$ are all smooth on $K_r \cap U$ and $A(x, y)e_1, \cdots, A(x, y)e_s$ are linearly independent, the coefficients $\lambda_1(x, y), \cdots, \lambda_s(x, y)$ are smooth functions. Then, $\lambda(x, y) = (\lambda_1(x, y), \cdots, \lambda_s(x, y), 0, \cdots, 0)$ is a smooth solution of (2.4) defined on $K_r \cap U$. □

For a smooth solution $\lambda(x, y)$ of (2.4) defined on $U \cap K_r$, let $X_{\lambda(x,y)}$ denote a vector field defined by

$$X_{\lambda(x,y)} = \sum_{j=1}^{n} \left(\frac{\partial b}{\partial y_j}(x, y) + \sum_{\ell=1}^{k} \frac{\partial a_\ell}{\partial y_j}(x, y)\lambda_\ell(x, y) \right) \frac{\partial}{\partial x_j} - \sum_{j=1}^{n} \left(\frac{\partial b}{\partial x_j}(x, y) + \sum_{\ell=1}^{k} \frac{\partial a_\ell}{\partial x_j}(x, y)\lambda_\ell(x, y) \right) \frac{\partial}{\partial y_j}.$$

Proposition 8 *Let $\lambda(x, y)$ be a smooth solution of (2.4) defined on $U \cap K_r$. Then, the following properties hold:*

(1) The vector field $X_{\lambda(x,y)}$ is a smooth tangent vector field on $U \cap K_r$ and its integral curves $\gamma(t)$ are solutions of $S_F\ |_{K_r \cap U}$. Moreover, integral curves of $X_{\lambda(x,y)}$ with $\gamma(0) \in K$ are solutions of $S_F\ |_{K \cap U}$.

(2) For a point $(x_0, y_0) \in K \cap U$, let $(x_0, y_0, \dot{x}_0, \dot{y}_0) = \phi(x_0, y_0, \lambda(x_0, y_0))$. Then, an integral curve $\gamma(t)$ of $X_{\lambda(x,y)}$ such that $\gamma(0) = (x_0, y_0)$ is a solution of $S_F\ |_{K \cap U}$ satisfying

$$(\gamma(0), d\gamma/dt(0)) = (x_0, y_0, \dot{x}_0, \dot{y}_0).$$

(3) Let $(x_0, y_0, \dot{x}_0, \dot{y}_0) = \phi(x_0, y_0, \lambda_0) \in S_F\ |_{K \cap U}$ and let $\mu(x, y)$ be a smooth section of the smooth bundle $N_{U \cap K_r}$ such that $\lambda(x_0, y_0) + \mu(x_0, y_0) = \lambda_0$. Then,

(3-1) $(\lambda + \mu)(x, y)$ is a smooth solution of (2.4) defined on $K_r \cap U$ and

(3-2) there is an integral curve $\gamma(t)$ of the vector field $X_{(\lambda+\mu)(x,y)}$, which is also a solution of $S_F\ |_{K \cap U}$, such that $(\gamma(0), d\gamma/dt(0)) = (x_0, y_0, \dot{x}_0, \dot{y}_0)$.

Proof The first assertion 1) follows already from Proposition 6. Since $X(a_i) = 0$ for $i = 1, \cdots, k$ on K, from Lemma 5, integral curves of $X_{\lambda(x,y)}$ with $\gamma(0) \in K$ preserve K. Hence, they are solutions of $S_F\ |_{K \cap U}$. The fact that the integral curve $\gamma(t)$ of $X_{\lambda(x,y)}$ such that $\gamma(0) = (x_0, y_0)$ is a solution of $S_F\ |_{K \cap U}$ is already proved in the proof of Proposition 6. Now $(x_0, y_0, \dot{x}_0, \dot{y}_0) = \phi(x_0, y_0, \lambda(x_0, y_0))$ and $\gamma(t)$ is an integral curve of $X_{\lambda(x,y)}$ with $\gamma(0) = (x_0, y_0)$. Therefore, we have

$$(\gamma(0), d\gamma/dt(0)) = (x_0, y_0, X_{\lambda(x,y)}(x_0, y_0)) = \phi(x_0, y_0, \lambda(x_0, y_0)) = (x_0, y_0, \dot{x}_0, \dot{y}_0),$$

which proves assertion 2) of the proposition.

For assertion 3), let $\lambda(x, y)$ be a smooth solution of (2.4) defined on $K_r \cap U$ and let $\mu(x, y)$ be a smooth cross section of the kernel bundle $N_{U \cap K_r}$ such that $\lambda(x_0, y_0) + \mu(x_0, y_0) = \lambda_0$. Then, (3-1) is obvious. Now let $\gamma(t)$ be an integral curve of $X_{\lambda(x,y)+\mu(x,y)}$ with $\gamma(0) =$

$(x_0, y_0) \in K$. Since $\lambda(x_0, y_0) + \mu(x_0, y_0) = \lambda_0$, Applying 2) to this integral curve $\gamma(t)$ of $X_{\lambda(x,y)+\mu(x,y)}$, we see that $\gamma(t)$ is a solution of $S_F \mid_{K\cap U}$ such that

$$(\gamma(0), d\gamma/dt(0)) = \phi(x_0, y_0, \lambda(x_0, y_0) + \mu(x_0, y_0)) = \phi(x_0, y_0, \lambda_0) = (x_0, y_0, \dot{x}_0, \dot{y}_0).$$

This completes the proof of 3). □

4 Proof of the main theorem

The last statement of the theorem saying that any smoothly solvable subset of L_F is a subset of S_F follows from Proposition 2.

Now let us prove the first conclusion that S_F *is smoothly solvable*, verifying that $Q = S_F$ satisfies the conditions in Definition 5. Let

$$(q_0, \dot{q}_0) = (x_0, y_0, \dot{x}_0, \dot{y}_0) = \phi(q_0, \lambda_0) \in S_F, \quad (q_0, \lambda_0) \in \tilde{S}_F$$

and let U be a small neighbourhood of (x_0, y_0) in $\mathbb{R}^{2n}$. We take $\tilde{S}_F \mid_K$ as $\tilde{N}$ in Definition 5, and $U \times \mathbb{R}^k$ as $\tilde{W}$;

$$\tilde{N} = \tilde{S}_F \mid_{K_r}, \quad \tilde{W} = U \times \mathbb{R}^k, \quad \tilde{N} \cap \tilde{W} = \tilde{S}_F \mid_{K_r\cap U}.$$

By the assumptions of Main Theorem, the generating function $F(x, y, \lambda)$ satisfies the condition in Proposition 6. Then by Proposition 7, there is a smooth solution $\lambda(x, y)$ of (2.4) defined on $K_r \cap U$. Therefore from Propositions 2 and 3, $\tilde{S}_F \mid_{K_r\cap U}$ is a smooth vector bundle on $K_r \cap U$. Hence, $\tilde{N} \cap \tilde{W} = \tilde{S}_F \mid_{K_r\cap U}$ is a smooth submanifold of $\mathbb{R}^{2n} \times \mathbb{R}^k$.

Lemma 9 *There exist a positive number* $\epsilon > 0$ *and a smooth map*

$$\tilde{\gamma} : (\tilde{N} \cap \tilde{W}) \times (-\epsilon, \epsilon) = \tilde{S}_F \mid_{K_r\cap U} \times (-\epsilon, \epsilon) \to \mathbb{R}^{2n}$$

such that for every $(q, \dot{q}) \in S_F \cap \phi(\tilde{W}) = S_F \mid_{K\cap U}$, *with* $(q, \dot{q}) = \phi(q, \lambda)$ *and*

$$(q, \lambda) \in \tilde{Q} \cap \tilde{W} = \tilde{S}_F \mid_{K\cap U},$$

the curve $\gamma_{(q,\lambda)} : (-\epsilon, \epsilon) \to \mathbb{R}^{2n}$ *defined by* $\gamma_{(q,\lambda)}(t) = \tilde{\gamma}(q, \lambda, t)$ *is a solution of* $Q = S_F \mid_{K\cap U}$ *with initial condition*

$$(\gamma_{(q,\lambda)}(0), \frac{d\gamma_{(q,\lambda)}}{dt}(0)) = (q, \dot{q}) = \phi(q, \lambda).$$

Proof Let $\lambda(x, y)$ be the smooth solution of (2.4) defined on $K_r \cap U$. Then, from Proposition 4,

$$\tilde{S}_F \mid_{K_r\cap U} = \lambda + N_{K_r\cap U} := \bigcup_{(x,y)\in K_r\cap U} \{(x, y)\} \times (\lambda(x, y) + N(x, y)).$$

Since, by Proposition 3, $N_{K_r\cap U}$ is a smooth vector bundle on $K_r \cap U$, it has locally $(k - r)$ independent smooth cross sections

$$e_1(x, y), \cdots, e_{k-r}(x, y),$$

which span $N_{(x,y)}$ at each point $(x, y) \in K_r \cap U$. So, elements of $N_{(x,y)}$ can be written as a linear combination of $e_1(x, y), \cdots, e_{k-r}(x, y)$,

$$\sum_{i=1}^{k-r} \mu_i e_i(x, y), \quad \mu_i \in \mathbb{R}.$$

Then for each $\mu = (\mu_1, \cdots, \mu_{k-r}) \in \mathbb{R}^{k-r}$, $\lambda(x, y) + \sum_{i=1}^{k-r} \mu_i e_i(x, y)$ is a smooth solution of (2.4). At each point $(x, y) \in K_r \cap U$, considering $\sum_{i=1}^{k-r} \mu_i e_i(x, y)$ as an element of $\mathbb{R}^k$, let $(\nu_1(x, y), \cdots, \nu_k(x, y))$ be its coordinates in $\mathbb{R}^k$;

$$\sum_{i=1}^{k-r} \mu_i e_i(x, y) = (\nu_1(x, y), \cdots, \nu_k(x, y)) = \nu(x, y) \in \mathbb{R}^k. \tag{4.1}$$

Now for $\mu = (\mu_1, \cdots, \mu_{k-r}) \in \mathbb{R}^{k-r}$, letting $\nu(x, y)$ be defined by (4.1), consider a vector field $X_{\lambda(x,y)+\mu}$ defined by

$$\begin{aligned} X_{\lambda(x,y)+\mu} &:= X_{\lambda(x,y)+\nu(x,y)} \\ &= \sum_{j=1}^{n} \left(\frac{\partial b}{\partial y_j}(x, y) + \sum_{\ell=1}^{k} \frac{\partial a_\ell}{\partial y_j}(x, y)(\lambda_\ell(x, y) + \nu_\ell(x, y)) \right) \frac{\partial}{\partial x_j} \\ &\quad - \sum_{j=1}^{n} \left(\frac{\partial b}{\partial x_j}(x, y) + \sum_{\ell=1}^{k} \frac{\partial a_\ell}{\partial x_j}(x, y)(\lambda_\ell(x, y) + \nu_\ell(x, y)) \right) \frac{\partial}{\partial y_j}. \end{aligned}$$

Since $\lambda(x, y) + \sum_{i=1}^{k-r} \mu_i e_i(x, y)$ is a smooth solution of (2.4), by Proposition 8, the vector field $X_{\lambda(x,y)+\mu} = X_{\lambda(x,y)+\nu(x,y)}$ is a smooth tangent vector field to $K_r \cap U$ and its integral curves are solutions of $S_F\ |_{K_r \cap U}$. By $\gamma_{\lambda(x,y)+\mu}(t)$, we denote the integral curve of $X_{\lambda(x,y)+\mu}$ passing through (x, y) at $t = 0$; $\gamma_{\lambda(x,y)+\mu}(0) = (x, y)$.

Now consider the map $\tilde{\gamma} : \tilde{S}_F\ |_{K_r \cap U} \times (-\epsilon, \epsilon) \to \mathbb{R}^{2n}$ defined by

$$\tilde{\gamma}(x, y, \lambda(x, y) + \sum_{i=1}^{k-r} \mu_i e_i(x, y), t) = \gamma_{\lambda(x,y)+\mu}(t).$$

Recall that every element of $\tilde{S}_F\ |_{K_r \cap U}$ can be written in the form $\lambda(x, y) + \sum_{i=1}^{k-r} \mu_i e_i(x, y)$. Now the map $\tilde{\gamma} : \tilde{S}_F\ |_{K_r \cap U} \times (-\epsilon, \epsilon) \to \mathbb{R}^{2n}$ satisfies the conditions in Lemma 9 as follows.

First, by Proposition 8-1), integral curves of $\gamma_{\lambda(x,y)+\mu}(t)$ of $X_{\lambda(x,y)+\mu}$ satisfying initial condition $\gamma_{\lambda(x,y)+\mu}(0) = (x, y) \in K$ of $X_{\lambda(x,y)+\mu}$ are solutions of $S_F\ |_{K \cap U}$. Let $(q, \dot{q}) = (x, y, \dot{x}, \dot{y}) \in S_F\ |_{K_r \cap U}$. Then, $(x, y, \dot{x}, \dot{y}) = \phi(x, y, \lambda(x, y) + \sum_{i=1}^{k-r} \mu_i e_i(x, y)$ for some $\mu \in \mathbb{R}^{k-r}$. The integral curve $\gamma_{\lambda(x,y)+\mu}(t)$ of $X_{\lambda(x,y)+\mu}$ satisfies the condition $\gamma_{\lambda(x,y)+\mu}(0) = (x, y)$. Then by Proposition 8-3),

$$(\dot{x}, \dot{y}) = d\gamma_{\lambda(x,y)+\mu}/dt(0).$$

This completes the proof of Lemma 9. □

Now by Lemma 9 and Definition 5, $(q_0, \dot{q}_0)$ is a smoothly solvable point of S_F. Thus, every point of S_F is a smoothly solvable point of $S_F \cap \tau^{-1}(U)$ and $S_F \cap \tau^{-1}(U)$ is smoothly solvable. This completes the proof of Main Theorem. □

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Data availability
Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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