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RESIDUAL ALGEBRAIC RESTRICTIONS OF DIFFERENTIAL FORMS*

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Dedicated to Professor Henry Laufer on the occasion of his 70th birthday

Abstract. We study germs of differential forms over singular varieties. The geometric restriction of differential forms to singular varieties is introduced and algebraic restrictions of differential forms with vanishing geometric restrictions, called residual algebraic restrictions, are investigated. Residues of plane curves-germs, hypersurfaces, Lagrangian varieties as well as the geometric and algebraic restriction via a mapping were calculated.

Key words. Differential forms, singularities, geometric restriction, parametric curves.

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1. Introduction.

For a smooth manifold M and the space $\Lambda^p(M)$ of all differential p-forms on M the restriction $\omega|_N$ of $\omega \in \Lambda^p(M)$ to a smooth submanifold $N \subset M$ is well defined by the geometry of N. If N is any subset of M then the forms $\alpha + d\beta$, $\alpha \in \Lambda^p(M)$, $\beta \in \Lambda^{p-1}(M)$, where α and β annihilates any p - tuple (and $p-1$ - tuple respectively) of vectors in T_xM, $x \in N$, are called algebraically vanishing on N or having zero algebraic restriction to N (see [6][7]). Now the restriction (algebraic restriction) of $\omega \in \Lambda^p(M)$ to N is defined as an equivalence class of ω modulo forms with zero algebraic restriction to N. The notion of algebraic restrictions was introduced by M. Zhitomirskii [19] for contact structures and in [6][7] for general differential forms. The idea goes back to V.I. Arnold's study (see [1]) of singular curves in the presence of symplectic structure. Restriction of symplectic two-form to the regular part of N is not complete symplectic invariant. It was proved in [7] that the complete invariant, Arnold's ghost invariant, is the singularity of the algebraic restriction of the symplectic form to N in the case N is quasi-homogeneous ([6][7]). We may show a familiar example of this phenomena for A_k-type singularities of plane curves.

$$N = A_k = \{x \in \mathbf{R}^{2n}; x_1^{k+1} - x_2^2 = x_{\geq 3} = 0\}, \quad k \geq 1.$$

Restrictions of two-forms to the regular part of N are vanishing but the algebraic restrictions (pure singularity effect - residual element) form a finite dimensional space. The space of algebraic restrictions of all two-forms on $\mathbf{R}^{2n}$ to A_k-singularity is spanned by algebraic restrictions of basic, symplectic forms

$$[\theta^i]_{A_k} = [x_1^i dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + \ldots + dx_{2n-1} \wedge dx_{2n}]_{A_k}, \quad i = 0, \ldots, k-1.$$

For given symplectic Darboux structure $(\mathbf{R}^{2n}, \omega)$ we have a local diffeomorphism Φ_i such that $\Phi_i^*\theta^i = \omega$. Then we get the symplectic classes of curves $A_k^i = \Phi_i^{-1}(A_k), i =$

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$0, \dots, k$ distinguished by algebraic restriction. In parametric form they were classified in [1] for A_{2k} singularity:

$$A_{2k,0} : t \mapsto (t^2, t^{2k+1}, 0, \dots, 0),$$
$$A_{2k,r} : t \mapsto (t^2, t^{2k+1+2r}, t^{2k+1}, 0, \dots, 0), r = 1, \dots, 2k.$$

The residual elements of two-forms on A_k, D_k, E_6, E_7, E_8 planar singularities where classified in [7] and in principle in [6] and [7], the notion of algebraic restrictions of differential forms was introduced and were established its basic properties. As we see the spaces of algebraic restrictions of contact forms and symplectic forms are effectively applied to contact and symplectic classifications of singularities [19][4][5][8][9].

In this paper we introduce the notion of "geometric restrictions" of differential forms and study its general properties (see §2). In particular we study the difference (or the quotient) of geometric restrictions and algebraic restrictions. In fact we study the space of algebraic restrictions with null geometric restrictions, which we call "residual module".

In [19] one can find the notion of geometric restrictions of the contact structure to singular varieties, as the restrictions to the regular parts of the varieties. The notions of geometric and algebraic restrictions of differential forms were studied under different names much earlier by many authors in the context of the generalization of de Rham's theorem for singular varieties (see for examples [16][10][11][6]). In particular Ferrari (Lemma 1.1, p.67 of [10]) proved that the notion of the geometric restrictions used in [19] and used in this paper agree for holomorphic differential forms and complex analytic spaces (cf. Lemma 4.2 in this paper).

The difference of geometric restrictions and algebraic restrictions are compered with the following general situation: A "variety" Z in a manifold M is regarded as the image of a mapping (parametrization) $f : N \to M$, $f(N) = Z$, while Z is regarded as a zero-set of a mapping (a system of defining equations) $F : M \to \mathbf{R}^p, F^{-1}(0) = Z$. If f and F satisfy certain conditions respectively, then the space of geometric restrictions is described in terms of f and the space of algebraic restrictions is described in terms of F.

Of course it is a fundamental but a difficult problem to give a general method choosing f and F as above from an arbitrary subset $Z \subset M$. Nevertheless we give the general framework of the theory and provide several useful observations for general Z to be effective in concrete calculations of residual modules for important examples which are shown also in this paper. In Section 2 we introduce the basic notions of geometric and algebraic restrictions to any subset of a smooth manifold. The deeper understanding of geometric restrictions goes through several constructions and mainly construction of a kind of tangent bundle - geometric tangents and co-normals to any subset of a manifold in Section 3, and stratified subsets in Section 4. The similar results for algebraic tangents and co-normals were obtained in Section 5. We then exploit these constructions in Section 6 and investigate the geometric and algebraic restrictions to any subset of a manifold represented by a mapping. Finally, in Sections 7, 8, 9 we conclude with the exact calculations of residues for hypersurfaces, for Lagrangian varieties and for plane curve-germs. Note that from the latter half of Section 6, we treat local cases.

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2. Space of geometric and algebraic restrictions. Let $\Lambda^\bullet(M) = \sum_k \Lambda^k(M)$ denote the space (total) of C^∞ differential forms on a C^∞ manifold M and $(\Lambda^\bullet(M), d)$

de Rham complex on M. Here $\bullet$ indicates the natural graduation. We set $\Lambda^k(M) = 0$ if $k < 0$ or $\dim(M) < k$. Given a subset $Z \subset M$, the notion of algebraic restrictions of differential forms is introduced in [6]: Let $\Lambda^\bullet_Z(M)$ denote the subspace of $\Lambda^\bullet(M)$ consisting of differential forms vanishing on Z. Note that $\Lambda^\bullet_Z(M)$ is not necessarily d-closed. Let $\mathcal{A}^\bullet(Z, M)$ denote the differential ideal of $(\Lambda^\bullet(M), d)$ generated by $\Lambda^\bullet_Z(M)$:

$$\mathcal{A}^k(Z, M) = \{\alpha + d\beta\,;\, \alpha \in \Lambda^k_Z(M), \beta \in \Lambda^{k-1}_Z(M)\}.$$

For an $\omega \in \Lambda^\bullet(M)$, the residue class $[\omega]^a_Z \in \Lambda^\bullet(M)/\mathcal{A}^\bullet(Z, M)$ is called the *algebraic restriction* of ω to Z.

In this paper we introduce the notion of *geometric restrictions* for any subset Z in a C^∞ manifold M as follows: Define

$$\mathcal{G}^\bullet(Z, M) := \{\omega \in \Lambda^\bullet(M)\,;\, f^*\omega = 0 \text{ for any } C^\infty \text{ mapping } f : N \to M \\ \text{from any } C^\infty \text{ manifold } N \text{ with } f(N) \subset Z\}.$$

Note that $\mathcal{G}^0(Z, M) = \mathcal{A}^0(Z, M) = \{h \in \Lambda^0(M)\,;\, h|_Z = 0\}$.

For an $\omega \in \Lambda^\bullet(M)$, the residue class $[\omega]^g_Z \in \Lambda^\bullet(M)/\mathcal{G}^\bullet(Z, M)$ is called the *geometric restriction* of ω to Z.

Accordingly we introduce the vector space

$$\mathcal{A}^\bullet(Z) := \Lambda^\bullet(M)/\mathcal{A}^\bullet(Z, M)$$

of algebraic restrictions to Z, and the vector space

$$\mathcal{G}^\bullet(Z) := \Lambda^\bullet(M)/\mathcal{G}^\bullet(Z, M)$$

of geometric restrictions to Z.

LEMMA 2.1. *For any subset Z in a C^∞ manifold M, we have*
(1) *$\mathcal{G}^\bullet(Z, M)$ is d-closed.*
(2) *$\mathcal{G}^\bullet(Z, M) \supset \mathcal{A}^\bullet(Z, M)$.*

Proof. (1) Let $\omega \in \mathcal{G}^\bullet(Z, M)$. Then for any C^∞ map $f : N \to M$ from any manifold N with $f(N) \subset Z \subset M$, we have $f^*\omega = 0$. Then $f^*(d\omega) = d(f^*\omega) = 0$. Therefore $d\omega \in \mathcal{G}^\bullet(Z, M)$. (2) Let $\alpha \in \Lambda^\bullet_Z(M)$. Then for any $f : N \to M$ with $f(N) \subset Z$, we have $f^*\alpha = 0$. Therefore we have $\mathcal{G}^\bullet(Z, M) \supset \Lambda^\bullet_Z(M)$. By (1), we have required result. $\square$

Now we introduce the space

$$\mathcal{R}^\bullet(Z) := \mathcal{G}^\bullet(Z, M)/\mathcal{A}^\bullet(Z, M)\ (\subset \mathcal{A}^\bullet(Z)\,),$$

of algebraic restrictions with null geometric restrictions to Z. Then there arises the natural exact sequence

$$0 \to \mathcal{R}^\bullet(Z) \to \mathcal{A}^\bullet(Z) \to \mathcal{G}^\bullet(Z) \to 0.$$

The space $\mathcal{A}^\bullet(Z)$ of algebraic restrictions of differential forms to Z has the natural module structure over de Rham exterior algebra $\Lambda^\bullet(M)$, which is defined by

$$\beta \wedge [\alpha]^a_Z := [\beta \wedge \alpha]^a_Z,$$

with the differential

$$\overline{d} : \mathcal{A}^\bullet(Z) \to \mathcal{A}^{\bullet+1}(Z)$$

defined by $\overline{d}[\alpha]^a_Z := [d\alpha]^a_Z$ and satisfying

$$\overline{d}(\beta \wedge [\alpha]^a_Z) = d\beta \wedge [\alpha]^a_Z + (-1)^k \beta \wedge \overline{d}[\alpha]^a_Z,$$

whenever $\beta \in \Lambda^k(M)$.

Also the space $\mathcal{G}^\bullet(Z)$ (resp. $\mathcal{R}^\bullet(Z)$) has the natural module structure over the de Rham exterior algebra $\Lambda^\bullet(M)$ as well.

REMARK 2.2. The non-zero algebraic restrictions of symplectic forms to a curve in a symplectic space was called "ghost" according to [1]. The symplectic forms has null geometric restrictions on parametric curves. Since we are regarding all algebraic restrictions with null geometric restrictions of differential forms, we may call our residues "pure ghosts".

3. Geometric tangents and conormals. To understand geometric restrictions generally, we introduce a kind of "tangent bundle" for any subset of a manifold.

Let E be a finite dimensional vector bundle over a topological space X. A subset $L \subset E$ is called *fibrewise linear*, if for any $x \in X$, $L_x := L \cap E_x$ is a linear subspace of the fibre E_x of E over x. Let $K \subset E$ be any subset. Then the *closed linear hull* $\widetilde{K}$ of K in E is defined as the smallest closed and fibrewise linear subset in E containing K, which is given by $\widetilde{K} = \cap L$ for all closed fibrewise linear subsets $L \subset E$ with $K \subset L$.

Let Z be any subset of a C^∞ manifold M. Let $p \in Z$. Then first we consider the set of "geometric" tangent vectors to Z at p, which is defined by

$$(T^g Z)^\circ_p := \{[\gamma]_0 \, ; \, \gamma : (\mathbf{R}, 0) \to (M, p) \ C^\infty\text{curve}, \ \gamma(\mathbf{R}, 0) \subset Z\} \subset T_p M.$$

Here $[\gamma]_0$ means the tangent vector represented by the curve γ at 0: $[\gamma]_0 \in T_{\gamma(0)}M$. Note that $(T^g Z)^\circ_p \subset T_p M$ is not necessarily a linear subspace.

Moreover we set $(T^g Z)^\circ = \cup_{p \in Z} (T^g Z)^\circ_p$. Then the *geometric tangent bundle* $T^g Z \subset TM|_{\overline{Z}}$ is defined by the closed linear hull in $TM|_{\overline{Z}}$ of the set $(T^g Z)^\circ$.

Note that, in general, $T^g Z$ is not necessarily a subbundle of $TM|_{\overline{Z}}$ (not necessarily locally trivial).

We define the *geometric conormal bundle* $T^*_Z M$ as the "dual" of $T^g Z$:

$$T^*_Z M := \left\{\alpha \in T^*M|_{\overline{Z}} \ ; \ \alpha|_{(T^g Z)_p} = 0, \text{ if } \ \alpha \in T^*_p M \text{ for some } p \in \overline{Z}\right\},$$

where $(T^g Z)_p \subset T_p M$ is the fibre of $T^g Z$ over $p \in \overline{Z}$.

EXAMPLE 3.1. Set $Z := \{(x_1, x_2) \, ; \, x_1^3 - x_2^2 = 0\} \subset \mathbf{R}^2$. Set $Z_0 = \{\mathbf{0}\}$ and $Z_1 := \overline{Z \setminus Z_0}$. Then $(T^g Z)^\circ = TZ_1 \cup TZ_0$, which is fibrewise-linear. We have $T^g Z = \overline{TZ_1 \cup TZ_0} = \{(t^2, t^3, v_1, v_2) \in T\mathbf{R}^2 \, ; \, 3tv_1 - 2v_2 = 0, t \in \mathbf{R}\}$. In particular we have a parametrization of $T^g Z$ by $\mathbf{R}^2 \to T\mathbf{R}^2, (t, s) \mapsto (t^2, t^3, s, \frac{3}{2}ts)$. Moreover we have $T^*_Z \mathbf{R}^2 = \{(t^2, t^3, p_1, p_2) \in T^*\mathbf{R}^2 \, ; \, p_1 = -\frac{3}{2}tp_2, t \in \mathbf{R}, p_2 \in \mathbf{R}\}$, that is called the open Whitney umbrella ([11][12]).

Let $\wedge^k(TM)$ be the exterior product bundle of the tangent bundle TM. Generally we define the *geometric k tangent bundle* $T^{g,k} Z \subset \wedge^k(TM)$ for any $k \geq 1$ as follows: First we set, for $p \in Z$,

$$\begin{aligned}(T^{g,k} Z)^\circ_p := \{\rho \in \wedge^k(T_p M) \, ; \, \rho = (\wedge^k f_*)(u_1 \wedge \cdots \wedge u_k), \\ \text{for some } f : (\mathbf{R}^k, 0) \to (M, p), \ f(\mathbf{R}^k, 0) \subset Z, \ u_1, \ldots, u_k \in T_0 \mathbf{R}^k\}.\end{aligned}$$

Then we set $(T^{g,k}Z)^\circ = \cup_{p\in Z}(T^{g,k}Z)^\circ_p$. Finally we define $T^{g,k}Z$ by the closed linear hull of $(T^{g,k}Z)^\circ$ in $\wedge^k(TM)|_{\overline{Z}}$.

For $k = 0$, we set $T^{g,0}Z := \overline{Z} \times \mathbf{R} \subset M \times \mathbf{R}$.

Note that $T^{g,1}Z = T^gZ$. Also note that, if $Z \subset M$ is a closed submanifold of M, then $T^{g,k}Z = \wedge^k(TZ)$.

The natural paring $\langle\ ,\ \rangle : \wedge^k(T_p^*M) \times \wedge^k(T_pM) \to \mathbf{R}$ is defined by

$$\langle \theta_1 \wedge \cdots \wedge \theta_k, v_1 \wedge \cdots \wedge v_k\rangle := \det(\theta_i(v_j))_{1\le i,j\le k},$$

for $\theta_1, \dots, \theta_k \in T_p^*M$ and $v_1, \dots, v_k \in T_pM$. Then any k-form $\omega \in \Lambda^k(M)$ is regarded as a fibrewise linear and continuous function

$$\omega : \wedge^k(TM) \to \mathbf{R}.$$

For any subset $L \subset \wedge^k(TM)$, we write $\omega|_L$ the restricted function $\omega|_L : L \to \mathbf{R}$. If $L = \wedge^k(TM)|_Z$ for a subset $Z \subset M$, then the restriction $\omega|_L$ is written also by $\omega|_Z$ as usual. If $L = T^{g,k}Z$, then $\omega|_L$ is written by $\omega\|_Z$ to distinguish with $\omega|_Z$ in this paper. Therefore if $Z \subset M$ be a submanifold of M, then $\omega\|_Z = i^*\omega$, the pull-back for the inclusion $i : Z \hookrightarrow M$.

Example 3.2. In Example 3.1, we have $T^{g,2}Z = Z \times \{0\} \subset \wedge^2(T\mathbf{R}^2)$, the zero-section.

In §2 we have introduced the space $\mathcal{G}^k(Z, M)$ with zero geometric restrictions. To give its characterization, we first show the following.

Lemma 3.3.

$$\mathcal{G}^k(Z, M) = \{\omega \in \Lambda^k(M)\,;\, g^*\omega = 0 \text{ for any } g : (\mathbf{R}^k, 0) \to M \text{ with } g(\mathbf{R}^k, 0) \subset Z\}.$$

Proof. The inclusion "$\subset$"is clear by the definition. To show the reverse inclusion, we take ω from the right hand side and let $f : N \to M$ be any C^∞ map with $f(N) \subset Z$. Let $p \in N$ and $v_1, \dots, v_k \in T_pN$. Suppose $v_1 \wedge \cdots \wedge v_k \neq 0$. Take a C^∞ immersion-germ $h : (\mathbf{R}^k, 0) \to (N, p)$ such that $v_1, \dots, v_k \in h_*(T_0(\mathbf{R}^k))$. Take $w_1, \dots, w_k \in T_0(\mathbf{R}^k)$ such that $h_*(w_i) = v_i (1 \le i \le k)$. Set $g = f \circ h$. Then $0 = (g^*\omega)(w_1 \wedge \cdots \wedge w_k) = (f^*\omega)(v_1 \wedge \cdots \wedge v_k)$. Therefore we have $f^*\omega = 0$ at any $p \in N$. Thus we have $f^*\omega = 0$. Therefore $\omega \in \mathcal{G}^k(Z, M)$. □

Then we have a description of the space $\mathcal{G}^k(Z, M)$ with zero geometric restrictions:

Proposition 3.4. *We have*

$$\begin{aligned}\mathcal{G}^k(Z, M) &= \{\omega \in \Lambda^k(M)\,;\, \omega|_{T^{g,k}Z} = 0\}\\ &= \{\omega \in \Lambda^k(M)\,;\, \omega|_{T^{g,k}Z} = 0, d\omega|_{T^{g,k+1}Z} = 0\}.\end{aligned}$$

Proof. The inclusion "$\subset$" of the first equality: Take any $\omega \in \mathcal{G}^k(Z, M)$. Take any germ $f : (\mathbf{R}^k, 0) \to M$ with $f(\mathbf{R}^k, 0) \subset Z$. Then $f^*\omega = 0$. This means that ω vanishes on $(\wedge^k f_*)(\wedge^k(T_0\mathbf{R}^k))$. Therefore ω vanishes on $(T^{g,k}Z)^\circ$. Since ω is fibrewise linear and continuous, we have $\omega|_{T^{g,k}Z} = 0$.

The inclusion "$\supset$" of the first equality: Suppose ω has a non-null geometric restriction to Z. Then there exists a map-germ $f : (\mathbf{R}^k, 0) \to M$ with $f(\mathbf{R}^k, 0) \subset Z$ and $f^*\omega \neq 0$ by Lemma 3.3. Then $\omega((\wedge^k f_*)(\wedge^k(T_0\mathbf{R}^k)) \neq 0$. This means that $\omega|_{T^{g,k}Z} \neq 0$, which leads to a contradiction. Thus we have the first equality.

Since $\mathcal{G}^k(Z, M)$ is d-closed, we have the second equality. □

We show a general property of geometric k tangent bundles.

LEMMA 3.5. *Let Z be any subset of a C^∞ manifold M. Let $T^{g,k}Z \wedge T^{g,\ell}Z$ denote the closed linear hull of*

$$\{v \wedge w\,;\ v \in T^{g,k}_p Z,\ w \in T^{g,\ell}_p Z,\ (p \in \overline{Z})\}$$

in $\wedge^{k+\ell}(TM)|_{\overline{Z}}$. Then we have

$$T^{g,k+\ell}Z \subset T^{g,k}Z \wedge T^{g,\ell}Z\ (\subset \wedge^{k+\ell}(TM)).$$

In particular we have

$$T^{g,2}Z \subset T^{g,1}Z \wedge T^{g,1}Z\ (\subset \Lambda^2(TM)).$$

Proof. Take any $\rho \in (T^{g,k+\ell}Z)^\circ$. Then, for some $p \in Z$, there exist $f : (\mathbf{R}^{k+\ell}, 0) \to (M, p)$ with $f(\mathbf{R}^{k+\ell}, 0) \subset Z$ and $u \in \wedge^{k+\ell}T_0\mathbf{R}^{k+\ell}$ such that $\rho = (\wedge^{k+\ell} f_*)(u)$. Then there exist $v_1, \dots, v_k \in T_0(T^k \times \{0\})$ and $v_{k+1}, \dots, v_{k+\ell} \in T_0(\{0\} \times T^\ell)$ such that $u = v_1 \wedge \cdots v_k \wedge v_{k+1} \wedge \cdots \wedge v_{k+\ell}$. Define $f_k : (\mathbf{R}^k, 0) \to (M, p)$ by $f_k(x) = f(x, 0)$ and $f_\ell : (\mathbf{R}^\ell, 0) \to (M, p)$ by $f_\ell(y) = f(0, y)$. Let $v = (f_k)_*(v_1) \wedge \cdots \wedge (f_k)_*(v_k)$ and $w = (f_\ell)_*(v_{k+1}) \wedge \cdots \wedge (f_\ell)_*(v_{k+\ell})$. Then $\rho = v \wedge w$. Therefore $\rho \in T^{g,k}Z \wedge T^{g,\ell}Z$. Thus we have $(T^{g,k+\ell}Z)^\circ \subset T^{g,k}Z \wedge T^{g,\ell}Z$. Taking the closed linear hull of both sides of the inclusion, we have the required inclusion. □

Let Ω be the canonical symplectic form on T^*M. Since Ω is a 2-form on the cotangent bundle T^*M, it is regarded as a function $\Omega : \wedge^2(T(T^*M)) \to \mathbf{R}$.

PROPOSITION 3.6. *Let M be a C^∞ manifold and Z a closed subset of M. Consider the geometric conormal $T^*_Z M \subset T^*M$ of Z in M. Then the Liouville* 1-*form $\Theta \in \Lambda^1(T^*M)$ vanishes on $T^{g,1}(T^*_Z M)$. The symplectic form $\Omega \in \Lambda^2(T^*M)$ vanishes on $T^{g,1}(T^*_Z M) \wedge T^{g,1}(T^*_Z M)$. In particular $\Theta \in \mathcal{G}^1(T^*_Z M, T^*M)$ and $\Omega \in \mathcal{G}^2(T^*_Z M, T^*M)$.*

Proof. Let $(x_0, \alpha_0) \in T^*_Z M$ and $v \in T^{g,1}(T^*_Z M)^\circ_{(x_0,\alpha_0)}$. Let v be represented by a curve $\gamma : (\mathbf{R}, 0) \to (T^*M, (x_0, \alpha_0))$ with $\gamma(\mathbf{R}, 0) \subset T^*_Z M$. Set $\gamma(t) = (x(t), \alpha(t))$. Then $x(0) = x_0$ and $\alpha(0) = \alpha_0$. Note that $x : (\mathbf{R}, 0) \to \overline{Z}$. Therefore $x'(0) \in (T^{g,1}\overline{Z})^\circ$. Since $T^{g,1}Z = T^{g,1}\overline{Z} \supset (T^{g,1}\overline{Z})^\circ$, we have $\alpha_0(x'(0)) = 0$. Then $\Theta(v) = \alpha_0(x'(0)) = 0$. Therefore we have $\Theta|_{T^{g,1}(T^*_Z M)^\circ} = 0$ and thus we have $\Theta|_{T^{g,1}(T^*_Z M)} = 0$. By Proposition 3.4, we have $\Theta \in \mathcal{G}^1(T^*_Z M, T^*M)$. Since $\Omega = d\Theta$, we have $\Omega \in \mathcal{G}^2(T^*_Z M, T^*M)$. To see the last result in another way, we take another $w \in T^{g,1}(T^*_Z M)^\circ_{(x_0,\alpha_0)}$. Let w be represented by a curve $\delta : (\mathbf{R}, 0) \to (T^*M, (x_0, \alpha_0))$ with $\delta(\mathbf{R}, 0) \subset T^*_Z M$. Set $\delta(t) = (y(t), \beta(t))$. Then $y(0) = x_0, \beta(0) = \alpha_0$ and $\alpha_0(y'(0)) = 0$. Then $\Omega(v \wedge w) = \alpha_0(x'(0)) - \alpha_0(y'(0)) = 0$. Therefore Ω vanishes on $T^{g,1}(T^*_Z M) \wedge T^{g,1}(T^*_Z M)$. By Lemma 3.5, we see Ω vanishes on $T^{g,2}(T^*_Z M)$. Then, by Proposition 3.4, we have $\Omega \in \mathcal{G}^2(T^*_Z M, T^*M)$. □

4. Geometric restrictions to a stratified set. In the previous section we treat arbitrary subset in a manifold. Here we will give a simple description of the space of differential forms with null geometric restrictions for a stratified set.

Let M be a manifold and Z be a subset of M. We mean by a stratification $\mathcal{S} = \{S\}$ a locally finite collection of submanifolds of M giving a disjoint decomposition $Z = \cup_{S\in\mathcal{S}} S$ of Z.

LEMMA 4.1. (Geometric restrictions to a stratified set) *We have*

$$\mathcal{G}^\bullet(Z, M) = \{\alpha \in \Lambda^\bullet(M) \ ; \ \alpha\|_S = 0, \text{ for any stratum } S \in \mathcal{S}\}.$$

Here $\alpha\|_S := \alpha|_{\wedge^k(TS)}$, *if* α *is a k-form. Therefore, for any* $\alpha, \alpha' \in \Lambda^\bullet(M)$, $[\alpha]^g_Z = [\alpha']^g_Z \in \mathcal{G}^\bullet(Z)$ *if and only if* $\alpha\|_S = \alpha'\|_S$, *for any stratum* $S \in \mathcal{S}$.

Proof. The inclusion "$\subset$" is clear, by taking, as f in the definition of $\mathcal{G}^\bullet(Z, M)$, the inclusions of strata. To show "$\supset$", take $\alpha \in \Lambda^\bullet(M)$ from RHS, and take any $f : N \to M$ with $f(N) \subset Z$. Consider the decomposition $N = \cup_{S\in\mathcal{S}} f^{-1}(S)$. Take any point $t_0 \in N$. Since $\mathcal{S}$ is locally finite, there exists $S \in \mathcal{S}$ such that the closure of the interior $f^{-1}(S)^o$ of $f^{-1}(S)$ contains t_0. By the condition $\alpha\|_S = 0$, we have $f^*\alpha = 0$ on $f^{-1}(S)^o$. By the continuity, we have $(f^*\alpha)(t_0) = 0$. This shows that $f^*\alpha = 0$. Therefore we have that α belongs to $\mathcal{G}^\bullet(Z, M)$. The second statement is clear. □

LEMMA 4.2. *Suppose that the stratification* $\mathcal{S}$ *of* Z *satisfies the boundary condition, namely, for any* $S, S' \in \mathcal{S}, \overline{S} \cap S' \neq \emptyset$ *implies* $\overline{S} \supset S'$, *and the Whitney's regularity condition (a), namely, for any* $S, S' \in \mathcal{S}$, *for any* $x_0 \in S'$ *and for any sequence* $\{y_n\}$ *on* S *converging to* x_0, *if there exists a limit* $V = \lim_{n\to\infty} T_{y_n} S \subset T_{x_0} M$, *then* $V \supset T_{x_0} S'$. *If there exists a stratum* $S_{\max}$ *with* $\overline{S} = Z$. *Then,*

$$\mathcal{G}^\bullet(Z, M) = \{\alpha \in \Lambda^\bullet(M) \ ; \ \alpha\|_{S_{\max}} = 0\}.$$

Proof. By the boundary condition and Whitney regularity (a), $\alpha\|_{S_{\max}} = 0$ implies that $\alpha\|_S = 0$ for any $S \in \mathcal{S}$. Therefore, by Lemma 4.1, we have the equality. □

5. Algebraic conormals and tangents. Let Z be a subset of a manifold M. Let $p \in \overline{Z}$. We consider the "algebraic conormals" to Z at p:

$$(T^{a,*}_Z M)_p := \{dh \in T^*_p M \, ; \, h : (M, p) \to \mathbf{R}, h|_Z = 0\}.$$

Then $(T^{a,*}_Z M)_p$ is a linear subspace of $T^*_p M$. Consider its "dual":

$$(T^a Z)^\circ_p := \{v \in T_p M \, ; \, \langle dh, v\rangle = 0 \text{ for any function } h \in \Lambda^0(M) \text{ with } h|_Z = 0\}.$$

We call the linear subspace $(T^a Z)^\circ_p$ *Zariski tangent space* of Z at p, which is the set of "algebraic" tangent vectors to Z at p. Then *algebraic tangent bundle* $T^a Z \subset TM$ is defined by the closed linear hull in $TM|_{\overline{Z}}$ of the set $(T^a Z)^\circ = \cup_{p\in\overline{Z}} (T^a Z)^\circ_p$ of "algebraic" tangent vectors to Z. We call $T^a Z$ also *Zariski tangent bundle* of Z.

Note that, in general, $T^a Z$ is not necessarily a subbundle of $TM|_{\overline{Z}}$, as well as $T^g Z$.

Moreover, we define the *algebraic k tangent bundle* $T^{a,k}Z$ by the closed linear hull in $\wedge^k(TM)|_{\overline{Z}}$ of $(T^{a,k}Z)^\circ = \cup_{p\in\overline{Z}}(T^{a,k}Z)^\circ_p$, where

$$(T^{a,k}Z)^\circ_p := \{\rho \in \wedge^k(T_pM)\,;\, \langle d\beta, \rho\rangle = 0 \text{ for any } \beta \in \Lambda^{k-1}(M) \text{ with } \beta|_Z = 0\}.$$

If Z is a closed submanifold of M, then $T^{g,k}Z = T^{a,k}Z = \wedge^k(TZ)$, for any $k \geq 1$. Note that $T^{a,1}Z = T^aZ$. For $k = 0$, we set $T^{a,0}Z = T^{g,0}Z = \overline{Z} \times \mathbf{R} \subset M \times \mathbf{R}$. Then we have:

Lemma 5.1. *Let Z be any subset of a manifold M and $k \geq 1$. Then we have*
(1) *For any $p \in Z$, $\{0\} \subset (T^{g,k}Z)^\circ_p \subset (T^{a,k}Z)^\circ_p \subset \wedge^k(T_pM)$.*
(2) $Z \times \{0\} \subset (T^{g,k}Z)^\circ \subset (T^{a,k}Z)^\circ \subset \wedge^k(TM)|_{\overline{Z}} \subset \wedge^k(TM)$.
(3) $\overline{Z} \times \{0\} \subset T^{g,k}Z \subset T^{a,k}Z \subset \wedge^k(TM)|_{\overline{Z}} \subset \wedge^k(TM)$.

Proof. (1) Let $\rho \in (T^{g,k}Z)^\circ_p$. Then $\rho = (\wedge^k f_*)(u_1 \wedge \cdots \wedge u_k) = f_*u_1 \wedge \cdots \wedge f_*u_k$ for some $f : (\mathbf{R}^k, 0) \to (M, p)$ with $f(\mathbf{R}^k, 0) \subset Z$ and $u_i \in T_0\mathbf{R}^k, 1 \leq i \leq k$. Let $\beta \in \Lambda^{k-1}(M)$ with $\beta|_Z = 0$. Then β is expressed, on a coordinate neighbourhood U of p, as a sum of forms $b\gamma$ with $b \in \Lambda^0(U), b|_Z = 0, \gamma = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \Lambda^{k-1}(U)$. Then $d\beta$ is the sum of forms $db \wedge \gamma$. Then the paring $\langle d\beta, \rho\rangle$ is the sum of $\langle db \wedge \gamma, \rho\rangle$. Since $f^*b = 0$, we have $\langle db, f_*u_i\rangle = \langle df^*b, u_i\rangle = 0$ for any $1 \leq i \leq k$. Therefore we have $\langle db \wedge \gamma, \rho\rangle = 0$ and we have $\langle d\beta, \rho\rangle = 0$. Thus we have $\rho \in (T^{a,k}Z)^\circ_p$. Therefore we have $(T^{g,k}Z)^\circ_p \subset (T^{a,k}Z)^\circ_p$. Other inclusions are clear. The assertions (2) and (3) follow from (1). □

Recall that $\mathcal{A}^k(Z, M)$ denotes the set of differential k-forms with null algebraic restrictions. Then we have:

Proposition 5.2.

$$\mathcal{A}^k(Z, M) \subset \{\omega \in \Lambda^k(M)\,;\, \omega|_{T^{a,k}Z} = 0,\ d\omega|_{T^{a,k+1}Z} = 0\}.$$

Proof. Let $\omega \in \mathcal{A}^k(Z, M)$. Then $\omega = \alpha + d\beta$, for a k-form α vanishing on Z and a $(k-1)$-from β vanishing on Z. Take $\rho \in (T^{a,k}Z)^\circ$. Let $\rho \in (T^{a,k}Z)^\circ_p$ for some $p \in \overline{Z}$. Then $\langle \alpha, \rho\rangle = 0$ since $\alpha(p) = 0$. Moreover $\langle d\beta, \rho\rangle = 0$ since $\rho \in (T^{a,k}Z)^\circ_p$. Thus we have $\langle \omega, \rho\rangle = 0$. Therefore we have $\omega|_{(T^{a,k}Z)^\circ} = 0$ and thus we have $\omega|_{T^{a,k}Z} = 0$.

Furthermore, for any $\rho' \in (T^{a,k+1}Z)^\circ$, $\langle d\omega, \rho'\rangle = \langle d\alpha, \rho'\rangle = 0$ by the definition of $(T^{a,k+1}Z)^\circ$. Thus we have $d\omega|_{T^{a,k+1}Z} = 0$. □

Remark 5.3. In Proposition 5.2, the equality does not hold in general. See Example 5.4.

Example 5.4. In Example 3.1, we have

$$(T^{a,1}Z)^\circ = \{(x_1, x_2, v_1, v_2)\,;\, x_1^3 - x_2^2 = 0, 3x_1^2v_1 - 2x_2v_2 = 0\} = TZ_1 \cup T_0\mathbf{R}^2,$$

which is closed in $T\mathbf{R}^2$. Therefore we have $T^{a,1}Z = (T^{a,1}Z)^\circ$. Further we have $(T^{a,2}Z)^\circ = \wedge^2(T_0\mathbf{R}^2) \cup (Z_1 \times \{0\}) = T^{a,2}Z \subset \wedge^2(T\mathbf{R}^2)|_Z$.

Let $\omega = -3x_1x_2dx_1 + 2x_1^2dx_2$. Then $d\omega = 7x_1dx_1 \wedge dx_2$. Then $\omega|_{T^{a,1}Z} = 0$, $d\omega|_{T^{a,2}Z} = 0$. However $\omega \notin \mathcal{A}^1(Z, \mathbf{R}^2)$.

REMARK 5.5. By Proposition 5.2 and Remark 5.3, it is interesting to study the space

$$\widetilde{\mathcal{A}}^k(Z,M) := \{\omega \in \Lambda^k(M)\,;\, \omega|_{T^{a,k}Z} = 0,\ d\omega|_{T^{a,k+1}Z} = 0\}.$$

6. Geometric and algebraic restrictions via a mapping. Let $f : N \to M$ be a C^∞ mapping from a C^∞ manifold N.

Let $\omega \in \Lambda^\bullet(M)$ be a differential form on M. Then we call the pull-back $f^*\omega$ the *geometric restriction of* ω *by* f. Then, regarding the morphism $f^* : \Lambda^\bullet(M) \to \Lambda^\bullet(N)$, we consider the subspace consisting of differential forms with null geometric restrictions by f:

$$(\mathrm{Ker} f^*)^\bullet := \{\omega \in \Lambda^\bullet(M)\,;\, f^*\omega = 0\}.$$

Then we have

$$(\mathrm{Ker} f^*)^\bullet \supset \mathcal{G}^\bullet(f(N), M).$$

Let $Z \subset M$ be any subset of M. We say that a C^∞ map $f : N \to M$ *dominates* $Z \subset M$ *geometrically*, if $f(N) \subset Z$ and the closed linear hull of $\wedge^k f_*(\wedge^k(TN))$ in $\wedge^k(TM)$ contains $T^{g,k}Z$ for any $k \geq 1$. See §3.

LEMMA 6.1. *Suppose* $f : N \to M$ *dominates* $Z \subset M$ *geometrically. Then we have*

$$(\mathrm{Ker} f^*)^\bullet = \mathcal{G}^\bullet(Z, M).$$

Proof. The inclusion "$\supset$" is clear by the definition.

To show "$\subset$", take any $\omega \in \Lambda^k(M)$ with $f^*\omega = 0$. Then ω restricted to $(\wedge^k f_*)(\wedge^k(TN))$ vanishes, so, by the assumption it is on the closed linear hull of $(\wedge^k f_*)(\wedge^k(TN))$, so it is on $T^{g,k}Z$. By Lemma 3.4, we have $\alpha \in \mathcal{G}^k(f(N), M)$. □

The space of geometric restrictions by f of differential forms, which is identified with

$$\mathcal{G}^\bullet(f) := \Lambda^\bullet(M)/(\mathrm{Ker} f^*)^\bullet,$$

has the natural module structure over the de Rham exterior algebra $\Lambda^\bullet(M)$.

In the case $Z = f(N)$, we describe $\mathcal{A}^k(Z,M)$ in terms of mapping f.

First we introduce the space

$$\Lambda^k(f) = \{\beta : N \to \wedge^k(T^*M)\,;\, \beta \text{ covers } f \text{ via the projection } \pi : \wedge^k(T^*M) \to M\},$$

the space of differential k-forms along f, and a morphism $\omega f : \Lambda^\bullet(M) \to \Lambda^\bullet(f)$ defined by $\alpha \mapsto \alpha \circ f$. Here $\wedge^k(T^*M)$ is the exterior product of the cotangent bundle T^*M. The notion ωf is used, based on the classical Mather's notation. As for Mather's notation, we define also a morphism $t^*f : \Lambda^\bullet(f) \to \Lambda^\bullet(N)$, by

$$(t^*f(\beta))(x) = \wedge^k(f_{*x})^*(\beta(x))$$

where $\wedge^k (f_{*x})^* : \wedge^k T^*_{f(x)} M \to \wedge^k T^*_x N$ is the wedge of the dual linear map of the differential map $f_{*x} : T_x N \to T_{f(x)} M$.

We have the commutative diagram for $k \geq 1$,

$$\begin{array}{ccccc} \Lambda^{k-1}(M) & \xrightarrow{\omega^{k-1} f} & \Lambda^{k-1}(f) & \xrightarrow{t^{*k-1} f} & \Lambda^{k-1}(N) \\ d \downarrow & & & & \downarrow d \\ \Lambda^k(M) & \xrightarrow{\omega^k f} & \Lambda^k(f) & \xrightarrow{t^{*k} f} & \Lambda^k(N). \end{array}$$

Note that $t^{*0} f$ gives the identification of $\Lambda^0(f)$ and $\Lambda^0(N)$, which is the space of sections of the trivial line bundle.

The following is clear by the definition of $\mathcal{A}(f(N), M)$:

LEMMA 6.2. *Let $f : N \to M$ be a C^∞ mapping. Then we have, for any $k \geq 0$,*

$$\text{Ker } \omega^k f + d(\text{Ker } \omega^{k-1} f) = \mathcal{A}^k(f(N), M).$$

We study the quotient space

$$\mathcal{R}^k(f) := (\text{Ker } f^*)^k / \mathcal{A}^k(f(N), M) = (\text{Ker } f^*)^k / (\text{Ker } \omega^k f + d(\text{Ker } \omega^{k-1} f)),$$

which is the space of algebraic restrictions to the image of f with null geometric restrictions by f.

The constructions above are localized, i.e. they are formulated in terms of sheaves naturally. From now on we treat the local cases only.

The following is clear:

LEMMA 6.3. *If $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ is an immersion-germ, then*

$$\mathcal{R}^k(f) = \text{Ker } f^{*k} / (\text{Ker } \omega^k f + d(\text{Ker } \omega^{k-1} f)) = 0,$$

for $k \geq 0$.

Moreover we have,

PROPOSITION 6.4. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$, $2n \leq m$, be a finitely determined map-germ, Z the germ of the image of f. Then the $\mathbf{R}$-vector space $\mathcal{R}^\bullet(Z) = \mathcal{R}^\bullet(f)$ is of finite dimension.*

Proof. We may suppose f is an analytic map-germ. Then f dominates Z geometrically. Therefore, by Lemma 6.1, we have $\mathcal{R}^\bullet(Z) = \mathcal{R}^\bullet(f)$. Consider the complexification $f_{\mathbf{C}} : (\mathbf{C}^n, 0) \to (\mathbf{C}^m, 0)$ of f. Then $\text{Ker} f_{\mathbf{C}}{}^*$ and $\text{Ker } \omega f_{\mathbf{C}} + d(\text{Ker } \omega f_{\mathbf{C}})$ are coherent submodules of $\Lambda^\bullet(\mathbf{C}^m, 0)$ over $\Lambda^0(\mathbf{C}^m, 0)$. Therefore $\mathcal{R}^\bullet(f_{\mathbf{C}})$ is also coherent. By Lemma 6.3, the support of $\mathcal{R}^\bullet(f_{\mathbf{C}})$ is just the origin. Then by using Nullstellensatz in the form of [18], we have that $\mathcal{R}^\bullet(f_{\mathbf{C}})$ is a finite dimensional vector space. Consider $\mathcal{R}^{\bullet,\omega}(f)$ which is defined similarly as $\mathcal{R}^\bullet(f)$ but by real analytic forms. Then we have that $\mathcal{R}^{\bullet,\omega}(f)$ is also of finite dimension. Moreover we can show that $\mathcal{R}^\bullet(f)$ is formally generated by $\mathcal{R}^{\bullet,\omega}(f)$, so it is generated by $\mathcal{R}^{\bullet,\omega}(f)$ over $\Lambda^0(\mathbf{R}^m, 0)$ (see [3]). Therefore we have that $\mathcal{R}^\bullet(f)$ is also of finite dimensional. □

REMARK 6.5. W. Domitrz [4] shows that the subspace of algebraic restrictions of closed 2-forms in $\mathcal{R}^2(C)$ on any analytic curve C is a finite dimensional vector space. Proposition 6.4 generalizes Domitrz's theorem under the assumption of finite determinacy.

Let $Z \subset (\mathbf{R}^m, 0)$ be a subset-germ in $\mathbf{R}^m$ at 0. The *embedding dimension* of Z is defined as the minimum of the dimensions of submanifold-germs $S \subset (\mathbf{R}^m, 0)$ with $Z \subset S$.

LEMMA 6.6. *Suppose the embedding dimension of $Z \subset (\mathbf{R}^m, 0)$ is equal to r. Let $S \subset (\mathbf{R}^m, 0)$ be a submanifold-germ of dimension r with $Z \subset S$. Let $h : (\mathbf{R}^m, 0) \to \mathbf{R}$ be a function-germ vanishing on Z. Then we have $dh|_{T_0S} = 0$. Therefore the tangent space T_0S to a submanifold-germ S of $(\mathbf{R}^m, 0)$ of dimension r containing Z is uniquely determined. In fact T_0S coincides with the Zariski tangent space $(T^aZ)^\circ_0$ of Z at 0 in $\mathbf{R}^m$ (see §5).*

Proof. Assume $dh|_{T_0S} \neq 0$. Then $h^{-1}(0) \subset (\mathbf{R}^m, 0)$ is a C^∞ hypersurface which is transverse to S. Then $h^{-1}(0) \cap S$ is a submanifold of $r - 1$ which contains Z. This leads to a contradiction with the assumption that the embedding dimension of Z is r. Thus we have

$$\begin{array}{rcl} T_0S & \subset & \{v \in T_0\mathbf{R}^m\,;\, \langle dh, v\rangle = 0 \text{ for any function-germ } h : (\mathbf{R}^m, 0) \to \mathbf{R} \text{ with } h|_Z = 0\} \\ & = & (T^aZ)^\circ_0. \end{array}$$

For any vector $v \notin T_0S$, there exists a function-germ $h : (\mathbf{R}^m, 0) \to \mathbf{R}$ with $h|_S = 0$ and $\langle dh, v\rangle \neq 0$. Therefore we have the equality $T_0S = (T^aZ)^\circ_0$. □

LEMMA 6.7. *For any $k = 1, 2, \dots, r$, any k-form α in $\mathcal{A}^k(Z, M)$, α vanishes on $\wedge^k(T^aZ)^\circ_0$.*

Proof. We remark that $\wedge^k(T^aZ)^\circ_0 \subset T^{a,k}Z_0$. Then by Proposition 5.2 we have the result. □

Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ be a germ of a proper mapping. Then the germ of the image of f is well-defined as a subset-germ in $(\mathbf{R}^m, 0)$. Therefore the embedding dimension of f is defined via the image of f.

PROPOSITION 6.8. *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ be a proper map-germ. Suppose the embedding dimension of f is equal to $r > n$. Let $S \subset (\mathbf{R}^m, 0)$ be a minimal dimensional submanifold-germ containing the image of f with* $\dim S = r$. *Then, for any $k = 1, 2, \dots, r$, any k-form in* Ker $\omega^k f + d($Ker $\omega^{k-1} f)$ *vanishes at T_0S. In particular we have*

$$\mathcal{R}^r(f) \neq 0.$$

Proof. The first half follows from Proposition 6.7. To show the second half, we take an r-form ω on $(\mathbf{R}^m, 0)$ such that $(\omega||_S)(0) \neq 0$. Then the geometric restriction of ω to the image of f is not equal to zero. Thus we see that the class of ω in $\mathcal{R}^r(f)$ is not equal to zero. □

EXAMPLE 6.9. Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^3, 0)$, $f(t) = (\frac{1}{3!}t^3, \frac{1}{4!}t^4, \frac{1}{5!}t^5)$. Then the embedded dimension of f is equal to 3. Then, in fact, the geometric restriction $[dx_1 \wedge dx_2 \wedge dx_3]^g = 0$ and the algebraic restriction $[dx_1 \wedge dx_2 \wedge dx_3]^a \neq 0$. Therefore the residue of volume form $[dx_1 \wedge dx_2 \wedge dx_3]^r \neq 0$ in $\mathcal{R}^3(f)$.

7. Residues for hypersurfaces. Let $F : (\mathbf{R}^m, 0) \to (\mathbf{R}, 0)$ be a non-zero analytic function-germ and consider the set-germ $Z \subset (\mathbf{R}^m, 0)$. Suppose the ideal $I_Z := \Lambda_Z(\mathbf{R}^m, 0) \subset \mathcal{O}_m := \Lambda^0(\mathbf{R}^m, 0)$ of function-germs vanishing on Z is generated by F. Then we have on the residues of top degree:

Proposition 7.1. *$\mathcal{R}^m(Z) \cong \mathcal{O}_m / \langle F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_m} \rangle_{\mathcal{O}_m}$. In particular $\dim_{\mathbf{R}} \mathcal{R}^m(Z)$ is given by the Turina number of F at 0.*

Proof. Let α be any m-form on $M = (\mathbf{R}^m, 0)$. Then $\alpha \in \mathcal{G}^m(Z, M)$. We have that $\alpha \in \mathcal{A}^m(Z, M)$ if and only if there exist an m-form β and an $(m-1)$-form γ such that $\alpha = F\beta + d(F\gamma)$. Take the volume form $\omega = dx_1 \wedge \cdots \wedge dx_m$. There exists a unique $h \in \mathcal{O}_m$ with $\alpha = h\omega$. Then $\alpha \in \mathcal{A}^m(Z, M)$ if and only if $h \in \langle F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_m} \rangle_{\mathcal{O}_m}$. Thus we have the result. ∎

For the residue of degree 1 of hypersurface, we have:

Proposition 7.2. *Let $F : M = (\mathbf{R}^m, 0) \to (\mathbf{R}, 0)$ be a C^∞ function-germ. Let Z denote the germ of zero-locus of F in $(\mathbf{R}^m, 0)$. Suppose the ideal $I(T^g Z)$ of function-germs on $(T\mathbf{R}^m, (0, 0))$ vanishing on the geometric tangent bundle $T^g Z \subset T\mathbf{R}^m$ is generated by*

$$\sum_{i=1}^{m} v_i \frac{\partial F}{\partial x_i}(x) \text{ and } F(x).$$

Here (x, v) denote the system of coordinate functions on $T\mathbf{R}^m$. Then we have $\mathcal{R}^1(Z) = 0$.

Proof. Let $\alpha \in \mathcal{G}^1(Z, M)$. Write $\alpha = \sum_{i=1}^m a_i dx_i$. By Proposition 3.4, α vanishes on the geometric tangents $T^g Z$. Then by the assumption, there exist C^∞ functions $B(x, v), C(x, v)$ such that

$$\sum_{i=1}^{m} a_i(x) v_i = B(x, v) \left(\sum_{i=1}^{m} v_i \frac{\partial F}{\partial x_i}(x) \right) + C(x, v) F(x),$$

on $(T\mathbf{R}^m, (0, 0))$. By differentiating by v_i, we have

$$a_i = \frac{\partial B}{\partial v_i} \left(\sum_{i=1}^{m} v_i \frac{\partial F}{\partial x_i} \right) + B \frac{\partial F}{\partial x_i} + \frac{\partial C}{\partial v_i} F.$$

Setting $v = 0$, we have

$$a_i(x) = B(x, 0) \frac{\partial F}{\partial x_i}(x) + \frac{\partial C}{\partial v_i}(x, 0) F(x).$$

Then

$$\alpha = \sum_{i=1}^{m} a_i dx_i = B(x, 0) dF(x) + F(x) \left(\frac{\partial C}{\partial v_i}(x, 0) dx_i \right) \in \mathcal{A}^1(Z, M).$$

∎

By the similar proof of Proposition 7.2, we have:

Proposition 7.3. *Let $F_1, \dots, F_r : M = (\mathbf{R}^m, 0) \to (\mathbf{R}, 0)$ be C^∞ function-germs. Let Z denote the germ of zero-locus of $F = (F_1, \dots, F_r) : M \to (\mathbf{R}^r, 0)$ in*

M. Suppose the ideal $I(T^gZ)$ of function-germs on $(T\mathbf{R}^m,(0,0))$ vanishing on the geometric tangent bundle $T^gZ \subset T\mathbf{R}^m$ is generated by

$$\sum_{i=1}^{m} v_i \frac{\partial F_j}{\partial x_i}(x), 1 \leq j \leq r, \text{ and } F_j(x), 1 \leq j \leq r.$$

Then we have $\mathcal{R}^1(Z) = 0$.

8. Residues for Lagrangian varieties. Now we suppose M is a symplectic manifold of dimension $2n$ with a symplectic form Ω. A subset $Z \subset M$ is called a *Lagrangian variety* if the geometric restriction $[\Omega]_Z^g = 0$ and the maximal rank of the geometric tangent bundle $T^gZ \subset TM$ is equal to n (see §3).

We describe $\mathcal{R}^1(Z)$ in terms of vector fields via the symplectic duality. The space of vector fields $V(M)$ over M corresponds to the space of 1-forms $\Lambda^1(M)$ by

$$X \mapsto X^\sharp := i_X\Omega \in \Lambda^1(M),\ (X \in V(M)).$$

The inverse of the correspondence is written, for any $\alpha \in \Lambda^1(M)$, by $\alpha \mapsto \alpha^\flat \in V(M)$. If $\alpha = dH$ for some $H \in \Lambda^0(M)$, then $X_H := (dH)^\flat$ is the Hamiltonian vector field with the Hamiltonian H.

If $M = \mathbf{R}^{2n}$ with the symplectic coordinates (x,p), then a vector filed $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n b_j \frac{\partial}{\partial p_j}$ corresponds to the 1-form $\omega = -\sum_{j=1}^n b_j dx_j + \sum_{i=1}^n a_i dp_i$.

The tangent bundle TM is identified with the cotangent bundle T^*M. Therefore to any subset $S \subset TM$, there corresponds a subset $S^\flat \subset T^*M$.

Similarly the space of 2-vector fields $V^2(M)$ corresponds to the space of functions $\Lambda^0(M)$ by

$$X \wedge Y \mapsto i_Y i_X\Omega \in \Lambda^0(M),\ (X \wedge Y \in V^2(M)).$$

The space of 0-vector fields (i.e. functions) $V^0(M)$ corresponds to $\Lambda^2(M)$ simply by

$$h \mapsto h\Omega \in \Lambda^2(M),\ (h \in V^0(M)).$$

Let $Z \subset M$ be a Lagrangian variety. Let $\omega \in \Lambda^1(M)$. Then $\omega \in \mathcal{G}^1(Z,M)$ if and only if the corresponding vector field $X = \omega^\flat$ (satisfying $i_X\Omega = \omega$) is tangent to Z_{reg}. In fact $0 = \omega(T_pZ) = i_X\Omega(T_pZ) = \Omega(X, T_pZ)$, so $X(p) \in T_pZ$, for any regular point $p \in Z_{\mathrm{reg}}$. A vector field X over M is called *logarithmic* if it is tangent to Z_{reg}. The 1-form ω belongs to $\mathcal{A}^1(Z,M)$ if and only if $\omega^\flat = X + X_H$ for a vector field X vanishing on Z and the Hamiltonian vector field X_H of a Hamiltonian function H vanishing on Z. In fact $\omega \in \mathcal{A}^1(Z,M)$ if and only if there exist a 1-form α vanishing on Z and a function H vanishing on Z such that $\omega = \alpha + dH$. Then $\omega^\flat = \alpha^\flat + (dH)^\flat$ and $\alpha^\flat$ vanishes on Z if and only if α vanishes on Z. Moreover we have $(dH)^\flat = X_H$. Thus we have:

Proposition 8.1. *The first order residue $\mathcal{R}^1(Z)$ is isomorphic as $\Lambda^0(M)$-module to the space of logarithmic vector fields modulo Hamiltonian vector fields restricted to $TM|_Z$.*

9. Residues for plane curve-germs. In this section we study the residues $\mathcal{R}^1(Z)$ for a germ of plane curve Z in $\mathbf{R}^2$, as a special case of our arguments discussed in the previous section. We also treat $\mathcal{R}^1(f)$ on the parametric case $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$.

Now our idea to treat plane curves is to fix a symplectic form (an area form) Ω on $\mathbf{R}^2$, say,

$$\Omega = dx_1 \wedge dx_2,$$

and apply the classification established in [13][14]. We conclude the paper by showing several examples from our previous classification result. Though the classification was performed in complex analytic case in [14], we can give the real classification by adding necessary $\pm$ to the lists.

Proposition 9.1. (cf. [13][14]) *Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a simple or a unimodal map-germ under diffeomorphism equivalence to the symplectic plane with the symplectic form $\Omega = dx_1 \wedge dx_2$. Then f is symplectomorphic to one of the following normal forms of map-germs $(x_1(t), x_2(t)) : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$:*

$$\begin{array}{ll}
A_{2\ell}: & (t^2, t^{2\ell+1}), \\
E_{6\ell}: & (t^3, (\pm)^{\ell+1} t^{3\ell+1} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1}), \\
E_{6\ell+2}: & (t^3, (\pm)^{\ell} t^{3\ell+2} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1}), \\
W_{12}: & (t^4, t^5 + \lambda_1 t^7), \\
W_{18}: & (t^4, t^7 + \lambda_1 t^9 + \lambda_2 t^{13}), \\
W^{\#}_{1,2\ell-1}: & (t^4, \pm t^6 + \lambda_1 t^{2\ell+5} + \lambda_2 t^{2\ell+9}), \lambda_1 \neq 0, \quad (\ell = 1, 2, \dots)
\end{array}$$

$$\begin{array}{ll}
N_{20}: & (t^5, t^6 + \lambda_1 t^8 + \lambda_2 t^9 + \lambda_3 t^{14}), \\
N_{24}: & (t^5, \pm t^7 + \lambda_1 t^8 + \lambda_2 t^{11} + \lambda_3 t^{13} + \lambda_4 t^{18}), \\
N_{28}: & (t^5, t^8 + \lambda_1 t^9 + \lambda_2 t^{12} + \lambda_3 t^{14} + \lambda_4 t^{17} + \lambda_5 t^{22}), \\
W_{24}: & (t^4, t^9 + \lambda_1 t^{10} + \lambda_2 t^{11} + \lambda_3 t^{15} + \lambda_4 t^{19}), \\
W_{30}: & (t^4, t^{11} + \lambda_1 t^{13} + \lambda_2 t^{14} + \lambda_3 t^{17} + \lambda_4 t^{21} + \lambda_5 t^{25}), \\
W^{\#}_{2,2\ell-1}: & (t^4, \pm t^{10} + \lambda_1 t^{2\ell+9} + \lambda_2 t^{2\ell+11} + \lambda_3 t^{2\ell+13} \\
 & \qquad + \lambda_4 t^{2\ell+17} + \lambda_5 t^{2\ell+21}), \lambda_1 \neq 0, \quad (\ell = 1, 2, \dots)
\end{array}$$

Remark 9.2. In any case, $T^g(Z)$ is the union of the tangent line ℓ at the origin and $T(Z_{\rm reg})$. Suppose f is not an immersion. Then $T^a(Z)$ is the union of $T_0\mathbf{R}^2$ and $T(Z_{\rm reg})$. Therefore $T^g(Z) \neq T^a(Z)$. Moreover both $T^g(Z)^\sharp$ and $T^a(Z)^\sharp$ are Lagrange varieties in $T^*\mathbf{R}^2$.

Let us consider the complexification $f_{\mathbf{C}} : (\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$ of f and $Z_{\mathbf{C}}$ the image of $f_{\mathbf{C}}$. A holomorphic vector field which is tangent to the regular part of $Z_{\mathbf{C}}$ is called a *logarithmic vector field* of $Z_{\mathbf{C}}$ (see [17][15]). Then it is known that the module $\mathrm{Der}(-Z_{\mathbf{C}})$ of logarithmic vector fields is free of rank 2. If $Z_{\mathbf{C}}$ is quasi-homogeneous, then it is generated by the Euler form and the Hamiltonian vector field X_F of the defining equation F of $Z_{\mathbf{C}}$. Here the Euler form is defined by

$$E^\sharp = -sx_2 dx_1 + rx_1 dx_2,$$

from the Euler vector field $E = rx_1 \frac{\partial}{\partial x_1} + sx_2 \frac{\partial}{\partial x_2}$, if $f = (f_1, f_2)$ is quasi-homogeneous by the weights $w(x_1) = r, w(x_2) = s$.

Proposition 9.3. (A_2) *In the case* A_2, $\dim_{\mathbf{R}} \mathcal{R}^1(Z) = 2$. *In fact* $\mathcal{R}^1(Z) = \mathcal{R}^1(f)$ *and it is generated over* $\mathbf{R}$ *by the classes of* $E^\sharp$ *and of* $x_1 E^\sharp$.

Proof. Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a map-germ defined by $f(t) = (t^2, t^3)$. Let $\alpha = a_1(x_1, x_2)dx_1 + a_2(x_1, x_2)dx_2 \in \Lambda^1(\mathbf{R}^2, 0)$. The condition that $f^*\alpha = 0$ is equivalent to that $2a_1(t^2, t^3) + 3a_2(t^2, t^3)t = 0$. Take $a_1 = -3x_2, a_2 = 2x_1$. Then we have the Euler form $E^\flat = -3x_2dx_1 + 2x_1dx_2 \in \mathrm{Ker} f^*$. The denominator $\mathcal{A}^1(Z, \mathbf{R}^2) = \mathrm{Ker}\ \omega f + d(\mathrm{Ker}\ \omega f)$ is generated by

$$(x_1^3 - x_2^2)dx_1,\ (x_1^3 - x_2^2)dx_2,\ x_1^2dx_1 - 3x_2dx_2.$$

Then $[E^\sharp] \neq 0$ in $\mathcal{R}^1(Z) = \mathcal{R}^1(f) = \mathrm{Ker}\ f^*/\mathrm{Ker}\ \omega f + d(\mathrm{Ker}\ \omega f)$. Therefore $\mathcal{R}^1(Z) \neq 0$. Moreover we see that $\mathcal{R}^1(Z)$ is generated over $\mathbf{R}$ by the classes of $E^\sharp$ and of $x_1E^\sharp$. In fact, $x_2E^\sharp$ and $x_1^2E^\sharp$ belong to $\mathcal{A}^1(Z, \mathbf{R}^2)$. □

Remark 9.4. Note that, in Proposition 9.3, the Lagrange variety $T^gZ^\sharp$ is the *open Whitney umbrella*, the conormal bundle of $(2,3)$ cusp and $T^aZ^\sharp$ is the "*open Whitney full-umbrella*", the union of open Whitney umbrella and the fibre through the origin (see [11][12]).

Proposition 9.5. (A_4) *Let* $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ *be a map-germ defined by* $f(t) = (t^2, t^5)$. *Then we have the Euler form* $E^\sharp = -5x_2dx_1 + 2x_1dx_2 \in \mathrm{Ker} f^*$ *and* $[E^\sharp] \neq 0$ *in* $\mathcal{R}^1(Z) = \mathcal{R}^1(f)$.

Proof. Let $\alpha = a_1(x_1, x_2)dx_1 + a_2(x_1, x_2)dx_2 \in \Lambda^1(\mathbf{R}^2, 0)$. The condition that $f^*\alpha = 0$ is equivalent to that $2a_1(t^2, t^5) + 5a_2(t^2, t^5)t^3 = 0$. Take $a_1 = -5x_2, a_2 = 2x_1$. The fact $[E^\sharp] \neq 0$ is checked by a direct calculation. □

Proposition 9.6. (E_6) *Let* $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ *be a map-germ defined by* $f(t) = (t^3, t^4)$. *Then we have the Euler form* $E^\flat = -4x_2dx_1 + 3x_1dx_2 \in \mathrm{Ker} f^*$ *and* $[E^\flat] \neq 0$ *in* $\mathcal{R}^1(Z) = \mathcal{R}^1(f) = \mathrm{Ker}\ f^*/\mathrm{Ker}\ \omega f + d(\mathrm{Ker}\ \omega f)$. *Therefore* $\mathcal{R}^1(Z) \neq 0$. *Moreover we see that* $\mathcal{R}^1(Z)$ *is generated over* $\mathbf{R}$ *by the classes of* $E^\sharp, x_1E^\sharp, x_2E^\sharp, x_1^2E^\sharp$.

Proof. Let $\alpha = a_1(x_1, x_2)dx_1 + a_2(x_1, x_2)dx_2 \in \Lambda^1(\mathbf{R}^2, 0)$. The condition that $f^*\alpha = 0$ is equivalent to that $3a_1(t^3, t^4) + 4a_2(t^3, t^4)t = 0$. Take $a_1 = -4x_2, a_2 = 3x_1$. Then we have the Euler form. The fact $[E^\sharp] \neq 0$ is checked by a direct calculation. The remaining follows from that $x_1^3E^\sharp, x_1x_2E^\sharp$ and $x_2^2E^\sharp$ belong to $\mathcal{A}^1(Z, \mathbf{R}^2)$. □

Example 9.7. (E_{12}) Let $f : (\mathbf{R}, 0) \to (\mathbf{R}^2, 0)$ be a map-germ defined by $f(t) = (x, y) = (t^3, \pm t^7 + \lambda t^8)$, $(\lambda \in \mathbf{R})$. This is not of quasi-homogeneous, if $\lambda \neq 0$. The defining equation of the image Z of f is given by

$$F(x, y) = x^7 \pm 3\lambda x^5y \pm \lambda^3x^8 \mp y^3.$$

By direct calculations, we see that, at least formally, $\mathcal{G}^1(Z, \mathbf{R}^2)$ is generated by the 1-forms

$$\omega_1 = -(\pm 49xy - 15\lambda^2x^4 - 57\lambda^3x^2y + 8\lambda^4y^2 \mp \lambda^5x^5)dx + 3(7x^2 - \lambda y)(1 \mp \lambda^3x)dy,$$

and

$$\omega_2 = -(\pm 7x^2y \pm \lambda y^2 - 2\lambda^2x^5 - 8\lambda^3x^3y)dx + 3x^2(1 \mp \lambda^3x)dy,$$

if $\lambda \neq 0$.

Then residue $\mathcal{R}^1(Z)$ of degree 1 is generated by $[\omega_1]^a$ and $[\omega_2]^a$ over $\Lambda^0(\mathbf{R}^2)$. If $\lambda = 0$, $\mathcal{R}^1(Z)$ is generated by only $[\omega_1]^a$.

Proof. The calculations, for the case $f(t) = (x, y) = (t^3, t^7 + \lambda t^8)$, go like this: Let $\omega = a(x, y)dx + b(x, y)dy$. Then $f^*\omega = 0$ if and only if $a \circ f = (b \circ f)(7t^4 + 8\lambda t^5)$. We set the ideal

$$I = \{b \in \mathcal{O}_{\mathbf{R}^2,0}\,;\,(b \circ f)(7t^4 + 8\lambda t^5) \in f^*\mathcal{O}_{\mathbf{R}^2,0}\},$$

and find the generator of I. Regarding that $\mathcal{O}_{\mathbf{R},0}/f^*\mathcal{O}_{\mathbf{R}^2,0}$ is generated by $1, t^2, t^4, t^5, t^8, t^{11}$ over $\mathbf{R}$, we find that $7x^2 - \lambda y \in I$ and $x^3 \in I$. In fact, since $y(7t^4 + 8\lambda t^5) = 7t^{11} + 15\lambda x^4 + 8\lambda^2 t^{13}$ and $x^2(7t^4 + 8\lambda t^5) = 7xy + \lambda t^{11}$, we have

$$\begin{aligned}
&(7x^2 - \lambda y)(7t^4 + 8\lambda t^5) = 49xy - 15\lambda^2 x^4 - 8\lambda^3 t^{13} = 49xy - 15\lambda^2 x^4 - 8\lambda^3 x^2 y + 8\lambda^4 t^{14} \\
&= 49xy - 15\lambda^2 x^4 - 8\lambda^3 x^2 y + 8\lambda^4 y^2 - 16\lambda^5 x^5 - 8\lambda^6 t^{16} \\
&= 49xy - 15\lambda^2 x^4 - 8\lambda^3 x^2 y + 8\lambda^4 y^2 - 16\lambda^5 x^5 - 8\lambda^6 x^3 y + 8\lambda^7 t^{17} \\
&= 49xy - 15\lambda^2 x^4 - 8\lambda^3 x^2 y + 8\lambda^4 y^2 - 16\lambda^5 x^5 - 8\lambda^6 x^3 y + 8\lambda^7 xy^2 - 16\lambda^8 x^6 - 8\lambda^9 t^{19} \\
&\cdots \\
&= 49xy + \lambda^2 x^4 + \frac{-16\lambda^2 - 8\lambda^3 x^2 y + 8\lambda^4 y^2}{1 - \lambda^3 x} = \frac{49xy - 15\lambda^2 x^4 - 57\lambda^3 x^3 y + 8\lambda^4 y^2 - \lambda^5 x^5}{1 - \lambda^3 x}.
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&x^3(7t^4 + 8\lambda t^5) = 7t^{13} + 8\lambda t^{14} = 7x^2 y + t^{14} = \cdots \\
&= 7x^2 y + \frac{\lambda y^2 - 2\lambda^2 x^5 - \lambda^3 x^3 y}{1 - \lambda^3 x} = \frac{7x^2 y + \lambda y^2 - 2\lambda^2 x^5 - 8\lambda^3 x^3 y}{1 - \lambda^3 x}.
\end{aligned}$$

For $f(t) = (t^3, -t^7 + \lambda t^8)$, we may replace y by $-y$ and λ by $-\lambda$ in the above calculations.

The determinant of coefficients of ω_1, ω_2 (and those of the corresponding logarithmic vector fields) is given by

$$\begin{vmatrix} -(\pm 49xy - 15\lambda^2 x^4 - 57\lambda^3 x^2 y + 8\lambda^4 y^2 \mp \lambda^5 x^5) & 3(7x^2 - \lambda y)(1 \mp \lambda^3 x) \\ -(\pm 7x^2 y \pm \lambda y^2 - 2\lambda^2 x^5 - 8\lambda^3 x^3 y) & 3x^2(1 \mp \lambda^3 x) \end{vmatrix} = 3(1 \mp \lambda^3 x)\lambda^2 F.$$

Thus we see that $\mathcal{R}^1(Z)$ is generated by $[\omega_1]^a$ and $[\omega_2]^a$ over $\Lambda^0(\mathbf{R}^2)$.

If $\lambda = 0$, then $\mathcal{G}^1(Z, \mathbf{R}^2)$ is generated by the Euler form $\omega_1 = -7ydx + 3xdy$ and the exterior differential $\omega_2 = dF$ of F. Then $\mathcal{R}^1(Z)$ is generated by only $[\omega_1]^a$, since $[dF]^a = 0$. ∎

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