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Polynomial symplectomorphisms

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Abstract

Let $\mathbb{K}$ be the field of real or complex numbers. Let $(X \cong \mathbb{K}^{2n}, \omega)$ be a symplectic affine space. We study the group of polynomial symplectomorphisms of X. We show that for an arbitrary $k \in \mathbb{N}$ the group of polynomial symplectomorphisms acts k-transitively on X. Moreover, if $2 \leqslant l \leqslant 2n-2$ then elements of this group can be characterized by polynomial automorphisms which preserve the symplectic type of all algebraic l-dimensional subvarieties of X.

1. *Introduction*

Throughout, (X, ω) will be a symplectic affine space over $\mathbb{K}$ (the field of real or complex numbers) of dimension $2n$, that is, $X \cong \mathbb{K}^{2n}$ (unless mentioned otherwise) and $\omega = \sum_i dx_i \wedge dy_i$ is the standard non-degenerate skew-symmetric form on X. Linear symplectomorphisms of (X, ω) are characterized (cf., for example, [**5**]) as linear automorphisms of X preserving some minimal, complete data defined by ω on systems of linear subspaces. In this way the linear symplectic group $\mathbf{Sp}(X)$ may be characterized geometrically together with its natural conformal and anti-symplectic extensions.

The purpose of this paper is to put the linear considerations of symplectic invariants into the more general context of polynomial automorphisms. We say that a polynomial automorphism $F : X \to X$ is a *symplectomorphism* (or is *symplectic* on X) if $F^*\omega = \omega$, that is,

$$\omega(u, v) = \omega(d_x F(u), d_x F(v))$$

for every $x \in X$ and every $u, v \in T_x X$. The group $\mathbf{PlSp}(X)$ of polynomial symplectomorphisms is an important tool in affine algebraic geometry (see [**2**, **3**, **9**, **10**]). In particular the group of polynomial symplectomorphisms of $\mathbb{C}^{2n}$ is conjectured to be isomorphic to the group of automorphisms of Weyl algebra $A_n(\mathbb{C})$ (see [**3**]).

The first property of $\mathbf{PlSp}(X)$ we prove is its transitivity (k-transitivity) on finite collections of points of X. Fix an arbitrary $k \in \mathbb{N}$; then we prove the following theorem.

Theorem 1. *For any two sets $\{a_1, \ldots, a_k\}$, $\{b_1, \ldots, b_k\}$ of points of (X, ω) there is an element $g \in \mathbf{PlSp}(X)$ such that $g(a_i) = b_i$, $i = 1, \ldots, k$. In other words $\mathbf{PlSp}(X)$ acts k-transitively on X.*

Let $\mathcal{A}_{l,2r}$ be the subset of the Grassmannian $G(l, 2n)$ of all l-dimensional linear subspaces of X on which the form ω has rank at most $2r$. In [**5**] it was proved that in the linear case the conformal symplectic group coincides with the set of all linear automorphisms preserving the non-empty stratum $\mathcal{A}_{l,2r}$ for fixed $0 < l < 2n$ and $2r + 2 \leqslant l$.

We say that an l-dimensional subspace of X, $2 \leqslant l \leqslant 2n-2$, has *type* σ if it is isotropic (if $2 \leqslant l \leqslant n$), symplectic or pseudo-symplectic (if l is odd). As a preparatory result used later in the polynomial case we prove the following theorem.

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Theorem 2. *Let (X, ω) be a symplectic vector space of dimension $2n > 2$, and let $F : X \to X$ be a linear automorphism. Let $2 \leqslant l \leqslant 2n-2$, and assume that F transforms l-dimensional subspaces of type σ onto subspaces of the same type. Then there is a non-zero constant c such that $F^*\omega = c\omega$.*

The very first step of the algebraic (polynomial) symplectic geometry starts with the analog of Theorem 2 in the case of polynomial automorphisms reconstructing the group of polynomial symplectomorphisms (conformal symplectomorphisms [**11**]).

Let $Y \subset X$ be a smooth k-dimensional $(2 \leqslant k \leqslant 2n-2)$ algebraic variety in an affine symplectic space X. We prove that there are even numbers $r_1 > \ldots > r_s$ $(s \geqslant 1)$ and disjoint algebraic locally closed subvarieties $Y_{r_1}, \ldots, Y_{r_s}$ covering Y and such that the form ω has rank r_i on Y_{r_i}. The sequence $\{r_1, \ldots, r_s\}$ is a symplectic invariant which we will call the symplectic type of the variety. Fix an arbitrary number $2 \leqslant k \leqslant 2n-2$; then we prove the following theorem.

Theorem 3. *Let X be an affine symplectic space of dimension $2n > 2$. A polynomial automorphism $\Phi : X \to X$ is a conformal symplectomorphism if and only if it preserves the symplectic types of all algebraic k-dimensional subvarieties of X.*

2. *Linear symplectomorphisms*

Here we recall some basic notions about the linear symplectic group. Let (X, ω) be as before. The symplectic complement of a linear subspace $L \subset X$ is defined as the subspace

$$L^\omega = \{x \in X : \omega(x, y) = 0 \ \ \forall y \in L\}.$$

This space may not be transversal to L. A subspace $L \subset X$ is called: *isotropic* if $L \subset L^\omega$, *coisotropic* if $L^\omega \subset L$, *symplectic* if $L \cap L^\omega = \{0\}$ and *Lagrangian* if $L^\omega = L$. The subspace L is symplectic if and only if $\omega\,|_L$ is a non-degenerate form. Moreover, L is *pseudo-symplectic* if $\dim L = 2k+1$ is an odd number and rank $\omega|_L = 2k$. For any subspace L we have

$$\dim L + \dim L^\omega = \dim X \quad \textit{and} \quad (L^\omega)^\omega = L.$$

There exists a basis of X, called a symplectic basis, $u_1, \ldots, u_n, v_1, \ldots, v_n$ such that

$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0, \ \ \omega(u_i, v_j) = \delta_{ij}.$$

If $L \subset X$ is a subspace, then there is a basis $u_1, \ldots, u_k, v_1, \ldots, v_k, w_1, \ldots, w_l$ of L such that $\omega\,|_L\,(u_j, v_k) = \delta_{jk}$ and the values of $\omega\,|_L$ on all other pairs vanish. This basis extends to a symplectic basis for (X, ω), and the integer $2k$ is the rank of $\omega\,|_L$ (cf. [**12**]).

We say that a linear automorphism $F : X \to X$ is a *symplectomorphism* (or is *symplectic* on X) if $F^*\omega = \omega$, that is, $\omega(x, y) = \omega(F(x), F(y))$ for every $x, y \in X$. If $L \subset X$ is a linear subspace, then we say that F is symplectic on L if $\omega(x, y) = \omega(F(x), F(y))$ for every $x, y \in L$. The group of automorphisms of (X, ω) is called the *symplectic group* and is denoted by $\mathbf{Sp}(X, \omega)$. Via a symplectic basis, $\mathbf{Sp}(X, \omega)$ can be identified with the group of $2n \times 2n$ matrices A which satisfy $A^T J_0 A = J_0$, where J_0 is the $2n \times 2n$ matrix of ω (in a symplectic basis). In the following we identify such matrices with linear symplectomorphisms. It can be proved (see, for example, [**8**]) that for $n \times n$ matrices X and Y the matrix

$$\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix},$$

where $X^TY = Y^TX$ and $XX^T + YY^T = \mathbf{1}$, is symplectic. In particular if X is a real orthogonal matrix then the matrices

$$\begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \tag{2.1}$$

and

$$\begin{bmatrix} 0 & -X \\ X & 0 \end{bmatrix} \tag{2.2}$$

are symplectic. We will need the following result (we give a proof here because of lack of a direct reference).

Theorem 2.1. *The group* $\mathbf{Sp}(X)$ *is an irreducible variety.*

Proof. Let (X, ω) be an affine symplectic space of dimension $2n$. We argue by induction on n. If $n = 1$ then $\mathbf{Sp}(X)$ is a non-singular quadric in $\mathbb{K}^4$, and hence it is an irreducible variety. Let $n > 1$ and let $H \subset X$ be a symplectic plane in X. The group $\mathbf{Sp}(X)$ acts on the Grassmannian $G = G(2, 2n)$ and the orbit of H is a Zariski dense open subset of G. In particular this orbit is an irreducible variety. Moreover, the isotropy group of H is isomorphic to the product $\mathbf{Sp}(\mathbb{K}^2)\times\mathbf{Sp}(\mathbb{K}^{2n-2})$, and hence it is irreducible by the induction hypothesis. This implies that the group $\mathbf{Sp}(X)$ is also irreducible. □

Definition 2.1. Let $\mathcal{A}_{l,2r} \subset G(l, 2n)$ denote the set of all l-dimensional linear subspaces of X on which the form ω has rank at most $2r$.

Of course $\mathcal{A}_{l,2r} \subset \mathcal{A}_{l,2r+2}$ if $2r + 2 \leqslant l$. We have the following proposition (see [**5**, Corollaries 6.3 and 6.4]).

Proposition 2.1. *Let* (X, ω) *be a symplectic vector space of dimension* $2n$*, and let* $F : X \to X$ *be a linear automorphism. Let* $0 < l < 2n$ *and let* $2r + 2 \leqslant l$*. Assume that the set* $\mathcal{A}_{l,2r}$ *is non-empty and* F *transforms* $\mathcal{A}_{l,2r}$ *into* $\mathcal{A}_{l,2r}$*. Then there is a non-zero constant* c *such that* $F^*\omega = c\omega$*.*

Proposition 2.2. *Let* (X, ω) *be a symplectic vector space of dimension* $2n$*, and let* $F : X \to X$ *be a linear automorphism. Let* $2 \leqslant l \leqslant n$*, and assume that* F *transforms* l*-dimensional isotropic* (*for example, Lagrangian*) *subspaces onto subspaces of the same type. Then there is a non-zero constant* c *such that* $F^*\omega = c\omega$*.*

Definition 2.2. Let H be a linear subspace of an odd dimension l. We say that H is pseudo-symplectic if rank $\omega|_H = l - 1$.

From Proposition 2.1 we can deduce the following interesting fact (which is not contained in [**5**]).

Proposition 2.3. *Let* (X, ω) *be a symplectic vector space of dimension* $2n$*, and let* $F : X \to X$ *be a linear automorphism. Let* $2 \leqslant l \leqslant 2n - 2$*, and assume that* F *transforms* l*-dimensional symplectic* (*or pseudo-symplectic if the number* l *is odd*) *subspaces onto subspaces of the same type. Then there is a non-zero constant* c *such that* $F^*\omega = c\omega$*.*

Proof. Let $l = 2r$ if the number l is even, otherwise let $l = 2r + 1$. By assumption the set $\mathcal{A}_{l,2r-2}$ is non-empty and the mapping F^* induced by F transforms the set $A = \mathcal{A}_{l,2r} \setminus \mathcal{A}_{l,2r-2}$ into the same set A. Of course $F^* : A \to A$ is an injection. Since A is a smooth algebraic variety and the mapping F^* is regular, it is a bijection by the Ax theorem (see [**1**] if $\mathbb{K} = \mathbb{C}$ and [**4**] if $\mathbb{K} = \mathbb{R}$). This means that F transforms $\mathcal{A}_{l,2r-2}$ into the same set, and we conclude the proof by Proposition 2.1. □

3. *Transitivity of the group* **PlSp***(2n)*

First we show that the group $\mathbf{PlSp}(X)$ is quite large. We start with the following lemma.

LEMMA 3.1. *Let $a_i = (\alpha_{i,1}, \ldots, \alpha_{i,2n}) \in \mathbb{K}^{2n}$, and let $\mathcal{A} = \{a_1, \ldots, a_m\}$ be a finite family of points. Let $\pi_k : \mathbb{K}^{2n} \ni (\alpha_1, \ldots, \alpha_{2n}) \to \alpha_k \in \mathbb{K}$ be the projection. There is a linear symplectomorphism L such that if $\mathcal{A}' = L(\mathcal{A})$, then all projections π_k, $k = 1, \ldots, 2n$, restricted to $\mathcal{A}'$ are one-to-one, that is, if*

$$L(a_i) = (\alpha'_{i,1}, \ldots, \alpha'_{i,2n})$$

then for every $\{i, j\} \subset \{1, \ldots, m\}$, $\alpha'_{i,s} = \alpha'_{j,s}$ for some s implies that $\alpha'_{i,s} = \alpha'_{j,s}$ for all s.

Proof. For $i \neq j$ consider the vectors $v_{ij} = a_i - a_j$. Let the desired symplectomorphism L have matrix $[l_{ij}]$, and let $l_i = [l_{i1}, \ldots, l_{i2n}]$ be the ith row. Let $(\cdot, \cdot)$ denote the scalar product. For given i, j consider the set $A(1, i, j) \subset \mathbf{Sp}(\mathbb{K}^{2n})$ of all symplectomorphisms L such that $(l_1, v_{ij}) = 0$. It is a proper algebraic subset of $\mathbf{Sp}(\mathbb{K}^{2n})$. Indeed, it is an algebraic subset of $\mathbf{Sp}(\mathbb{K}^{2n})$. Moreover, it is a proper subset of $\mathbf{Sp}(\mathbb{K}^{2n})$, because we can easily find a matrix of type (2.1) or (2.2) which is not contained in $A(1, i, j)$. Indeed, every unit vector $e_s = (0, \ldots, 1_s, \ldots, 0)$ can be realized as a row of some matrix Q_s of type (2.1) or (2.2). In particular if ${v_{ij}}_s \neq 0$, then $(e_s, v_{ij}) \neq 0$ and the matrix Q_s does not belong to the set $A(1, i, j)$.

In the same way for every $s \in \{1, \ldots, 2n\}$ the set $A(s, i, j)$ of symplectomorphisms L which satisfy $(l_s, v_{ij}) = 0$ is a nowhere dense algebraic subset of $\mathbf{Sp}(\mathbb{K}^{2n})$. However, if

$$L \in \mathbf{Sp}(\mathbb{K}^{2n}) \setminus \bigcup_{s,i,j} A(s, i, j),$$

then for every i, j we have $L(a_j) = L(a_i)$ if and only if for some $s \in \{1, \ldots, n\}$ we have $\alpha'_{i,s} = \alpha'_{j,s}$. □

Let us recall the following definition.

DEFINITION 3.1. Let G be a group which acts on a set X. We say that G acts *k-transitively* on X if for any two k-element subsets $\{a_1, \ldots, a_k\}$, $\{b_1, \ldots, b_k\}$ of X, there is a $g \in G$ such that $g(a_i) = b_i$ for $i = 1, \ldots, k$.

We have the following basic result.

THEOREM 3.1. *Let (X, ω) be an affine symplectic space of dimension $2n$. For every $m \in \mathbb{N}$ the group $\mathbf{PlSp}(X)$ acts m-transitively on X.*

Proof. Let $a_i = (\alpha_i, \beta_i) \in \mathbb{K}^{2n}$ and let $b_i = (\gamma_i, \delta_i) \in \mathbb{K}^{2n}$, $i = 1, \ldots, m$, be finite families of distinct points. By Lemma 3.1 there are linear symplectomorphisms L and T such that

$$L(a_i) = (\alpha'_i, \beta'_i) \quad \text{and} \quad T(b_i) = (\gamma'_i, \delta'_i),$$

where for every i, j, s we have $L(a_j) = L(a_i)$ if and only if $\alpha'_{is} = \alpha'_{js}$ for some $s \in \{1, \ldots, n\}$, and $T(b_j) = T(b_i)$ if and only if $\delta'_{is} = \delta'_{js}$ for some $s \in \{1, \ldots, n\}$.

Let $\phi_i(t)$ be a polynomial of one variable such that

$$\phi_i(\alpha'_{is}) = \beta'_{is} \quad \text{for } s = 1, \ldots, n.$$

Consider the polynomial symplectomorphism

$$\Phi(x, y) = (x, y_1 - \phi_1(x_1), y_2 - \phi_2(x_2), \ldots, y_n - \phi_n(x_n)).$$

By construction we have

$$\Phi \circ L(a_i) = (\alpha'_i, 0) \ \text{ for } i = 1, \ldots, m.$$

In a similar way we can construct a polynomial symplectomorphism

$$\Psi(x, y) = (x, y_1 + \psi_1(x_1), y_2 + \psi_2(x_2), \ldots, y_n + \psi_n(x_n))$$

such that

$$\Psi(\alpha'_i, 0) = (\alpha'_i, \delta'_i) \ \text{ for } i = 1, \ldots, m.$$

Further there exists a polynomial symplectomorphism

$$\Lambda(x, y) = (x_1 - \lambda_1(y_1), x_2 - \lambda_2(y_2), \ldots, x_n - \lambda_n(y_n), y)$$

such that

$$\Lambda(\alpha'_i, \delta'_i) = (0, \delta'_i) \ \text{ for } i = 1, \ldots, m.$$

Finally, we can construct a polynomial symplectomorphism

$$\Sigma(x, y) = (x_1 + \sigma_1(y_1), x_2 + \sigma_2(y_2), \ldots, x_n + \sigma_n(y_n), y)$$

such that

$$\Sigma(0, \delta'_i) = (\gamma'_i, \delta'_i) \ \text{ for } i = 1, \ldots, m.$$

Set

$$P = T^{-1} \circ \Sigma \circ \Lambda \circ \Psi \circ \Phi \circ L.$$

Then

$$P(a_i) = b_i \ \text{ for } i = 1, \ldots, m.$$

□

EXAMPLE 3.1. Theorem 3.1 does not hold for an arbitrary symplectic algebraic variety. We construct a smooth rational algebraic symplectic manifold Y with trivial automorphism group, in particular $\mathbf{PlSp}(Y) = \{id\}$. Let $G \subset \mathbb{C}^{2n}$ be a sufficiently generic hypersurface of degree $d > 2n$. Set $Y = \mathbb{C}^{2n} \setminus G$ and equip Y with the symplectic structure induced by the inclusion $i : Y \to \mathbb{C}^{2n}$.

We show that $\mathrm{Aut}(Y) = \{id\}$. Let $F : Y \to Y$ be a polynomial automorphism of Y. Since the hypersurface G is not uniruled (for details see, for example [**6**, **7**]), F has a unique extension to a polynomial automorphism $\overline{F} : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$. Moreover, by [**6**] we have $\mathrm{Aut}(G) = \{id\}$, and we know that the hypersurface G is the identity set for automorphisms, that is, if $\overline{F}|_G = \{id\}$, then $\overline{F} = id$. Altogether this implies that $\mathrm{Aut}(Y) = \{id\}$.

4. *Geometric characterization*

DEFINITION 4.1. Let (X, ω) be an affine symplectic space, and let $Y \subset X$ be a smooth algebraic subvariety. We say that Y is a *Lagrangian variety* if for every $y \in Y$ the linear space T_yY is a Lagrangian subspace of T_yX. In an analogous way we define a symplectic, pseudo-symplectic and isotropic variety.

Of course in X there are affine linear isotropic (or symplectic) subvarieties — these are varieties of the form $a + H$, where H is a linear isotropic (or symplectic) linear subspace of X. We show that there are also quite a lot of non-linear ones. The measure of non-linearity is the degree of a variety. Let us recall the following definition.

DEFINITION 4.2. Let $Y \subset \mathbb{C}^n$ be a complex variety of dimension k. By the degree of Y ($\deg Y$) we mean the number

$$\#(L^{n-k} \cap Y),$$

where L^{n-k} is an $(n-k)$-dimensional sufficiently general affine-linear subspace of $\mathbb{C}^n$. If $Y \subset \mathbb{R}^n$ is a real variety, then by $\deg Y$ we mean $\deg Y_{\mathbb{C}}$, where $Y_{\mathbb{C}}$ denotes the Zariski closure of Y in $\mathbb{C}^n$.

It is not difficult to prove the following proposition.

PROPOSITION 4.1. *Let $Y \subset \mathbb{K}^n$ be an algebraic variety. Assume that there is an affine line l which intersects Y in precisely D points. Then*

$$\deg Y \geqslant D.$$

Proof. We use induction on n. If $n = 1$ or $n = 2$ then Y is a set of points or a curve and the result is clear. Now let $n > 2$ and assume that our result holds for $n - 1$. Take a general hyperplane H which contains l. Then by the Bézout theorem $\deg Y \cap H \leqslant \deg Y$, and by the induction hypothesis we have $\deg Y \cap H \geqslant D$. □

Now we can prove the following proposition.

PROPOSITION 4.2. *Let (X, ω) be a symplectic $2n$-dimensional affine space. For any positive integers $s \leqslant n$ and D there is an algebraic isotropic s-dimensional subvariety $Y \subset X$ such that*

$$\deg Y \geqslant D.$$

Proof. Fix a linear isotropic s-dimensional subvariety $H \subset X$. Choose D points $a_1, \ldots, a_D$ on H and additionally a point $a_0 \notin H$.

Now take a line $l \subset X$ and choose distinct points $b_1, \ldots, b_D, b_0$ on it. By Theorem 3.1 there is a polynomial symplectomorphism Φ of X such that

$$\Phi(a_j) = b_j \text{ for } j = 0, 1, \ldots, D.$$

Now set $Y = \Phi(H)$. By construction the line l intersects Y in at least D points and it is not contained in Y. This implies that $\deg Y \geqslant D$. □

In the same way we can prove the following proposition.

Proposition 4.3. *For any even integer $0 < s < 2n$ and any positive integer D there is an algebraic symplectic s-dimensional subvariety $Y \subset X$ such that*

$$\deg Y \geqslant D.$$

Similarly for any odd integer $0 < s < 2n$ and any positive integer D there is an algebraic pseudo-symplectic s-dimensional subvariety $Y \subset X$ such that

$$\deg Y \geqslant D.$$

Finally, we show that a polynomial symplectomorphism can be described as one which preserves symplectic, pseudo-symplectic or isotropic algebraic subvarieties of X.

Proposition 4.4. *Let (X, ω) be an affine symplectic space of dimension $2n > 2$. Fix an integer $2 \leqslant s \leqslant n$. Assume that $\Phi : X \to X$ is a polynomial automorphism which preserves the family of all s-dimensional isotropic subvarieties of X. Then Φ is a conformal symplectomorphism, that is, there exists a non-zero number $c \in \mathbb{K}$ such that*

$$\Phi^*(\omega) = c\omega.$$

Proof. Fix $x \in H \subset X$, where H is an affine-linear s-dimensional isotropic subvariety of X. Let $x' = \Phi(x)$ and let $H' = \Phi(H)$. By assumption the variety H' is isotropic. This means that the space $d_x \Phi(T_x H) = T_{x'} H'$ is also isotropic. Hence $d_x\Phi$ transforms all linear l-dimensional isotropic subspaces of $T_x X$ onto subspaces of the same type. By Proposition 2.2 this implies that $d_x \Phi$ is a conformal symplectomorphism, that is,

$$(d_x\Phi)^*\omega = \lambda(x)\omega,$$

where $\lambda(x) \neq 0$. This means that there is a smooth (even polynomial) function $\lambda : X \to \mathbb{K}^*$ ($=\mathbb{K}\backslash\{0\}$) such that

$$\Phi^*(\omega) = \lambda\omega.$$

since the form ω is closed, so is $\Phi^*(\omega)$. Since $n > 1$ this implies that the derivative $d\lambda$ vanishes, that is, the function λ is constant. □

In a similar way (we now use Proposition 2.3) we can prove the following theorem.

Proposition 4.5. *Let (X, ω) be an affine symplectic space of dimension $2n > 2$. Fix an integer $0 < s < n$. Assume that $\Phi : X \to X$ is a polynomial automorphism which preserves the family of all $2s$-dimensional symplectic subvarieties of X or (if $1 < s < n-1$) the family of all $(2s+1)$-dimensional pseudo-symplectic subvarieties of X. Then Φ is a conformal symplectomorphism, that is, there exists a non-zero $c \in \mathbb{K}$ such that*

$$\Phi^*(\omega) = c\omega.$$

To end this section, we introduce the notion of symplectic type of an algebraic variety.

Proposition 4.6. *Let (X, ω) be an affine symplectic space of dimension $2n$. Let $Y \subset X$ be a smooth k-dimensional algebraic variety. Then there are even integers $r_1 > \ldots > r_s$ (where $s \geqslant 1$) and disjoint algebraic locally closed subvarieties $Y_{r_1}, \ldots, Y_{r_s}$ which cover Y such that the form ω has rank r_i on Y_i. Moreover $Y_{r_{i+1}} \subset \mathrm{cl}(Y_{r_i})$. The sequence $\{r_1, \ldots, r_s\}$ is a symplectic invariant, we call it the symplectic type of the variety Y.*

Proof. Consider the Gauss mapping

$$G : Y \ni y \longrightarrow T_yY \in G(k, 2n).$$

This is a regular (locally polynomial) mapping. By [5] the linear spaces in Grassmannian $G(k, 2n)$ on which the rank of ω is equal to r form a smooth locally closed (in the Zariski topology) subset S_r, and $S_{r-2} \subset \mathrm{cl}(S_r)$. Now it is enough to take $Y_r = G^{-1}(S_r)$ if this set is not empty. □

EXAMPLE 4.1. (a) A $2k$-dimensional subvariety $Y \subset X$ is a symplectic subvariety if and only if the symplectic type of Y is $\{2k\}$.

(b) A $(2k+1)$-dimensional subvariety $Y \subset X$ is a pseudo-symplectic subvariety if and only if the symplectic type of Y is $\{2k\}$.

(c) A subvariety $Y \subset X$ is an isotropic subvariety if and only if the symplectic type of Y is $\{0\}$.

Now the following statement is obvious.

THEOREM 4.1. *Let (X, ω) be an affine symplectic space of dimension $2n > 2$. Fix an integer $2 \leqslant k \leqslant 2n-2$. A polynomial automorphism $\Phi : X \to X$ is a conformal symplectomorphism if and only if it preserves the symplectic types of all algebraic k-dimensional subvarieties of X.*

It seems that a complex symplectic geometry has a special flavor. To see this we conclude this paper by the following theorem.

THEOREM 4.2. *Let $X = (\mathbb{C}^{2n}, \omega)$ be a complex affine symplectic vector space. Let $Y \subset X$ be algebraic submanifold of dimension $0 < 2s < 2n$. Let*

$$S = \{y \in Y : T_yY \text{ is not a symplectic space}\}.$$

Then only the following three cases are possible:

(1) $S = Y$;

(2) *S is a hypersurface;*

(3) $S = \emptyset$.

Proof. Assume that $S \neq Y$ and $S \neq \emptyset$. Consider the Gauss mapping

$$G : Y \ni y \longrightarrow T_yY \in G(2s, 2n).$$

This is a regular (locally polynomial) mapping. By [5] the linear spaces in Grassmannian $G(2s, 2n)$ on which the rank of ω is less than $2s$ form a divisor, let us denote it by D. Now $S = G^{-1}(D)$, consequently the set S is a divisor in Y. □

COROLLARY 4.1. *Let X and Y be as above. If Y is symplectic at every point outside a subset of codimension at least two, then Y is a symplectic submanifold of X.*

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