SYMPLECTIC BIFURCATIONS OF PLANE CURVES AND ISOTROPIC LIFTINGS

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Abstract

We study the classification of varieties in the Marsden–Weinstein reduction and their liftability. In particular the complete symplectic classification of the Bruce–Gaffney plane curve singularites is provided and is applied to obtain naturally the Lagrangian openings.

1. Introduction

Symplectic structures arise naturally in diverse contexts such as Hamiltonian mechanics, field theory, geometrical optics, algebraic geometry, etc. In all these theories the bifurcations of various symplectic objects, like isotropic or Lagrangian varieties, representing the states of the systems play an important role. The purpose of this article is twofold. First we formulate the theory of symplectic bifurcations with the symplectic group actions on the reduced spaces. Secondly we provide the complete classification of simple symplectic bifurcations of curves and determine the possible differential and symplectic invariants, in particular, the symplectic defect.

We start with a coisotropic fibration $H : M^{2n} \to \mathbb{R}^{n-k}$, $0 \leqslant k \leqslant n-1$, of the symplectic space (M^{2n}, ω) of dimension $2n$, $n \geqslant 2$. By the Jacobi–Liouville theorem, locally there exist relative Darboux coordinates $p_1, \dots, p_n, q_1, \dots, q_n$ of M such that $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$, and

$$H(p, q) = (\bar{q}) = (q_{k+1}, \dots, q_n)$$

cf. [**1**, p. 301]. Then we consider the family of canonical reductions (Marsden–Weinstein symplectic reduction)

$$\bar{q} \mapsto \pi_{\bar{q}} : H^{-1}(\bar{q}) \to H^{-1}(\bar{q})/\sim_{\bar{q}} \cong T^*\mathbb{R}^k,$$
$$\pi_{\bar{q}} : (p_1, \dots, p_k, q_1, \dots, q_k, p_{k+1}, \dots, p_n) \mapsto (p_1, \dots, p_k, q_1, \dots, q_k),$$

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where $\sim_{\bar{q}}$ denotes the equivalence relation on $H^{-1}(\bar{q})$ induced by the characterisic foliation of H. Then we have the total projection

$$\pi : M^{2n} \to N^{n+k} = T^*\mathbb{R}^k \times \mathbb{R}^{n-k},$$
$$\pi(p, q) = (p_1, \ldots, p_k, q_1, \ldots, q_k; q_{k+1}, \ldots, q_n).$$

Let $L^m \subset M^{2n}$ be an isotropic variety ($\omega|_L = 0$, $m \leqslant n$). Then the projection $\pi(L) \subset N = T^*\mathbb{R}^k \times \mathbb{R}^{n-k}$ provides a bifurcation of $(m-n+k)$-dimensional isotropic varieties $\pi_{\bar{q}}(H^{-1}(\bar{q}) \cap L)$ in $T^*\mathbb{R}^k$ with the parameter space $\mathbb{R}^{n-k}$. In this paper we consider the bifurcation of curves, namely, the case $m = n - k + 1$.

Any map-germ $F : (\mathbb{R}^{n-k+1}, 0) \to N = T^*\mathbb{R}^k \times \mathbb{R}^{n-k}$ is called a symplectic bifurcation problem of curves. The germ F is called *transverse* if F is transverse to $T^*\mathbb{R}^k \times \{0\}$ at 0. Moreover any symplectic bifurcation problem of curves is called *an isotropic bifurcation of curves* if there exists an isotropic map-germ $\widetilde{F} : (\mathbb{R}^{n-k+1}, 0) \to ((M^{2n}, \omega), 0)$ such that $\pi \circ \tilde{F} = F$, and $\tilde{F}^*\omega = 0$; cf. [**17**, p. 29]. We show that *any transverse symplectic bifurcation problem of curves is isotropic* (Proposition 2.3).

In the paper [**22**], V. M. Zakalyukin classified the simple stable Lagrangian submanifold-germs ($m = n$) by symplectomorphisms which preserve a given coisotropic fibration. Then, admitting Lagrangian or isotropic varieties, we study the liftability and the classification problem of varieties in the reduced space. In other words, we consider the ‘bottom-up’ construction. The idea appeared earlier in the Ph.D. Thesis of M. Mikosz and part of it is published in [**19**]. We see that there exist, even in the simplest case $n = 2, k = 1$, many examples of non-transverse bifurcation problems of curves $F : (\mathbb{R}^2, 0) \to N = \mathbb{R}^2 \times \mathbb{R}$ which are not liftable to isotropic mappings into M (Examples 2.6 and 2.7).

To make clear the problem we are going to study, let us consider the bifurcation problem of ‘cross caps’ or ‘Whitney umbrellas’ in the three space $\mathbb{R}^2 \times \mathbb{R} = T^*\mathbb{R} \times \mathbb{R}$ with the symplectic foliation. Then we observe the four typical examples of bifurcations $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2 \times \mathbb{R})$, namely

(1) transverse immersions or no bifurcation,

(2) transverse Whitney umbrellas or cusp bifurcations,

(3) hyperbolic Whitney umbrellas or X-pinch bifurcations, and

(4) elliptic Whitney umbrellas or figure-8 bifurcations;

see Fig. 1; see also section 5. The classification is similar to the classification of functions on cross-caps; see [**7**]. However, we remark that the non-transverse immersions are never isotropically liftable (Proposition 2.5), so they do not enter into our list, contrary to ordinary singularity theory. The first two are transverse and the last two are not. Note that changes of irreducible components occur through the bifurcations of types (3) and (4), while the transverse bifurcations of curves provide just irrreducible curves, like the cases (1) and (2).

In this paper we study the transverse bifurcation problem of curves in detail. In section 2, we show the unique isotropic liftability of transverse bifurcations. In section 3, we study the general classification problem of parametrized varieties in the reduced space under a natural equivalence relation, *liftable equivalence*, that corresponds to the classification of isotropic varieties by symplectomorphisms preserving π-fibres for the given coisotropic fibration $H : \mathbb{R}^{2n} \to \mathbb{R}^{n-k}$. We show that a liftable diffeomorphism is actually a diffeomorphism preserving the symplectic

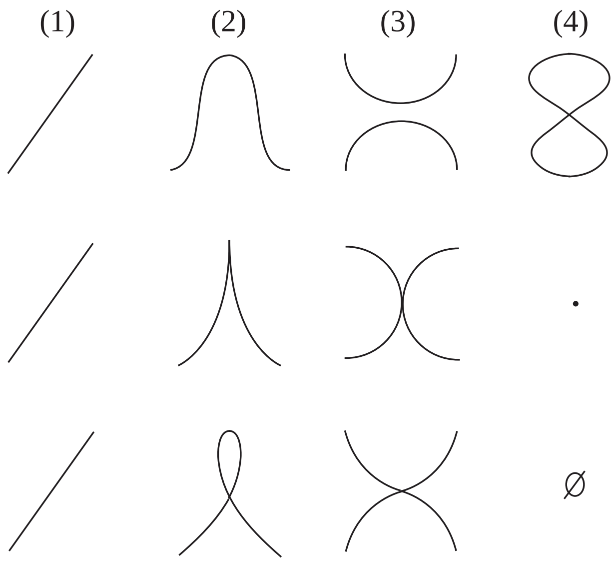

Fig. 1 Typical isotropic bifurcations

foliation of $N = T^*\mathbb{R}^k \times \mathbb{R}^{n-k}$. Thus the classification under the liftable equivalence turns out to be reduced completely to the classification and deformation problem of curves on the symplectic space $T^*\mathbb{R}^k = \mathbb{R}^{2k}, k \geqslant 1$.

In section 4, we give the classification of symplectic bifurcation problems of curves under the liftable equivalence.

In particular, we study the simplest case $k = 1$, namely, the bifurcation problem of curves in the symplectic plane, in detail.

Now we try to clarify the fundamental problem we face. Suppose two plane curve-germs are transformed to each other by an orientation preserving diffeomorphism on the plane. Then, it is natural to ask: Are they transformed by a symplectomorphism of the plane? Is singularity theory using symplectomorphisms different from ordinary singularity theory even for plane curves? Then we encounter ‘symplectic ghosts’ in the sense of Arnold [**3**]. After several preliminaries in sections 6 and 7 answering this question, we describe, in section 8, for a plane curve-germ $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$, the difference sp-codim(f) − codim(f) between the codimension (sp-codim(f)) of f under the ‘symplectic equivalence’ and the right–left (that is, $\mathcal{A}_e$) codimension codim(f) of f in an explicit way. We call this difference the *symplectic defect* or the *symplectic ghost number* of f. We remark that our ghosts have moduli, while the ghosts which appeared in [**3**] are discrete.

Obviously the symplectic defect is an invariant under symplectic equivalence, namely a symplectic invariant. However, we show that the symplectic defect is, in fact, an $\mathcal{A}$-invariant of f, not just a symplectic invariant. *If the symplectic defect is positive, then the classification of plane curves by symplectomorphisms differs from the classification by diffeomorphisms, and the difference depends only on the $\mathcal{A}$-equivalence class of the plane curves.* Therefore the symplectic codimension itself is an $\mathcal{A}$-invariant. If two plane curve-germs f and f' are $\mathcal{A}$-equivalent, then

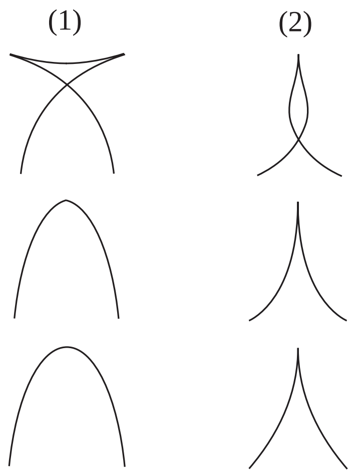

Fig. 2 Swallowtail and folded umbrella

sp-codim(f) = sp-codim(f'). In fact we show that sp-codim(f) is equal to the δ-invariant (the number of complex double points after a perturbation) of f. Note that, a priori, the δ-invariant has nothing to do with symplectic equivalence. We also remark that, in the case $k \geqslant 2$, the symplectic codimension of $f : (\mathbb{R}, 0) \to T^*\mathbb{R}^k$ is, in fact, not necessarily an $\mathcal{A}$-invariant (Remark 8.5).

Moreover, we show in section 10, that if the symplectic defect equals zero, the symplectic classification of plane curves coincides with their isotopy classification. As an application, we obtain normal forms under liftable equivalence for certain map-germs. Also, we calculate symplectic defects for $\mathcal{A}$-simple plane curves

$$A_{2\ell}, E_{6\ell}, E_{6\ell+2}, W_{12}, W_{18}, W^{\#}_{1,2q-1}$$

classified by Bruce and Gaffney [**5**] and give the complete symplectic classification of $\mathcal{A}$-simple plane curves (Theorems 9.2 and 9.6). Moreover we obtain the symplectically mini-versal unfoldings of them (Proposition 9.9). As a byproduct of our approach, we give the classification $(A_{2\ell}, E_6, E_8^{\pm})$ of 'symplectically simple' plane curves (Corollary 9.8).

As typical well-known examples of Lagrangian varieties, there are open Whitney umbrellas and open swallowtails. In this paper, in a systematic way, we characterize the opening processes of Whitney umbrellas and swallowtails; see sections 7, 9, 11 and 12. We show that the Lagrangian liftings are obtained, in many cases, from open Whitney umbrellas by the reduction process (Corollary 7.4). Also we study the polynomial construction of open swallowtails from the viewpoint of the present paper (section 11). In section 12, we treat the construction of open swallowtails (Fig. 2 (1)) and also 'open folded umbrellas' (Fig. 2 (2)), via the notion of 'frontal-symplectic' versality.

2. Lagrangian liftability

Let $F : (\mathbb{R}^{n-k+1}, 0) \to (N, 0) = (T^*\mathbb{R}^k \times \mathbb{R}^{n-k}, 0)$ be a smooth map germ.

DEFINITION 2.1 By an *isotropic lifting* of F, we mean a smooth map-germ $\widetilde{F} : (\mathbb{R}^{n-k+1}, 0) \to$

$(M, 0)$ which is isotropic, that is, $\widetilde{F}^*\omega = 0$, and for which $\pi \circ \widetilde{F} = F$:

$$\begin{array}{ccc} (\mathbb{R}^{n-k+1}, 0) & \xrightarrow{\widetilde{F}} & ((M, 0), \omega) \\ \| & & \downarrow \pi \\ (\mathbb{R}^{n-k+1}, 0) & \xrightarrow{F} & (N, 0). \end{array}$$

When $k = 1$, we call an isotropic lifting also a *Lagrangian lifting*.

Now we have the following sufficient condition for the isotropic liftability of f.

DEFINITION 2.2 A smooth map-germ $F : (\mathbb{R}^{n-k+1}, 0) \to (N, 0) = (T^*\mathbb{R}^k \times \mathbb{R}^{n-k}, 0)$ is called *transverse* if F is transverse to $T^*\mathbb{R}^k \times \{0\}$.

The result mentioned in the Introduction is proved in the following precise form.

PROPOSITION 2.3 *Let $F : (\mathbb{R}^{n-k+1}, 0) \to (N, 0)$ be a transverse smooth map-germ. Then there exists a smooth isotropic lifting $\widetilde{F} : (\mathbb{R}^{n-k+1}, 0) \to M = \mathbb{R}^{2n}$ for F. In fact, for some coordinates $x = (t, \lambda_1, \ldots, \lambda_{n-k})$ with $F(t, \lambda) = (f_\lambda(t), \lambda)$, the remaining components of $\widetilde{F}$ are given by*

$$p_j(x) = \int_0^t \left(\sum_{i=1}^k \frac{\partial p_i}{\partial \lambda_j} \frac{\partial q_i}{\partial t} - \frac{\partial p_i}{\partial t} \frac{\partial q_i}{\partial \lambda_j} \right) dt \quad (k + 1 \leqslant j \leqslant n).$$

Moreover the Lagrangian liftings of F are equivalent to each other by symplectomorphisms on M preserving the fibres of H.

Proof. From the equation $\widetilde{F}^*\omega = 0$ we get the necessary condition for the germ to be a solution of a first-order partial differential equation. The details are as follows. (See also [**12**].)

By assumption, there exist coordinates

$$x = (x_1, x_2, \ldots, x_{n-k+1}) = (t, \lambda_1, \ldots, \lambda_{n-k})$$

such that

$$F(t, \lambda_1, \ldots, \lambda_{n-k}) = (q_1(t, \lambda), p_1(t, \lambda), \ldots, q_k(t, \lambda), p_k(t, \lambda), \lambda_1, \ldots, \lambda_{n-k}),$$
$$\widetilde{F}(t, \lambda_1, \ldots, \lambda_{n-k}) = (F(t, \lambda), p_{k+1}(t, \lambda), \ldots, p_n(t, \lambda)).$$

Set $\theta = \sum_{i=1}^n p_i dq_i$, the Liouville form on $M = T^*\mathbb{R}^n$. Then $\omega = d\theta$. Since the condition $\widetilde{F}^*\omega = 0$ is equivalent to $\widetilde{F}^*\theta$ being closed, we set, locally, $\widetilde{F}^*\theta = de$ for some function e on $(\mathbb{R}^{n-k+1}, 0)$. Then we get

$$\frac{\partial e}{\partial t} = \sum_{i=1}^k p_i(x) \frac{\partial q_i}{\partial t}, \quad \frac{\partial e}{\partial \lambda_j} = p_{k+j}(t, \lambda) + \sum_{i=1}^k p_i(t, \lambda) \frac{\partial q_i}{\partial \lambda_j}$$

$(j = 1, \ldots, n - k)$. If we take a function $e(x)$ satisfying the first equation, then we obtain $p_{k+j}(x)$ $(j = 1, \ldots, n - k)$ by the second equality. Now we set

$$e = \int_0^t \sum_{i=1}^k p_i(x) \frac{\partial q_i}{\partial t} dt + \varphi(\lambda), \quad \lambda = (\lambda_1, \ldots, \lambda_{n-k}),$$

for some function φ which is independent of t. Then we have

$$\frac{\partial e}{\partial \lambda_j} = \int_0^t \left(\sum_{i=1}^k \frac{\partial p_i}{\partial \lambda_j} \frac{\partial q_i}{\partial t} + p_i \frac{\partial^2 q_i}{\partial t \partial \lambda_j} \right) dt + \frac{\partial \varphi}{\partial \lambda_j}$$
$$= \int_0^t \left(\sum_{i=1}^k \frac{\partial p_i}{\partial \lambda_j} \frac{\partial q_i}{\partial t} - \frac{\partial p_i}{\partial t} \frac{\partial q_i}{\partial \lambda_j} \right) dt + \sum_{i=1}^k p_i \frac{\partial q_i}{\partial \lambda_j} + \frac{\partial \varphi}{\partial \lambda_j}.$$

Thus we have

$$p_{k+j}(t, \lambda) = \int_0^t \left(\sum_{i=1}^k \frac{\partial p_i}{\partial \lambda_j} \frac{\partial q_i}{\partial t} - \frac{\partial p_i}{\partial t} \frac{\partial q_i}{\partial \lambda_j} \right) dt + \frac{\partial \varphi}{\partial \lambda_j} \quad (1 \leqslant j \leqslant n-k).$$

Note that $H \circ \widetilde{F}(x) = (\lambda_1, \dots, \lambda_{n-k})$. Then the local symplectomorphism

$$(p, q) \mapsto \left(p_1, \dots, p_k, p_{k+1} - \frac{\partial \varphi(\bar{q})}{\partial q_{k+1}}, \dots, p_n - \frac{\partial \varphi(\bar{q})}{\partial q_n}, q \right),$$

$\bar{q} = (q_{k+1}, \dots, q_n)$, eliminates the terms $\partial \varphi / \partial \lambda_j$.

REMARK 2.4 (The coincidence of singular loci for the lifting and the original map) The singular locus of $\widetilde{F}$ coincides with the singular locus of F:

$$\mathrm{Sing}(\widetilde{F}) = \mathrm{Sing}(F) = \left\{ (t, \lambda) \in (\mathbb{R}^{n-k+1}, 0) \,\middle|\, \frac{\partial q_i}{\partial t} = \frac{\partial p_i}{\partial t} = 0, 1 \leqslant i \leqslant k \right\}.$$

In the case $k = 1$, moreover, we have the following.

PROPOSITION 2.5 *If* $F : (\mathbb{R}^n, 0) \to N = T^*\mathbb{R} \times \mathbb{R}^{n-1}$ *is an immersive germ, then the transversality condition is the necessary and sufficient condition for the existence of Lagrangian liftings for* F.

Proof. Assume that F is an immersion and not transverse to $T^*\mathbb{R} \times \{0\}$. Then we see that the image $F_*(T_0\mathbb{R}^n)$ contains $T_0(T^*\mathbb{R} \times \{0\})$ by a simple argument of dimension. This means that, for any lifting $\widetilde{F}$ of F, $\widetilde{F}_*(T_0\mathbb{R}^n)$ contains the (p_1, q_1)-plane in $T_0 M$. Then it is impossible that $\widetilde{F}$ is isotropic. □

EXAMPLE 2.6 (Non-transversal and liftable germs) There exist non-transverse and liftable map-germs. For example, if a Lagrangian immersion $L : (\mathbb{R}^n, 0) \to M^{2n}$ is not transverse to the H-level $H^{-1}(H(L(0)))$ for $H : M^{2n} \to \mathbb{R}^{n-1}$, then the tangent space $L_*(T_0\mathbb{R}^n)$ contains a characteristic direction and $F = \pi \circ L$ is not an immersion, while F is liftable to L. In particular, a Lagrange surface in $\mathbb{R}^4$ projects to a Whitney umbrella. This observation is related to the study of smooth perturbations of singular surfaces in $\mathbb{R}^4$ [**10**].

EXAMPLE 2.7 (Non-transverse and non-liftable germs) Let $F : (\mathbb{R}^2, 0) \to (N, 0) = (\mathbb{R}^2 \times \mathbb{R}, 0)$ be a non-transverse map-germ of the form

$$F(x_1, x_2) = (q_1, p_1, q_2) = (q_1(x), p_1(x), \tfrac{1}{2}(x_1^2 \pm x_2^2)).$$

Consider the second-order differential operator $\Delta = \partial^2/\partial x_1^2 \pm \partial^2/\partial x_2^2$ associated to $\frac{1}{2}(x_1^2 \pm x_2^2)$. If F

has a Lagrangian lifting, then $dp_1 \wedge dq_1 + d\varphi \wedge d(\frac{1}{2}(x_1^2 \pm x_2^2)) = 0$ for some function $\varphi = \varphi(x_1, x_2)$. Then we see, by simple formal calculations, that the Jacobian $J(p_1, q_1)$ satisfies an infinite number of conditions $(\Delta^\ell J(p_1, q_1))(0, 0) = 0$ for $\ell = 0, 1, 2, \dots$. So, if $(\Delta^\ell J(p_1, q_1))(0, 0) \neq 0$ for some ℓ, then F is never liftable. For example,

$$F(x_1, x_2) = (q_1, p_1, q_2) = (x_1, x_2^3, \tfrac{1}{2}(x_1^2 \pm x_2^2))$$

is not liftable. On the other hand, $(x_1, x_1x_2, \frac{1}{2}(x_1^2 \pm x_2^2))$ is liftable.

3. Liftable equivalence

Let $\phi : (N, 0) \to (N, 0)$ be a diffeomorphism-germ. We call ϕ a *symplectically liftable* diffeomorphism if there exists a symplectomorphism-germ $\Phi : ((M, 0), \omega) \to ((M, 0), \omega)$ such that the following diagram commutes:

$$\begin{array}{ccc} ((M,0),\omega) & \xrightarrow{\Phi} & ((M,0),\omega) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ (N,0) & \xrightarrow{\phi} & (N,0). \end{array}$$

DEFINITION 3.1 We say that the two map germs $F_1, F_2 : (\mathbb{R}^{n-k+1}, 0) \to (N, 0)$ are *liftably equivalent* if the following diagram commutes:

$$\begin{array}{ccccc} (\mathbb{R}^{n-k+1},0) & \xrightarrow{F_1} & (N,0) & \xleftarrow{\pi} & ((M,0),\omega) \\ {\scriptstyle \psi}\downarrow & & {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ (\mathbb{R}^{n-k+1},0) & \xrightarrow{F_2} & (N,0) & \xleftarrow{\pi} & ((M,0),\omega), \end{array}$$

where ψ, ϕ are diffeomorphism-germs and Φ is a symplectomorphism-germ, that is, ϕ is a symplectically liftable diffeomorphism.

The symplectically liftable diffeomorphisms of $(N, 0)$ form a subgroup $\mathcal{G}_{\text{symp}}$ of the group $\mathcal{G}$ of diffeomorphism-germs. Classification of singularities of F according to the equivalence group $\mathcal{G}_{\text{symp}}$ is similar to the standard right–left classification of singularities of map-germs $(\mathbb{R}^{n-k+1}, 0) \to (\mathbb{R}^{n+k}, 0)$; however, this is restricted to the space of map-germs which have Lagrangian liftings.

Now we describe symplectically liftable diffeomorphisms in an explicit way.

PROPOSITION 3.2 *For a diffeomorphism-germ $\phi : (N, 0) = (T^*\mathbb{R}^k \times \mathbb{R}^{n-k}, 0) \to (T^*\mathbb{R}^k \times \mathbb{R}^{n-k}, 0)$, the following conditions are equivalent:*

(1) *ϕ is a symplectically liftable diffeomorphism;*

(2) *ϕ is a Poisson diffeomorphism (for the Poisson structure on N induced from M by π);*

(3) *ϕ is a family of symplectic diffeomorphisms on $T^*\mathbb{R}^k$ with parameter $\bar{q} = (q_{k+1}, \dots, q_n)$. Namely, if we set*

$$\phi(q_1, p_1, \dots, q_k, p_k, \bar{q}) = (Q_1, P_1, \dots, Q_k, P_k, \bar{Q}),$$

then $\overline{Q} = (Q_{k+1}, \dots, Q_n)$ *depends only on* $\bar{q}$*, and*

$$(q_1, p_1, \dots, q_k, p_k) \mapsto (Q_1, P_1, \dots, Q_k, P_k)$$

is a symplectomorphism on $(T^*\mathbb{R}^k, 0)$ *for each fixed* $\bar{q} = (q_{k+1}, \dots, q_n)$*;*

(4) ϕ *has a symplectic lifting* $\Phi : (M, 0) \to (M, 0)$ *preserving fibres of* H*, namely, there exists a diffeomorphism-germ* $\sigma : (\mathbb{R}^{n-k}, 0) \to (\mathbb{R}^{n-k}, 0)$ *such that the following diagram commutes.*

$$\begin{array}{ccccc} (N,0) & \xleftarrow{\pi} & ((M,0),\omega) & \xrightarrow{H} & (\mathbb{R}^{n-k},0) \\ \phi\downarrow & & \Phi\downarrow & & \downarrow\sigma \\ (N,0) & \xleftarrow{\pi} & ((M,0),\omega) & \xrightarrow{H} & (\mathbb{R}^{n-k},0) \end{array}$$

Proof. (1) $\Rightarrow$ (2): We denote by $\{\ ,\ \}_M$ the Poisson bracket on the symplectic manifold M. Then the Poisson bracket $\{\ ,\ \}_N$ on N is defined by $\{h, k\}_N = \{h \circ \pi, k \circ \pi\}_M$ for any functions on $(N, 0)$. Let Φ be a symplectic lifting of ϕ on $(M, 0)$. Then, for any functions h, k on $(N, 0)$, we have $\{h\circ\phi, k\circ\phi\}_N = \{h\circ\phi\circ\pi, k\circ\phi\circ\pi\}_M = \{h\circ\pi\circ\Phi, k\circ\pi\circ\Phi\}_M = \{h\circ\pi, k\circ\pi\}_M = \{h, k\}_N$. Thus ϕ is a Poisson diffeomorphism.

(2) $\Rightarrow$ (3): The Poisson structure on N induces the foliation by symplectic leaves, $T^*\mathbb{R}^k \times \{\bar{q}\}$, intrinsically. So naturally ϕ induces a family of symplectomorphisms on $T^*\mathbb{R}^k$. Also we can argue in a more direct way as follows. The derivations $\{P_i, \cdot\}_N\{Q_i, \cdot\}_N$, $(1 \leqslant i \leqslant k)$ generate the tangent spaces to the leaves $T^*\mathbb{R}^k \times \{\bar{q}\}$. Since we have, for each j with $1 \leqslant j \leqslant n-k$, $\{P_i, Q_{k+j}\}_N = \{Q_i, Q_{k+j}\}_N = 0 (1 \leqslant i \leqslant k)$, we see that each Q_{k+j} is independent of $(q_1, p_1, \dots, q_k, p_k)$.

(3) $\Rightarrow$ (4): Let $\Phi : (M, 0) \to (M, 0)$,

$$\Phi(q_1, p_1, \dots, q_k, p_k, \bar{q}) = (Q_1, P_1, \dots, Q_k, P_k, \overline{Q}, \overline{P}),$$

$\overline{P} = (P_{k+1}, \dots, P_n)$ be a diffeomorphism-germ covering ϕ with respect to π. The condition that Φ is a symplectomorphism is the existence of a smooth function $E : (M, 0) \to \mathbb{R}$, called a generating function of Φ, satisfying

$$\Phi^*\theta - \theta = dE,$$

where $\theta = \sum_{i=1}^n p_i dq_i$. Then the condition is equivalent to

$$\sum_{i=1}^k P_i \frac{\partial Q_i}{\partial q_j} - p_j = \frac{\partial E}{\partial q_j} \qquad (1 \leqslant j \leqslant k),$$

$$\sum_{i=1}^k P_i \frac{\partial Q_i}{\partial p_j} = \frac{\partial E}{\partial p_j} \qquad (1 \leqslant j \leqslant k),$$

$$\sum_{i=1}^k P_i \frac{\partial Q_i}{\partial q_j} + \sum_{i=k+1}^n P_i \frac{\partial Q_i}{\partial q_j} - p_j = \frac{\partial E}{\partial q_j} \qquad (k+1 \leqslant j \leqslant n)$$

for some function $E : (N, 0) \to \mathbb{R}$. The first and second equalities mean that $E_{\bar{q}}$ is a generating function of the symplectomorphism $\phi_{\bar{q}}$ for each $\bar{q} \in (\mathbb{R}^{n-k}, 0)$. Therefore the function

$E : (N, 0) \to \mathbb{R}$ is uniquely determined by these two equalities up to the addition of functions on $\bar{q}$. Then, by the last equality, the H-level preserving symplectic lifting Φ of ϕ is given by setting

$$\overline{P} = \left(\frac{\partial E}{\partial \bar{q}} + \bar{p} - \sum_{i=1}^{k} P_i \frac{\partial Q_i}{\partial \bar{q}} \right) \left(\frac{\partial Q_\ell}{\partial q_j} \right)^{-1}_{k+1 \leqslant \ell \leqslant n, k+1 \leqslant j \leqslant n},$$

where

$$\frac{\partial E}{\partial \bar{q}} = \left(\frac{\partial E}{\partial q_{k+1}}, \ldots, \frac{\partial E}{\partial q_n} \right), \text{ and } \frac{\partial Q_i}{\partial \bar{q}} = \left(\frac{\partial Q_i}{\partial q_{k+1}}, \ldots, \frac{\partial Q_i}{\partial q_n} \right).$$

(4) $\Rightarrow$ (1): This implication is trivial.

REMARK 3.3 In the case when $k = 1$, diffeomorphisms of type

$$\phi(q_1, p_1, \bar{q}) = (\delta^{-1} q_1 + \beta(\bar{q}), \delta p_1 + \alpha(q), \gamma(\bar{q}))$$

and

$$\phi(q_1, p_1, \bar{q}) = (-p_1, q_1, \bar{q}),$$

and their compositions are all symplectically liftable. (Here α, β, γ are smooth functions, $\delta \in \mathbb{R} - \{0\}$, and $\bar{q} = (q_2, \ldots, q_n)$.)

REMARK 3.4 We have also the description of the infinitesimal deformations corresponding to liftable and lifted equivalences. Any vector field X over N generating a liftable equivalence is given by

$$X(q_1, p_1, \ldots, q_k, p_k, \bar{q}) = X_{h_{\bar{q}}}(q_1, p_1, \ldots, q_k, p_k) + \sum_{i=1}^{n-k} a_i(\bar{q}) \frac{\partial}{\partial q_{k+i}}$$

for some functions $h(q_1, p_1, \ldots, q_k, p_k, \bar{q})$ and $a_i(\bar{q})$, $1 \leqslant i \leqslant n-k$, where $X_{h_{\bar{q}}}$ is the Hamiltonian vector field over $T^*\mathbb{R}^k$ with the Hamiltonian $h_{\bar{q}}$ for each fixed $\bar{q} \in (\mathbb{R}^{n-k}, 0)$. The lifted Hamiltonian vector field $\widetilde{X}$ over $M = T^*\mathbb{R}^n$ has the Hamiltonian $\tilde{h} = h + \sum_{i=1}^{n-k} p_{k+i} a_i(\bar{q})$; $\widetilde{X} = X_{\tilde{h}}$.

DEFINITION 3.5 Two isotropic map-germs $L, L' : (\mathbb{R}^{n-k+1}, 0) \to (M, 0)$ are called *H-symplectically equivalent* if there exist a symplectomorphism $\Phi : (M, 0) \to (M, 0)$ and diffeomorphisms $\psi : (\mathbb{R}^{n-k+1}, 0) \to (\mathbb{R}^{n-k+1}, 0)$, $\sigma : (\mathbb{R}^{n-k}, 0) \to (\mathbb{R}^{n-k}, 0)$ such that the following diagram commutes.

$$\begin{array}{ccccc} (\mathbb{R}^{n-k+1}, 0) & \xrightarrow{L} & (M, 0) & \xrightarrow{H} & (\mathbb{R}^{n-k}, 0) \\ \psi \downarrow & & \Phi \downarrow & & \downarrow \sigma \\ (\mathbb{R}^{n-k+1}, 0) & \xrightarrow{L'} & (M, 0) & \xrightarrow{H} & (\mathbb{R}^{n-k}, 0) \end{array}$$

Now the following is clear.

COROLLARY 3.6 *Let $F, F' : (\mathbb{R}^{n-k+1}, 0) \to (T^*\mathbb{R}^k \times \mathbb{R}^{n-k}, 0)$ be transverse map-germs. If F and F' are liftably equivalent, then their isotropic liftings $\widetilde{F}$ and $\widetilde{F'}$ are H-symplectically equivalent.*

4. Classification under liftable equivalence

In what follows, we concentrate on the symplectic bifurcation problem of curves on the symplectic plane ($k = 1$). Thus we consider map-germs $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^2 \times \mathbb{R}^{n-1}, 0)$.

PROPOSITION 4.1 *For a transverse map-germ* $F : (\mathbb{R}^n, 0) \to (N, 0) = (\mathbb{R}^2 \times \mathbb{R}^{n-1}, 0)$*, we have the following.*

(a1) *If F is an immersion at* 0*, then F is liftably equivalent to*

$$(0, x_1, x_2, \dots, x_n).$$

(a2) *Suppose F is not an immersion at* 0*. Then F is liftably equivalent to the germ $\phi \circ F \circ \psi$ such that the* 2*-jet* $j^2(\phi \circ F \circ \psi)$ *is equal to* (x_1^2, x_1x_2, x'), $(x_1^2, 0, x')$, $(x_1x_2, 0, x')$ *or* $(0, 0, x')$, *where* $x' = (x_2, \dots, x_n)$.

Proof. (a1) This follows from Proposition 4.2 below.

(a2) By Proposition 2.5 (a) and using a right equivalence ψ we can assume that,

$$j^2F(0) = (c_0x_1^2 + c_1(x')x_1 + c_2(x'), b_0x_1^2 + b_1(x')x_1 + b_2(x'), x'),$$

where $b_1(0) = 0, c_1(0) = 0$ and $x' = (x_2, \dots, x_n)$.

Let $c_0 \neq 0$. Then by using right equivalence, we can assume $c_0 = 1, c_1(x') = 0$. Then, by applying the symplectically liftable diffeomorphism

$$(q_1, p_1, q') \mapsto (q_1 - c_2(q'), p_1 - b_2(q'), q'),$$

the 2-jet is transformed to $(x_1^2, b_0x_1^2 + b_1(x')x_1, x')$.

Moreover, using the symplectically liftable diffeomorphism $(q_1, p_1, q') \mapsto (q_1, p_1 - b_0q_1, q')$, we get $(x_1^2, b_1(x')x_1, x')$ as the 2-jet. If the linear form $b_1(x')$ is not identically zero, by a coordinate change of x' we obtain (x_1^2, x_1x_2, x'). If $b_1(x') = 0$, then we have $(x_1^2, 0, x')$.

If $c_0 = 0$ and $b_0 \neq 0$, we can proceed as above using the symplectically liftable diffeomorphism $(q_1, p_1, q') \mapsto (-p_1, q_1, q')$. If $b_0 = c_0 = 0$, then we get $(x_1x_2, 0, x')$ or $(0, 0, x')$ as the 2-jet within the liftable equivalence classes.

Now we have the following prenormal form for F.

PROPOSITION 4.2 *Let* $F : (\mathbb{R}^n, 0) \to (N, 0) = (\mathbb{R}^2 \times \mathbb{R}^{n-1}, 0)$, $F(x_1, \dots, x_n) = (q_1(x), p_1(x), q'(x))$*, be a smooth map-germ. Assume that F is transverse to* $\mathbb{R}^2 \times \{0\}$ *and that F is finite, namely, the ideal generated by components of F is of finite codimension. Then F is liftably equivalent to one of the following forms, for some* $m \geqslant 2$*:*

$$F_m(x) = \left(x_1^m + \sum_{i=1}^{m-2} a_i(x')x_1^i, x_1c(x), x'\right),$$

where $x' = (x_2, \dots, x_n)$*, and* $a_i(x')$, $c(x)$ *are smooth function-germs.*

Proof. By the transversality assumption, using right equivalence, one can reduce F to the form

$$(x_1, x') \mapsto (a(x_1, x'), c_1(x_1, x'), x').$$

Since F is finite, we can assume, $\partial^m a(0)/\partial x_1^m \neq 0$, for some m, up to liftable equivalence. Then by the classification of A_m-type singularities of functions, we obtain the liftably equivalent form

$$(x_1, x') \mapsto \left(x_1^m + \sum_{i=1}^{m-2} a_i(x') x_1^i + a_0(x'), c_2(x_1, x'), x' \right).$$

We write $c_2(x_1, x') = x_1 c(x_1, x') + c_3(x')$. Then the liftable diffeomorphism $(q_1, p_1, q') \mapsto (q_1 - a_0(q'), p_1 - c_3(q'), q')$ yields the required form.

REMARK 4.3 In the case when $m = 2$, we get the form

$$F_2(x_1, x') = (x_1^2, x_1 c(x_1, x'), x').$$

Then, by setting $c(x_1, x') = x_1 \phi(x_1^2, x') + \psi(x_1^2, x')$ and by taking a liftable diffeomorphism $(q_1, p_1, q') \mapsto (q_1, p_1 - \psi(q_1, q'), q')$, we see that F_2 is liftable equivalent to the form

$$F_2'(x_1, x') = (x_1^2, x_1 \phi(x_1^2, x'), x').$$

For the case $m = 2$, we have the normal forms, by using the versality theorem in the symplectic case [**8**, pp. 223–254; **9**].

PROPOSITION 4.4 *Let $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^2 \times \mathbb{R}^{n-1}, 0)$ be a finite and transverse map germ. Assume the* 2*-jet of F is equal to $(x_1^2, x_1 x_2, 0)$ or $(x_1^2, 0, 0)$. Then F is liftably equivalent to*

$$(q_1, p_1, q') = (x_1^2, x_1^{2\ell+1} + \lambda_1(x') x_1^{2\ell-1} + \lambda_2(x') x_1^{2\ell-3} + \cdots + \lambda_\ell(x') x_1, x')$$

for some positive integer ℓ, and for some functions $\lambda_1(x'), \ldots, \lambda_\ell(x')$ of $x' = (x_2, \ldots, x_n)$ with $\lambda_j(0) = 0, 1 \leqslant j \leqslant \ell$.

Proof. We may assume $F|\mathbb{R} \times \{0\}$ is of type $A_{2\ell} : f(t) = (q_1, p_1) = (t^2, t^{2\ell+1})$, by using a liftable diffeomorphism. Note that the right–left equivalence class and the symplectic equivalence class coincide for a plane curve of type $A_{2\ell}$; see section 10. Then

$$G(t, \lambda_1, \ldots, \lambda_\ell) = (t^2, t^{2\ell+1} + \lambda_1 t^{2\ell-1} + \lambda_2 t^{2\ell-3} + \cdots + \lambda_\ell t, \lambda_1, \ldots, \lambda_\ell)$$

is a versal unfolding of $F|\mathbb{R} \times \{0\}$ ([**6**]). Then G is a symplectically versal unfolding of $F|\mathbb{R} \times \{0\}$; see section 7. Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a plane curve-germ. Recall that an unfolding $G : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^\ell, 0)$ of f is called *symplectically versal* if any unfolding $F : (\mathbb{R} \times \mathbb{R}^s, 0) \to (\mathbb{R}^2 \times \mathbb{R}^s, 0)$ of f is symplectically equivalent to $\varphi^* G$ for some smooth map-germ $\varphi : (\mathbb{R}^s, 0) \to (\mathbb{R}^\ell, 0)$.

Therefore, there exists a smooth mapping $x' \mapsto \lambda(x')$ such that the pull-back $\lambda^* G$ is liftably equivalent to F.

5. Non-transverse bifurcations of curves

In this section we consider map-germs $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0), F(x_1, x_2) = (q_1(x), p_1(x), q_2(x))$ such that $q_2 : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ is a submersion or a Morse function at 0. We call such map-germs *of Morse type*. We consider the generic classification problem of liftable germs among the class of map-germs of Morse type. Then we have the following.

PROPOSITION 5.1 *Let $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ be a generic liftable map-germ of Morse type, namely, a generic one-parameter bifurcation problem of plane curves of Morse type. Then F is liftably equivalent to one of following types* (*Fig.* 3).

(1) $(x_1, x_2) \mapsto (x_1, 0, x_2)$*: The transverse immersion.*

(2) $(x_1, x_2) \mapsto (x_1^2, x_1^3 + x_1x_2, x_2)$*: The transverse Whitney umbrella, or the cusp bifurcation.*

(3) $(x_1, x_2) \mapsto (x_1, x_1x_2 + O(3), \frac{1}{2}(x_1^2 - x_2^2))$*: The hyperbolic Whitney umbrella, or the X-pinch bifurcation.*

(4) $(x_1, x_2) \mapsto (x_1, x_1x_2 + O(3), \frac{1}{2}(x_1^2 + x_2^2))$*: The elliptic Whitney umbrella, or the figure-eight bifurcation.*

Here $O(3)$ means the terms in x_1, x_2 at least of third order.

Moreover the germ $F(x_1, x_2) = (x_1, x_1x_2 + \varphi(x_1, x_2), \frac{1}{2}(x_1^2 \pm x_2^2))$, $\mathrm{ord}\varphi \geqslant 3$, is liftable if and only if φ is of the form

$$\varphi(x_1, x_2) = \int_0^{x_2} \left(\mp x_2 \frac{\partial \psi}{\partial x_1} + x_1 \frac{\partial \psi}{\partial x_2} \right) dx_2 + \kappa(x_1)$$

for some smooth function $\psi(x_1, x_2)$ of order at least 2 and $\kappa(x_1)$ of order at least 3.

Proof. Let F be transverse. Then F is approximated by transverse immersions and a transverse Whitney umbrella. A generic transversal Whitney umbrella is a versal one-parameter unfolding of a plane curve of type A_2. Then F is liftably equivalent to the above normal form. If q_2 is a non-submersive Morse function, then F is liftably equivalent to $(q_1(x), p_1(x), \frac{1}{2}(x_1^2 \pm x_2^2))$. Moreover by the genericity assumption we may assume that $q_1(x) = x_1$ using liftable equivalence. Then we see that F is liftably equivalent to $(x_1, x_2) \mapsto (x_1, ax_1x_2 + O(3), \frac{1}{2}(x_1^2 \pm x_2^2))$, for $a \in \mathbb{R} - \{0\}$, which is liftably equivalent to $(x_1, x_1x_2 + O(3), \frac{1}{2}(x_1^2 \pm x_2^2))$.

The last statement is clear. We see F is liftable if and only if there exists a function ϕ such that

$$x_1 + \frac{\partial \varphi}{\partial x_2} \pm x_2 \frac{\partial \phi}{\partial x_1} - x_1 \frac{\partial \phi}{\partial x_2} = 0.$$

If such ϕ exists, then ϕ must have the form $x_2 + \psi$, $\mathrm{ord}\psi \geqslant 2$.

Note that the lifting is an ordinary open Whitney umbrella in the case (2), while the lifting is an immersion in each of the cases (3) and (4). Also note that the non-transverse immersions are never liftable by Proposition 2.5. We do not have yet the exact normal forms for hyperbolic and elliptic Whitney umbrellas under liftable equivalence.

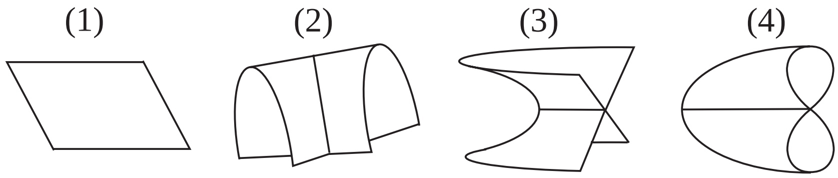

Fig. 3 Liftable germs of Morse type

6. Symplectic equivalence of plane curves

A transversal map-germ $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^2 \times \mathbb{R}^{n-1}, 0)$ is liftably equivalent to an unfolding $(t, \lambda) \mapsto (f_\lambda(t), \lambda)$, where $\lambda \in (\mathbb{R}^{n-1}, 0)$ and f_λ is a family of parametrized curves in the symplectic plane $\mathbb{R}^2$, t being the inner variable and $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ the outer variables. Therefore we proceed to consider the classification problems of bifurcations (unfoldings) of curves in the symplectic plane.

Two families of plane curves $f_\lambda, f'_\lambda, (\lambda \in (\mathbb{R}^\ell, 0))$ are called *symplectically equivalent* if there exist a family of diffeomorphisms $\Sigma = (\sigma_\lambda) : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}, 0)$, a family of symplectomorphisms $T = (\tau_\lambda) : (\mathbb{R}^2 \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2, 0)$, and a diffeomorphism $\varphi : (\mathbb{R}^\ell, 0) \to (\mathbb{R}^\ell, 0)$ such that $\tau_\lambda \circ f'_\lambda \circ \sigma_\lambda = f_{\varphi(\lambda)}$, for some representatives of germs. Then, setting $F : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^\ell, 0)$, $F(t, \lambda) = (f_\lambda(t), \lambda)$ and $F' : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^\ell, 0)$, $F(t, \lambda) = (f'_\lambda(t), \lambda)$, we see that if f_λ and f'_λ are symplectically equivalent then F and F' are liftably equivalent.

In ordinary singularity theory, the versal unfolding of a singularity dominates any other unfoldings. To seek the versal unfolding of curves on the symplectic plane for symplectic equivalence, we must first study the symplectic classification problem of plane curves.

For example, consider the simple cusp (A_2) $f = (t^2, t^3) : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$. Then the unfolding $F : (\mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ defined by $F(t, \lambda) = (t^2, t^3 + \lambda t, \lambda)$ is versal with respect to the right–left equivalence. Then we ask: Is it a symplectically versal unfolding?

Now first we consider the basic problem. Let $C, C' \subset (\mathbb{R}^2, 0)$ be two curve germs. Assume that there exist a diffeomorphism-germ $\sigma : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with $\sigma(C) = C'$. Then does there exist a symplectic (area-preserving) diffeomorphism $\sigma' : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ with $\sigma'(C) = C'$?

We call two map germs $f, f' : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *isotopic* (resp. *equivalent*) if there exist a smooth family $\tau_s : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ of diffeomorphism-germs starting from the identity τ_0 (resp. a diffeomorphism-germ $\tau : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$) and a diffeomorphism-germ $\sigma : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $f' \circ \sigma = \tau_1 \circ f$ (resp. $f' \circ \sigma = \tau \circ f$). Moreover f and f' are called *symplectically isotopic* (resp. *symplectically equivalent*) if we can take, in the above definitions, τ_s (resp. τ) to be symplectic.

A map-germ $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$, is called *achiral* (resp. *chiral*) if f and $\bar{f}$ are isotopic (resp. non-isotopic). Here we denote by $\bar{f}$ the map-germ $(\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ defined by $\bar{f}(t) = (f_1(t), -f_2(t))$.

Here we give several examples illustrating the notions introduced above.

EXAMPLE 6.1 (About the definition of isotopy) Consider curves $f(t) = (t^2, t^3)$ and $f'(t) = (t^2, -t^3)$ of type A_2 (resp. $f(t) = (t^3, t^4)$ and $f'(t) = (t^3, -t^4)$ of type E_6) . Then we see that f and f' are symplectically isotopic, by just taking τ_s identity (resp. the rotation by $s\pi$) and

$\sigma(t) = -t$. Therefore germs (t^2, t^3) and (t^3, t^4) are achiral. However, there does not exist a smooth family of pairs of diffeomorphism-germs (σ_s, τ_s) starting from $(\mathrm{id}_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}^2})$ with $f' \circ \sigma_1 = \tau_1 \circ f$.

EXAMPLE 6.2 (The difference between equivalence and isotopy) Consider curves $f(t) = (t^3, t^5)$ and $f'(t) = (t^3, -t^5)$ of type E_8. Then f and f' are equivalent but not isotopic. Therefore the germ (t^3, t^5) is chiral.

LEMMA 6.3 *Let m, k be positive integers and k even. Then the two curve-germs $f = (t^m, t^{m+k} + o(t^{m+k}))$ and $f' = (t^m, -t^{m+k} + o(t^{m+k}))$ are not isotopic.*

Proof. Assume $\sigma(t) = \alpha t + \cdots, \alpha \neq 0$, $\tau(p_1, q_1) = (ap_1 + bq_1 + \cdots, cp_1 + dq_1 + \cdots)$ and that $\tau \circ f \circ \sigma = f'$. Then we see that, first, $a\alpha^m = 1, c\alpha^m = 0$. So $c = 0, a = \alpha^{-m}$. Then we see $\alpha^{m+k} d = -1$, so $d = -\alpha^{-m-k}$. Therefore the linear term of τ must have the negative determinant $-\alpha^{-2m-k} < 0$. Then it is impossible to connect τ and the identity by a smooth family of diffeomorphism-germs.

Since any symplectomorphism-germ can be connected to the identity through symplectomorphism-germs, we see that f and f' are symplectically isotopic if and only if they are symplectically equivalent. Therefore the following is clear.

LEMMA 6.4 *If $f, f' : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ are symplectically equivalent, then they are isotopic.*

Now naturally we are led to the following question: Are $f, f' : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ symplectically equivalent if they are isotopic? We answer in detail in the following sections.

7. Symplectic versality and stability

Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a map-germ. Recall the codimension of f is defined by

$$\mathrm{codim}(f) := \dim_{\mathbb{R}} \left(V_f / tf(V_1) + wf(V_2) \right),$$

where $V_f := \{v : (\mathbb{R}, 0) \to T\mathbb{R}^2 \mid \pi \circ v = f\}$ is the space of vector field-germs along f, V_1 (resp. V_2) is the space of vector field-germs over $(\mathbb{R}, 0)$ (resp. $(\mathbb{R}^2, 0)$), and $tf : V_1 \to V_f$ (resp. $wf : V_2 \to V_f$) is the homomorphism defined by $tf(\xi) := f_*(\xi)$ (resp. $wf(\eta) := \eta \circ f$). A plane curve f is called $\mathcal{A}$-finite if $\mathrm{codim}(f) < \infty$. Then f has an $\mathcal{A}$-versal unfolding with the parameter dimension $\mathrm{codim}(f)$. If f is analytic, the condition of $\mathcal{A}$-finiteness is equivalent to, for instance, that the complexification of f has an injective representative.

Moreover, in general, we define

$$\text{sp-codim}(f) := \dim_{\mathbb{R}} \left(V_f / tf(V_1) + wf(VH_2) \right),$$

where $VH_2 \subseteq V_2$ the space of Hamiltonian vector field-germs over the symplectic plane $(\mathbb{R}^2, 0)$. Then clearly

$$\text{sp-codim}(f) \geqslant \mathrm{codim}(f).$$

Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a plane curve-germ. An unfolding $F : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^\ell, 0)$ of f is called *symplectically versal* if any unfolding $G : (\mathbb{R} \times \mathbb{R}^s, 0) \to (\mathbb{R}^2 \times \mathbb{R}^s, 0)$ of f is symplectically equivalent to $\varphi^* F$ for some smooth map-germ $\phi : (\mathbb{R}^s, 0) \to (\mathbb{R}^\ell, 0)$. The following result is a special case of the versality theorem in [**9**].

PROPOSITION 7.1 *An unfolding* $F : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^\ell, 0)$ *of* $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *is symplectically versal if and only if* F *is infinitesimally symplectically versal, that is,*

$$V_f = \left\langle \left. \frac{\partial \bar{F}}{\partial \lambda_1} \right|_{\mathbb{R}\times 0}, \dots, \left. \frac{\partial \bar{F}}{\partial \lambda_\ell} \right|_{\mathbb{R}\times 0} \right\rangle_{\mathbb{R}} + tf(V_1) + wf(VH_2).$$

Moreover two versal unfoldings F *and* F' *of* f *with the same parameter dimension are liftably equivalent.*

A map-germ $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *has a symplectically versal unfolding if and only if* $\text{sp-codim}(f) < \infty$.

REMARK 7.2 By Damon's theory [**9**], we have the characterization of 'symplectic finite determinacy'. A map-germ $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ is called *symplectically finitely determined* if there exists a positive integer k such that any $f' : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ with $j^k f'(0) = j^k f(0)$ is symplectically equivalent to f. Then f is symplectically finitely determined if and only if $\text{sp-codim}(f) < \infty$.

We have a close relation between symplectic versality and symplectic stability [**15**] via the notion of Lagrangian liftings.

THEOREM 7.3 (Symplectic versality and stability) *Let* $F : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^\ell, 0)$ *be a symplectically versal unfolding. Then the Lagrangian lifting* $\widetilde{F} : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^{2\ell}, 0)$ *is symplectically stable, that is, any isotropic deformation of* $\widetilde{F}$ *is trivialized by symplectic equivalences. Therefore* $\widetilde{F}$ *is symplectically equivalent to an open Whitney umbrella* [**14**]. *In particular* $\widetilde{F}$ *has an injective representative.*

Proof. To see the symplectic stability of $\widetilde{F}$, we apply [**15**, Proposition 5.1]. Then it suffices to show that $\widetilde{F}$ is right–left equivalent to an analytic map-germ, that $R_{\widetilde{F}} = \widetilde{F}^*\mathcal{E}_{2+2\ell}$ and that $\text{codim}(\text{Sing}\widetilde{F}_{\mathbf{C}}) \geqslant 2$. Here $R_{\widetilde{F}}$ is the space of function-germs $h \in \mathcal{E}_{1+\ell}$ such that the exterior differential dh of h is a linear combination of the exterior differentials of components in $\widetilde{F}$ with coefficients from $\mathcal{E}_{1+\ell}$.

Since $f = F|_{\mathbb{R}\times 0}$ is symplectically finitely determined, f (resp. F) is symplectically equivalent to a polynomial map-germ. Moreover, we have that $\text{Sing}\widetilde{F}_{\mathbf{C}} = \text{Sing}F_{\mathbf{C}} = \{q_1 \circ F/\partial t = p_1 \circ F/\partial t = 0\}$ is of codimension 2.

Since F is finite map-germ of corank one, by [**15**, Corollary 2.4], we have $R_{\widetilde{F}} = R_F$ is a finite $\mathcal{E}_{2+\ell}$-module via F. Therefore it is a finite $\mathcal{E}_{2+2\ell}$-module via $\widetilde{F}$. On the other hand, we see that $p_{j+1} \circ \widetilde{F}$ is a generating function of $\partial\bar{F}/\partial\lambda_j|_{\mathbb{R}\times 0}$, setting $F(t, \lambda) = (\bar{F}(t, \lambda), \lambda)$. Thus, by the symplectic versality of F, we see $R_{\widetilde{F}} \subseteq \widetilde{F}^*\mathcal{E}_{2+2\ell} + m_\ell R_F \subseteq \widetilde{F}^*\mathcal{E}_{2+2\ell} + m_{2+2\ell} R_F$. Then, by Nakayama's lemma, we see that $R_{\widetilde{F}} \subseteq \widetilde{F}^*\mathcal{E}_{2+2\ell}$. Since the converse inclusion is clear, we have the equality $R_{\widetilde{F}} = \widetilde{F}^*\mathcal{E}_{2+2\ell}$.

Thus we see $\widetilde{F}$ is symplectically stable. Then, by [**15**, Proposition 5.1], we deduce that $\widetilde{F}$ is symplectically equivalent to an open Whitney umbrella.

For a mapping $\varphi : (\mathbb{R}^s, 0) \to (\mathbb{R}^\ell, 0)$, and an unfolding $F : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^\ell, 0)$, $F(t, \lambda) = (f_\lambda(t), \lambda)$, we define the pull-back unfolding $\varphi^*F : (\mathbb{R} \times \mathbb{R}^s, 0) \to (\mathbb{R}^2 \times \mathbb{R}^s, 0)$ by $(\varphi^*F)(t, \mu) = (f_{\varphi(\mu)}(t), \mu)$.

PROPOSITION 7.4 *Let* $F : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^\ell, 0)$ *be an unfolding of* $f = F|_{\mathbb{R}\times 0}$ *and* $\widetilde{F}$

a Lagrangian lifting of F. *Let* $\varphi : (\mathbb{R}^s, 0) \to (\mathbb{R}^\ell, 0)$ *be a map-germ. Then the lifting* $\widetilde{\varphi^* F} : (\mathbb{R} \times \mathbb{R}^s, 0) \to (\mathbb{R}^2 \times \mathbb{R}^s, 0)$ *of* $\varphi^* F$ *defined by*

$$p_j := \sum_{1 \leqslant k \leqslant \ell} \frac{\partial \varphi_k}{\partial \mu_j} (p_{k+1} \circ \widetilde{F}), \ (2 \leqslant j \leqslant s+1)$$

is a Lagrangian lifting of $\varphi^* F$. *In fact we have*

$$\widetilde{\varphi^* F}^* \theta_{\mathbb{R}^2 \times \mathbb{R}^{2s}} = (\mathrm{id}_{\mathbb{R}} \times \varphi)^* \widetilde{F}^* \theta_{\mathbb{R}^2 \times \mathbb{R}^{2\ell}}$$

for the Liouville form $\theta_{\mathbb{R}^2 \times \mathbb{R}^{2s}}$ *on* $\mathbb{R}^2 \times \mathbb{R}^{2s} = T^*(\mathbb{R} \times \mathbb{R}^s)$ *(respectively* $\theta_{\mathbb{R}^2 \times \mathbb{R}^{2\ell}}$ *on* $\mathbb{R}^2 \times \mathbb{R}^{2\ell} = T^*(\mathbb{R} \times \mathbb{R}^\ell)$*).*

The above Proposition 7.4 means the Lagrangian lifting of the pull-back unfolding can be obtained by reduction from the Lagrangian lifting of the original unfolding. In particular we have the following.

COROLLARY 7.5 *Let* $G : (\mathbb{R}^n, 0) \to (N, 0) = (\mathbb{R}^2 \times \mathbb{R}^{n-1}, 0)$ *be a transverse map-germ to* $\mathbb{R}^2 \times \{0\}$. *Assume the restriction* $(G^{-1}(\mathbb{R}^2 \times \{0\}), 0) \to \mathbb{R}^2 \times \{0\}$ *is* $\mathcal{A}$*-finite; then* G *is obtained from an open Whitney umbrella by a reduction process.*

Proof. We may suppose that G is an unfolding of an $\mathcal{A}$-finite map-germ $g : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$. Let $F : (\mathbb{R} \times \mathbb{R}^k, 0) \to (\mathbb{R}^2 \times \mathbb{R}^k, 0)$ be a symplectically versal unfolding of g. Then there exists a map-germ $\varphi : (\mathbb{R}^{n-1}, 0) \to (\mathbb{R}^k, 0)$ such that G is symplectically equivalent to the pull-back unfolding $\varphi^* F$. Then the Lagrangian lifting $\widetilde{G}$ is H-symplectically equivalent to $\widetilde{\varphi}^* F$, that is a reduction of the open Whitney umbrella $\widetilde{F}$.

8. Symplectic defect

Set, for a map-germ $f = (f_1, f_2) : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$,

$$G_f := \{h \in \mathcal{E}_1 \mid dh \in \langle df_1, df_2 \rangle_{f^* \mathcal{E}_2}\} = \{h \in \mathcal{E}_1 \mid dh \in f^*(\Omega_2^1)\},$$

where $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) is the $\mathbb{R}$-algebra of C^∞ map-germs on $(\mathbb{R}, 0)$ (resp. $(\mathbb{R}^2, 0)$), Ω_2^1 is the space of differential 1-forms on $(\mathbb{R}^2, 0)$ and the homomorphism $f^* : \mathcal{E}_2 \to \mathcal{E}_1$ (resp. $f^* : \Omega_2^1 \to \Omega_1^1$) is defined by the pull-back by f. Moreover we set

$$R_f := \{h \in \mathcal{E}_1 \mid dh \in \langle df_1, df_2 \rangle_{\mathcal{E}_1}\}$$

(cf. [**15**]). Thus we have defined intrinsically the sequence of vector spaces

$$\mathcal{E}_1 \supseteq R_f \supseteq G_f \supseteq f^* \mathcal{E}_2$$

for the right–left equivalence class of f.

For each element $h \in G_f$, the exterior differential dh is written as $(b \circ f) df_1 - (a \circ f) df_2 = f^*(b dq_1 - a dp_1)$ for some functions $a, b \in \mathcal{E}_2$. Through the symplectic structure $dp_1 \wedge dq_1$ on the (q_1, p_1)-plane $\mathbb{R}^2$, the 1-form $b dq_1 - a dp_1$ on $(\mathbb{R}^2, 0)$ corresponds to the vector field $\eta = a \partial/\partial q_1 + b \partial/\partial p_1$ over $(\mathbb{R}^2, 0)$. The vector field $wf(\eta)$ along f is regarded as an infinitesimal isotropic deformation of f. In this case we say that h is a generating function of $wf(\eta)$.

In general, any function $h(t)$ is called *a generating function* of a vector field

$$v = v_1(t)\left(\frac{\partial}{\partial q}\circ f\right) + v_2(t)\left(\frac{\partial}{\partial p}\circ f\right) : (\mathbb{R},0) \to T\mathbb{R}^2$$

along $f : (\mathbb{R},0) \to (\mathbb{R}^2,0)$ if $dh = v_2 df_1 - v_1 df_2 (= v^*\tilde{\theta})$, the pull-back by the isotropic map $v : (\mathbb{R},0) \to T\mathbb{R}^2 \cong T^*\mathbb{R}^2$ of the Louville 1-form on $T\mathbb{R}^2$.

Thus, G_f is the space of generating functions of infinitesimal deformations of f induced from diffeomorphisms on the plane $\mathbb{R}^2$. Then we see that G_f is an $\mathbb{R}$-vector subspace of $\mathcal{E}_1$ and that G_f contains $f^*\mathcal{E}_2$. Similarly, $f^*\mathcal{E}_2$ is regarded as the space of generating functions of infinitesimal deformations of f induced from symplectomorphisms on the plane $\mathbb{R}^2$. Moreover, R_f is the space of generating functions of all infinitesimal deformations of f.

Then the following is clear.

LEMMA 8.1 *Let $f : (\mathbb{R},0) \to (\mathbb{R}^2,0)$ be a map-germ. For a diffeomorphism $\tau : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$, we have $R_{\tau\circ f} = R_f$ and $G_{\tau\circ f} = G_f$. Morover, for a diffeomorphism $\sigma : (\mathbb{R},0) \to (\mathbb{R},0)$, $\sigma^* : \mathcal{E}_1 \to \mathcal{E}_1$ maps R_f to $R_{f\circ\sigma}$ and G_f to $G_{f\circ\sigma}$ respectively.*

The next lemma is the main lemma of this paper.

LEMMA 8.2 *There exists a vector-space isomorphism*

$$\frac{tf(V_1) + wf(V_2)}{tf(V_1) + wf(VH_2)} \cong \frac{G_f}{f^*\mathcal{E}_2}.$$

Proof. Taking generating functions (mod $f^*\mathcal{E}_2$) we define a linear map $\Phi : tf(V_1) + wf(V_2) \to \mathcal{E}_1/f^*\mathcal{E}_2$. Note that, for each $tf(\xi) = (a(t)f_1', a(t)f_2') \in tf(V_1)$, we have $a(t)f_2'df_1 - a(t)f_1'df_2 = 0$, and $tf(V_1)$ maps to 0 mod $f^*\mathcal{E}_2$; see also [**15**]. The image of Φ coincides with $G_f/f^*\mathcal{E}_2$. We will show that the kernel of Φ is equal to $tf(V_1) + wf(VH_2)$. Using a symplectic equivalence, we may assume $\text{ord}(f_1) < \text{ord}(f_2) \notin \mathbf{Z}(\text{ord}(f_1))$. Now suppose, for $\eta = a(q,p)\partial/\partial q + b(q,p)\partial/\partial p \in V_2$, a generating function of $wf(\eta)$ belongs to $f^*\mathcal{E}_2$. This means $b(f_1,f_2)df_1 - a(f_1,f_2)df_2 = dH(f_1,f_2)$ for some $H \in \mathcal{E}_2$. Then $b(f_1,f_2)f_1' - a(f_1,f_2)f_2' = H_q(f_1,f_2)f_1' + H_p(f_1,f_2)f_2'$. So

$$b(f_1,f_2) = \big(a(f_1,f_2) + H_p(f_1,f_2)\big)\frac{f_2'}{f_1'} + H_q(f_1,f_2).$$

Since $\text{ord}\left(f_2'/f_1'\right) = \text{ord}(f_2) - \text{ord}(f_1) \neq \text{ord}(f_1)$, we see that $a(0,0) + H_p(0,0) = 0$. So the quotient $c(t) := (a(f_1,f_2) + H_p(f_1,f_2))/f_1'$ belongs to $\mathcal{E}_1$. Then

$$\begin{pmatrix} a(f_1,f_2) \\ b(f_1,f_2) \end{pmatrix} = c(t)\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} + \begin{pmatrix} -H_p \\ H_q \end{pmatrix} \in tf(V_1) + wf(VH_2).$$

It is clear that $tf(V_1) + wf(VH_2)$ is included in the kernel of Φ. Thus we have the required isomorphism.

Note that the dimension of $G_f/f^*\mathcal{E}_2$ depends only on the right–left equivalence class of f. Thus we have the following theorem.

THEOREM 8.3 *Let* $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *be an* $\mathcal{A}$*-finite map-germ. Then the symplectic defect*

$$\mathrm{sd}(f) := \text{sp-codim}(f) - \mathrm{codim}(f)$$

is equal to $\dim(G_f/f^*\mathcal{E}_2)$*, and depends only on the right–left equivalence class of* f*, that is, the symplectic defect is an* $\mathcal{A}$*-invariant. Hence* sp-codim(f) *is an* $\mathcal{A}$*-invariant.*

REMARK 8.4 If f is $\mathcal{A}$-finite, then, by the Mather–Gaffney theorem, we see f is $\mathcal{L}$-finite [**21**, p. 494]. Then we have that the vector space $\mathcal{E}_1/f^*\mathcal{E}_2$ is of finite dimension. So, if f is $\mathcal{A}$-finite, namely, if codim(f) is finite, then sp-codim(f) is necessarily finite.

REMARK 8.5 The symplectic codimension is not an $\mathcal{A}$-invariant (a diffeomorphism invariant) for map-germs $\mathbb{R} \to \mathbb{R}^4$. For example, consider map-germs

$$A_{2,0} : (q_1 = t^2,\ p_1 = t^3, q_2 = 0,\ p_2 = 0)$$

and

$$A_{2,1} : (q_1 = t^2,\ p_1 = t^5, q_2 = t^3,\ p_2 = 0),$$

from Arnold's classification [**3**]. Then $A_{2,0}$ and $A_{2,1}$ are clearly $\mathcal{A}$-equivalent. However we have sp-codim$(A_{2,0}) = 3$, and sp-codim$(A_{2,1}) = 4$. In fact, when $f = A_{2,0} : (\mathbb{R}.0) \to (\mathbb{R}^4, 0)$, we can take ${}^t(0, t, 0, 0), {}^t(0, 0, t, 0), {}^t(0, 0, 0, t)$ as a basis of the vector space $V_f/(tf(V_1) + wf(VH_4))$. For $f = A_{2,1}$ we need ${}^t(0, t^2, 0, 0)$ in addition.

From the definition of the symplectic defect, we have the following.

PROPOSITION 8.6 *Let* $F : (\mathbb{R} \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^\ell, 0)$ *be an* $\mathcal{A}$*-versal unfolding of* $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *with* $\mathrm{sd}(f) = 0$*. Then* F *is a symplectically versal unfolding of* f*.*

Finally we show that the symplectic codimension of an $\mathcal{A}$-finite map-germ is, actually, equal to the classical δ-invariant.

Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be an $\mathcal{A}$-finite map-germ. Then we set $\delta(f) := \dim_{\mathbb{R}} \mathcal{E}_1/f^*\mathcal{E}_2$.

THEOREM 8.7 *For an* $\mathcal{A}$*-finite map-germ* $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$*,*

$$\text{sp-codim}(f) = \delta(f).$$

Proof. We have an exact sequence of vector spaces:

$$0 \to \frac{V'_f}{tf(V_1)} \to \frac{V_f}{tf(V_1) + wf(VH_2)} \to \frac{R_f}{f^*\mathcal{E}_2} \to 0,$$

where V'_f is the space of vector fields along f having zero generating functions. Note that $wf(VH_2) \cap V'_f \subseteq tf(V_1)$. Now we have

$$\dim_{\mathbb{R}} \frac{V'_f}{tf(V_1)} = \dim_{\mathbb{R}} \frac{\mathcal{E}_1}{R_f}.$$

To see this, we may assume $f_1 = t^k$ and $\mathrm{ord}(f_2) > k$, for some positive integer. Then $R_f = \mathbb{R} + m_1^k$.

So $\dim_{\mathbb{R}}(\mathcal{E}_1/R_f) = k-1$. On the other hand, if we set $\varphi(t) = f_2'/f_1'$, the vector space $V_f'/tf(V_1)$ has basis

$$\begin{pmatrix} 1 \\ \varphi \end{pmatrix}, \begin{pmatrix} t \\ t\varphi \end{pmatrix}, \ldots, \begin{pmatrix} t^{k-2} \\ t^{k-2}\varphi \end{pmatrix}.$$

Therefore we see that also $\dim_{\mathbb{R}} V_f'/tf(V_1)$ is equal to $k-1$.

Thus

$$\begin{aligned} \text{sp-codim}(f) &= \dim_{\mathbb{R}} \frac{V_f}{tf(V_1) + wf(VH_2)} \\ &= \dim_{\mathbb{R}} \frac{V_f'}{tf(V_1)} + \dim_{\mathbb{R}} \frac{R_f}{f^*\mathcal{E}_2} \\ &= \dim_{\mathbb{R}} \frac{\mathcal{E}_1}{R_f} + \dim_{\mathbb{R}} \frac{R_f}{f^*\mathcal{E}_2} \\ &= \dim_{\mathbb{R}} \frac{\mathcal{E}_1}{f^*\mathcal{E}_2} = \delta(f). \end{aligned}$$

REMARK 8.8 The vector space $V_f'/tf(V_1)$ has a clear geometric meaning: The space V_f' consists of vector fields $v \in V_f$ along f such that, for any regular point $t \in \mathbb{R}$ of f, $v(t) \in f_*(T_t\mathbb{R})$. Such a vector field may not come from a vector field over $\mathbb{R}$ via f_*. Then $V_f'/tf(V_1)$ measures its gap. Also it has the clear algebraic meaning as the cohomology of a complex $0 \to V_1 \to V_f' \to 0$ defined by the Jacobi matrix of f.

REMARK 8.9 For an $\mathcal{A}$-finite map-germ $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ the Milnor number μ is equal to 2δ (cf. [**20**]). So we have $\text{sp-codim}(f) = \frac{1}{2}\mu$.

9. Symplectic defects and symplectically versal unfoldings of simple plane curves

In this section, we calculate the symplectic defects, defined in the previous section, in several cases.

First we give examples of plane-curves without symplectic defect

PROPOSITION 9.1 *If a plane curve is right–left equivalent to* $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$, $f(t) = (t^m, t^{m+k})$ *for some positive integers* m, k, *then its symplectic defect is equal to zero.*

Proof. It is sufficient to see $\text{sd}(f) = 0$. Let $h \in G_f$. Then

$$dh = a(t^m, t^{m+k})d(t^m) + b(t^m, t^{m+k})d(t^{m+k})$$

for some $a, b \in \mathcal{E}_2$. Note that there is a positive integer ℓ such that $f^*\mathcal{E}_2$ contains functions with order at least ℓ. Now it is easy to see that h is a function of t^m, t^{m+k} up to functions with order at least ℓ. Therefore we have $h \in f^*\mathcal{E}_2$. Thus $G_f/f^*\mathcal{E}_2 = 0$ and $\text{sd}(f) = 0$.

Bruce and Gaffney [**5**] classified simple plane curves. The $\mathcal{A}$-equivalence class of simple (0-modal) plane curves are given in the following list:

$A_{2\ell} : t \mapsto (t^2, t^{2\ell+1})$;
$E_{6\ell} : t \mapsto (t^3, t^{3\ell+1} \pm t^{3(\ell+p)+2})$, $0 \leqslant p \leqslant \ell - 2$; $t \mapsto (t^3, t^{3\ell+1})$;
$E_{6\ell+2} : t \mapsto (t^3, t^{3\ell+2} \pm t^{3(\ell+p)+4})$, $0 \leqslant p \leqslant \ell - 2$; $t \mapsto (t^3, t^{3\ell+2})$;
$W_{12} : t \mapsto (t^4, t^5 \pm t^7)$; $t \mapsto (t^4, t^5)$;

$W_{18} : t \mapsto (t^4, t^7 \pm t^9)$; $t \mapsto (t^4, t^7 \pm t^{13})$; $t \mapsto (t^4, t^7)$;
$W^{\#}_{1,2q-1} : t \mapsto (t^4, t^6 + t^{2q+5})$, $q \geqslant 1$.

Note that, in the above list, the germs $(t^3, t^4 \pm t^5)$ and (t^3, t^4) of type E_6 (resp. $(t^3, t^5 \pm t^7)$ and (t^3, t^5) of type E_8) are actually $\mathcal{A}$-equivalent; see also [**4**, pp. 57–59].

Then we have the following.

THEOREM 9.2 (1) *If f is equivalent to $A_{2\ell}$, E_6, E_8 or $E_{6\ell} : (t^3, t^{3\ell+1})$; $E_{6\ell+2} : (t^3, t^{3\ell+2})$; $W_{12} : (t^4, t^5)$; $W_{18} : (t^4, t^7)$ then* $\mathrm{sd}(f) = 0$.

(2) *If f is equivalent to $E_{6\ell} : (t^3, t^{3\ell+1} \pm t^{3(\ell+p)+2}), 0 \leqslant p \leqslant \ell - 2, \ell \geqslant 2$, then $t^{3(\ell+p+1)+2}, \dots, t^{6\ell-1}$ form a basis of $G_f/f^*\mathcal{E}_2$ and* $\mathrm{sd}(f) = \ell - p - 1$. *The family $(t^3, (\pm 1)^{\ell+1} t^{3\ell+1} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1})$ contains all symplectic classes of type $E_{6\ell}$. If f is equivalent to $E_{6\ell+2} : (t^3, t^{3\ell+2} \pm t^{3(\ell+p)+4})$, then $t^{3(\ell+p+1)+4}, \dots, t^{6\ell+1}$ form a basis of $G_f/f^*\mathcal{E}_2$ and* $\mathrm{sd}(f) = \ell - p - 1$. *The family $(t^3, (\pm 1)^{\ell} t^{3\ell+2} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1})$ contains all symplectic classes of type $E_{6\ell+2}$.*

(3) *If f is equivalent to $W_{12} : (t^4, t^5 \pm t^7)$, then t^{11} forms a basis of $G_f/f^*\mathcal{E}_2$ and* $\mathrm{sd}(f) = 1$. *The family $(t^4, t^5 + \lambda t^7)$ contains all symplectic classes of type W_{12}.*

(4) *If f is equivalent to $W_{18} : (t^4, t^7 \pm t^9)$, then t^{13}, t^{17} form a basis of $G_f/f^*\mathcal{E}_2$ and* $\mathrm{sd}(f) = 2$. *If f is equivalent to $W_{18} : (t^4, t^7 \pm t^{13})$, then t^{17} forms a basis of $G_f/f^*\mathcal{E}_2$ and* $\mathrm{sd}(f) = 1$. *The family $(t^4, t^7 + \lambda t^9 + \mu t^{13})$ contains all symplectic classes of type W_{18}.*

(5) *If f is equivalent to $W^{\#}_{1,2q-1} : (t^4, t^6 \pm t^{2q+5})$, then t^{2q+9}, t^{2q+13} form a basis of $G_f/f^*\mathcal{E}_2$ and* $\mathrm{sd}(f) = 2$. *The family $(t^4, \pm t^6 + \lambda t^{2q+5} + \mu t^{2q+9})$, $\lambda \neq 0$, contains all symplectic classes of type $W^{\#}_{1,2q-1}$.*

To examine the symplectic equivalence classes, we first note the following results.

LEMMA 9.3 *Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be $\mathcal{A}$-finite. If* $\mathrm{ord}(f) = m$, *then f is symplectically equivalent to $(t^m, t^{m\ell+j} + o(t^{m\ell+j}))$ for some $\ell \geqslant 1$ and j with $1 \leqslant j \leqslant m - 1$. In particular, if* $\mathrm{ord}(f) = 2$, *then f is symplectically equivalent to $(t^2, t^{2\ell+1} + o(t^{2\ell+1}))$ for some $\ell \geqslant 1$. If* $\mathrm{ord}(f) = 3$, *then f is symplectically equivalent to $(t^3, t^{3\ell+1} + o(t^{3\ell+1}))$ or $(t^3, t^{3\ell+2} + o(t^{3\ell+2}))$ for some $\ell \geqslant 1$.*

If $\mathrm{ord}(f) = 4$ *and f is $\mathcal{A}$-simple, then f is symplectically equivalent to $(t^4, t^5 + o(t^5))$, $(t^4, t^6 + o(t^6))$ or $(t^4, t^7 + o(t^7))$.*

REMARK 9.4 If f is symplectically equivalent to $(t^4, t^\ell + o(t^\ell))$, $\ell \geqslant 9$, then f is not $\mathcal{A}$-simple. Moreover if $\mathrm{ord}(f) \geqslant 5$, then f is not $\mathcal{A}$-simple; see [**5**].

Proof of Theorem 9.2. We give the calculation for W_{18} in detail. Other cases can be treated in a similar way, so we omit the detail for them.

Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be the map-germ defined by $f(t) = (q_1, p_1) = (t^4, t^7 + t^{13})$ (the most degenerate case of type W_{18}). We examine the quotient space $\mathcal{E}_1/f^*\mathcal{E}_2$. The implicit equation of the curve is

$$y^4 = x^7 + 4x^5y^2 - 2x^{10} + x^{13}.$$

Then examining monomials $x^i y^j$, $i \geqslant 0$, $0 \leqslant j \leqslant 3$ pulled back by f, we see

$$f^*\mathcal{E}_2 = \langle 1, t^4, t^8, t^{12}, t^{16}, t^{14}, t^{15}, t^7 + t^{13}, t^{11} + t^{17} \rangle_{\mathbb{R}} + m_1^{18}.$$

So the projections of $t, t^2, t^3, t^5, t^6, t^9, t^{10}, t^{13}, t^{17}$ form a basis of $\mathcal{E}_1/f^*\mathcal{E}_2$ over $\mathbb{R}$. Further, we see

$$G_f = \langle t^{11}, t^{17} \rangle_{\mathbb{R}} + f^*\mathcal{E}_2.$$

In fact, note that a function $h \in \mathcal{E}_1$ with $dh = (t^7 + t^{13})d(t^4)$ belongs to G_f. Thus $h = \frac{4}{11}t^{11} + \frac{4}{17}t^{17} \in G_f$. Since $t^{11} + t^{17} \in f^*\mathcal{E}_2 \subseteq G_f$, we see both t^{11} and t^{17} belong to G_f. Morever we easily see that any polynomial with just monomials $t, t^2, t^3, t^5, t^6, t^9, t^{10}$ belonging to G_f must be zero. Therefore we see t^{17} forms a basis of $G_f/f^*\mathcal{E}_2$. So $\dim_{\mathbb{R}} G_f/f^*\mathcal{E}_2 = 1$, and $\mathrm{sd}(f) = 1$.

Similarly, for $f = (t^4, t^7 + t^9)$, we have

$$f^*\mathcal{E}_2 = \langle 1, t^4, t^8, t^{12}, t^{16}, t^{14}, t^7 + t^9, t^{11} + t^{13}, t^{15} + t^{17}\rangle_{\mathbb{R}} + m_1^{18}$$

and

$$G_f = \langle t^{11}, t^{13}, t^{15}, t^{17}\rangle_{\mathbb{R}} + f^*\mathcal{E}_2.$$

Here we remark, in this case, $\int (t^7 + t^9)d(t^4) = \frac{4}{11}t^{11} + \frac{4}{13}t^{13} \in G_f$, and also $\int t^4(t^7 + t^9)d(t^4) = \frac{4}{15}t^{15} + \frac{4}{17}t^{17} \in G_f$. Thus we see t^{13}, t^{17} form a basis of $G_f/f^*\mathcal{E}_2$, and $\mathrm{sd}(f) = 2$.

For the case $f = (t^4, t^7)$, we see $\mathrm{sd}(f) = 0$ by Proposition 9.1.

Now consider the two-parameter family $f_{\lambda,\mu}(t) = (t^4, t^7 + \lambda t^9 + \mu t^{13})$.

We show that, if a germ f is of type W_{18}, namely, if f is $\mathcal{A}$-equivalent to (t^4, t^7), $(t^4, t^7 + t^9)$ or $(t^4, t^7 + t^{13})$, then f is symplectically equivalent to $f_{\lambda,\mu} = (t^4, t^7 + \lambda t^9 + \mu t^{13})$ for some $(\lambda, \mu) \in \mathbb{R}^2$. First assume f is of type W_{18}. Then f is symplectically equivalent to $f' = (t^4, t^7 + o(t^7))$ by Lemma 9.3. We write

$$f'(t) = (t^4, t^7 + \lambda t^9 + \mu t^{13} + \Phi(t)),$$

where $\Phi \in \langle t^8, t^{10}, t^{11}, t^{12}\rangle_{\mathbb{R}} + m_1^{14}$. Set $v = \Phi(t)\left(\frac{\partial}{\partial p} \circ f'\right) \in V_{f'}$. Consider the generating function $e \in m_1$ of v so that $de = \Phi(t)d(t^4)$. Then $e \in \langle t^{12}, t^{14}, t^{15}, t^{16}\rangle_{\mathbb{R}} + m_1^{18}$. We remark that $\langle t^{12}, t^{14}, t^{15}, t^{16}\rangle_{\mathbb{R}} + m_1^{18} \subseteq f'^* m_2^2$. Therefore, $e = f'^*H$, for some $H \in m_2^2$. Consider the Hamiltonian vector field X_H with Hamiltonian H. Note that $X_H \in VH_2 \cap m_2V_2$. Then the generating function of the vector field $v - X_H \circ f'$ along f' is zero. So, $v - X_H \circ f' \in V'_{f'}$. Moreover $\mathrm{ord}(v - X_H \circ f') \geqslant 4$. By Lemma 9.5 below, we see that there exists $\xi \in m_1V_1$ such that $v - X_H \circ f' = tf'(\xi)$. Thus we have $v \in tf(m_1V_1) + wf(VH_2 \cap m_2V_2)$. Then, using the homotopy method we see f is symplectically equivalent to $f_{\lambda,\mu}$.

LEMMA 9.5 *Let $w \in V'_f$. If $\mathrm{ord}(w) \geqslant \mathrm{ord}(f) - 1$, then $w \in tf(V_1)$. If $\mathrm{ord}(w) \geqslant \mathrm{ord}(f)$, then $w \in tf(m_1V_1)$.*

Proof. Let $w = a\partial/\partial q + b\partial/\partial p$. Since $w \in V'_f$, we see that $bdf_1/dt - adf_2/dt = 0$. Set $c(t) = a/(df_1/dt)$. Then we see ${}^t(a, b) = c(t)\,{}^t(df_1/dt, df_1/dt)$.

In Theorem 9.2, each symplectic normal form (family) may have finite redundancy. In fact, we know, by the infinitesimal method, that the intersection of the family and an orbit under the symplectic equivalence forms, in a jet space, a zero-dimensional algebraic set, so a finite set.

Actually, by direct and formal calculations, we have the exact description of the symplectic moduli spaces of $\mathcal{A}$-simple plane curves. Indeed the exact determination of symplectic normal form turns out to be a surprisingly simple task, after the infinitesimal consideration stated in Theorem 9.2.

THEOREM 9.6 (1) *Let $\ell \geqslant 2$. Then any plane curve germ of type $E_{6\ell}$ is symplectically equivalent to*

$$f_\lambda = \left(t^3, (\pm 1)^{\ell+1}t^{3\ell+1} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1}\right)$$

for some $\lambda = (\lambda_1, \dots, \lambda_{\ell-1}) \in \mathbb{R}^{\ell-1}$. *Moreover* f_λ *and* $f_{\lambda'}$ *are symplectically equivalent if and only if* $\lambda' = (\pm 1)^{\ell-1}\lambda$.

(2) *Let* $\ell \geqslant 2$. *Then any plane curve germ of type* $E_{6\ell+2}$ *is symplectically equivalent to*

$$f_\lambda = \left(t^3, (\pm 1)^\ell t^{3\ell+2} + \sum_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1}\right)$$

for some $\lambda = (\lambda_1, \dots, \lambda_{\ell-1}) \in \mathbb{R}^{\ell-1}$. *Moreover* f_λ *and* $f_{\lambda'}$ *are symplectically equivalent if and only if* $\lambda' = (\pm 1)^{\ell}\lambda$.

(3) *Any plane curve germ of type* W_{12} *is symplectically equivalent to*

$$f_\lambda = (t^4, t^5 + \lambda t^7)$$

for some $\lambda \in \mathbb{R}$. *Moreover* f_λ *and* f_λ' *are symplectically equivalent if and only if* $\lambda' = \lambda$.

(4) *Any plane curve germ of type* W_{18} *is symplectically equivalent to*

$$f_{\lambda,\mu} = (t^4, t^7 + \lambda t^9 + \mu t^{13})$$

for some $(\lambda, \mu) \in \mathbb{R}^2$. *Moreover* $f_{\lambda,\mu}$ *and* $f_{\lambda',\mu'}$ *are symplectically equivalent if and only if* $(\lambda', \mu') = (\lambda, \mu)$.

(5) *Let* $q \geqslant 1$. *Then any plane curve germ of type* $W^{\#}_{1,2q-1}$ *is symplectically equivalent to*

$$f_{\lambda,\mu} = (t^4, \pm t^6 + \lambda t^{2q+5} + \mu t^{2q+9})$$

for some $(\lambda, \mu) \in (\mathbb{R} - \{0\}) \times \mathbb{R}$. *Moreover* $f_{\lambda,\mu}$ *and* $f_{\lambda',\mu'}$ *are symplectically equivalent if and only if* $(\lambda', \mu') = \pm(\lambda, \mu)$.

For the proof of Therem 9.6, we need the following lemma.

LEMMA 9.7 *Let* $Q = q + bp + h_1 q^2 + h_2 qp + h_3 p^2 + \cdots$, $P = p + k_1 q^2 + k_2 qp + k_3 p^2 + \cdots$ *indicate the* 2*-jet of a symplectomorphism-germ* $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. *Then we have* $2h_1 - 2bk_1 + k_2 = 0$ *and* $h_2 - bk_2 + 2k_3 = 0$.

Proof. Since we have $dP \wedge dQ = (1 + 2h_1 q + h_2 p - 2bk_1 q + k_2 q - bk_2 p + 2k_3 p + \dots) dp \wedge dq$, the result is straightforward.

Proof of Therem 9.6. We give the proof just in the case W_{18}. The remaining cases, more or less, can be treated similarly.

Suppose $(t^4, t^7 + \lambda t^9 + \mu t^{13})$ and $(t^4, t^7 + \lambda' t^9 + \mu' t^{13})$ are symplectic equivalent by a difeomorphism-germ $\sigma : (\mathbb{R}, 0) \to (\mathbb{R}, 0), \sigma(t) = a_1 t + a_2 t^2 + a_3 t^3 + \cdots$, and a symplectomorphism-germ $\tau : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, $\tau(p, q) = (P, Q)$ with $Q = aq + bp + h_1 q^2 + h_2 qp + h_3 p^2 + \dots$, $P = cq + ep + k_1 q^2 + k_2 qp + k_3 p^2 + \dots$. Namely we suppose that

$$at^4 + b(t^7 + \lambda t^9 + \mu t^{13}) + h_1 t^8 + h_2 t^4 (t^7 + \lambda t^9 + \mu t^{13}) + \cdots = (a_1 t + a_2 t^2 + a_3 t^3 + \cdots)^4$$

and

$$\begin{aligned} ct^4 + e(t^7 + \lambda t^9 + \mu t^{13}) + k_1 t^8 + k_2 t^4 (t^7 + \lambda t^9 + \mu t^{13}) + \cdots \\ = (a_1 t + a_2 t^2 + a_3 t^3 + \cdots)^7 + \lambda' (a_1 t + a_2 t^2 + a_3 t^3 + \cdots)^9 \\ + \mu' (a_1 t + a_2 t^2 + a_3 t^3 + \cdots)^{13}. \end{aligned}$$

Table 1 The symplectic classification of simple plane curves

	Diff. normal form	Defect	Sym. normal form
$A_{2\ell}$	$(t^2, t^{2\ell+1})$	0	$(t^2, t^{2\ell+1})$
E_6	(t^3, t^4)	0	(t^3, t^4)
$E_{6\ell}(\ell \geqslant 2)$	$(t^3, t^{3\ell+1} \pm t^{3(\ell+p)+2}), 0 \leqslant p \leqslant \ell - 2$	$\ell - p - 1$	$(t^3, (\pm 1)^{\ell+1} t^{3\ell+1} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)-1})$
	$(t^3, t^{3\ell+1})$	0	
E_8	(t^3, t^5)	0	$(t^3, \pm t^5)$
$E_{6\ell+2}(\ell \geqslant 2)$	$(t^3, t^{3\ell+2} \pm t^{3(\ell+p)+4}), 0 \leqslant p \leqslant \ell - 2$	$\ell - p - 1$	$(t^3, (\pm 1)^{\ell} t^{3\ell+2} + \Sigma_{j=1}^{\ell-1} \lambda_j t^{3(\ell+j)+1})$
	$(t^3, t^{3\ell+2})$	0	
W_{12}	$(t^4, t^5 \pm t^7)$	1	$(t^4, t^5 + \lambda t^7)$
	(t^4, t^5)	0	
W_{18}	$(t^4, t^7 \pm t^9)$	2	$(t^4, t^7 + \lambda t^9 + \mu t^{13})$
	$(t^4, t^7 \pm t^{13})$	1	
	(t^4, t^7)	0	
$W^{\#}_{1,2q-1}$	$(t^4, t^6 + t^{2q+5}), q \geqslant 1$	2	$(t^4, \pm t^6 + \lambda t^{2q+5} + \mu t^{2q+9})$

Then we have $a = a_1^4, c = 0, e = a_1^7$. Thus we have $1 = ae = a_1^{11}$, so we have $a_1 = 1$ and $a = 1, c = 0, e = 1$. In particular, the 2-jet of τ has the form as in Lemma 9.7. Then, from the first equation, we see $a_2 = 0, a_3 = 0, b = 4a_4, h_1 = 4a_5, b\lambda = 4a_6$. Moreover, from the second equation, we have $k_1 = 0, \lambda = \lambda', a_4 = 0$. Then we have $k_2 = 7a_5$ and $\mu + k_2\lambda = \mu' + 9a_5\lambda'$, besides $b = 0$. By Lemma 9.7, we see $0 = 2h_1 - 2bk_1 + k_2 = 8a_5 + 7a_5 = 15a_5$, hence $a_5 = 0$, as well as $k_1 = 0$. Therefore we have $\lambda = \lambda'$ and $\mu = \mu'$.

We summarize the result in Table 1. We have the following corollary.

COROLLARY 9.8 *The symplectically simple* (0-*modal*) *plane curves are symplectically classified into* $A_{2\ell}, \ell = 1, 2, 3, \ldots$; $E_6 : t \mapsto (t^3, t^4)$; $E_8^{\pm} : t \mapsto (t^3, \pm t^5)$.

Proof. By Theorem 9.2, the germs of type $E_{6\ell}$ ($\ell \geqslant 2$), $E_{6\ell+2}$ ($\ell \geqslant 2$), W_{12}, W_{18}, or $W^{\#}_{1,2q-1}$ are never symplectically simple. Since $A_{2\ell}$ is adjacent to just A_{2m} with $m \leqslant \ell$, E_6 is adjacent to just E_6, A_4, A_2, A_0, and $E_8^{\pm}$ is adjacent to just $E_8^{\pm}$, E_6, A_4, A_2, A_0. So they have nearby just finitely many symplectic equivalence classes.

For the symplectically versal unfoldings we have the following results.

PROPOSITION 9.9 *The symplectically versal unfolding with the minimal number of parameters for each $\mathcal{A}$-simple plane curve is given by*

$A_{2\ell}$ (sp-codim $= \ell$):

$$\left(t^2, t^{2\ell+1} + \sum_{j=1}^{\ell} \lambda_j t^{2\ell-2j+1}\right),$$

$(\lambda_1, \ldots, \lambda_\ell) \in (\mathbb{R}^\ell, 0)$.

$E_{6\ell}$ (sp-codim $= 3\ell$) :

$$\left(\begin{array}{l} t^3 + \lambda t, \\ (\pm 1)^{\ell+1} t^{3\ell+1} + \sum_{j=1}^{\ell} \mu_j t^{3\ell-3j+1} + \sum_{j=1}^{2\ell-1} \nu_j t^{6\ell-3j-1} \end{array}\right),$$

$(\nu_1, \ldots, \nu_{\ell-1}) \in \mathbb{R}^{\ell-1}$, $(\lambda, \mu_1, \ldots, \mu_\ell, \nu_\ell, \ldots, \nu_{2\ell-1}) \in (\mathbb{R}^{2\ell+1}, 0)$.

$E_{6\ell+2}$ (sp-codim $= 3\ell + 1$) :

$$\left(\begin{array}{l} t^3 + \lambda t, \\ (\pm 1)^{\ell} t^{3\ell+2} + \sum_{j=1}^{\ell} \mu_j t^{3\ell-3j+2} + \sum_{j=1}^{2\ell} \nu_j t^{6\ell-3j+1} \end{array}\right),$$

$(\nu_1, \ldots, \nu_{\ell-1}) \in \mathbb{R}^{\ell-1}$, $(\lambda, \mu_1, \ldots, \mu_\ell, \nu_\ell, \ldots, \nu_{2\ell}) \in (\mathbb{R}^{2\ell+2}, 0)$.

W_{12} (sp-codim $= 6$) :

$$\left(\begin{array}{l} t^4 + \lambda_1 t^2 + \lambda_2 t, \\ t^5 + \mu_1 t^7 + \mu_2 t^3 + \mu_3 t^2 + \mu_4 t \end{array}\right),$$

$\mu_1 \in \mathbb{R}$, $(\lambda_1, \lambda_2, \mu_2, \mu_3, \mu_4) \in (\mathbb{R}^5, 0)$.

W_{18} (sp-codim $= 9$) :

$$\left(\begin{array}{l} t^4 + \lambda_1 t^2 + \lambda_2 t, \\ t^7 + \mu_1 t^{13} + \mu_2 t^9 + \mu_3 t^6 + \mu_4 t^5 + \mu_5 t^3 + \mu_6 t^2 + \mu_7 t \end{array}\right),$$

$(\mu_1, \mu_2) \in \mathbb{R}^2$, $(\lambda_1, \lambda_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7) \in (\mathbb{R}^7, 0)$.

$W^{\#}_{1,2q-1}$ (sp-codim $= q + 7$) :

$$\left(\begin{array}{l} t^4 + \lambda t^2 + \rho t, \\ \pm t^6 + t^{2q+5} + \mu t^{2q+9} + \sum_{j=0}^{q+2} \nu_j t^{2q+5-2j} + \theta t^2 + \rho t^{2q+2} \end{array}\right),$$

$(\nu_0, \mu) \in \mathbb{R}^2$, $\nu_0 \neq -1$, $(\lambda, \nu_1, \ldots, \nu_{q+2}, \theta, \rho) \in (\mathbb{R}^{q+5}, 0)$.

Remarkably the symplectic versal unfolding can be taken uniformly for each class of simple plane curves; this is not the case for the $\mathcal{A}$-versal unfoldings. This is natural because the A-E-W-$W^{\#}$-classification is based on the constancy of the Milnor number μ, and the μ-constant strata coincide with the sp-codim constant strata (cf. Theorem 8.7).

PROPOSITION 9.10 *Let* $F : (\mathbb{R}^n, 0) \to (\mathbb{R}^2 \times \mathbb{R}^{n-1})$ *be a symplectically versal unfolding of* $A_{2\ell} : (t^2, t^{2\ell+1})$, $\ell \leqslant n-1$. *Then* F *is liftable equivalent to*

$$(x_1, x_2, \ldots, x_n) = (t, \lambda_1, \ldots, \lambda_{n-1}) \mapsto$$
$$(q_1, p_1, q_2, \ldots, q_n) = (t^2, t^{2\ell+1} + \lambda_1 t^{2\ell-1} + \lambda_2 t^{2\ell-3} + \cdots + \lambda_\ell t, \lambda_1, \ldots, \lambda_{n-1}).$$

EXAMPLE 9.11 (The opening of Whitney umbrella) Any symplectically versal unfolding of $A_1 : (t^2, t^3)$ is liftable equivalent to

$$(x_1, x_2, \ldots, x_n) = (t, \lambda_1, \ldots, \lambda_{n-1}) \mapsto (q_1, p_1, q_2, \ldots, q_n) = (t^2, t^3 + \lambda_1 t, \lambda_1, \ldots, \lambda_{n-1}).$$

The Lagrangian lifting is symplectically equivalent to

$$(x_1, x_2, \ldots, x_n) = (t, \lambda_1, \ldots, \lambda_{n-1}) \mapsto$$
$$(q_1, p_1, q_2, \ldots, q_n, p_2, \ldots, p_n) = (t^2, t^3 + \lambda_1 t, \lambda_1, \tfrac{2}{3} t^3, 0, \ldots, 0).$$

10. Isotopy and symplectic classifications

We show that if the symplectic defect vanishes, then the classifications by isotopy and by symplectomorphism coincide.

LEMMA 10.1 *There are isomorphisms of the vector spaces*

$$\frac{tf(m_1 V_1) + wf(m_2 V_2)}{tf(m_1 V_1) + wf(m_2 V_2 \cap VH_2)} \cong \frac{G'_f}{f^* m_2^2} \cong \frac{G_f}{f^* \mathcal{E}_2},$$

where m_1 *(resp.* m_2*) is the maximal ideal of* $\mathcal{E}_1$ *(resp.* $\mathcal{E}_2$*) consisting of functions* H *with* $H(0) = 0$, *and* $G'_f = \{h \in m_1 \mid dh \in \langle df_1, df_2 \rangle_{f^* m_2}\}$.

Proof. In the proof of Lemma 8.2, assume $a(0,0) = b(0,0) = 0$. Then, from $a(0,0) = 0$, we have that $H_p(0,0) = 0$, and from $b(0,0) = 0$, we have that $H_q(0,0) = 0$. So we have $c(0) = 0$. Therefore we have the first isomorphism.

To get the second isomorphism, first we remark that if we have $(A \circ f)df_1 + (B \circ f)df_2 = 0$, then we have $A(0,0) = B(0,0) = 0$. (This is proved by comparing orders of terms easily.) Then we show $G'_f \cap f^*\mathcal{E} = f^* m_2^2$. In fact, the inclusion $G'_f \cap f^*\mathcal{E} \supseteq f^* m_2^2$ is clear. Let $h = a(f_1, f_2) \in G'_f \cap f^*\mathcal{E}$. Then $dh = (C \circ f)df_1 + (D \circ f)df_2$ with $C(0,0) = 0$, $D(0,0) = 0$. Besides we have $dh = \left(\frac{\partial a}{\partial q} \circ f\right) df_1 + \left(\frac{\partial a}{\partial p} \circ f\right) df_2$. So we have

$$\left\{\left(\frac{\partial a}{\partial q} - C\right) \circ f\right\} df_1 + \left\{\left(\frac{\partial a}{\partial p} - D\right) \circ f\right\} df_2 = 0.$$

Therefore we have $\partial a(0,0)/\partial q = \partial a(0,0)/\partial p = 0$, namely $a \in m_2^2$. So $h = a \circ f \in f^* m_2^2$.

Lastly we show for any $h \in G_f$, there exist $\alpha, \beta, \gamma \in \mathbb{R}$ with $h - (\alpha + \beta f_1 + \gamma f_2) \in G'_f$. In fact, $dh = (R \circ f)df_1 + (S \circ f)df_2$, for some $R, S \in \mathcal{E}_2$. Then $d(h - h(0) - R(0,0) f_1 - S(0,0) f_2) \in \langle df_1, df_2 \rangle_{f^* m_2}$.

COROLLARY 10.2 *The symplectic defect of a plane curve-germ measures the codimension of the symplectic equivalence orbit in the $\mathcal{A}$-equivalence orbit of the germ* (*in the jet space of sufficiently high order*).

REMARK 10.3 Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a map-germ. Then, for diffeomorphisms $\tau : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $\sigma : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, we have $G'_{\tau\circ f} = G'_f$ and $\sigma^*(G'_f) = G'_{f\circ\sigma}$.

THEOREM 10.4 *Let* $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *be an $\mathcal{A}$-finite map-germ with* $\mathrm{sd}(f) = 0$*. If a map-germ* $f' : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ *is isotopic to* f*, then* f' *is symplectically equivalent to* f*.*

Proof. Using a right equivalence, we may assume $f' = \tau_1 \circ f$, for a family of diffeomorphisms $\tau_s : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ starting from $\tau_0 = \mathrm{id}_{\mathbb{R}}$. Set $f_s = \tau_s \circ f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$. Then $\mathrm{sd}(f_s) = 0$. Thus $tf_s(m_1V_1) + wf_s(m_2V_2) \subseteq tf_s(m_1V_1) + wf_s(m_2V_2 \cap VH_2)$. By the homotopy method, we have the required result.

EXAMPLE 10.5 The plane curves of type E_8 are classified up to isotopy into $E_8^+ : t \mapsto (t^3, t^5)$ and $E_8^- : t \mapsto (t^3, -t^5)$, because they are chiral. Then, since the symplectic defect vanishes in this case, this gives also the symplectic classification.

11. Lagrangian liftings of the swallowtails

Let M^{2k} be the space of polynomials of degree $2k + 1$ of the form (cf. [**2**, **11**, **13**, **18**])

$$M^{2k} = \left\{ \frac{x^{2k+1}}{(2k+1)!} + q_1 \frac{x^{2k-1}}{(2k-1)!} + \cdots + q_k \frac{x^k}{(k)!} - p_k \frac{x^{k-1}}{(k-1)!} + \cdots + (-1)^k p_1 \right\}$$

endowed with the symplectic Darboux form $\sum_{i=1}^{k} dp_i \wedge dq_i$ (reduction of the sl_2-invariant symplectic form on the space of binary forms of degree $2k + 3$).

The canonical projection into N is given by the derivative

$$D^{k-1} = \frac{d^{k-1}}{dx^{k-1}},$$

which projects M^{2k} into the space of polynomials

$$N = \left\{ \frac{x^{k+2}}{(k+2)!} + q_1 \frac{x^k}{(k)!} + \cdots + q_k x - p_k \right\}.$$

The standard (generalized) swallowtail in N is defined as the space $\Sigma_k \subset N$ of polynomials having at least one root of multiplicity at least 2.

The derivative d/dx of the polynomial decreases the multiplicities of its roots, however, the difference of the degree of polynomial and the multiplicity of the root, called the comultiplicity, is not affected by the derivative. So the polynomials of Σ_k have roots of comultiplicity at most k.

The canonical Lagrangian variety, which is a Lagrangian lifting of Σ_k, is defined by V. I. Arnold as the space $\tilde{\Sigma}_k$ of polynomials in M^{2k} having a root of multiplicity at least $k + 1$. This lifting is most regular (stabilization in the sense of Arnold) because the multiplicity is at least $k + 1$ and the degree of the polynomial is $2k + 1$ and finally the polynomials of $\tilde{\Sigma}_k$ have only one unique root of this multiplicity. So the intersection points of Σ_k are avoided.

A parametrization of Σ_k is given in the form

$$F : (\mathbb{R}^k, 0) \to (N, 0),$$

$$F(s) = \Bigg(s_1, \ldots, s_{k-1}, -\frac{s_k^{k+1}}{(k+1)!} - \sum_{i=1}^{k-1} s_i \frac{s_k^{k-i}}{(k-i)!}, \\ -\frac{s_k^{k+2}}{(k+2)k!} - \sum_{i=1}^{k-1} s_{k-i} \frac{s_k^{k-i+1}}{(k-i+1)(k-i-1)!} \Bigg).$$

Its Lagrangian lifting $\widetilde{\Sigma}_k$, $\widetilde{F} : (\mathbb{R}^k, 0) \to (M^{2k}, \omega)$ is generated by the following generating family (cf. [**16**, p. 106]):

$$P_k(q, \lambda) = \frac{1}{2} \int_0^l \left(\frac{k+2}{(k+1)!} x^{k+1} + \sum_{i=1}^{k} q_i \frac{x^{k-i}}{(k-i)!} \right)^2 dx.$$

Thus the associated symplectic bifurcating family of curves (swallowtail bifurcation family) in $(\mathbb{R}^2, dp_k \wedge dq_k)$ is defined by

$$q_k = -\frac{k+2}{(k+1)!} x^{k+1} - \sum_{i=1}^{k-1} \frac{1}{(k-i)!} q_i x^{k-i},$$
$$p_k = -\frac{1}{k!} x^{k+2} - \sum_{i=1}^{k-1} \frac{1}{(k-i+1)(k-i-1)!} q_i x^{k-i},$$

where x is the curve parameter and $(q_1, \ldots, q_{k-1})$ are the bifurcation parameters of the family. We see that this is an unfolding of the curve

$$(q_k, p_k) = \left(-\frac{k+2}{(k+1)!} x^{k+1}, -\frac{1}{k!} x^{k+2} \right).$$

In a more general setting, this result may be formulated in the following way.

PROPOSITION 11.1 *Let $G : (\mathbb{R} \times N^{k+1}, 0) \to \mathbb{R}$ be a function family germ with A_{k+1}-type singularity. Let Σ_k be the discriminant set of G,*

$$\Sigma_k = \{u \in N^{k+1} \mid G(x, u) = 0, G'_x(x, u) = 0, \text{for some } x \in (\mathbb{R}, 0)\}.$$

Then there exist a symplectic space $(\mathbb{R}^{2k}, \omega)$ and an isotropic fibration

$$\pi : ((\mathbb{R}^{2k}, \omega), 0) \to (N^{k+1}, 0); (p, q) \mapsto (q_1, \ldots q_k, p_k)$$

and a Lagrangian lifting $\widetilde{\Sigma}$ of Σ. Moreover $\widetilde{\Sigma}$ is uniquely defined by the conditions

$$\widetilde{\Sigma} = \left\{ \bar{u} \in \mathbb{R}^{2k} \mid D^{-(k-l)} G(x, u) + \sum_{i=1}^{k-l} (-1)^{i-1} p_{k-i} \frac{x^{k-i-l}}{(k-i-l)!} = 0, 1 \leqslant \ell \leqslant k \right\},$$

where $D^0 G(x, u) = G(x, u)$ and $\bar{u} = (u, p_1, \ldots, p_{k-1})$.

12. Frontal-symplectic versality and open swallowtails

In the case $k = 2$, we interpret Givental's construction from the versality viewpoint of 'frontal-symplectic' category, based on the fact that the swallowtail surface provides the versal unfolding of plane curve of type $E_6 : t \mapsto (t^3, t^4)$, among wave-front curves.

Here we give a direct method to construct a versal unfolding in the frontal-symplectic category.

Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a non-flat map-germ. After using a symplectomorphism of $\mathbb{R}^2$, we assume $\mathrm{ord} f_1 < \mathrm{ord} f_2$. Let $(f, \varphi) : (\mathbb{R}, 0) \to (\mathbb{R}^3, 0)$ be the Legendrian liftings of f for the contact form $\alpha = dy - pdx$. In fact $\varphi = (df_2/dt)/(df_1/dt)$, and then $df_2 - \varphi df_1 = 0$. Note that $\mathrm{ord}\varphi = \mathrm{ord} f_2 - \mathrm{ord} f_1$. Let $w = (f, \varphi; \xi, \eta, \psi) : (\mathbb{R}, 0) \to T\mathbb{R}^3$ be an infinitesimal deformation of (f, φ) among Legendrian (integral) mappings. Then $d\eta - \psi df_1 - \varphi d\xi = 0$, that is, $d(\eta - \varphi\xi) = -\xi d\varphi + \psi df_1$. Set $k = \eta - \varphi\xi$. Then $k - k(0)$ has order at least $\min\{\mathrm{ord} f_1, \mathrm{ord}\varphi\}$. For the induced infinitesimal deformation $v = (f; \xi, \eta) : (\mathbb{R}, 0) \to T\mathbb{R}^2$ of f, take a function h with $dh = \eta df_1 - \xi df_2$, a *generating function* of v. Then $dh = \eta df_1 - \xi\varphi df_1 = (\eta - \xi\varphi)df_1 = kdf_1$. So $h - h(0)$ is a sum of a monomial of order $\mathrm{ord} f_1$ and a function of order at least $\min\{2\mathrm{ord} f_1, \mathrm{ord} f_2\}$.

Set $m = \mathrm{ord} f_1$, $k = \min\{2\mathrm{ord} f_1, \mathrm{ord} f_2\}$ and set

$$S = \mathbb{R} + \mathbb{R}t^m + m_1^k.$$

Then S is a vector subspace of $\mathcal{E}_1$ containing $f^*\mathcal{E}_2$.

LEMMA 12.1 *Let* $F : (\mathbb{R}^1 \times \mathbb{R}^\ell, 0) \to (\mathbb{R}^2 \times \mathbb{R}^\ell, 0)$, $F(t, \lambda) = (\bar{F}(t, \lambda), \lambda)$ *be a frontal unfolding of a non-flat map-germ* $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$. *Assume* $\mathrm{ord}(f_1) < \mathrm{ord}(f_2)$. *If* $\left.\frac{\partial \bar{F}}{\partial \lambda_1}\right|_{\mathbb{R}\times 0}, \dots, \left.\frac{\partial \bar{F}}{\partial \lambda_\ell}\right|_{\mathbb{R}\times 0}$ *generate* $S/f^*\mathcal{E}_2$ *via generating functions over* $\mathbb{R}$, *and also generate vector fields*

$$t^i \frac{\partial}{\partial q_1} \circ f + \varphi t^i \frac{\partial}{\partial p_1} \circ f$$

$(2\mathrm{ord}(f_1) - \mathrm{ord}(f_2) \leqslant i \leqslant \mathrm{ord}(f_1) - 2)$ *over* $\mathbb{R}$, *then* F *is a frontal-symplectically versal unfolding of* f. *Frontal-symplectically versal unfoldings are unique up to liftable equivalence.*

EXAMPLE 12.2 (The open swallowtail) Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$, $f(t) = (t^3, t^4)$ be a map-germ of type E_6. Then the one-parameter unfolding $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, $F(t, \lambda) = (q_1, p_1, q_2) = (t^3 + 3\lambda t, t^4 + 2\lambda t^2, \lambda)$ of f is a frontal-symplectic versal unfolding of f. The image of F is the swallowtail surface and has the double point locus. The Lagrangian lifting

$$\widetilde{F}(t, \lambda) = (q_1, p_1, q_2, p_2) = (t^3 + 3\lambda t, t^4 + 2\lambda t^2, \lambda, \tfrac{6}{5}t^5 + 2\lambda t^3)$$

of F coincides with the open swallowtail surface, which has no self-intersections.

EXAMPLE 12.3 (The open folded umbrella) Let $f : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$, $f(t) = (t^2, t^5)$ be a map-germ of type A_4. Then the one-parameter unfolding $F : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$, $F(t, \lambda) = (q_1, p_1, q_2) = (t^2, t^5 + \lambda t^3, \lambda)$ of f is a frontal-symplectic versal unfolding of f. The image of F is the folded umbrella and has the double point locus. The Lagrangian lifting

$$\widetilde{F}(t, \lambda) = (q_1, p_1, q_2, p_2) = (t^2, t^5 + \lambda t^3, \lambda, \tfrac{2}{5}t^5)$$

of F has no selfintersections and may be called 'the open folded umbrella'.

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