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# Symmetrization of convex plane curves

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**Abstract:** Several point symmetrizations of a convex curve $\Gamma$ are introduced and one, the affinely invariant 'central symmetric transform' (CST) with respect to a given basepoint inside $\Gamma$, is investigated in detail. Examples for $\Gamma$ include triangles, rounded triangles, ellipses, curves defined by support functions and piecewise smooth curves. Of particular interest is the region of basepoints for which the CST is convex (this region can be empty but its complement in the interior of $\Gamma$ is never empty). The (local) boundary of this region can have cusps and in principle it can be determined from a geometrical construction for the tangent direction to the CST.

**Keywords:** Affine invariants, convex plane curves, singularities, symmetrization, higher inflections.

## 1. Introduction

There are several ways in which a convex plane curve $\Gamma$ can be transformed into one which is, in some sense, symmetric. Here are three possible constructions.

PT Assume that $\Gamma$ is at least $C^1$. For each pair of parallel tangents of $\Gamma$ construct two parallel lines with the same separation and the origin (or any other fixed point) equidistant from them. The resulting envelope, the *parallel tangent transform*, will be symmetric about the chosen fixed point. All choices of fixed point give the same transform up to translation. This construction is affinely invariant. A *curve of constant width*, where the separation of parallel tangent pairs is constant, will always transform to a circle. Another example is given in Figure 1(a).
This construction can also be interpreted in the following way: Reflect $\Gamma$ in the chosen fixed point to give $\Gamma^*$ say. Then the parallel tangent transform is $\frac{1}{2}(\Gamma + \Gamma^*)$ where this is the Minkowski sum.

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CPL Given a straight line $\ell$ in the plane, for each chord of $\Gamma$ perpendicular to the line $\ell$ move the chord along its length so that its midpoint lies on $\ell$. The resulting curve will be symmetric by reflexion about $\ell$, This construction is not affinely invariant. It is not hard to prove that the transformed curve is smooth and has no inflexions. When $\Gamma$ is an ellipse then the transform is another ellipse, whereas a circle always transforms into an identical circle. Another example is given in Figure 1(b).

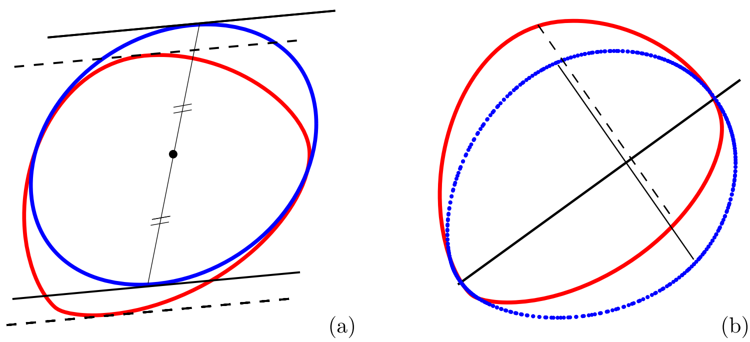



Figure 1: (a): a convex curve and its central symmetric transform by parallel tangents (PT above), using the marked point as fixed point. The dashed parallel tangents to the initial curve are moved parallel to themselves to be equidistant from the marked point. (b): the same convex curve and its transform CPL translating each chord which is perpendicular to the marked line parallel to itself so that the marked line bisects the translated chord.

CST Given a point $P$, usually interior to $\Gamma$, move each chord through $P$ along its length so that $P$ is its midpoint. The result will be a curve symmetric about $P$. This construction is affinely invariant and is the construction we shall investigate in this article. We call the transformed curve the 'central symmetric transform' or CST of $\Gamma$ relative to $P$ and we refer to $P$ as the *basepoint* of the CST. An example is given in Figure 2(a), (b). In the case of (b) the transform of a convex curve has become nonconvex. In fact the transform may be nonconvex for all choices of interior point $P$. One of the problems we wish to consider is identifying the boundary of the region, if non-empty, consisting of $P$ for which the CST is convex. We call this the *convex CST boundary* (CCST boundary) of $\Gamma$.

In this article we begin a detailed study of the last construction, the central symmetric transform, and we shall always assume that $\Gamma$ is smooth or

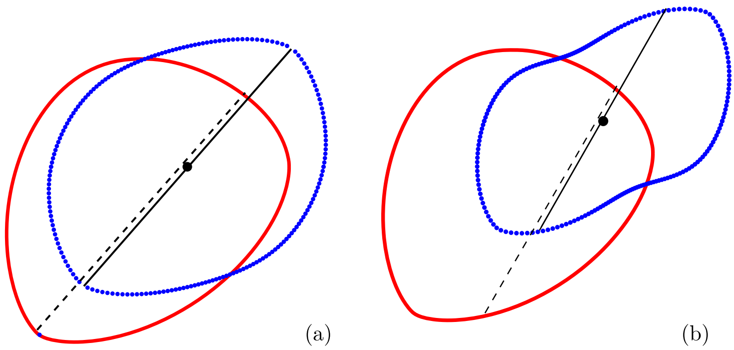



Figure 2: (a), (b): the same convex curve as in Figure 1 and its central symmetric transform CST, relative to the marked points. For (b) the CST has become nonconvex. Chords through the marked point are moved parallel to themselves so that they are bisected by that point.

piecewise smooth. Calculation of the CST relative to a given basepoint $(p, q)$ is relatively straightforward, and we give many examples of this throughout the article. The calculation of the CCST boundary is, except for some simple examples, computationally much more intensive; in §5 we show how to plot the basepoints inside a convex curve which result in a *nonconvex* CST. The CCST boundary then becomes 'visible', and plotting it point by point, is in principle possible by the same method. We give several explicit or numerical calculations of the CST: in the next section for a triangle, a rounded triangle and an ellipse, and in §5 for a curve defined by a support function or in polar coordinates. Whenever a curve is given both by parametric equations *and* by an algebraic equation then a similar method will apply. In §6 we give an example where $\Gamma$ is not smooth but consists of two smooth arcs meeting at an angle. This allows two additional ways in which the CST can become nonconvex as the basepoint moves around, besides the appearance of two inflexions via a degenerate inflexion. We give local criteria for a degenerate inflexion in §4 and we use this to show that the boundary can be *singular* in §4.2. We show that the region consisting of points inside $\Gamma$ relative to which the CST is convex can be *empty* in §2.1 and that it is always bounded away from the curve $\Gamma$ itself in §3. We also give some directions for further work in §7.

## 2. Examples of the central symmetric transform (CST)

In this section we shall give several examples where the CST can be calculated simply and which also illustrate some general properties of the CST.

### 2.1. CST of two intersecting straight lines, triangle and rounded triangle

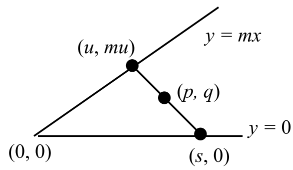



Figure 3: The setup for calculating the CST of two lines.

With the notation of Figure 3, the condition for the three points to be collinear is $(u-p)q = (mu-q)(p-s)$. Solving this for $u$ in terms of $s, p, q$ and substituting in the formula for the CST which is (translating to the point $(p, q)$), $(x, y) = \left(\frac{1}{2}(u-s)+p, \frac{1}{2}mu+q\right)$, we find that the CST is parametrized by $s$:

$$x = \frac{ms^2 - 3mps + 2mp^2 - 2mpq}{2(mp-q-ms)}, \quad y = \frac{q(2mp-2q-3ms)}{2(mp-q-ms)}.$$

The equation of the CST is then (eliminating $s$)

$$2mxy - 2y^2 - 3mqx + (4q-mp)y + 2mpq - 2q^2 = 0.$$

This is a hyperbola through $(p, q)$ (corresponding to $s = 0$) and it is easy to check that the centre is $\left(\frac{1}{2}p + \frac{q}{m}, \frac{1}{2}q\right)$. Translating to the centre the equation becomes $2mx'y' - 2y'^2 = \frac{1}{2}q(q-mp)$ (using $x', y'$ as coordinates relative to parallel axes through the centre). As we would expect, there is degeneracy when $(p, q)$ lies on one of the given lines $y = 0, y = mx$, that is $q = 0$ or $q = mx$. The asymptotes are parallel to the two lines $y = 0, y = mx$ that we started with and the axes of the hyperbola are the bisectors of the angles between these asymptotes.

Since the CST of two lines consists of hyperbolic arcs, when we consider three line segments forming a triangle the CST will always be a sequence of hyperbolic arcs joined together and hence will never be convex. That is to say the region consisting of points $(p, q)$ for which the CST relative to $(p, q)$ is convex, is *empty*. The same must apply to any smooth convex curve sufficiently close to a triangle.

We may assume the triangle is isosceles right-angled since the CST construction is affinely invariant. An example is shown in the Figure 4, left, where

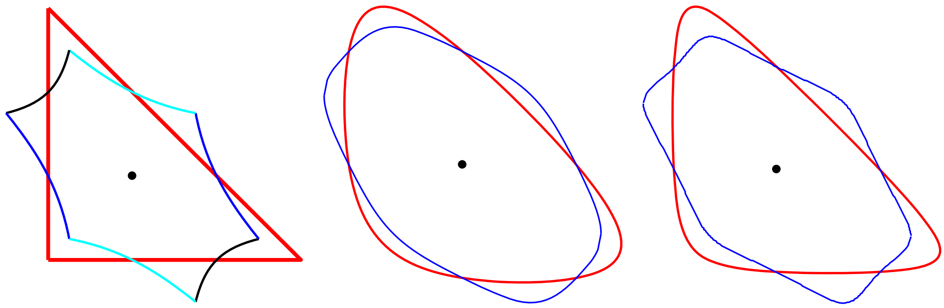


Figure 4: CST of a triangle (left) and of two 'near triangles', all relative to the marked point.

the lines are $x = 0, y = 0$ and $x + y = 1$ and $(p, q) = (\frac{1}{3}, \frac{1}{3})$, the centroid of the triangle. The lines forming this triangle are $xy(x+y-1) = 0$ and we can therefore consider the family of curves $C_k : xy(x+y-1)+k = 0$ where $k > 0$ is constant. For small enough $k$ this curve has a loop whose CST will always be nonconvex and therefore the region of $(p, q)$ for which the CST relative to $(p, q)$ is convex is *empty*. As $k$ increases there is the possibility of convex CSTs. For example with $k = \frac{1}{50}$ and $(p, q) = (\frac{1}{3}, \frac{1}{3})$ the CST is as shown in the Figure 4, centre. For $k = \frac{1}{130}$ the CST relative to the same point has become noticeably nonconvex (Figure 4, right).

The way in which these CSTs are calculated is as follows. Let us fix values of $k, p, q$. The equation of $C_k$ is quadratic in each variable so we can solve for $y$ as a function of $x$, giving two solutions, say $y_1(x), y_2(x)$. We then consider points $(X, Y) = (x, y) + \lambda(p - x, q - y)$ on the line from $(x, y) \in C_k$ to $(p, q)$, and the equation $C_k(X, Y) = 0$. There will be one of these equations for each solution $y = y_1, y = y_2$. One solution for $\lambda$ is clearly $\lambda = 0$ and there will be two others, the smaller of which is the value of $\lambda$ giving the 'other side' of the oval on the line through $(x, y)$ and $(p, q)$. (So care must be exercised that this solution for $\lambda$ is selected.) For such $(X, Y)$ we have the CST consisting of points $(p, q) \pm \frac{1}{2}(X - x, Y - y)$.

### 2.2. CST of a circle or ellipse

Since the CST is affinely invariant, it is enough to consider the circle. Let $\Gamma$ be a unit circle centred at $(0, 0)$ we consider, without loss of generality, the CST relative to the point $(p, 0), 0 < p < 1$. The point $(p + r\cos t, r\sin t)$, on a chord through $P$, lies on the circle if and only if $r^2 + 2pr\cos t + p^2 - 1 = 0$. Solving this for $r$, say $r = r_1, r = r_2$ (note that $r_1r_2 < 0$), the CST, translated to have centre at the origin, is parametrized $(\frac{1}{2}|r_1 - r_2|\cos t, \frac{1}{2}|r_1 - r_2|\sin t)$, which comes to

$$\left(\sqrt{1 - p^2\sin^2 t}\cos t, \sqrt{1 - p^2\sin^2 t}\sin t\right).$$

The condition for an inflexion on this curve at the point with parameter $t$ is

$$\sin^2 t = \frac{1 + p^2}{2p^2(2 - p^2)}.$$

Given $0 < p < 1$, the condition for this to lie in the interval $[0, 1]$ is $p \geq \frac{1}{\sqrt{2}}$. Hence we have the following.

**Proposition 2.1.** *Let $\Gamma$ be an ellipse, and $R$ the region inside $\Gamma$ consisting of points $P$ such that the CST of $\Gamma$ with respect to $P$ is strictly convex (that is, without inflexions). Then $R$ is the interior of a concentric ellipse, the scale factor being $\frac{1}{\sqrt{2}}$. In other words, this concentric ellipse is the (global) CCST boundary of $\Gamma$.* □

Figure 5 shows a unit circle and two CSTs corresponding to $P = (0.6, 0)$ (convex) and $P = (0.8, 0)$ (nonconvex). In this figure, the CST is centred at the point $P$, marked with a dot. The boundary of the convex CST region is shown dashed. Note that if $p^2 = \frac{1}{2}$ then $\sin^2 t = 1$, that is $t = \pm\frac{1}{2}\pi$: in the figure, as $P$ moves across the dashed circle, the inflexions form in pairs directly above and below $P$ on the CST.

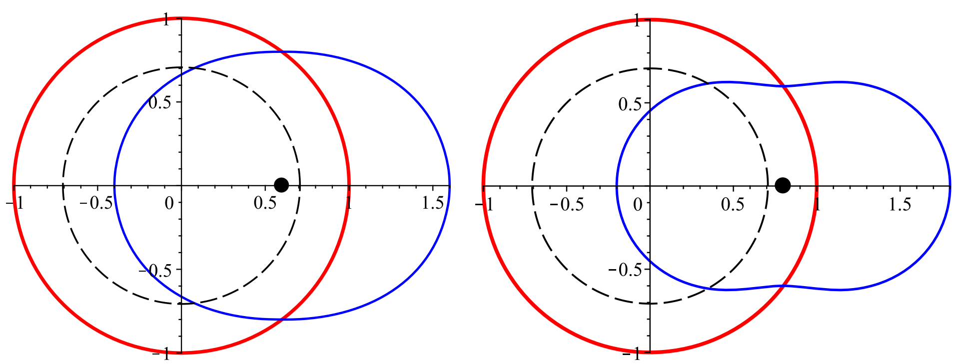



Figure 5: CST of a circle relative to the marked points. On the left, the point is inside the dashed circle of radius $\frac{1}{2}\sqrt{2}$ which is the convex CST boundary, while on the right it is outside and the CST is nonconvex.

This is a very exceptional example in that *all* the chords which give double-inflexion points—that is, points of the CST at which the tangent line has four-point contact, two inflexions having coincided—and which as a result contribute to the CCST boundary are tangent to the CCST boundary itself.

**Remark 2.2.** We can also regard this as a special case of Example 4.2 below. Writing the circle as $(x+d)^2+y^2=1$, with centre $(-d,0)$ where $0<d<1$, and radius 1, it is easy to verify that the upper curve has expansion

$$y = c - \frac{d}{c}x - \frac{1}{2c^3}x^2 + \text{ h.o.t., where } c = \sqrt{1-d^2},$$

and the condition $g_1^2 = -r_0 g_2$ of Proposition 4.1 holds exactly when $2d^2=1$. Thus the chord along the $y$-axis and at a distance $\frac{1}{2}\sqrt{2}$ from the centre of the circle is a double-inflexion chord and also tangent to the local CCST boundary, confirming the result of the calculation above. Indeed this calculation shows that, in the case of the circle (or the ellipse), the chord is tangent to the global CCST boundary.

We do not know how to determine the full set of curves $\Gamma$ for which all double-inflexion chords are tangent to the CCST boundary. The criterion of Proposition 4.1 could in principle be used for this.

## 3. Local structure of the CST for basepoints $(p,q)$ approaching the curve

Figure 6 shows the CST of a circle for basepoints $P$ crossing the circle. Thus the 'local' CST generated by chords both of whose ends are close to $P$, which

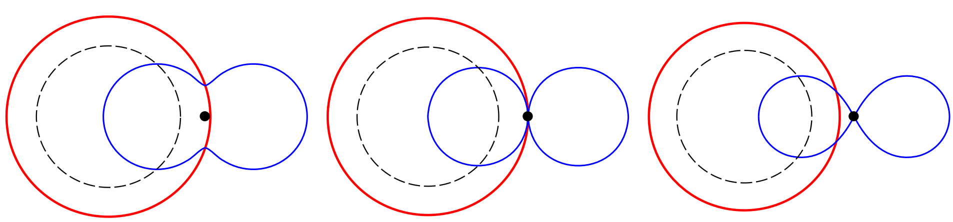


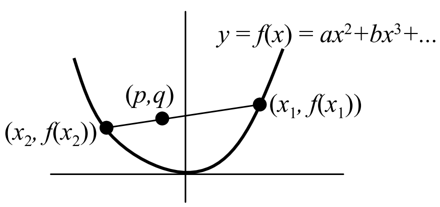
Figure 6: CST of a circle for basepoints just inside, on, and outside the circle. The dashed circle is the convex CST boundary. The CST has four inflexions in the left figure, a tacnode in the middle figure and a crossing with inflexions on both branches in the right figure. These are general local phenomena (Proposition 3.2).

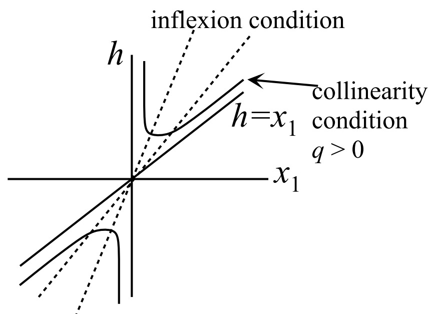
is itself close to the circle, changes from two branches with two inflexions each, through two branches with ordinary contact to a crossing where both branches have an inflexion at the crossing point. In fact these observations are general, as we now proceed to show. It is clear that given a point $P = (p, q)$ close to any smooth curve $C$ without an inflexion the chords through $P$ with one end close to $P$ may have the other end very far from $P$, so in what follows we assume that both ends are close to $P$.

We shall take $C$ in Monge form, tangent to the $x$-axis at the origin, with equation $y = f(x) = ax^2 + bx^3 + cx^4 + \dots$, $a > 0$. First let $P$ be $(0, 0)$; then clearly the CST consists locally of exactly the points $\pm\frac{1}{2}(x, f(x))$ that is two curves having ordinary (2-point) contact at $P$.

If we take $P = (p, 0)$, that is a point on the tangent line at the origin, then a short calculation shows that the local CST, translated to the origin, has branches $y = \pm\frac{1}{8}\frac{ax^3}{p}$ + h.o.t., that is two branches with inflexions and inflexional tangents along the tangent to $C$. This is as observed in the case of the circle above.

In order to investigate the case when $P$ is 'just inside' $C$ consider a point $(x_1, f(x_1))$ on the curve joined to the given point $P = (p, q)$, close to the origin and with $q > f(p)$ so that $(p, q)$ is 'above' the curve. The line from $(x_1, f(x_1))$ to $(p, q)$ meets the curve again at say $(x_2, f(x_2))$, where $x_2 \neq x_1$. We then wish to consider the set of CST points $\pm\frac{1}{2}(x_1 - x_2, f(x_1) - f(x_2))$ as the point $(p, q)$ approaches the curve. See the left-hand diagram below. We shall take $p = 0, q > 0$ for simplicity in what follows.




The collinearity condition is

$$(f(x_2) - q)(x_1 - p) = (f(x_1) - q)(x_2 - p).$$

But the left-hand side is zero when $x_1 = x_2$ so by Hadamard's lemma $x_1 - x_2$ is a factor of the left-hand side, and removing it leaves a term $-q$. Let $h = x_1 - x_2$, so that substituting $x_2 = x_1 - h$ we are removing a factor $h$ and the

CST for $x_1, h$ small and fixed small $q > 0$ take the form (omitting now the factor $\frac{1}{2}$)

$$(h, f(x_1) - f(x_1 - h)), \text{ subject to } \tfrac{1}{h}\left(x_1 f(x_1 - h) - (x_1 - h) f(x_1)\right) - q = 0.$$

We want to detect inflexions on this curve, for small $x_1, h, q$ where $q > 0$. In particular this will prove that for a basepoint sufficiently close to and inside an oval, the CST can never be convex.

**Lemma 3.1.** *For a smooth curve parametrized as $(t, A(t, u))$ where $t, u$ are subject to the nonsingular constraint $B(t, u) = 0$ the condition for an inflexion at $(t, A(t, u))$ is as follows. We use the abbreviation $A_1$ for differentiation of $A$ with respect to its first variable, $A_{12}$ for second derivative with respect to first and second variables, etc.*

$$A_{11}B_2^2 - 2A_{12}B_1B_2^2 + A_{22}B_1^2B_2 = A_2B_{11}B_2^2 - 2A_2B_{12}B_1B_2 + A_2B_{22}B_1^2,$$

*all derivatives being evaluated at a point $(t, A(t, u))$ on the curve.*

This lemma is proved by taking either $u = U(t)$ or $t = T(u)$ to solve $B(t, u) = 0$ locally (the two give the same result) and using say $B(t, U(t)) \equiv 0$ to find $U', U''$. The curve is now parametrized by $t$ as $(t, A(t, U(t)))$ and the condition for an inflexion is $A'' = 0$. This gives the displayed formula. □

In our case we need to describe the constraint $B$ (the collinearity condition as a function of $h$ and $x_1$) and the function $A$ (the CST parametrized by $h$), and use these to find the condition of the lemma for an inflexion on the CST, which will be independent of $q$. For this we take the curve as before in Monge form $y = ax^2 + bx^3 + \dots$ where $a > 0$. In fact, a straightforward but tedious calculation shows that, expanding in terms of $x_1, h$ and $q$, the lowest terms of the CST take the form $(h, -q + ahx_1 - ax_1^2)$ and the lowest terms of the resulting inflexion condition are $-2a^4h(h^2 - 6hx_1 + 6x_1^2)$. Bearing in mind that $a$ and $q$ are $> 0$ these curves are drawn in the right-hand diagram above. Note that for the inflexion condition we ignore $h = 0$ which means $x_1 = x_2$ and the slopes of the two real branches are $3 \pm \sqrt{3}$, both of which are $> 1$. hence the branches of the inflexion condition will intersect the other curve in the diagram in two points on each side of the origin, resulting in two inflexions for small $q$.

Hence we have shown the following.

**Proposition 3.2.** *As the basepoint $P$ moves across a smooth curve $C$ without an inflexion, the CST as defined locally by chords both of whose endpoints are close to $P$ changes from two smooth branches with two inflexions each*

*($P$ inside $C$) through two smooth branches with ordinary tangency ($P$ on $C$) to a crossing with each branch having an inflexion, the inflexional tangent being also tangent to $C$ ($P$ outside $C$).*

*In particular for a smooth closed strictly convex curve $C$ and a basepoint $P$ sufficiently close to and inside $C$ the CST is never convex.* □

**Remark 3.3.** We may use the collinearity condition to solve for $q$ in terms of $x_1, x_2$ and consider the map (germ), with basepoint $P = (0, q)$ as above

$$\mathbb{R}^2 \to \mathbb{R}^3,\ F(x_1, x_2) = (q(x_1, x_2),\ x_1 - x_2,\ f(x_1) - f(x_2)).$$

Substituting $h = x_1 - x_2$ as above gives the map

$$F : \mathbb{R}^2 \to \mathbb{R}^3, \quad F(x_1, h) = (q(x_1, h), h, f(x_1) - f(x_1, h)). \tag{1}$$

A short calculation shows that the 2-jet of $F$ is

$$j^2(F) = (ahx_1 - ax_1^2,\ h,\ 2ahx_1 - ah^2),\ \text{where } a > 0. \tag{2}$$

Writing $(u, v, w)$ for the coordinates in $\mathbb{R}^3$ use the 'left' change of coordinates $(u, v, w) \mapsto (u, v, w + av^2)$ to change (2) to $(ahx_1 - ax_1^2, h, 2ahx_1)$ and then $(u, v, w) \mapsto (u - \frac{1}{2}w, v, w)$ to turn (2) into the $\mathcal{A}$-equivalent form, after scaling and permuting the coordinates by $(u, v, w) \mapsto (v, w, u)$

$$(h, hx_1, x_1^2), \tag{3}$$

which is the normal form for a Whitney umbrella, or crosscap (see for example [2, Th. 4:3], also [3]) and is 2-$\mathcal{A}$-determined. Thus the local CSTs as $q$ passes through 0 form the cross-sections of a surface diffeomorphic to a Whitney umbrella. Of course this tells us nothing about inflexions.

Reversing the above left transformations, that is diffeomorphisms of the target $\mathbb{R}^3$, gives $(u, v, w) \mapsto (-aw + av, u, 2av - u^2)$ and the projection on the first coordinate of the 2-jet in (2) becomes, on the standard Whitney umbrella $(x, xy, y^2)$, the function $(u, v, w) \mapsto v - w$. The level sets of this function are diffeomorphic to the local CSTs as the basepoint moves across the curve $C$. This function is not among the normal forms of functions on the Whitney umbrella classified in [1, Th. 3.13] but, as the authors show, it is equivalent to having 1-jet $w$, giving the 2-determined function $(u, v, w) \mapsto w - u^2$, that is $WU_1^-$ in their notation. Figure 7 shows the sections of the standard Whitney umbrella by planes $v - w =$ constant, the centre one being $v - w = 0$.

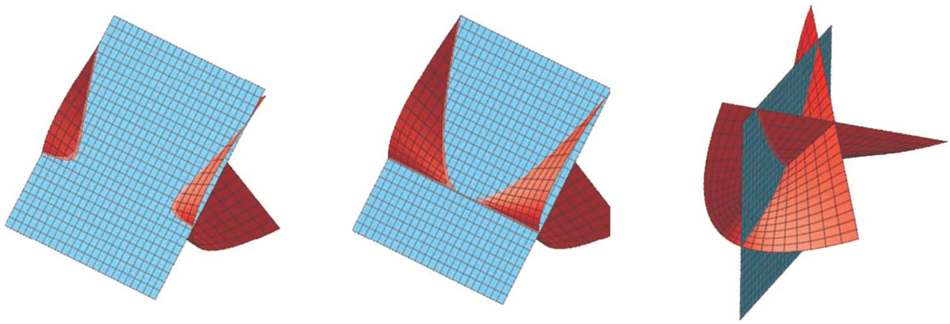


Figure 7: Sections of a standard Whitney umbrella (crosscap) which correspond with the local evolution of a CST as the basepoint crosses the curve $\Gamma$.

## 4. Local conditions for higher inflexions on the CST

For calculating the CST with centre $(0,0)$ the diagram shows the setup used to make local calculations. Write a general point on the lower curve as

$$(u,\ -r_0 + f_1u + f_2u^2 + f_3u^3 + \ldots),$$

and a general point on the upper curve as

$$(x,\ s_0 + g_1x + g_2x^2 + g_3x^3 + \ldots).$$

Note that $r_0 > 0$ and $s_0 > 0$.

Taking points on the two curves with parameters $u, x$ respectively, we write down the condition for these two points to be collinear with the origin, namely

$$u(s_0 + g(x)) = x(-r_0 + f(u)).$$

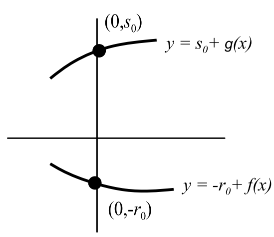



Figure 8: The local setup for finding inflexions and degenerate inflexions on the CST.

Now we solve this as a power series for $u$ as a function of $x$, say $u = U(x)$ and substitute into $f$ to obtain say $F(x) = f(U(x))$, again as a power series in $x$ up to a suitable order (order 4 is enough). The first terms are

$$U(x) = -\frac{r_0}{s_0}x - \frac{r_0(f_1 - g_1)}{s_0^2}x^2 + \ldots,$$
$$F(x) = -\frac{r_0 f_1}{s_0}x - \frac{r_0(f_1^2 - f_1 g_1 - f_2 r_0)}{s_0^2}x^2 + \ldots.$$

The parametrization of one side of the CST, translated to the origin, is then

$$(X(x), Y(x)) = \left(\tfrac{1}{2}(x - U(x)), \quad \tfrac{1}{2}(r_0 - F(x) + s_0 + g(x))\right).$$

The other is $(-X(x), -Y(x))$ of course. In practice when making calculations or examining examples we often omit the factor $\frac{1}{2}$.

The conditions for a *higher inflexion* on the CST at $x = 0$ can then be written as

$$X'Y'' - X''Y' = 0; \; X'Y''' - X'''Y' = 0,$$

where these derivatives are evaluated at $x = 0$. After some manipulation these conditions become the following.

$$r_0 s_0 (f_1 - g_1)^2 = (r_0 + s_0)(f_2 r_0^2 - g_2 s_0^2), \tag{4}$$
$$(r_0 + s_0)(f_3 r_0^3 + g_3 s_0^3) = (f_1 - g_1)(f_2 r_0^3 + 2 f_2 r_0^2 s_0 + 2 g_2 r_0 s_0^2 + g_2 s_0^3). \tag{5}$$

It would be interesting to have direct interpretations of these conditions in terms of affinely invariant quantities associated to the pair of curve segments.

A *double inflexion chord* of $\Gamma$, relative to a given basepoint, is a chord for which the endpoints occur at degenerate inflexions on the CST. These will usually be inflexions of order two and not higher, but could be of order 3, a case we consider below (§4.2) in the context of a singular CCST boundary.

The following proposition gives the criterion for a double-inflexion chord to be tangent to the local CCST boundary, in the general situation above. We stress the "local" nature of this result: we identify the local CCST boundary with the CST centres close to the given centre (here $(0, 0)$) through which there is a double inflexion chord close to the given double-inflexion chord (which is here along the $y$-axis). In principle the local CCST boundaries for different starting chords may be intersecting curves, so that local boundary points are not in fact global.

**Proposition 4.1.** *Suppose that, in the general situation above, conditions (4) and (5) hold so that the chord joining* $(0, -r_0)$ *and* $(0, s_0)$ *is a double-inflexion chord for the CST relative to* $(0, 0)$*. Then the chord is tangent to the local CCST boundary if and only if* $f_2 r_0^2 + g_2 s_0^2 = 0$.

The proof of this proposition is (at present) a direct calculation. That is, we take points $P = (p, q)$ close to $(0, 0)$ and construct the CST with centre $(p, q)$, then require this to have a double inflexion for some chord close to the $y$-axis. To do this, we take a point with first coordinate $x$ on the upper curve, join it to $P$ and extend to meet the lower curve in a point $(u, -r_0 + f(u))$ where $u$ is a function of $x, p, q$. Then the parametrization of the CST relative to $P$ is computed and the conditions that it has a double inflexion give two equations in $x, p, q$. These generally have a smooth solution $p = p(x), q = q(x)$ and the first terms can be computed. The condition that the tangent is 'vertical', that is parallel to the starting chord along the $y$-axis, comes to

$$(f_1 - g_1)^2 s_0 = 2r_0(r_0 + s_0) f_2,$$

the imbalance between $f, r_0$ and $g, s_0$ arising because of the parametrization by $x$ above. However, using (4) this is easily reduced to the symmetrical equation in the statement of the proposition. □

**Example 4.2.** As an example of the proposition, consider two curve segments which are symmetric with respect to reflexion in the $x$-axis, and take $r_0 = s_0$ in the general case above. Symmetry means $g_i = -f_i$ for $i \geq 1$, so that (5) and the condition of the proposition automatically hold. If in addition (4) holds, that is if $f_1^2 = r_0 f_2$ for the lower curve, or $g_1^2 = -r_0 g_2$ for the upper curve, then the chord along the $y$-axis is a double-inflexion chord and is tangent to the local CCST boundary. (We do not know a geometrical interpretation of the condition $f_1^2 = r_0 f_2$.) See §2.2 for the special case of a circle.

**Remark 4.3.** The curve $\Gamma$ is assumed to be convex. Thus the upper curve has $g_2 < 0$ (concave downwards at $(0, s_0)$ and the lower curve has $f_2 > 0$ (concave upwards at $(0, -r_0)$). The CST at the upper point (above the origin) will be concave downwards if and only if its curvature, oriented as above by $x$ on the lower curve, and therefore *right to left*, is $> 0$. The curvature has the sign of $X'Y'' - X''Y'$ and hence the upper branch of the CST is concave downwards, towards the chosen point, if and only if $r_0 s_0 (f_1 - g_1)^2 > (r_0 + s_0)(f_2 r_0^2 - g_2 s_0^2)$, by (4). This is also the condition for the lower branch to be concave upwards, towards the chosen point. If the CST is convex then this condition must hold.

### 4.1. The "Parallelogram Construction" of the CST tangent

In the situation of Figure 8 it is easy to check that a tangent vector to the CST (at either of the two ends of the chord) is parallel to $(r_0+s_0,\ f_1r_0+g_1s_0)$. To see this we solve locally for $u$ as a function of $x$ and find the linear terms of the CST as

$$\pm\frac{1}{2}\left(\frac{r_0+s_0}{s_0}x,\ r_0+s_0+\frac{f_1r_0+g_1s_0}{s_0}x\right),$$

so that the tangent direction to either branch is $(r_0+s_0,\ f_1r_0+g_1s_0)$. This direction therefore depends only on the ratio of distances of the basepoint from the ends of the chord of $\Gamma$ and the slopes of the tangents to $\Gamma$ there. In fact this gives a simple geometrical construction for the tangent direction to the CST, shown in Figure 9. We take lines through the endpoints of the chord parallel to the tangents at the *other ends* of the chord; their point of intersection, in the notation above, is $\frac{1}{g_1-f_1}(r_0+s_0, f_1r_0+g_1s_0)$ so that the line joining this point to the basepoint (here, the origin) has the same slope as the tangent to the CST. See the figure. This construction is affinely invariant.

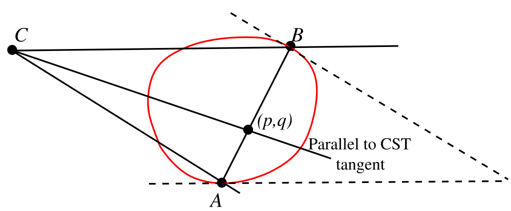



Figure 9: Construction of the tangent direction to the CST with a given basepoint. The dashed lines are tangents to the given curve at the endpoints $A, B$ of the chord and $(p,q)$ is the basepoint. Lines are drawn through $A$ parallel to the tangent at $B$ and vice versa. These meet at $C$ and the line $L$ joining $C$ to the basepoint is parallel to the tangent lines to the CST at either of the corresponding points. (These are given by $\pm\frac{1}{2}(B-A)$ for the CST centred at the origin or these points added to $(p,q)$ for the CST when translated to have its centre at $(p,q)$.)

### 4.2. Singular point of the CCST boundary

In this section we construct an example where the local boundary of the convex CST region (the CCST boundary) has a cusp singularity. We take the local boundary to consist of those points $(p, q)$ for which some chord through $(p, q)$ gives a double inflexion on the CST. It is of course possible that such a point is not on the true CCST boundary, depending on the global structure of the boundary.

The additional condition which we impose on the CST beyond (4) and (5) in §4 is that the inflexion is *triple*, that is also $X'Y^{(4)} - X^{(4)}Y' = 0$. After some manipulation and use of the condition (4) the triple inflexion condition can be written in the following form:

$$\begin{aligned} 2(r_0+s_0)(f_4r_0^4-g_4s_0^4) &= (f_1-g_1)(3f_3r_0^4+5f_3r_0^3s_0-5g_3r_0s_0^3-3g_3s_0^4) \\ &\quad -(f_1-g_1)^2(f_2r_0^3+2f_2r_0^2s_0-2g_2r_0s_0^2-g_2s_0^3) \\ &\quad +2(r_0+s_0)(f_2r_0+g_2s_0)(f_2r_0^2+g_2s_0^2) \end{aligned} \tag{6}$$

An example fulfilling the conditions (4), (5) and (6) is

$$r_0 = 2,\ \ s_0 = 5,\ f(u) = u^2,\ \ g(x) = 2x - \frac{12}{175}x^2 - \frac{456}{6125}x^3 + \frac{81584}{1071875}x^4.$$

Note that the lower branch is concave upwards close to $(0, -r_0)$ and the upper branch concave downwards close to $(0, s_0)$; compare Figure 8.

In order to study the nearby chords which give double inflexions, and which therefore contribute to the CCST boundary, we use the same procedure as in the proof of Proposition 4.1. Thus we take a point $(p, q)$ close to $(0, 0)$ and chords through $P = (p, q)$ meeting the upper branch in $X = (x, s_0+f(x))$. The corresponding point on the lower branch, collinear with $P$ and $X$, has first coordinate $u$ which is solved locally in terms of $x, p, q$ and the result substituted in $f(u)$ to obtain the second coordinate of the other end of the chord $XP$. Then the corresponding CST is calculated for this choice of $P$ and the two conditions (4), (5) are applied, giving two equations in three variables $x, p, q$. The resulting curve in the $(p, q)$ plane is parametrized by $x$: writing say $p = p_1x + p_2x^2 + \dots, q = q_1x + q_2x^2 + \dots$ we substitute into the two equations and equate coefficients. The result is that $p_1 = q_1 = 0$ and

$$p = 0.197x^2 - 0.946x^3 + \dots,\ \ q = -2.281x^2 - 0.132x^3 + \dots,$$

working to 3 significant figures. This is an ordinary cusp at the origin $p = q = 0$; see Figure 10. Note that the cuspidal tangent is *not* aligned with the initial chord, which lies along the $y$-axis.

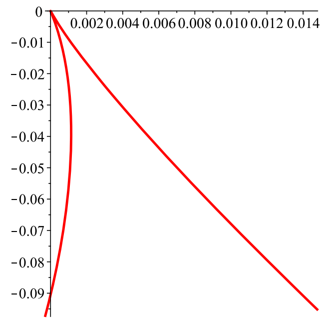



Figure 10: A cusp on the local CCST boundary, the result of a triple inflexion chord of $\Gamma$.

## 5. Computing the region in which the CST is not convex

There are several approaches to this and we give here one which is based on the geometrical construction of the tangent line to the CST in 4.1; see Figure 9.

Consider the family of tangents to the CST for a given basepoint $(p, q)$, following one of the branches as chords through the basepoint sweep round through an angle of $2\pi$ (after an angle of $\pi$ the endpoints of the chord, and the branches of the CST, reverse). The tangents to a curve $(t, f(t))$ form a family of lines $y = xf'(t) + f(t) - tf'(t) = xm(t) + c(t)$ say, where $m$ is the gradient and $c$ the intercept on the $y$-axis. The condition for an inflexion is $f''(t) = 0$, that is $m'(t) = 0$. Thus we can just consider the gradients of the lines.

In practice then we have two points of a given curve $\gamma$, say $\gamma(t)$ and $\gamma(u)$, and a point $(p, q)$ inside the curve. We calculate the intersection point of lines given by the parallelogram construction in Figure 9 above, say $(\xi(t, u, p, q), \eta(t, u, p, q))$; this involves solving only linear equations. Then the gradient of the line joining this to $(p, q)$ is $m = \dfrac{q - \eta}{p - \xi}$. We want to find the condition for "$m' = 0$". However we must restrict attention to pairs of points of $\gamma$ for which $\gamma(t), \gamma(u)$ and $(p, q)$ are collinear. Call the condition for collinearity $C(t, u, p, q) = 0$. To obtain the required condition assume that this equation is solved as $u = U(t, p, q)$ and fix the point $(p, q)$, so that $C(t, U, p, q) \equiv 0$. Then $C_t + C_u U' = 0$, the prime indicating $\frac{d}{dt}$ for fixed $p, q$. We want the condition $m'(t, U(t)) = 0$, that is $m_t + m_u U' = 0$, which gives $C_u m_t - C_t m_u = 0$. Substituting for $m_u, m_t$ this can be written in the form $I = 0$ where

$$I(t, u, p, q) = (p - \xi)(C_t \eta_u - C_u \eta_t) + (q - \eta)(C_u \xi_t - C_t \xi_u).$$

This is the form which we use in computer programs implementing this method. The steps are then as follows.

(i) Given a parametrized smooth closed curve $\Gamma$, usually with parameter from 0 to $2\pi$.
(ii) Compute $\xi, \eta$, the intersections of lines parallel to the two tangents, in terms of $t, u, p, q$. This involves solving only linear equations.
(iii) Compute the collinearity condition $C(t, u, p, q) = 0$, which is clearly linear in $p$ and $q$, and solve for $q$ as a function $q = q_1(p, t, u)$ which is linear in $p$.
(iv) Compute the inflexion condition $I = 0$ above and use only its numerator. The degree of this expression in $p$ and in $q$ will always be 2. In fact for fixed $t, u$ the equation $I = 0$ represents a *conic* in the $p, q$ plane. The collinearity condition $C = 0$ represents a line and (real) intersections $(p, q)$ between these two mean that the CST with basepoint $(p, q)$ has an inflexion for the chord given by $t, u$.
(v) Substitute $q = q_1(t, u, p)$ into $I$ to obtain a condition $I_1(t, u, p) = 0$ say. This has degree 2 in $p$.
(vi) Take a large number of pairs of points given by $(t, u)$ on the curve and for each one determine a corresponding pair of solutions of the quadratic equation $I_1(t, u, p) = 0$, discarding any $(t, u)$ for which this quadratic equation in $p$ has equal or non-real roots.
(vii) Determine the corresponding $q$ using $q = q_1(t, u, p)$ and discard any $(p, q)$ which do not lie within the curve $\gamma$.
(viii) Plot the points $(p, q)$ which have survived this process.
(ix) Remark 1: Plotting the points $(p, q)$ where the quadratic equation $I_1 = 0$ in $p$ has coincident roots (or where the conic and line in (iv) above are tangent) gives a reasonable approximation to the boundary of the convex CST region but is subject to some numerical instability.
(x) Remark 2: Curves $\Gamma$ will usually be defined in terms of $\cos t$ and $\sin t$ and for simplification of the equations above it is useful to translate into algebraic expressions using the standard substitution $T = \tan(\frac{1}{2}t)$.

### 5.1. Examples

Three examples are given in Figure 11, the given convex curves being defined either by a support function $h$ or a polar equation $r$. For a support function $h$ the curve is $(x(t), y(t)) = (h(t)\cos t - h'(t)\sin t, h(t)\sin t + h'(t)\cos t)$, while for a polar curve $(x(t), y(t)) = (r(t)\cos t, r(t)\sin t)$.

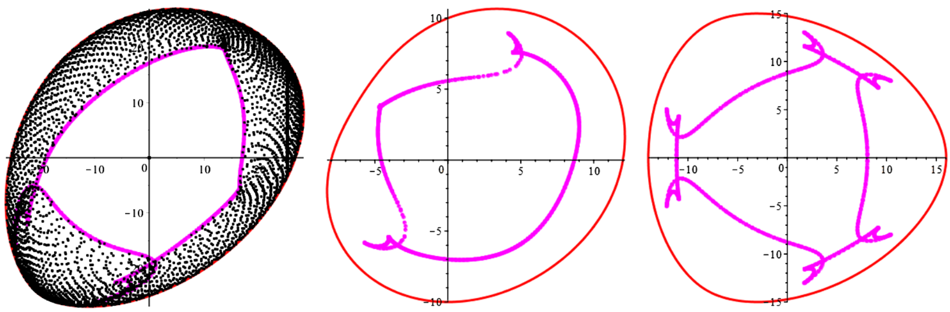


Figure 11: Left: The curve has support function $h(t) = \cos 3t + 4 \sin 2t + 27$. The dots show basepoints which give a con-convex CST and the “boundary” of this region of dots is drawn. By “boundary” we mean that crossing the boundary changes the number of inflexions by two. Centre: the curve has polar equation $r(t) = \sin 2t + 2 \cos t + 10$ and just the boundary is drawn. See Figure 12 for examples of the resulting CSTs. Right: a curve of constant width with support function $h(t) = \cos 3t + 15$ and just the boundary of the convex CST region drawn.

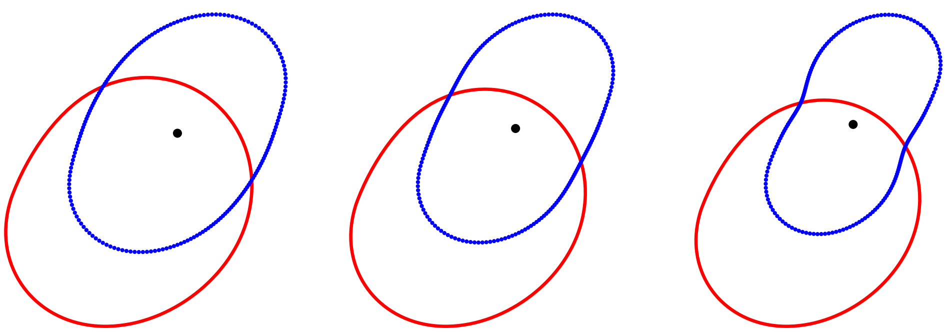


Figure 12: Basepoints moving across the convex CST boundary in the centre example of Figure 11 showing the appearance of inflexions. The basepoints are $p = 6$ and $q = 6, 7, 8$ from left to right.

## 6. A piecewise smooth example

If the given curve $\Gamma$ is piecewise smooth then this allows additional ways in which the CST can cease to be locally convex as the basepoint $(p, q)$ moves inside $\Gamma$. The local transitions are illustrated schematically in Figure 13. In studying an example we must therefore find the positions of the basepoint which allow the transition moments in the cases (b) and (c) of the figure to

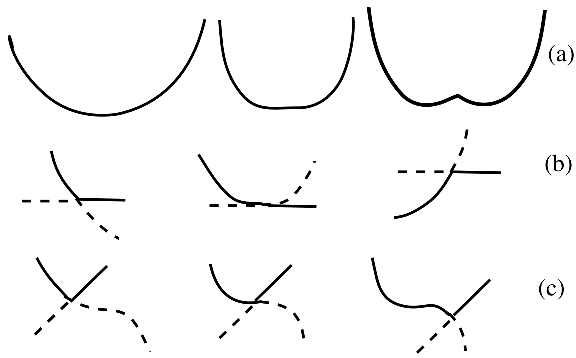



Figure 13: The three generic ways in which local nonconvexity can appear on the CST of a piecewise smooth curve, as the basepoint moves. (a) This is the double-inflexion transition which we have already used above. (b) Two intersecting branches of the CST become tangent. The region with local boundary the solid lines becomes nonconvex. (c) An inflexion on one branch of the CST is crossed transversally by another branch of the CST, again creating a nonconvexity in the region with local boundary the solid lines.

occur, as well as the double-inflexion transition moment of (a), used above for smooth curves $\Gamma$. An example of $\Gamma$ consisting of an arc of the unit circle and an arc of the circle centre $(a, 0)$, radius $r$, where $a > 0, r > 0$ and $a-1 < r < a+1$, enclosing a convex region, is given in Figure 14. The transition curves for (b) and (c) are calculated point-by-point in a similar way to those for type (a) (see §5) and all are shown in the top left figure.

## 7. Directions for further investigation

In this first study of the Central Symmetric Transform we have raised a number of questions which we hope to explore further elsewhere. All the examples studied have a *connected* region comprising those basepoints $(p, q)$ inside a convex plane curve $\Gamma$ for which the associated CST is convex. We do not have a general argument to show this is always the case. Detailed consideration of examples is hampered by the computational difficulty of calculating the boundary of the convex CST region. For a smooth curve $\Gamma$ this boundary can be computed in principle by finding those basepoints lying on some chord which corresponds to a double inflexion on the CST; we have seen in §6 that when $\Gamma$ is piecewise smooth, there are several ways in which the CST can become nonconvex, and this makes the determination of the boundary more difficult.

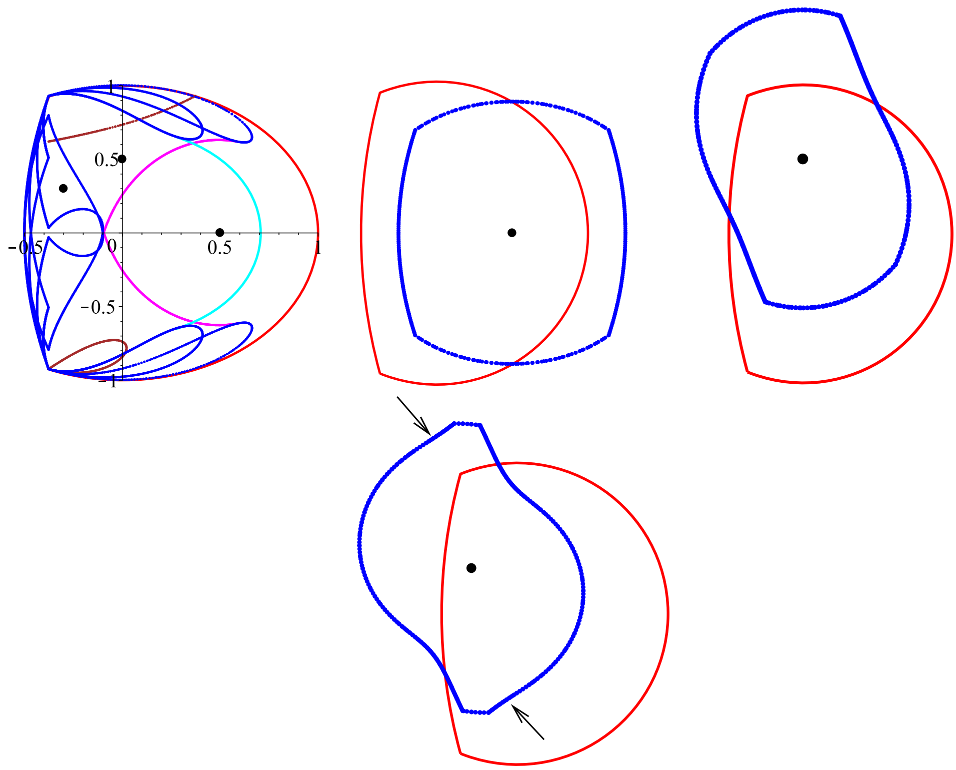



Figure 14: Top left: circular arcs from the unit circle and the circle centre $(a, 0)$ radius $r$ where $a = 3, r = 3.5$ enclose a convex region. The figure shows (with unequal scales on the axes for clarity) the network of curves consisting of basepoints where a transition of one of the types in Figure 13 occurs. The region containing the origin consists of basepoints for which the CST is convex and in this example no other basepoints give a convex CST, so that the CCST boundary is the boundary of this region. Other figures: a convex CST becomes nonconvex when the basepoint $(p, q)$ moves from $(0.5, 0)$ to $(0, 0.5)$, crossing a transition curve of type (a) in Figure 13. It then acquires additional inflexions, marked approximately by the arrows, when $(p, q)$ moves to $(-0.3, 0.3)$, crossing a transition curve of type (c).

A more efficient and computationally practical method of computing the CST for a given basepoint would be a considerable help in studying examples.

It is clear that the definition of the CST can be extended to a smooth convex closed surface $\Gamma$ in 3-space. Given a basepoint $\mathbf{p}$ inside $\Gamma$ we consider all chords through $\mathbf{p}$ and for each one shift the chord along its length so that

**p** is the midpoint of the shifted chord. Then the CST is the locus of endpoints of the shifted chords. In this case we will be looking for basepoints containing a chord for which the CST acquires a point with Gauss curvature zero, that is a parabolic point. Again it is likely that explicit calculation will need both a parametrization of $\Gamma$ and a global equation defining it.

An extension of the definition of CST in the plane case would be to a nonconvex $\Gamma$. Here, we cannot choose any interior point for a basepoint since chords through an interior point will not necessarily meet $\Gamma$ in exactly two points. The region of basepoints will be limited to a subset of the interior where the chords do meet $\Gamma$ in exactly two points.

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