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SYMMETRY DEFECT OF ALGEBRAIC VARIETIES*

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Abstract. Let $X, Y \subset k^m (k = \mathbb{R}, \mathbb{C})$ be smooth manifolds. We investigate the central symmetry of the configuration of X and Y. For $p \in k^m$ we introduce a number $\mu(p)$ of pairs of points $x \in X$ and $y \in Y$ such that p is the center of the interval $\overline{xy}$. We show that if X, Y (including the case $X = Y$) are algebraic manifolds in a general position, then there is a closed (semi-algebraic) set $B \subset k^m$, called *symmetry defect* set of the X and Y configuration, such that the function μ is locally constant and not identically zero outside B. If $k = \mathbb{C}$, we estimate the number μ (in fact we compute it in many cases) and show that the symmetry defect is an algebraic hypersurface and consequently the function μ is constant and positive outside B. We also show that in the generic case the topological type of the symmetry defect set of a plane curve is constant, i.e. the symmetry defect sets for two generic curves of the same degree are homeomorphic (by the same method we can prove similar statement for any irreducible family of smooth varieties $Z^n \subset \mathbb{C}^{2n}$). Moreover, for $k = \mathbb{R}$, we estimate the number of connected components of the set $U = k^m \setminus B$. In the last section we give an algorithm to compute the symmetry defect set for complex smooth affine varieties in general position.

Key words. polynomial mapping, fibration, bifurcation points, center symmetry set, Wigner caustic.

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1. Introduction.

Over the last two decades numerous methods have been developed to study affine geometry of curves and surfaces especially their affinely invariant symmetry characteristics. The symmetry sets [4, 5, 6] and the center symmetry sets were investigated extensively in [13, 10, 11]. Several constructions of the set corresponding to the point of central symmetry for perturbed centrally symmetric ovals were presented in the literature and resulted in the kind of symmetry defect called center symmetry set. The center symmetry set directly appears in the construction of the so-called Wigner caustic. This caustic is obtained by the stationary phase method applied to the semiclassical Wigner function which completely describes a quantum state in the symplectic phase space [2]. This set is a subset of all midpoints of chords connecting pairs of points of the curve or surface Z, and it can be described as a bifurcation set in the following way: for each pair of points $(x, y) \in Z$, we let $a = \frac{x+y}{2}$ be the midpoint of the chord connecting x and y, and $\mu(a)$ the number of chords whose midpoint is a (let us note that for $x \neq y$, the chord (x, y) is different than (y, x)). For most points, $\mu(a)$ is locally constant. The caustic consists of the set of points a where $\mu(a)$ changes.

We call this set a symmetry defect or bifurcation set. We generalize this construction for n-dimensional algebraic varieties $Z \subset \mathbb{C}^{2n}$ and investigate the symmetry defect set $B(Z)$ corresponding in the real compact case to the center symmetry set.

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(See also [12], [7], [8] for recent results in the real compact case).

In the general algebraic setting we introduce the bifurcation set $B(f)$ of a dominant generically finite polynomial mapping $f : X \to \mathbb{C}^n$ of an affine n-dimensional variety X onto $\mathbb{C}^n$ as the set of values $z \in \mathbb{C}^n$ for which the cardinality of the set $f^{-1}(z)$ differs on the geometric degree of f. It has two components: the set of non-properness of f, S_f and the set of critical values of f, $K_0(f)$. The fundamental result on the structure of $B(f)$ which we prove in Section 2 says that $B(f)$ is a closed hypersurface or the empty set (Theorem 2.4) and its degree is estimated by the formula involving the degrees of the system of parameters on X, the degrees of the polynomial functions f_i, $f = (f_1, \dots, f_n)$ and the geometric degree of f (Theorem 2.9). For real affine manifolds X, Y and generically finite polynomial mappings $f : X \to Y$ the set $B(f) \subset Y$ is closed and semi-algebraic. Moreover outside of $B(f)$, f is a differentiable covering (Theorem 2.14).

The aim of this paper is to investigate geometric properties of the bifurcation set $B(\Phi)$ in the case of the special midpoint map Φ which sends any two points of the set Z to the midpoint of the chord connecting them. In this case the set $B(\Phi)$ is a generalization of the "center symmetry set" $B(Z)$ which we mentioned before.

More generally, let $X, Y \subset k^m$ be smooth manifolds of dimensions r and s respectively, where $r + s = m$ ($k = \mathbb{R}$ or $k = \mathbb{C}$). For a given point $a \in k^m$ we are interested in the number $\mu(a)$ of pairs of points $x \in X$ and $y \in Y$ such that a is the center of the interval $\overline{xy}, a = \frac{x+y}{2}$ (midpoint map). We show that if X, Y (we do not exclude the case $X = Y$) are algebraic manifolds in a general position, then there is a closed (semi-algebraic) set $B \subset k^m$, such that the function $\Phi : X \times Y \ni (x, y) \mapsto (x + y)/2 \in k^m$ is a differentiable covering outside B. The minimal such a set $B = B(X, Y)$ we will call *symmetry defect set* of X and Y. If $k = \mathbb{C}$, we show that the symmetry defect is an algebraic hypersurface and consequently the function μ is constant and positive outside B (Theorem 3.8), this constant being denoted by $\mu_{X,Y}$. We estimate the number $\mu_{X,Y}$ and the degree of the hypersurface $B(X, Y)$.

More precisely let $X, Y \subset \mathbb{C}^m$ be smooth algebraic manifolds of dimensions r and s and degrees p, q respectively ($r + s = m$). Assume that X and Y have no common points at infinity. Then $\deg B(X, Y) \le pq(sp+rq-m)$ and $\mu_{X,Y} = pq$. Moreover, for every P we have $\mu(P) \le pq$.

For $X = Y = Z$ assume that $Z \subset \mathbb{C}^{2n}$ is a n-manifold in a general position, and denote $\mu_{Z,Z}$ simply by μ_Z. If $\deg\, Z = p$ then $0 < \mu_Z \le p^2 - p - n + 1$ and $\deg B(Z) < p^2(1 + 2n(p-1))$. Moreover, $Z \subset B(Z)$.

For $k = \mathbb{R}$, we estimate the number of connected components of the set $U = k^m \setminus B$. In the case $X = Y = Z$ and X is a projectively smooth variety, we show that the asymptotic part of the symmetry defect set $B(Z)$ coincides with the asymptotic variety of Z (Theorem 3.15) which is the union of all tangent spaces at points at infinity of Z.

For smooth algebraic plane curves $Z, W \subset \mathbb{C}^2, \deg Z = p, \deg W = q$, which meet transversally at r common points at infinity we get the formula for the generic symmetry of Z and W, i.e. we show that for a generic point $a \in \mathbb{C}^2$, the number of pairs $\{z, w\}, z \in Z, w \in W$ such that a is the center of an interval $\overline{zw}$ is equal to $\mu_{Z,W} = pq - r$. Moreover the degree of a symmetry defect set is estimated $degB(Z, W) \le pq(p+q-2)+r$ (Theorem 4.1). In particular if $X = Y = Z$ and a curve Z is transversal to the line at infinity we get $\mu_Z = p^2-p$ and $\deg B(Z) \le 2p^2(p-1)+p$. For the projectively smooth curve $Z \subset \mathbb{C}^2$, i.e. with the smooth projective closure of Z in $\mathbb{P}^2$, it was proved that the asymptotic part of $B(Z)$ is the union of lines $L \subset \mathbb{C}^2$

(asymptotes) which are tangent to Z at infinity (Theorem 4.6). In this special case of curves we show that in generic case the topological type of the symmetry defect set is constant, i.e. the symmetry defect sets for two generic curves of the same degree are homeomorphic (Theorem 5.2). By the same method we can prove similar statement for any irreducible family of smooth n-dimensional varieties $Z \subset \mathbb{C}^{2n}$. In the last section the algorithm to compute the symmetry defect set for complex smooth affine varieties in general position was constructed (Theorem 6.5). The symmetry defect set (or rather its real part) was visualized for several examples of elliptic curves.

2. Bifurcation set. Let $k = \mathbb{C}$ or $k = \mathbb{R}$ and let X, Y be affine varieties over k. Recall the following (see [14], [15]):

DEFINITION 2.1. Let $f : X \to Y$ be a generically-finite (i.e. a generic fiber is finite) and dominant (i.e. $\overline{f(X)} = Y$) polynomial mapping of affine varieties. We say that f *is finite at a point* $y \in Y$, if there exists an open neighborhood U of y such that the mapping $f\mid_{f^{-1}(U)}: f^{-1}(U) \to U$ is proper.

If $k = \mathbb{C}$ it is well-known that the set S_f of points at which the mapping f is not finite, is either empty or it is a hypersurface (see [14], [15]). We say that the set S_f is *the set of non-properness* of the mapping f.

Let X be an affine variety of dimension n. We have the following sharp estimate for the degree of the hypersurface S_f for a generically-finite and dominant polynomial mapping $f = (f_1, ..., f_n) : X \to \mathbb{C}^n$:

THEOREM 2.2. *(see [17]) Let $X \subset \mathbb{C}^m$ be an affine $n-$dimensional variety of degree D and let $f = (f_1, ..., f_n) : X \to \mathbb{C}^n$ be a generically finite dominant mapping. Then the set S_f of non-properness of the mapping f is a hypersurface (or the empty set) and*

$$\deg\, S_f \leq \frac{D(\prod_{i=1}^n \deg\, f_i) - \mu(f)}{\min_{1\leq i\leq n} \deg\, f_i},$$

where $\mu(f) = (\mathbb{C}(X) : \mathbb{C}(f_1, ..., f_n))$ is the geometric degree of f.

DEFINITION 2.3. Let $k = \mathbb{C}$. Let X, Y be smooth affine $n-$dimensional varieties and let $f : X \to Y$ be a generically finite dominant mapping of geometric degree $\mu(f)$. The bifurcation set of the mapping f is the set

$$B(f) = \{y \in Y : \#f^{-1}(y) \neq \mu(f)\}.$$

We have the following fundamental theorem:

THEOREM 2.4. *Let $k = \mathbb{C}$. Let X, Y be smooth affine complex varieties of dimension n. Let $f : X \to Y$ be a polynomial dominant mapping. Then the set $B(f)$ is either empty (so f is an unramified topological covering) or it is a closed hypersurface.*

Proof. Let us note that outside the set S_f the mapping f is a (ramified) analytic cover of degree $\mu(f)$. By the Lemma 2.5 below if $y \notin S_f$ we have $\#f^{-1}(y) \leq \mu(f)$. Moreover, since f is an analytic covering outside S_f it is well known that the fiber $f^{-1}(y)$ counted with multiplicity has exactly $\mu(f)$ points. In particular, if $y \in K_0(f)$, the set of critical values of f, then $\#f^{-1}(y) < \mu(f)$.

Now let $y \in S_f$. There are two possible cases :

a) $\#f^{-1}(y) = \infty$.

b) $\#f^{-1}(y) < \infty$.

In case b) let U be an affine neighborhood of y over which the mapping f is quasi-finite. Let $V = f^{-1}(U)$. By Zariski Main Theorem in the version given by Grothendieck, there exists a normal variety $\overline{V}$ and a finite mapping $\overline{f} : \overline{V} \to U$, such that

1) $V \subset \overline{V}$,

2) $\overline{f}\mid_V = f$.

Since $y \in \overline{f}(\overline{V} \setminus V)$, it follows by the Lemma 2.5 below, that $\#f^{-1}(y) < \mu(f)$. Consequently, if $y \in S_f$, we have $\#f^{-1}(y) < \mu(f)$. Finally we have $B(f) = K_0(f) \cup S_f$.

Now we show that the set $B(f) = K_0(f) \cup S_f$ is a hypersurface. Let $J(f)$ be the set of singular points of f. The set $J(f)$ is a hypersurface (because locally it is the zero set of the Jacobian of f). Denote by J_i the irreducible components of $J(f)$. Let $W_i = \overline{f(J_i)}$. If all W_i are hypersurfaces then the theorem is true. If, for example $\dim W_1 < n_1$, then the mapping $f : J_1 \to W_1$ has non-compact generic fiber, this means in particular that $W_1 \subset S_f$. Thus the set $\bigcup W_i \cup S_f$ is a hypersurface. But $B(f) = \bigcup W_i \cup S_f$ (note that $B(f)$ is closed- see Theorem 2.14).

Moreover, if $B(f) = \emptyset$, then f is a surjective topological covering. □

LEMMA 2.5. *Let X, Y be affine normal varieties of dimension n. Let $f : X \to Y$ be a finite mapping. Then for every $y \in Y$ we have $\#f^{-1}(y) \leq \mu(f)$.*

Proof. Let $\#f^{-1}(y) = \{x_1, ..., x_r\}$. We can choose a function $h \in \mathbb{C}[X]$ which separates all x_i (in particular we can take as h the equation of a general hyperplane section). Since f is finite we have a monic polynomial $T^s + a_1(f)T^{s-1} + ... + a_s(f) \in f^*\mathbb{C}[Y][T]$, $s \leq \mu(f)$. If we substitute $f = y$ to this equation we get the desired result. □

Our next aim is to estimate the degree of the hypersurface $B(f)$. We start with the following:

DEFINITION 2.6. Let $X \subset \mathbb{C}^m$ be an affine irreducible variety, $\dim X = n$. We say that the polynomials $h_1, ..., h_{m-n}$ are a system of parameters on X if

1) $X \subset V(h_1, ..., h_{m-n})$,

2) rank $[\frac{\partial h_i}{\partial x_j}] = m - n$ on a dense subset of X.

PROPOSITION 2.7. *Let $X \subset \mathbb{C}^m$ be an affine irreducible n-dimensional variety. Then X has a system of parameters $h_1, ..., h_{m-n}$ of degrees bounded by $D = \deg X$.*

Proof. Let $r = m - n$. It is enough to construct polynomials $h_1, ..., h_r$ of degree $D = \deg X$, which vanish on X and for which rank $[\frac{\partial h_i}{\partial x_j}] = r$ on a dense subset of X.

Let us take a point $x \in X$. Let S be the closure of the union of all secants xy, where $y \in X$ is another point of X. It is easy to see that $\dim S \leq n + 1$ and the projective closure $\overline{S}$ of S contains the projective closure of X.

Now on the hyperplane at infinity H_∞ let us choose a system of homogeneous coordinates $x_1, ..., x_{m-r}, x_{m-r+1}, ..., x_m$ in the way that for every $j > m - r$ we have $\{x_1 = 0, ..., x_{m-r} = 0, x_j = 0\} \cap \overline{S} = \emptyset$. Of course every sufficiently general system of coordinates has this property. The coordinate system on H_∞ can be extended in an obvious way to a coordinate system on the whole of $\mathbb{P}^m$ (by adding a new variable x_0).

Now for every $j > m - r$ let us consider the projection $\pi_j : X \ni x \to (x_1, ..., x_{m-r}, x_j) \in \mathbb{C}^{m-r+1}$. By construction, the mapping π_j is proper and birational (the last property follows from the fact that $(\pi_j)^{-1}(\pi_j(x)) = \{x\}$ and that π_j

is smooth at x). The image $X_j := \pi_j(X)$ is a hypersurface in $\mathbb{C}^{m-r+1}$. Let h_s be a reduced equation of X_{m-r+s}. Then h_s vanishes on X and $\frac{\partial h_s}{\partial x_{m-r+s}}$ does not vanish identically on X. Now it is easy to check that the polynomials $h_1, ..., h_r$ (of degree $D = \deg X$) vanish on X and $Jac(h_1, ..., h_r)$ does not vanish identically on X. □

PROPOSITION 2.8. *Let $X \subset \mathbb{C}^m$, $\dim X = n$, be an affine irreducible variety. Let $h_1, ..., h_{m-n}$ be system of parameters on X. Assume that $F = (f_1, ..., f_n) : X \to \mathbb{C}^n$ is a generically finite polynomial mapping. Let*

$$W := det \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_{m-n}}{\partial x_1} & \frac{\partial h_{m-n}}{\partial x_2} & \cdots & \frac{\partial h_{m-n}}{\partial x_m} \\ \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}.$$

Then the set of critical points of the mapping F is contained in $X \cap \{W = 0\}$.

Proof. Let us note that for every $x \in X$ such that $W(x) \neq 0$, the polynomials $h_1, ..., h_{m-n}, f_1, ..., f_n$ form a local system of coordinates. Hence at such a point we have that $F|X$ is a local isomorphism. □

THEOREM 2.9. *Let $X \subset \mathbb{C}^m$ be an affine $n-$dimensional variety of degree D. Let $h_1, ..., h_{m-n}$ be a system of parameters on X of degrees $D_1, D_2, ..., D_{m-n}$ respectively. Let $f = (f_1, ..., f_n) : X \to \mathbb{C}^n$ be a generically finite dominant mapping. Assume that $\deg f_i = d_i$ and $d_1 \geq d_2 ... \geq d_n$. Then the set $B(f)$ is a hypersurface (or the empty set) and*

$$\deg\, B(f) \leq D \prod_{i=2}^{n} d_i [\sum_{i=1}^{n} (d_i - 1) + \sum_{j=1}^{m-n} (D_j - 1)] + \frac{D \prod_{i=1}^{n} d_i - \mu(f)}{d_n}.$$

Proof. We have $B(f) = K_0(f) \cup S_f$. First we estimate the degree of the $n-1$ dimensional part of $\overline{K_0(f)}$. Let H be the union of $n-1$ dimensional components of the set $\overline{K_0(f)}$. We estimate the degree of H. Let $L_i = \sum_{k=i}^{n} a_{ik} x_k,\ i = 1, 2, ..., n-1$, where a_{ik} are sufficiently general numbers. Then equations $L_i = 0, i = 1, ..., n-1$ give a generic line l. It is enough to estimate the number of points in $l \cap H$. The pre-image $l' := f^{-1}(l)$ is given on X by equations $L_i' = \sum_{k=i}^{n} a_{ik} f_k = 0$. Hence this pre-image is a curve of degree bounded by $D \prod_{i=2}^{n} d_i$. To every point $x \in l \cap H$ correspond at least one point on the set $l' \cap \{W = 0\}$, where W is a polynomial as in Proposition 2.8. Hence finally

$$\deg\, H \leq D \prod_{i=2}^{n} d_i [\sum_{i=1}^{n} (d_i - 1) + \sum_{j=1}^{m-n} (D_j - 1)].$$

Now note that $B(f) = H \cup S_f$. To finish the proof it is enough to use Theorem 2.2. □

COROLLARY 2.10. *Let $X \subset \mathbb{C}^m$ be an affine $n-$dimensional variety of degree D. Let $h_1, ..., h_{m-n}$ be a system of parameters on X of degrees $D_1, D_2, ..., D_{m-n}$ respectively. Let $f = (f_1, ..., f_n) : X \to \mathbb{C}^n$ be a generically finite dominant mapping.*

Assume that $\deg f_i = d_i$ *and* $d_1 \geq d_2 ... \geq d_n$. *Then the set* $B(f)$ *is a hypersurface (or the empty set) and*

$$\deg\ B(f) < D\prod_{i=2}^{n} d_i[1+\sum_{i=1}^{n}(d_i-1)+\sum_{j=1}^{m-n}(D_j-1)].$$

Corollary 2.11. *Let* $X \subset \mathbb{C}^m$ *be an affine* $n-$*dimensional variety of degree* D. *Let* $f = (f_1, ..., f_n) : X \to \mathbb{C}^n$ *be a generically finite dominant mapping. Assume that* $\deg f_i \leq d$. *Then the set* $B(f)$ *is a hypersurface (or the empty set) and*

$$\deg\ B(f) < Dd^{n-1}[1 + n(d-1) + (m-n)(D-1)].$$

In the real case we can also define the set $B(f)$. Let us note that in the complex case, an equivalent definition of $B(f)$ is $B(f) = K_0(f) \cup S_f$ (see the proof of Theorem 2.4). Hence the following definition coincides with the previous one:

Definition 2.12. Let $k = \mathbb{R}$. Let X, Y be affine $n-$dimensional varieties and let $f : X \to Y$ be a generically finite dominant mapping. The bifurcation set of the mapping f is the set

$$B(f) = K_0(f) \cup S_f.$$

Definition 2.13. Let X, Y be differentiable manifolds and let $f : X \to Y$ be a generically finite mapping. We say that f is a differentiable covering if for every point $y \in Y$ either $f^{-1}(y) = \emptyset$ or there is an open neighborhood U_y such that there is an integer r and open subsets $V_i \subset X$ and:

1) $f^{-1}(U) = \bigcup_{i=1}^{r} V_i$,
2) $f_{V_i} : V_i \to U_y$ is a diffeomorphism.

Theorem 2.14. *Let* X, Y *be affine manifolds. Let* $f : X \to Y$ *be a generically-finite polynomial mapping. Then the set* $B(f) = K_0(f) \cup S_f$ *is closed and semi-algebraic. Moreover, the set* $B(f)$ *is the smallest subset* $B \subset Y$, *such that the mapping* $f_{X\setminus f^{-1}(B)} : X \setminus f^{-1}(B) \to Y \setminus B$ *is a differentiable covering.*

Proof. The set S_f is closed. Indeed, let $y_j \in S_f$ and $y_j \to y$. By the definition of S_f we have that there are sequences $x_n^j \to \infty$, such that $f(x_n^j) \to y_j$. In particular we can assume that $||x_n^j|| > n$, $||f(x_n^j) - y_j|| < 1/n$. Now if we take $x_n = x_n^n$, then $f(x_n) \to y$. This means that $y \in S_f$.

Now let $y \in cl(K_0(f))$. This means that there is a sequence $y_j \to y$, $y_j \in K_0(f)$, in particular $y_j = f(x_j)$. If a sequence x_j is bounded, then there is subsequence $x_{j_k} \to x_0$ and $y = lim\ f(x_{j_k})$. This implies that $y \in K_0(f)$.

If a sequence x_j is unbounded, then there is subsequence $x_{j_k} \to \infty$ and $y = lim\ f(x_{j_k})$. This implies that $y \in S_f$. The rest of the proof is obvious. □

Directly from Corollary 2.11 we get:

Corollary 2.15. *Let* $X \subset \mathbb{R}^m$ *be an affine* $n-$*dimensional variety of degree* D. *Let* $f = (f_1, ..., f_n) : X \to \mathbb{R}^n$ *be a generically finite dominant mapping. Assume that* $\deg f_i \leq d$. *Then the set* $B(f)$ *is closed and semialgebraic (or the empty set) and there is a real hypersurface* B, *which contains* $B(f)$ *and*

$$\deg\ B < Dd^{n-1}[1 + n(d-1) + (m-n)(D-1)].$$

3. Symmetry defect.

DEFINITION 3.1. Let $k = \mathbb{C}$ or $k = \mathbb{R}$. Let $Z, W \subset k^m$ (it is possible that $Z = W$) be smooth algebraic manifolds of dimensions r and s respectively. We say that Z, W are in a general position if $r + s = m$ and there exist points $z \in Z$ and $w \in W$ such that $T_z Z + T_w W = k^m$.

PROPOSITION 3.2. *Let $Z, W \subset \mathbb{C}^m$ be smooth algebraic manifolds of dimensions r and s respectively, where $r + s = m$. If Z and W have no common points at infinity then they are in a general position.*

Proof. Let us consider the group G of all translations in $\mathbb{C}^m$. By Kleiman-Bertini Theorem (see [19]) for general $g \in G$, the varieties Z and gW meet transversally. Since they have no common points at infinity there is a point $x \in Z \cap gW$. It is easy to see that $T_x Z + T_{g^{-1}x} W = \mathbb{C}^m$. $\square$

REMARK 3.3. If $Z, W \subset \mathbb{R}^{r+s}$ are smooth algebraic manifolds of dimensions r and s respectively and Z, W have not common complex points at infinity then they are in a general position.

PROPOSITION 3.4. *Let $k = \mathbb{C}$ or $k = \mathbb{R}$.*

a) A curve X in k^2 is in a general position if and only if X is not a line.

b) A smooth algebraic surface X in k^4 is in a general position if and only if X is not a cylinder and it is not contained in any hyperplane.

Proof. For a subvariety $X \subset k^m$ we identify the tangent space $T_x X$ with the linear subspace of $T_x k^m = k^m$. Since the tangent bundle of k^m is trivial we can identify all $T_x k^m$ with one copy of k^m. We start with the following Lemma:

LEMMA 3.5. *Let $X \subset k^m$ be a smooth subvariety. Assume that all tangent spaces $T_x X$ are contained in some linear hyperplane $H \subset k^m$. Then X is also contained in a hyperplane.*

Proof. Let h be a linear polynomial which is the equation of H. Consider the function $g = h|_X : X \to k$. If g is not a constant, then by Sard's Theorem a generic fiber of h is transversal to X- a contradiction. $\square$

Now we can prove our Proposition. The case a) follows immediately from Lemma 3.5. We prove b). Assume that X is not a cylinder and it is not contained in a hyperplane. Choose two different tangent spaces $T_x X$ and $T_y X$ (it is possible by Lemma 3.5). If $T_x X$ and $T_y X$ are transversal we are done. Hence assume that they have a common line l. Note that a generic space $T_z X$ does not contain the line l. Indeed, otherwise we can take a vector $\mathbf{v} \in l$ and we would have a constant vector field $\mathbf{v}$ along X. We can integrate this vector field and we obtain a family of lines parallel to l which covers X. This means that $X = l \times X'$ is a cylinder (X' is the image of a generic projection $\pi : X \to k^{m-1}$ along l).

This implies that a generic tangent space $T_z X$ meets $T_x X$ and $T_y X$ either transversally or along two different lines $l_1 \subset T_x X$ and $l_2 \subset T_y X$. In the second case the space $T_z X$ is contained in the hyperplane determined by spaces $T_x X$ and $T_y X$ - a contradiction. $\square$

DEFINITION 3.6. Let $k = \mathbb{C}$ or $k = \mathbb{R}$. Let $Z, W \subset k^m$ be varieties of dimensions r and s respectively, such that $r + s = m$. For a point $P \in k^m$ we denote by $\mu(P)$ the number of pairs $\{z, w\}$ where $z \in Z$ and $w \in W$ such that P is the center of the interval

$\overline{zw}$. The number $\mu_{Z,W} := sup_{\{P:\mu(P)<\infty\}}\{\mu(P)\}$ we call the generic symmetry of Z and W (if $Z = W$ we write simply μ_Z). By the symmetry defect $B(Z,W)$ of Z and W we mean the bifurcation set of the mapping $\Phi : Z \times W \ni (z,w) \mapsto (z+w)/2 \in k^m$. Moreover, by an asymptotic part of the symmetry defect we mean S_Φ and by a proper part of this set we mean $cl(K_0(\Phi))$.

Remark 3.7. For $Z = W$ the number $\mu(P)$ is the number of ordered secants of Z for which P is the center.

We apply these results to our problem. We have:

Theorem 3.8. *Let $Z, W \subset \mathbb{C}^m$ be smooth algebraic manifolds of dimensions r and s respectively, $r + s = m$. Assume that Z and W are in a general position. Let deg $Z = p$ and deg $W = q$. Then there exists a number $0 < \mu = \mu_{Z,W} \leq pq$ and an algebraic hypersurface $B(Z,W) \subset \mathbb{C}^m$ (the symmetry defect of Z and W) of degree $d < pq(1 + sp + rq - m)$, such that a point $a \in \mathbb{C}^m \setminus B(Z,W)$ if and only if a is the center of exactly μ intervals of the type $\overline{xy}$, where $x \in Z$ and $y \in W$.*

Proof. Let $X = Z \times W \subset \mathbb{C}^m \times \mathbb{C}^m$ and $f : Z \times W \ni (x,y) \to (x+y)/2 \in \mathbb{C}^m$. Note that f is not a local biholomorphism at the point (x,y) exactly if $T_xZ + T_yW \neq \mathbb{C}^m$. Indeed, the mapping $df : T_xZ \oplus T_yW \ni (w,v) \to (w+v)/2 \in \mathbb{C}^m$ is not an isomorphism, if and only if there exists non-zero vectors $z \in T_xZ$ and $w \in T_yW$ such that $z + w = 0$. This implies that $\{z\} \in T_xZ \cap T_yW$, i.e., T_xZ is not transversal to T_yW i.e., $T_xZ + T_yW \neq \mathbb{C}^m$. Since Z and W are in a general position we obtain that f is generically finite.

Now it is enough to apply Corollary 2.10 to X and f. Of course deg $X = pq$. We can find a system of parameters $\{h_1, ..., h_s\}$ of Z and $\{g_1, ..., g_r\}$ of W. By Proposition 2.7 we can assume that deg $h_i \leq p$ and deg $g_j \leq q$. Note that the polynomials $h_1(x), ..., h_s(x), g_1(y), ..., g_r(y)$ form a system of parameters for $Z \times W$. Hence deg $B(Z,W) < pq(1 + sp + rq - m)$.

Moreover, by Bezout Theorem the mapping f has geometric degree bounded by deg $X = pq$. This implies that $0 < \mu_{Z,W} \leq pq$. $\square$

Remark 3.9. Let us note that for $b \in B(Z,W)$ either $\mu(b) < \mu_{Z,W}$ or $\mu(b)$ is infinite. This justifies the name of the hypersurface $B(Z,W)$.

Corollary 3.10. *Let $Z, W \subset \mathbb{C}^m$ be smooth algebraic manifolds of dimensions r and s respectively, where $r + s = m$. Assume that Z and W have no common points at infinity. Let deg $Z = p$ and deg $W = q$. Then $\deg B(Z,W) \leq pq(sp + rq - m)$ and $\mu_{Z,W} = pq$. Moreover, for every P we have $\mu(P) \leq pq$.*

Proof. Let $f : X := Z \times W \ni (x,y) \to (x+y)/2 \in \mathbb{C}^m$. For a point $O \in \mathbb{C}^m$ let T_O be a symmetry with a center in O, i.e.,

$$T_O(x) = O - \overrightarrow{Ox}.$$

Let $O \in \mathbb{C}^m$ be a generic point and let $Z \cap T_O(W) = \{x_1, ..., x_s\}$. Note that $f^{-1}(O) = (x_1, T_O(x_1)), ..., (x_s, T_O(x_s))$. By Sard's Theorem we have that O is a regular value of f, consequently the varieties Z and $T_O(W)$ meet only transversally. Moreover, since T_O is the identity at infinity, these varieties have no common points at infinity. Hence by Bezout theorem we get $\mu_{Z,W} = s = pq$. Moreover $\deg S_f \leq \deg X - \mu(f) = pq - \mu_{Z,W} = 0$. This implies that the mapping f is finite. In particular for every

$P \in \mathbb{C}^m$ we have $\mu(P) \le \mu(f) = pq$. Additionally, the set $B(Z,W)$ coincides with $K_0(F)$ and consequently deg $B(Z,W) \le pq(sp + rq - m)$. □

Corollary 3.11. *Let $Z, W \subset \mathbb{R}^m$ be smooth algebraic manifolds of dimensions r and s respectively, $r+s=m$. Assume that Z and W are in a general position. Let $B(Z,W)$ be the symmetry defect of Z and W. Then the set $\mathbb{R}^m \setminus B$ has at most $[(pq)(1+sp+rq-m)-1]^m$ connected components. Moreover, for every $a \in \mathbb{R}^m$ if $\mu(a)$ is finite, then $\mu(a) \le pq$.*

Proof. The number of connected components of the complement of a hypersurface $B \subset \mathbb{R}^m$ is bounded by D^m , where $D =$ deg B ([1], 3.9.6). □

Corollary 3.12. *Let $Z \subset \mathbb{C}^{2n}$ be a smooth algebraic manifold of dimension n which is transversal to the hyperplane at infinity. Assume that Z is in a general position. Let deg $Z = p$. Then $0 < \mu_Z \le p^2-p-n+1$ and $\deg B(Z) < p^2(1+2n(p-1))$. Moreover, $Z \subset B(Z)$.*

Proof. Take $X = Z \times Z \subset \mathbb{C}^{2n} \times \mathbb{C}^{2n}$ and $f : Z \times Z \ni (x,y) \to (x+y)/2 \in \mathbb{C}^{2n}$. For a point $O \in \mathbb{C}^{2n}$ let T_O be a symmetry with a center in O. Let $O \in \mathbb{C}^{2n}$ be a generic point and let $Z \cap T_O(Z) = \{x_1, ..., x_s\}$. Note that $f^{-1}(O) = (x_1, T_O(x_1)), ..., (x_s, T_O(x_s))$. By Sard's Theorem we have that O is a regular value of f, consequently the varieties Z and $T_O(Z)$ meets only transversally.

Let $R = \overline{Z} \cap \pi_\infty$ (where π_∞ is the hyperplane at infinity). By assumption R has degree p. Since T_O is the identity on π_∞ we have $\overline{Z} \cap T_O(\overline{Z}) \cap \pi_\infty = R$. Hence by the generalized Bezout theorem (see [9], Theorem 12.3, p.223) and result of Lazarsfeld (Example 12.3.5 in [9]) we get $\mu_Z \le p^2 - p - n + 1$. Additionally, since $\deg X = p^2$ we have (as above) deg $B < p^2(1 + 2n(p-1))$. Moreover, $Z \subset B(Z)$ because for $z \in Z$, the point (z,z) is a critical point of f ($d_{(z,z)} f(T_z Z \oplus T_z Z) = T_z Z \ne \mathbb{C}^{2n}$). □

Corollary 3.13. *Let $Z \subset \mathbb{R}^{2n}$ be smooth algebraic manifold of dimension n. Assume that Z is in a general position. Let deg $Z = p$. Then $0 < \mu_Z \le p^2 - p - n + 1$. Let $B(Z)$ be the symmetry defect of Z. Then the set $\mathbb{R}^{2n} \setminus B(Z)$ has at most $[p^2(1+2n(p-1))-1]^{2n}$ connected components.*

Proof. The number of connected components of the complement of a hypersurface $B \subset \mathbb{R}^m$ is bounded by D^m , where $D =$ deg B ([1], 3.9.6). □

To state the next result we need the following definition:

Definition 3.14. We say that an affine variety $X \subset \mathbb{C}^m$ is projectively smooth if its projective closure $\overline{X} \subset \mathbb{P}^m$ is smooth. If X is projectively smooth, then by an asymptotic set of X we mean the set $\mathcal{A}(X) := (\bigcup_{x \in \overline{X} \setminus X} T_x \overline{X}) \cap \mathbb{C}^m$.

Theorem 3.15. *Let $Z \subset \mathbb{C}^{2n}$ be a projectively smooth variety of dimension n, which is in a general position. Then the asymptotic part of $B(Z)$ is the asymptotic set of Z.*

Proof. Let $\Phi : Z \times Z \ni (x,y) \mapsto (x+y)/2 \in \mathbb{C}^{2n}$. If there exists a sequence $(x_n, y_n) \to \infty$ such that $\Phi(x_n, y_n) \to w \in Z$, then necessarily x_n and y_n have to tend to the same point $a \in \overline{Z} \setminus Z$ (because otherwise $w \in \pi_\infty$). But then the line $\overline{x_n y_n}$ tends to the line which is tangent at the point a to Z. The affine part of this line is contained in $T_a \overline{Z} \cap \mathbb{C}^{2n} \subset \mathcal{A}(Z)$.

Conversely, for any point $w \in L$, where $L = T_a \overline{Z} \cap \mathbb{C}^{2n} \subset \mathcal{A}(Z)$, we can choose a sequence of points x_n, y_n such that $x_n, y_n \to a$ and $\Phi(x_n, y_n) \to w$. Indeed, let l

be a projective line given by a and w. Let $\pi : Z \to L$ be a finite projection with a center at infinity. There exists an open neighborhood $U_a \subset \mathbb{P}^{2n}$ of the point a such that $U_a \cap \pi^{-1}\{a\} = \{a\}$. Let a vector $\mathbf{v}$ be parallel to the line l. Consider a sequence of points: $x'_n = n\mathbf{v} + w,\ y'_n = w - n\mathbf{v}$ and let $x_n \in U_a \cap \pi^{-1}\{x'_n\},\ y_n \in U_a \cap \pi^{-1}\{y'_n\}$ (such points there exist for $n >> 0$). It is easy to check that $\Phi(x_n, y_n) \to w \in Z$. □

Corollary 3.16. *Let $Z \subset \mathbb{C}^{2n}$ be a projectively smooth variety of dimension n, which is in a general position. Then the asymptotic set of Z is either a closed hypersurface in $\mathbb{C}^{2n}$ or the empty set. Moreover, if a point $a \in \mathbb{C}^{2n}$ does not belong to $\mathcal{A}(Z)$, then there exists a secant $\overline{xy}$ of Z (i.e. $x, y \in Z$), such that a is the midpoint of $\overline{xy}$.*

Definition 3.17. Let $Z \subset k^m$ be an algebraic variety. We say that the point $O \in k^m$ is a center of symmetry of Z if $T_O(Z) = Z$.

Proposition 3.18. *If a variety Z is not a cylinder then it has at most one center of symmetry.*

Proof. Indeed, if A and B are two different centers of symmetry of Z, then $T_A \circ T_B(Z) = Z$. Since the mapping $T = T_B \circ T_A(Z)$ is the translation $x \to x + 2\overrightarrow{AB}$, we have that Z is a cylinder. □

The following is obvious but interesting:

Proposition 3.19. *If $Z \subset k^{2n}$ is a $n-$dimensional variety in a general position and Z has a center of symmetry O, then the variety $B(Z)$ has also (the same) center of symmetry. Moreover, $O \in B(Z)$. More generally, if $T : k^{2n} \to k^{2n}$ is a linear isomorphism, then $T(B(Z)) = B(T(Z))$.*

4. Plane curves. In this section we will study more precisely the case $m = 2$.

Theorem 4.1. *Let $Z, W \subset \mathbb{C}^2$ be smooth algebraic curves. Assume that Z and W are not parallel lines and that they have r common points at infinity, at which they meet transversally. Let deg $Z = p$ and deg $W = q$. Then $\mu_{Z,W} = pq - r$ and* $\deg B(Z, W) \le pq(p + q - 2) + r$.

Proof. Let $X = Z \times W \subset \mathbb{C}^2 \times \mathbb{C}^2$ and $f : Z \times W \ni (x, y) \to (x + y)/2 \in \mathbb{C}^2$. For a point $O \in \mathbb{C}^2$ let T_O be a symmetry with a center in O. Let $O \in \mathbb{C}^2$ be a generic point and let $Z \cap T_O(W) = \{x_1, ..., x_s\}$. As before the curves Z and $T_O(W)$ meet only transversally. Moreover, Z and $T_O(W)$ have r common points at infinity, at which they meet transversally (note that the point O is sufficiently general). Hence by Bezout theorem we get $\mu_{Z,W} = pq - r$. Moreover $\deg S_f \le \deg X - \mu(f) = pq - \mu_{Z,W} = r$. Thus deg $B(Z, W) \le pq(p + q - 2) + r$. □

Theorem 4.2. *Let $Z \subset \mathbb{C}^2$ be a smooth algebraic curve, which is not a line. Assume that Z is transversal to the line at infinity. Let deg $Z = p$. Then $\mu_Z = p^2 - p$ and* $\deg B(Z) \le 2p^2(p - 1) + p$.

Proof. Let $X = Z \times Z \subset \mathbb{C}^2 \times \mathbb{C}^2$ and $f : Z \times Z \ni (x, y) \to (x + y)/2 \in \mathbb{C}^2$. For a point $O \in \mathbb{C}^2$ let T_O be a symmetry with a center in O. Let $O \in \mathbb{C}^2$ be a generic point and let $Z \cap T_O(Z) = \{x_1, ..., x_s\}$. As before curves Z and $T_O(Z)$ meet only transversally. Moreover, Z and $T_O(Z)$ have p common points at infinity, at which they meet transversally (note that the point O is sufficiently general). Hence by

Bezout theorem we get $\mu_Z = p^2 - p$. Moreover $\deg S_f \leq \deg X - \mu(f) = pq - \mu_Z = p$. Thus $\deg B(Z) \leq 2p^2(p-1) + p$. □

PROPOSITION 4.3. *Let $Z \subset \mathbb{C}^2$ be a curve which is not a line. Then there exists at most one point O such that $\mu(O) = \infty$. If such a point O exists, it is a center of symmetry of Z.*

Proof. Indeed, if $\mu(O) = \infty$, then the curves Z and $T_O(Z)$ have infinitely many common points, and by the Bezout Theorem $Z = T_O(Z)$. Now we can apply Proposition 3.18. □

REMARK 4.4. It is not difficult to see that a generic curve $Z \subset \mathbb{C}^2$ of degree $d > 2$ does not have center of symmetry. However for every $d > 0$ the smooth curve

$$\Gamma_d = \{(x, y) \in \mathbb{C}^2 : x^{2d} + y^{2d} = 1\}$$

has a center of symmetry.

To formulate the next result let us recall:

DEFINITION 4.5. An asymptote $L \subset \mathbb{C}^2$ of a curve $Z \subset \mathbb{C}^2$ is a line which is tangent to the curve Z at infinity. More precisely, there exists a point $a \in \overline{Z} \setminus Z$ such that $L = T_a\overline{Z} \cap \mathbb{C}^2$.

By Theorem 3.15 we have:

THEOREM 4.6. *Let $Z \subset \mathbb{C}^2$ be a projectively smooth curve (i.e. the projective closure of Z in $\mathbb{P}^2$ is smooth), which is not a line. Then the asymptotic part of $B(Z)$ is the union of all asymptotes of Z.*

COROLLARY 4.7.
1. If a projectively smooth curve $Z \subset \mathbb{C}^2$ has no asymptotes, then $B(Z)$ has empty asymptotic part. In particular for every point $a \in \mathbb{C}^2$ there exists a secant of Z such that a is a mid point of this secant
2. More generally for every curve Z in general position if a point a does not belong to any asymptote of Z, then there exists a secant of Z such that a is a midpoint of this secant.
3. If a curve Z has a center of symmetry, then it has asymptotes.

Proof. The first two parts of the Corollary follow directly from Theorem 4.6. For the last part note that if Z has a center of symmetry, then the mapping $\Phi : Z \times Z \ni (x, y) \to (x+y)/2 \in \mathbb{C}^2$ has an infinite fiber, so it is not proper. Hence the asymptotic part of $B(Z)$ can not be empty. □

EXAMPLE 4.8. *Non-empty asymptotic part of the symmetry defect set.*
We check Theorem 4.6 directly. Let us consider $Z = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$. We describe the symmetry defect set $B(Z)$ of Z. By definition it is the bifurcation set of the mapping

$$\Phi : Z \times Z \to \mathbb{R}^2.$$

Z has a parametrization $Z = \{(s, \frac{1}{s}), s \in \mathbb{R}^*\}$. Thus we have a mapping

$$\Phi : \mathbb{R}^* \times \mathbb{R}^* \ni (s, t) \to (\frac{s+t}{2}, \frac{1}{2s} + \frac{1}{2t}).$$

We compute the bifurcation set of the mapping Φ. The set of critical points of Φ is given by equation $\frac{1}{s^2} = \frac{1}{t^2}$. This means that the set of critical values consists of origin and hyperbola Z itself. Now we compute the second component of the bifurcation set which is the non-properness set S_Φ of Φ. By definition we have

$$S_\Phi = \{(a, b) \in \mathbb{R}^2 : (a, b) = \lim \Phi(s_n, t_n), where\ (s_n, t_n) \to Bd(\mathbb{R}^* \times \mathbb{R}^*)\},$$

where Bd denotes the boundary. The condition $(s_n, t_n) \to Bd(\mathbb{R}^* \times \mathbb{R}^*)$ means that either $s_n \to 0$ or ∞ or $t_n \to 0$ or ∞. First we assume that $s_n \to \infty$. This implies that also $t_n \to \infty$ and $s_n + t_n \to a \in \mathbb{R}$. Consequently $\Phi(s_n, t_n) \to (a, 0)$ and conversely every such a point is in S_Φ. If $s_n \to 0$ we get points $(0, a)$. The same result (by symmetry) we get if we consider t. Finally $S_\Phi = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$. Now we have

$$B(Z) = \{(x, y) \in \mathbb{R}^2 : xy = 0\ or\ xy = 1\}.$$

The shadowed area in Figure 1 is an image of Φ. The integers indicate the number of points in a fiber of Φ. If we consider the complex case then the image of Φ is $\mathbb{C}^2 - \{xy = 0\} \cup \{(0, 0)\}$. By Theorem 4.2 the number of points in a generic fibre of Φ (i.e. the number of ordered secants for which a given point is a mid-point) is equal to two.

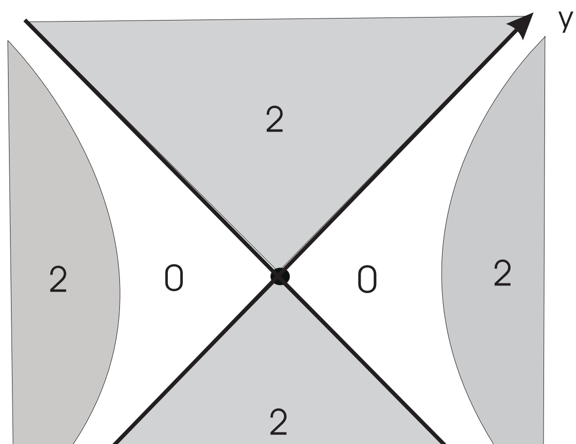

FIG. 1. *Symmetry defect set for hyperbola*

EXAMPLE 4.9. *The only asymptotic part of the symmetry defect set.* To show that the proper part (excluding the curve itself) of the symmetry defect set can have only asymptotic bifurcation points we consider $Z = \{(x, y) \in \mathbb{R}^2 : x^2y = 1\}$. Then $B(Z) - Z = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ is the set of asymptotic points.

Let us compute the number μ_Z in the complex case (note that we can not use Theorem 4.2 because the curve Z is not transversal to the line in infinity). Take a symmetry T_O with a sufficiently general center $O = (a, b)$. In coordinates this symmetry is given by $(x, y) \mapsto 2(a, b) - (x, y)$. Hence the curve $T_O(Z)$ in coordinates is given by an equation

$$-yx^2 + 2bx^2 + 4axy - 8abx - 4a^2y + 8a^2b - 1 = 0$$

To find μ_Z it is enough to find the number of common points of the curve Z and $T_O Z$. This can be done by substituting the parametrization $(t, 1/t^2)$ to the equation of the curve $T_O(Z)$. After this substitution we obtain a polynomial of degree 4, thus in general we have four common points. Then μ_Z is equal 4. Let us notice that this number is not achieved in the real case. Moreover in the complex case the set $B(f)$ contains also the line $\{x = 0\}$.

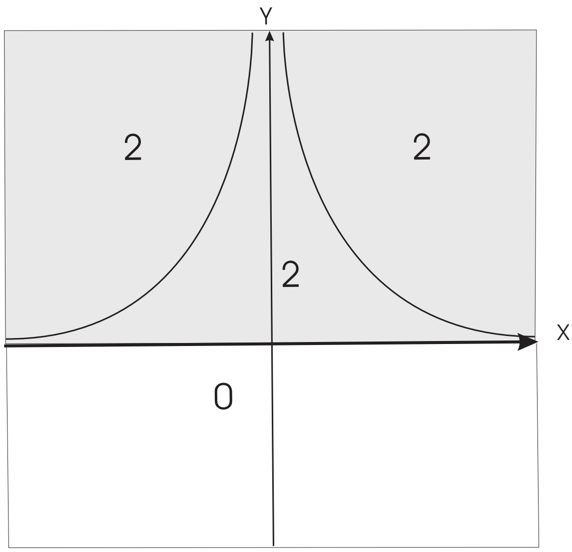

FIG. 2. *Symmetry defect set for $x^2y = 1$*

EXAMPLE 4.10. *Empty asymptotic part for non-compact variety.* Let us consider $Z = \{(x, y) \in \mathbb{R}^2 : y - x^2 = 0\}$. It is easy to check that the mapping Φ is proper. Hence the asymptotic part of symmetry defect set is empty. In fact $B(Z) = Z$. Of course Z has no asymptotes.

EXAMPLE 4.11. *Not pure codimension one symmetry defect set in the real case* For a circle the symmetry defect set reduces to a circle itself and its center. Hence in the real case the symmetry defect set may be not of pure codimension one. Let us note that in the complex case a circle is simply a hyperbola and we have the same situation as in Example 4.8

5. Topological stability of the symmetry defect set. We show in this section that for generic curves of a given degree the topological type of the symmetry defect set is constant. We begin with proving the lemma.

LEMMA 5.1. *Let X, Y be a complex algebraic variety and $f : X \to Y$ be a polynomial dominant mapping. Then two generic fibers of f are homeomorphic.*

Proof. Let X_1 be an algebraic completion of X. Take $X_2 = \overline{graph(f)} \subset X_1 \times \overline{Y}$, where $\overline{Y}$ is a smooth algebraic completion of Y. We can assume that $X \subset X_2$. Let $Z = X_2 \setminus X$. We have an induced mapping $\overline{f} : X_2 \to \overline{Y}$, such that $\overline{f}_X = f$.

There is a Whitney stratification $\mathcal{S}$ of the pair (X_2, Z). For every smooth strata $S_i \in \mathcal{S}$ let B_i be the set of critical values of the mapping $f|_{S_i}$.Take $B = \overline{\bigcup B_i}$. Take $X_3 = X_2 \setminus f^{-1}(B)$ and $Z_1 = Z \setminus f^{-1}(B)$. The restriction of the stratification $\mathcal{S}$ to X_3 gives a Whitney stratification of the pair (X_3, Z_1). We have a proper mapping $f_1 : X_3 \to \overline{Y} \setminus B$ which is submersion on each strata. By the Thom first isotopy theorem there is a trivialization of f_1, which preserves the strata. It is an easy observation that this trivialization gives a trivialization of the mapping $f : X \setminus f^{-1}(B) \to Y \setminus B$. □

THEOREM 5.2. *Symmetry defect sets B_1, B_2 for generic curves $C_1, C_2 \subset \mathbb{C}^2$ of the same degree $d> 1$ are homeomorphic.*

Proof. Any curve $C \subset \mathbb{C}^2$ of degree d can be identified with one point (given by coefficients of equation of C) in $\mathbb{P}^{N(d)}$. Observe that under this identification the set of smooth irreducible curves of degree d correspond to some open subset M of $\mathbb{P}^{N(d)}$. We can assume that M is an affine variety. Let us define

$$R = \{(c, x, y) \in M \times \mathbb{C}^2 \times \mathbb{C}^2 : \phi(c)(x) = 0, \quad \phi(c)(y) = 0\},$$

where $\phi(c)(x)$ denotes the equation of the curve C with coefficients c . Let us note that R is a smooth irreducible subvariety of $M \times \mathbb{C}^2 \times \mathbb{C}^2$ of codimension two. Indeed, R is given by two equations

$$\sum_{|\alpha| \le d} a(c)_\alpha x^\alpha = 0,$$
$$\sum_{|\beta| \le d} a(c)_\beta y^\beta = 0.$$

By assumption these equations define complete intersection. Moreover we have a projection $R \to M$ with irreducible fiber which is a product of two irreducible curves. This means that R is irreducible. Note that R is an affine variety. Consider the following morphism

$$\Psi : R \ni (c, x, y) \mapsto (c, \frac{x+y}{2}) \in M \times \mathbb{C}^2.$$

Since $d > 1$ the mapping Ψ is dominant. Indeed for every $c \in M$ the set $\phi(R) \cap c \times \mathbb{C}^2$ is dense in $c \times \mathbb{C}^2$. By Theorem 2.4 the mapping Ψ has constant number of points in the fiber outside the bifurcation set $B(\Psi) \subset M \times \mathbb{C}^2$. This implies that $B(C) = c \times \mathbb{C}^2 \cap B(\Psi)$. In particular the symmetry defect set of the curve C coincide with the fiber over c of the projection $\pi : B(\Psi) \ni a \mapsto c \in M$. Now we conclude the proof by Lemma 5.1. □

COROLLARY 5.3. *Up to homeomorphism there is only finite number of symmetry defect sets for curves of bounded degree.*

REMARK 5.4. In a similar way we can prove that for a smooth, connected, algebraic family $\mathcal{A}$ of $n-$dimensional varieties in general position in $\mathbb{C}^{2n}$, the symmetry defect sets for two generic members of $\mathcal{A}$ are homeomorphic.

6. The Algorithm. In this section we use Gröbner bases to compute the set $B(Z)$ effectively.

Let us recall the definition of Gröbner basis. Assume that in the set of monomials in $\mathbb{C}[x_1,...,x_n]$ we have the ordering induced by the lexicographic ordering in $\mathbb{N}^n$, i.e., $a_\alpha x^\alpha > a_\beta x^\beta$, if $\alpha > \beta$ (in this paper we consider only this ordering). By $inP = a_d x^d$ we will denote the initial form of a polynomial $P = \sum a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{C}[x_1, \ldots, x_n]$, where $d = max\{\alpha = (\alpha_1, \ldots, \alpha_n);\ a_\alpha \neq 0\}$. We have the following basic definition (see [20]):

Definition 6.1. A finite subset $\mathcal{B} \subset I \subset \mathbb{C}[x_1,...,x_n]$ of an ideal I is called a Gröbner basis of this ideal, if the set $\{inP;\ P \in \mathcal{B}\}$ generates the ideal generated by all initial forms of the ideal I.

The Gröbner basis of the ideal I is a basis of this ideal, moreover it can be easily computed by arithmetical operations only. We have the following basic fact (see [20]):

Theorem 6.2. *Consider the ring $\mathbb{C}[x_1,...,x_n;y_1,...,y_m]$. Let $V \subset \mathbb{C}^n \times \mathbb{C}^m$ be an algebraic set and let $p : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ denote the projection. Assume that $\mathcal{B}$ is a Gröbner basis of the ideal $I(V)$. Then $\mathcal{B} \cap \mathbb{C}[y_1,...,y_m]$ is a Gröbner basis of the ideal $I(p(V)) = I(cl(p(V)))$.*

Proof. Observe that $I(p(V)) = I(V) \cap \mathbb{C}[y_1,...,y_m]$ and then use [20], Proposition 4.3. $\square$

Now we show how to compute effectively the set S_f for a polynomial mapping $f : X \to \mathbb{C}^n$. We start with the following:

Lemma 6.3. *(see [18]): Let $f : X \to k^n$ be a dominant generically finite polynomial map and let $k(f_1, \ldots, f_n) \subset k(X)$ be the induced field extension. Let $k[X] = k[x_1,...,x_r]$ and*

$$x_i^{n_i} + \sum_{k=1}^{n_i} a_k(f)^i x_i^{n_i-k} = 0,$$

where the $a_k^i \in k(f_1,...,f_n)$ are rational functions, be the minimal equation of x_i over $k(f_1, \ldots, f_n)$. Let S denote the union of poles of all functions a_k^i. Then f is finite at a point y if and only if $y \in k^n \setminus S$.

Proposition 6.4. *Let $X \subset \mathbb{C}^m$ be a smooth affine $n-$dimensional variety. Let $f = (f_1,...,f_n) : X \to \mathbb{C}^n$ be a generically finite dominant mapping. Then the set S_f can be computed effectively.*

Proof. Let $F_i = a_0^i(f)x_i^{n_i} + \sum_{k=1}^{n_i} a_k^i(f)x_i^{n_i-k} = 0$ be a minimal polynomial of an element x_i over the field $\mathbb{C}(f_1,...,f_n)$, such that $GCD\{a_k^i\} = 1$. By Lemma 6.3, we have $S_f = \bigcup_{i=1}^m \{a_0^i = 0\}$. Hence if $H = LCM\{a_0^i\}$, then $S_f = \{H = 0\}$. To find a polynomial F_i, it is enough to find the image of the mapping $\Phi_i = (f_1,...,f_n,x_i) : X \to \mathbb{C}^{n+1}$. This can be done in a standard way: let us take the ring $\mathbb{C}[x_1,...,x_m,y_1,...,y_n,y_{n+1}]$ with the lexicographic order. Let $I(X) = \{g_1,...,g_r\} \subset \mathbb{C}[x_1,...,x_m]$. Consider the ideal

$$J = (g_1,...,g_r,y_1 - f_1,...,y_n - f_n,y_{n+1} - x_i) \subset \mathbb{C}[x_1,...,x_m,y_1,...,y_n,y_{n+1}].$$

If we compute the Gröbner basis $B(J)$ of J then $F_i = B(J) \cap \mathbb{C}[y_1,...,y_{n+1}]$. If $F_i = a_0^i(y_1,...,y_n)y_{n+1}^{n_i} + ... + a_{n_i}^i(y_1,...,y_n)$, then $H = LCM\{a_0^i\}$. $\square$

THEOREM 6.5. *Let $Z \subset \mathbb{C}^{2n}$ be a smooth affine $n-$dimensional variety, which is in a general position. Then the set $B(f)$ can be computed effectively.*

Proof. First we assume that Z is a complete intersection. Let $I(Z) = \{h_1, ..., h_n\}$. Let $\Phi : Z \times Z \ni (x, y) \mapsto (x + y)/2 \in \mathbb{C}^{2n}$. By definition we have $B(Z) = B(\Phi)$. We have

$$B(\Phi) = \overline{K_0(f)} \cup S_\Phi.$$

By Proposition 6.4 and Theorem 2.4 it is enough to compute the $n-1$-dimensional part of $\overline{K_0(f)}$. Consider the ring $R = \mathbb{C}[x_1, ..., x_{2n}, y_1, ..., y_{2n}, w_1, ..., w_{2n}]$ with the lexicographic order.

Let

$$W := det \begin{bmatrix} \frac{\partial h_1}{\partial x_1}(x) & \frac{\partial h_1}{\partial x_2}(x) & \dots & \frac{\partial h_1}{\partial x_{2n}}(x) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_n}{\partial x_1}(x) & \frac{\partial h_n}{\partial x_2}(x) & \dots & \frac{\partial h_n}{\partial x_{2n}}(x) \\ \frac{\partial h_1}{\partial y_1}(y) & \frac{\partial h_1}{\partial y_2}(y) & \dots & \frac{\partial h_1}{\partial y_{2n}}(y) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_n}{\partial y_1}(y) & \frac{\partial h_n}{\partial y_2}(y) & \dots & \frac{\partial h_n}{\partial y_{2n}}(y) \end{bmatrix}.$$

Define the ideal

$$J = (h_1(x), ..., h_n(x), h_1(y), ..., h_n(y), W, 2w_1 - (x_1 + y_1), ..., 2w_{2n} - (x_{2n} + y_{2n})) \subset R.$$

Let us compute the Gröbner basis $B(J)$ and put $B(J) \cap \mathbb{C}[w_1, ..., w_{2n}] = \{b_1, ..., b_s\}$. Now put $Q = GCD(b_1, ..., b_r)$. Then $B(\Phi) = \{Q = 0\} \cup S_\Phi$.

A general case we do similarly. Let $I(Z) = \{h_1, ..., h_m\}$. Of course $m \geq n$. Consider the set $\mathcal{C}$ of all combination of n elements chosen from the set $\{h_1, ..., h_m\}$. For $\alpha = (h_{i_1}, ..., h_{i_n}) \in \mathcal{C}$ and $\beta = (h_{j_1}, ..., h_{j_n}) \in \mathcal{C}$ consider a polynomial

$$W_{\alpha,\beta} := det \begin{bmatrix} \frac{\partial h_{i_1}}{\partial x_1}(x) & \frac{\partial h_{i_1}}{\partial x_2}(x) & \dots & \frac{\partial h_{i_1}}{\partial x_{2n}}(x) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_{i_n}}{\partial x_1}(x) & \frac{\partial h_{i_n}}{\partial x_2}(x) & \dots & \frac{\partial h_{i_n}}{\partial x_{2n}}(x) \\ \frac{\partial h_{j_1}}{\partial y_1}(y) & \frac{\partial h_{j_1}}{\partial y_2}(y) & \dots & \frac{\partial h_{j_1}}{\partial y_{2n}}(y) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_{j_n}}{\partial y_1}(y) & \frac{\partial h_{j_n}}{\partial y_2}(y) & \dots & \frac{\partial h_{j_n}}{\partial y_{2n}}(y) \end{bmatrix}.$$

Define the ideal

$$J = (h_1(x), ..., h_m(x), h_1(y), ..., h_m(y), \{W_{\alpha,\beta}\}_{\alpha,\beta \in \mathcal{C}}, 2w_1 - (x_1 + y_1), ..., 2w_{2n} - (x_{2n} + y_{2n})) \subset R.$$

Let us compute the Gröbner basis $B(J)$ and put $B(J) \cap \mathbb{C}[w_1, ..., w_{2n}] = \{b_1, ..., b_s\}$. Now put $Q = GCD(b_1, ..., b_r)$. Then $B(\Phi) = \{Q = 0\} \cup S_\Phi$. □

COROLLARY 6.6. *Let $Z \subset \mathbb{C}^{2n}$ be a smooth, $n-$dimensional affine variety, which is in a general position. Let σ be a subfield of $\mathbb{C}$ generated by all coefficients of* *polynomials g_j, which generates the ideal $I(Z) \subset \mathbb{C}[x_1, ..., x_{2n}]$. Then there exists a finite family $\{h_1, ..., h_s\}$ of polynomials from $\sigma[y_1, ..., y_n]$, such that*

$$B(f) = \{y \in \mathbb{C}^{2n} : g_i(y) = 0,\ i = 1, ..., s\}.$$

REMARK 6.7. Let us note that our method to compute the symmetry defect set works only in complex case. If we consider the real algebraic variety $Z \subset \mathbb{R}^{2n}$ and Z^c denote the complexification of Z then $B(Z)$ is contained in $B(Z^c) \cap \mathbb{R}^{2n}$. Hence our algorithm help us to understand the real symmetry defect set. In particular the real number μ_Z is constant in every connected component $\mathbb{R}^{2n} - B(Z^c) \cap \mathbb{R}^{2n}$.

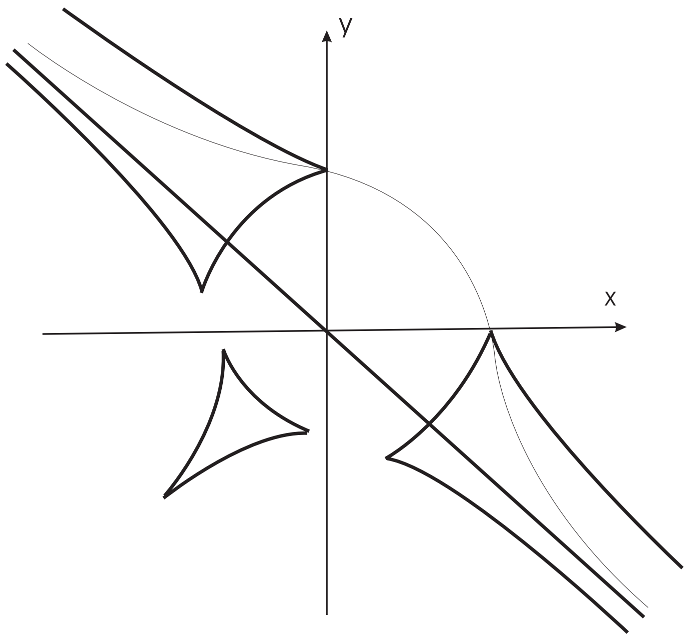

FIG. 3. *Symmetry defect set for elliptic curve $x^3 + y^3 = 1$.*

EXAMPLE 6.8. Let us compute the symmetry defect set for elliptic curve $Z = \{(x, y) \in \mathbb{C}^2 : x^3 + y^3 = 1\}$. We use Magma computer system. First we compute $K_0(\Phi)$. We have the following implementation of our algorithm:

$Q := RationalField(); R < x, x_1, y, y_1, u, v >:= PolynomialRing(Q, 6, "lex");$
$F := x^3 + x_1^3 - 1;$
$G := y^3 + y_1^3 - 1;$
$W := x^2 * y_1^2 - x_1^2 * y^2;$
$I := ideal < R|F, G, W, x + y - 2 * u, x_1 + y_1 - 2 * v >;$
$Groebner(I); I$

As a result we have $I \cap \mathbb{Q}[u, v] = \{u^{15} + 5 * u^{12} * v^3 - 11/4 * u^{12} + 10 * u^9 * v^6 + 5/2 * u^9 * v^3 + 145/64 * u^9 + 10 * u^6 * v^9 + 21/2 * u^6 * v^6 - 321/64 * u^6 * v^3 - 19/64 * u^6 +$

$5 * u^3 * v^12 + 5/2 * u^3 * v^9 - 321/64 * u^3 * v^6 - 73/32 * u^3 * v^3 - 13/64 * u^3 + v^15 - 11/4 * v^12 + 145/64 * v^9 - 19/64 * v^6 - 13/64 * v^3 - 1/64\}$. This is the equation of a proper part of $B(Z)$. To compute an asymptotic part of $B(Z)$ we use the following algorithm (which computes F_1):

$Q := RationalField(); R < x, x_1, y, y_1, u, v, z >:= PolynomialRing(Q, 7, "lex");$
$F := x^3 + x_1^3 - 1;$
$G := y^3 + y_1^3 - 1;$
$I := ideal < R|F, G, x + y - 2 * u, x_1 + y_1 - 2 * v, x - z >;$
$Groebner(I); I$

As a result we have $I \cap \mathbb{Q}[u, v, z] = \{u^9 - 9/2 * u^8 * z + 9 * u^7 * z^2 + 3 * u^6 * v^3 - 81/8 * u^6 * z^3 - 3/4 * u^6 - 9 * u^5 * v^3 * z + 27/4 * u^5 * z^4 + 9/4 * u^5 * z + 45/4 * u^4 * v^3 * z^2 - 81/32 * u^4 * z^5 - 45/16 * u^4 * z^2 + 3 * u^3 * v^6 - 81/8 * u^3 * v^3 * z^3 + 15/8 * u^3 * v^3 + 27/64 * u^3 * z^6 + 27/16 * u^3 * z^3 + 3/16 * u^3 - 9/2 * u^2 * v^6 * z + 27/4 * u^2 * v^3 * z^4 - 45/16 * u^2 * v^3 * z - 27/64 * u^2 * z^4 - 9/32 * u^2 * z + 9/4 * u * v^6 * z^2 - 81/32 * u * v^3 * z^5 + 45/32 * u * v^3 * z^2 + 9/64 * u * z^2 + v^9 - 3/4 * v^6 + 27/64 * v^3 * z^6 - 15/64 * v^3 - 1/64\}$, hence $a_0^1 = 27/64(u^3 + v^3)$. Similarly, we can check that $a_0^i = c_i(u^3 + v^3)$ for every i. This means that Z has three asymptotes (only one is real).

Hence $S_\Phi = \{u^3 + v^3 = 0\}$. Finally $B(Z) = \{u^{15} + 5 * u^{12} * v^3 - 11/4 * u^{12} + 10 * u^9 * v^6 + 5/2 * u^9 * v^3 + 145/64 * u^9 + 10 * u^6 * v^9 + 21/2 * u^6 * v^6 - 321/64 * u^6 * v^3 - 19/64 * u^6 + 5 * u^3 * v^12 + 5/2 * u^3 * v^9 - 321/64 * u^3 * v^6 - 73/32 * u^3 * v^3 - 13/64 * u^3 + v^15 - 11/4 * v^12 + 145/64 * v^9 - 19/64 * v^6 - 13/64 * v^3 - 1/64 = 0\} \cup \{u^3 + v^3 = 0\}$.

REMARK 6.9. Note that our algorithm gives an easy way to compute all asymptotes of a given curve Z. More generally, in this way we can compute an asymptotic set of a $n-$dimensional variety $Z \subset \mathbb{C}^{2n}$ which is in a general position.

EXAMPLE 6.10. In a similar way we can compute the symmetry defect set for the elliptic curve $Z = \{(x, y) \in \mathbb{C}^2 : x^2 - y^3 + y = 0\}$. We obtain that $B(Z)$ consists only of the proper part $K_0(\Phi) = \{(u, v) \in \mathbb{C}^2 : u^8 - 7 * u^6 * v^3 - u^6 * v + 18 * u^4 * v^6 - 5 * u^4 * v^4 - 8/3 * u^4 * v^2 + 1/27 * u^4 - 20 * u^2 * v^9 + 23 * u^2 * v^7 - 5 * u^2 * v^5 + 7/27 * u^2 * v^3 - 1/27 * u^2 * v + 8 * v^12 - 18 * v^10 + 44/3 * v^8 - 152/27 * v^6 + 28/27 * v^4 - 2/27 * v^2 = 0\}$ (note that Z has no asymptotes).

EXAMPLE 6.11. The symmetry defect set for the elliptic curve $Z = \{(x, y) \in \mathbb{C}^2 : x^2 - y^3 + y - 1 = 0\}$ is a curve $B(Z) = \{(u, v) \in \mathbb{C}^2 : u^8 - 7 * u^6 * v^3 - u^6 * v - 3 * u^6 + 18 * u^4 * v^6 - 5 * u^4 * v^4 - 2 * u^4 * v^3 - 8/3 * u^4 * v^2 + 2 * u^4 * v + 82/27 * u^4 - 20 * u^2 * v^9 + 23 * u^2 * v^7 - u^2 * v^6 - 5 * u^2 * v^5 - 12 * u^2 * v^4 + 250/27 * u^2 * v^3 + 3 * u^2 * v^2 - 28/27 * u^2 * v - 29/27 * u^2 + 8 * v^12 - 18 * v^10 + 7 * v^9 + 44/3 * v^8 - 8 * v^7 - 179/27 * v^6 + 10/3 * v^5 + 55/27 * v^4 - 16/27 * v^3 - 11/27 * v^2 + 1/27 * v + 1/27 = 0\}$ (note that Z has no asymptotes).

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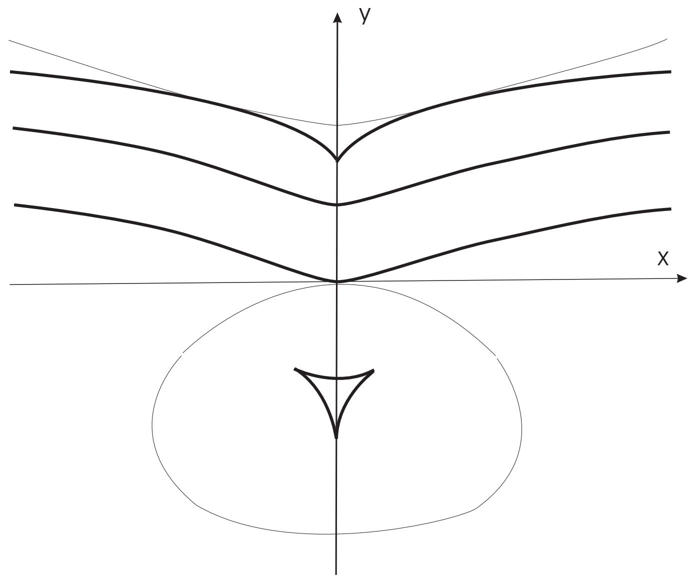

Fig. 4. *Symmetry defect set for elliptic curve* $x^2 = y^3 - y$.

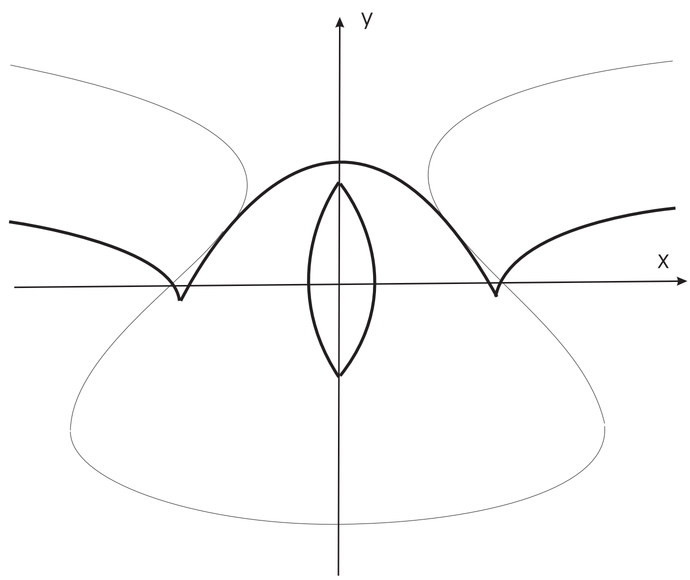

Fig. 5. *Symmetry defect set for elliptic curve* $y^2 = x^3 - x + 1$.

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