ON INTERACTION OF THE CLASSICAL FRACTALS

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Preprint Nr.41 June 1991

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by

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Abstract. *We graphically investigate the fractal properties of a coupling of a few quadratic mappings acting on different domains. Evolutions of the Julia sets for the simplest interactions through the moving line and the growing circular boundary are illustrated. Structure of the corresponding Mandelbrot set is discussed.*

([1,2]) *Part of this work was done while* ([1]) *in Mathematics Department, University of California at Santa Cruz, USA* and

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1. INTRODUCTORY REMARKS

Let us consider the quadratic map of the complex plane, $f_c(z) = z^2 + c$. The sequence of points of $\mathbf{C}$, $z_{n+1} = f_c(z_n)$, $n = 0, 1, ...$, is called the orbit of z_0 by f_c. Now there are few questions concerning the behaviour of that sequence. Does the sequence $\{z_n\}$ converge? How does it depend on z_0 and c? What the set of points of $\mathbf{C}$ to which the orbit is coming infinitely close look like? All these questions and answers, describe the complex dynamics of f_c (see [1], [2]). Dependence of f_c on c gives rise to questions concerning the structure of qualitative changes in dynamics with respect to c. This is the study of bifurcations of dynamics of f_c.

Let us observe that for $c = 0$ the asymptotic dynamics of f_c is governed by the circle $S^1 = \{|z| = 1\}$. It reduces to the chaotic map $\theta \to 2\theta$ on S^1. S^1 in that case is the boundary of the set of points that escape to infinity. In general if f_c has an attracting periodic orbit, there is a collection of points that are attracted to the cycle. These are points that lie in the basin of attraction of the cycle. There are also points whose orbits tend to infinity. The points which lie on the boundary between the basin of attraction and the escaping points form what is called **the Julia set** for f_c (cf. [1]). In general the Julia set of the polynomial $P : \mathbf{C} \to \mathbf{C}$, $J(P)$ is defined as the closure of the set of repelling periodic points of P, i.e. for repelling periodic point $z_0 = P^n(z_0)$, we have $|(P^n)'(z_0)| > 1$. The Julia set $J(f_c)$ is completely invariant, i.e. $J(f_c)$ contains all forward images as well as preimages of points in $J(f_c)$. So the complement of $J(f_c)$, denoted by $S(f_c)$ (called the stable set) is also completely invariant. The observation that for some $z_0 \in J(f_c)$ we can write $J(f_c)$ in the form $J(f_c) = closure(\bigcup_{k=0}^{\infty} f_c^{-k}(z_0))$ is useful in building an algorithm for plotting Julia sets graphically. There is a list of very interesting properties of Julia sets presented already in mathematical literature (see e.g. [1]), together with the beautiful pictures and applications. Our aim is just to recall some of the properties which may be useful in further research.

The next step in investigation of dynamics of quadratic maps on $\mathbf{C}$ is to investigate the bifurcations for varying $c \in \mathbf{C}$. In the study of bifurcations the important role is played by the behaviour of the critical point 0 of f_c under the iterations. We distinguish the set of points whose orbits do not escape to infinity under iteration of f_c, $K_c = \{z : f_c^n(z)$ is bounded for $n \in N\}$. This is the set of special importance (to investigate the dynamics), its boundary is a Julia set $J(f_c)$. Being a region of attraction is usually easier to investigate. Thus the bifurcation set on the complex plane of "c" should at least describe the transformation of its properties. The main structure of K_c is governed by the following **Mandelbrot set,**

$$M = \{c : f_c^n(0) \nrightarrow \infty\}.$$

For $c \in M$ the set K_c is connected. The Mandelbrot set of families of quadratic maps has already been investigated in [1] (see also Fig.6f.).

An important stimulation for the theory of fractals comes from applications [2]. There are a lot of fractal sets, strange attractors etc. found in physics, biology, medical sciences and phenomena in nature (see e.g. [2], [4]). An interesting question concerning the evolution of forms in nature is the following: Does in general, is it possible for one form with a prescribed symmetry to evolve into another one with a completely different symmetry? Do the intermediate forms exhibit any symmetry or are they completely chaotic? In this paper we do not intend to give a complete answer to this question but we will construct a family of models where this does happend and, where the breaking evolution of the forms is chaotic.

In applications, the dynamical models often consist of the couples of dynamical systems which operate on the separate domains. Then the resulting phase trajectory is a kind of mixture of influence of all attractors involved in that coupling. So this is a part of the motivation for our study of fractal interactions. By introducing the notion of interaction of fractals we graphically investigate the possible evolutions of Julia sets and the structure of Mandelbrot set. In Section 2, we generalize the notion of quadratic map by modify the domain of its action and taking the piecewise constant function which represents the standard constant "c" [1]. The concrete evolutions of two coexisting fractals throug the vertical line is investigated in Section 3. The secular domain of an appearing fractal and the corresponding evolution is investigated in Section 4. In Section 5, we try to describe the Mandelbrot-type set for the growth of a new fractal with circular domain. It is a fractal set in the five dimensional space. We investigate some of the natural, evolving intersections of that set.

2. INTERACTION OF FRACTALS BY GROWTH OF INFLUENCITY DOMAINS

Let us consider the family of quadratic mappings : $f_{c_1}, f_{c_2}, ..., f_{c_i}, ... : C \to C, \quad i \in I \subset N$

$$f_{c_i}(z) = z^2 + c_i, \quad i \in I,$$

where $c_i, i \in I$ are fixed complex numbers. We denote

$$f^n_{c_i}(z) = \underbrace{f_{c_i} \circ f_{c_i} \circ ... \circ f_{c_i}(z)}_{n \text{ times}}, \quad i \in I.$$

We refer to all fractal properties of the quadratic mappings f_{c_i}, (see Section 1). The corresponding fractals, Julia Sets associated to K_{c_i} we denote by J_{c_i}. Let $A_i, i \in I$ be a disjoint partition of C parametrized by I, $\bigcup A_i = C$, $A_k \cap A_l = \phi, k \neq l$.

DEFINITION 2.1 Fractal set $J_{(A_i,c_i)}$ defined as the Julia set for the mapping

$$f_{(A_i,c_i)} = \begin{cases} z^2 + c_1, & \text{for } z \in A_1; \\ .... & \\ z^2 + c_k, & \text{for } z \in A_k, \quad k \in I; \\ .... & \end{cases}$$

is called the interaction of fractals $\{J_{c_i}\}_{i \in I}$ acting on the domains $\{A_i\}_{i \in I}$.

We see that the interaction of fractals $\{J_{c_i}\}$ is defined by an ordinary quadratic mappings with the slightly generalized constant term. In our case this additional constant (cf [1]), is a piecewise constant function. In the rest of the paper we investigate some concrete forms for evolving of two domains and their corresponding fractal interactions.

3. EVOLUTION OF TWO COEXISTING FRACTALS THROUGH THE DIVIDING LINE

Let us take the infinite domain $A_h := \{z \in C : Rez > h\}$, parametrized by $h \in R$. Now our formula for interaction takes a form :

$$f_{(h,c_1,c_2)} = \begin{cases} z^2 + c_1, & \text{for } Rez \leq h; \\ z^2 + c_2, & \text{for } Rez > h. \end{cases}$$

Here we have the classical fractals operating throught the half-spaces of C. See Fig 1. below.

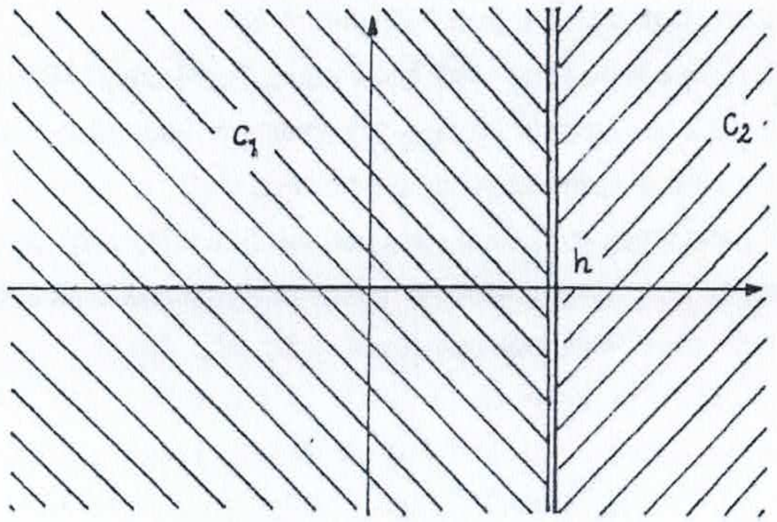

Figure 1. Interaction of fractals through the dividing line.

Remark 3.1 We see that:

(*i*) $$f_{(h,c_1,c_2)} \to f_{c_1}, \text{ for } h \to +\infty$$

(*ii*) $$f_{(h,c_1,c_2)} \to f_{c_2}, \text{ for } h \to -\infty$$

For the purposes of the computer calculations "infinity" is reached already if $|h| > 10$. In both cases (i),(ii) there is no interaction and the corresponding fractals look like J_{c_1} and J_{c_2} respectively (they are illustrated if Fig. 2 a,e). Changing h from $-\infty$ to $+\infty$ we obtain the evolution of fractals connecting J_{c_1} and J_{c_2}, (See Fig. 2 a,- g). It seems that the process of transformation of fractals depends on the kind of interaction of J_{c_1} and J_{c_2}. Analysis of bifurcations in evolution of forms of fractals is possible by investigation of parametrized classes of interactions.

Let us look at the proces of interaction of two classical fractals. For this investigation we will take fractals obtained by iteration of function $z^2 + c$ for $c_1 = 0.44 + 0.2i, c_2 = -0.2 + 0.6i$. We can see these fractals at the figure 2a,2g. All figures in between are obtained for diffrent values of h. We can start from sufficiently big h. Then we have shape

wich is exactely J_{c_1} Then we take h smaller and smaller. Thus we can see influence of the second fractal. Our figure is more and more "fat".
For $0.4 < h < 1.0$ we have shape with holes inside "solid body" (Fig. 2b).
For $-0.2 < h < 0.4$ we have "solid body " without any internal structure (Fig. 2c).
For $h < -0.2$ we loose compactness of our set (Fig. 2d).
For h=-0.5 we have shape with three pieces of the solid body (Fig. 2e).
Then again for the next $h = -0.5203$ we get the compactness back (Fig. 2f).
Finaly we get the shape wich looks exactly like J_{c_2} (Fig. 2g).

No we look at the proces of interaction of a classical fractal and the regular shape of a circle (which is a Julia set for iteration of z^2). For this investigation we will take fractals obtained by iteration of function $z^2 + c$ for $c_1 = 0.0 + 0.0i$ and $c_2 = 0.44 + 0.2i$. We can see these fractals at the figures 3a,3e. All figures between them are obtained for diffrent values of h (Fig 3a). We can start from big h. Then we obtain pure circle (wich is J_{c_1}). Now we take h smaller and smaller. We can see the influence of the second fractal. For $0.5 < h < 1.0$ we see the shape of a circle with a very strong border on h. In the region $Rez < h$ we have something wich looks like a circle.
In the region $Rez > h$ we have a small piece of J_{c_2}. We can observe the counterimages of that little piece of J_{c_2} under the mapping z^2 in the region $Re_z < h$. They look like the small "legs" of a circle (Fig. 3b). As h decreases these legs become biger (Fig 3c,3d). Finally our set looks like J_{c_2}.

Remark 3.2.

As far we defined the interaction of the fractals, which are "in contact" by the boundaries of the sets A_i. An interesting question is, how does the fractals interact being in some "distance"? Let A and B be two subsets of C, $\bar{A} \cap \bar{B} = \phi$, and $c_1, c_2, c_3 \in$ C. The fractal set $J_{(A,B,c_1,c_2,c_3)}$ defined by the iterations of the mapping

$$f_{(A,B,c_1,c_2,c_3)} = \begin{cases} z^2 + c_1, & \text{for } z \in A; \\ z^2 + c_2, & \text{for } z \in B; \\ c_3, & \text{for } z \in \mathrm{C} - A \cup B. \end{cases}$$

is called the interaction of fractals J_{c_1} and J_{c_2} on a distance through the fractal J_{c_3}.
If c_3 tends to infinity we say that J_{c_1} and J_{c_2} interact throug "vacuum". This interaction is quite poor because the set $\mathrm{C} - A \cup B$ is actually diverging all sequences touching with it. The further investigations of that concept we leave to the forthcoming paper.

4. LOCAL DISTURBANCE BY A CIRCULAR DOMAIN

Let us take a circular domain $A_r := \{z \in \mathbf{C} : |z| \leq r\}$, parametrized by $r \in R_+$. Now our formula for interaction takes a form:

$$f_{(r,c_1,c_2)} = \begin{cases} z^2 + c_1, & \text{for } |z| \leq r; \\ z^2 + c_2, & \text{for } |z| > r. \end{cases}$$

Here we have classical fractals operating throught the circle with center at the point $0+0i$, See Fig. 4, below.

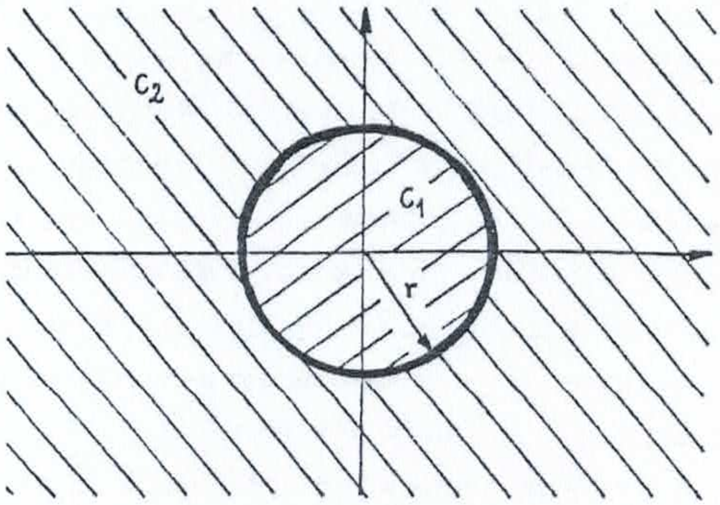

Figure 4. Growth of the fractal with circular domain.

Remark 4.1

We see that

(*) $\qquad f_{(r,c_1,c_2)} \to f_{c_1}$, for $r \to +\infty$

(**) $\qquad f_{(r,c_1,c_2)} \to f_{c_2}$, for $r \to 0$

It seems to be interesting to consider an influence of the fractal set on the regular shape of circle. For this purpose we take $c_2 = 0 + 0i$, then for the function $f_{c_2} = z^2 + c_2 = z^2$

we get J_{c_2} = circle at $0 + 0i$ and radius 1. For the second parameter c_1 we take $c_1 \approx 0$ (for instance $c_1 = 0.35 + 0.05i$ like in the Figure 4). If we take $r = 0$ then we get a circle. All points with $|z| < 1$ under infinite iterations tend to zero because $|z_{(n+1)}| = |z_n||z_n|$ and we have $|z_{(n+1)}| < |z_n|$.
All points with $|z| = 1$ under infinite iterations stay at the circle $|z_{(n+1)}| = |z_n||z_n| = 1$. All points with $|z| > 1$ under infinite iterations tend to $+\infty$ because $|z_{(n+1)}| = |z_n||z_n|$ and we have $|z_{(n+1)}| > |z_n|$. As r increases then we have a small region with $|z| < r$ where for iterations we use functon f_{c_2} instead of $f_0 = z^2$. This means that for all points with $|z| < 1$ we can not say that $|z_{(n+1)}| < |z_n|$, so we are not sure that all iterations for points $|z| < 1$ tend to zero. For some points with $|z_n| < r$ we have $|z_{(n+1)}| > r$. Then it is possible that for some points z_n we have $|z_{(n+1)}| > 1$ and for all the next iterations (in the region $|z| > r$ -thus iterated by the function $f_0 = z^2$) it will send our point to infinity. Now we look at the conditions when it may happen. Let us take $c_1 = 0 + 0i$, $c_2 = 0 + 0i$ and $0 < r < 1$. If we woud like to have some holes inside our circle we must assume $|z_{(n+1)}| > 1$ for some z_n. For points z, such that $r < |z_n| < 1$ we always obtain $|z_{(n+1)}| = |z_n|^2 < 1$. Now we look at the points with $|z_n| \leq r$. We have $z_{(n+1)} = z_n^2 + c_2$ and so $|z_{(n+1)}| \leq |z_n|^2 + |c_2|$, but $|z_n| < r$, so $|z_{(n+1)}| < r^2 + |c_2|$. Because we need $|z_{(n+1)}| > 1$ so we get the condition for r and $|c_2|$ namely $1 < r^2 + |c_2|$. In this case parabola $|c_2| = 1 - r^2$ is a bifurcation set.

As we increase r to make it close to 1 but still less than 1 we have that all points with $|z| = 1$ are iterated by function $z_{(n+1)} = z_n^2$. This preserves the circle $|z| = 1$, so those points belong to our figure.

5. THE MANDELBROT SET

Now we define the corresponding extention of the classical Mandelbrot set . We introduce the following notion

$$R \times \mathbf{C} \times \mathbf{C} \supset \bar{M} := \{(r, c_1, c_2);\ \text{for every } s > 0 \text{ there exists } n \in N \text{ such that } |f^n_{(r,c_1,c_2)}(0)| > s\}$$

DEFINITION 5.1.

The Mandelbrot set $\tilde{M}$ corresponding to the interaction of fractals is defined as follows

$$\tilde{M} = R \times \mathbf{C} \times \mathbf{C} - \bar{M}.$$

It is a fractal set in the five dimensional space.

We tend to investigate the structure of $\tilde{M}$ by taking the intersections of it with two and three dimensional subspaces of $R \times \mathbf{C} \times \mathbf{C}$. If we look on that set for $r \to +\infty$, and we make an intersection of it with the subspace of fixed r and c_2 we obtain an ordinary mandelbrot set M (cf. [ ]) on the complex plane parametrized by c_1. In this situation we start from $0 + 0i$ and we stay in the region where iterations are governed by the function f_{c_1} (if $r \to +\infty$ it is not posssible to get the region $|z| > r$).

It is interesting to consider the case when $r \to 0$. If we look at diffrent values of c_1 with fixed r and c_2 we have $0 + 0i = z_0$ tends to $z_0^2 + c_1 = c_1 = z_1$. Then for the next iterations we have $z_{(n+1)} = z_n^2 + c_2$. As r goes to 0 it is not possible to come back to the region where we use the second function for iterations. In this case, as we take all $c_1 \in \mathbf{C}$, it is equivalent to the case when all $z_1 \in \mathbf{C}$, and iterating it with f_{c_2}. It means that the intersection of $\tilde{M}$ with the plane $\{(r, c_1, c_2) : r = 0, c_2 = const\}$ is J_{c_2}. Now taking diffrent r from 0 to $+\infty$ we can get the evolution of the shape from J_{c_2} to the standard Mandelbrot set M (See Fig. 6. a-f).

Remark 5.2.

The above example shows that $\tilde{M}$ can not be used to classify $J_{(r,c_1,c_2)}$ sets (like in the case of ordinary Mandelbrot set). If we take more general version of our "circular" interaction, say instead of investigating the set $|z| < 0$ we can take the set $C(a + bi, r)$, and if we tend to 0 with r then for this more general situation we easily see that the function f_{c_1} operates inside the very small set $C(a + bi, r \to 0)$. It is so small that it has not influence on the global dynamics determined by the mapping f_{c_2}. It means that for very small r it does not matter whether we investigate the set $C(0+0i,r)$ or the set $C(a+bi,r)$ for some $a,b\neq 0$. However for the Mandelbrot set $\tilde{M}$, in both cases we obtain quite different outlooks. For $C(a+bi,r)$ we have $0+0i=z_0\to z_0^2+c_2=c_2$. Then for the subsequent iterations we have $z_{(n+1)}=z_n^2+c_2$. Changement of c_1 does not change anything else. For every c_1 we have the same iteration. For instance, if for some c_2; $f_{c_2}^n(0)\to 0$, we get for all c_1, $(r,c_1,c_2)\in\tilde{M}$ so we obtain all points from the plane $\mathbf{C}$ belonging to the intersection of M with the plane $\{(r,c_1,c_2) : r=0, c_2=const.\}$. If for another c_2; $f_{c_2}^n(0)\to\infty$, then we get that none of $c_1\in\mathbf{C}$ belongs to the intersection of $\tilde{M}$ with the plane $\{(r,c_1,c_2) : r=0, c_2=const.\}$. Hence $J_{(r,c_1,c_2)}=J_{(r,c_1,c_2')}$ and $\tilde{M}=M'$ it means that we can not use the set $\tilde{M}$ for classification of $J_{(r,c_1,c_2)}$.

Remark 5.3. We define the Mandelbrot set for the half-plane domain in an analogous way;

$$\tilde{M}=R\times\mathbf{C}\times\mathbf{C}-\tilde{M}.$$

where

$$R\times\mathbf{C}\times\mathbf{C}\supset\tilde{M}:=\{(h,c_1,c_2);\ \text{for every } s>0 \text{ there exists } n\in N \text{ such that } |f^n_{(h,c_1,c_2)}(0)|>s\}.$$

In this case for $h\to\infty$ we get an ordinary Mandelbrot set in the c_1-plane. If $h\to-\infty$ we get an ordinary Mandelbrot set in c_2-plane. An intersection of $\tilde{M}$ with the subspace $\{c_1=c_2\}\subset R\times\mathbf{C}\times\mathbf{C}$ gives also an ordinary Mandelbrot set.

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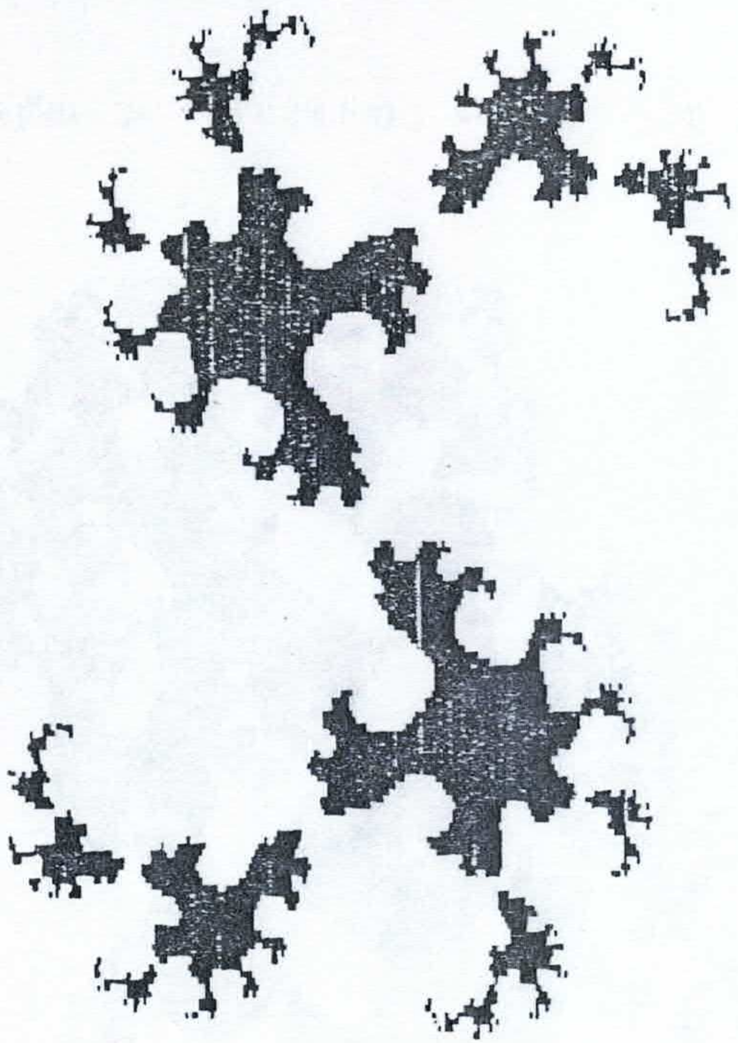

Figure 2a. $c_1 = 0.44 + i0.2,\ c_2 = -0.2 + i0.6,\ h = 1.5$

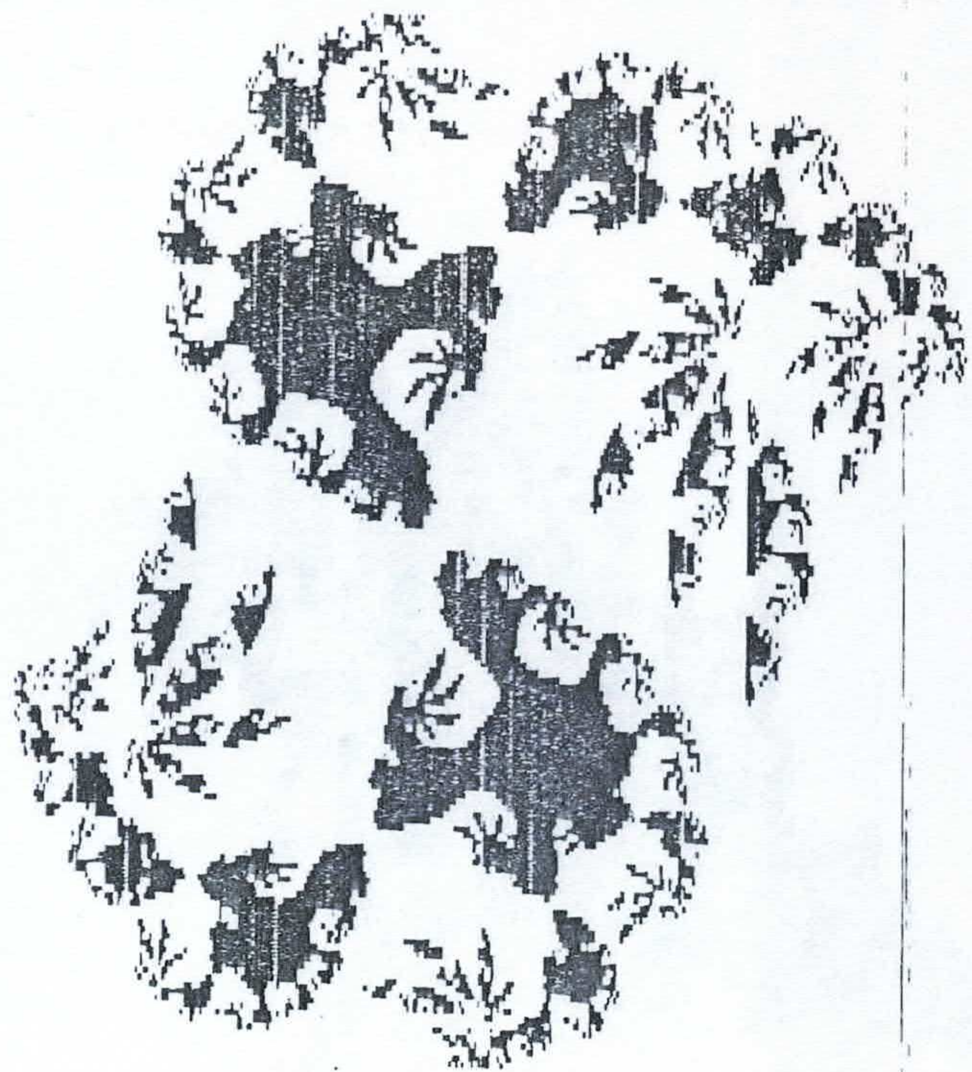

Figure 2b. $c_1 = 0.44 + i0.2,\ c_2 = -0.2 + i0.6,\ h = 0.75$

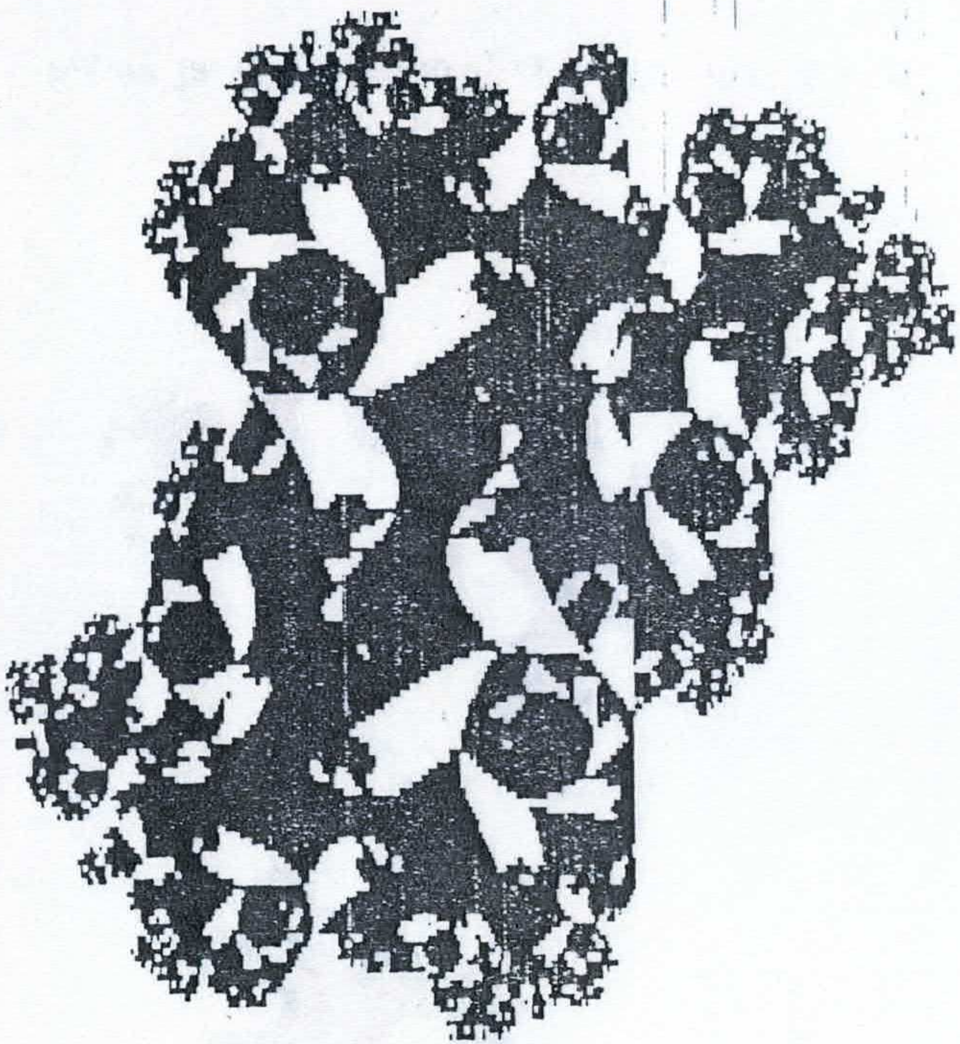

Figure 2c. $c_1 = 0.44 + i0.2$, $c_2 = -0.2 + i0.6$, $h = 0.5$

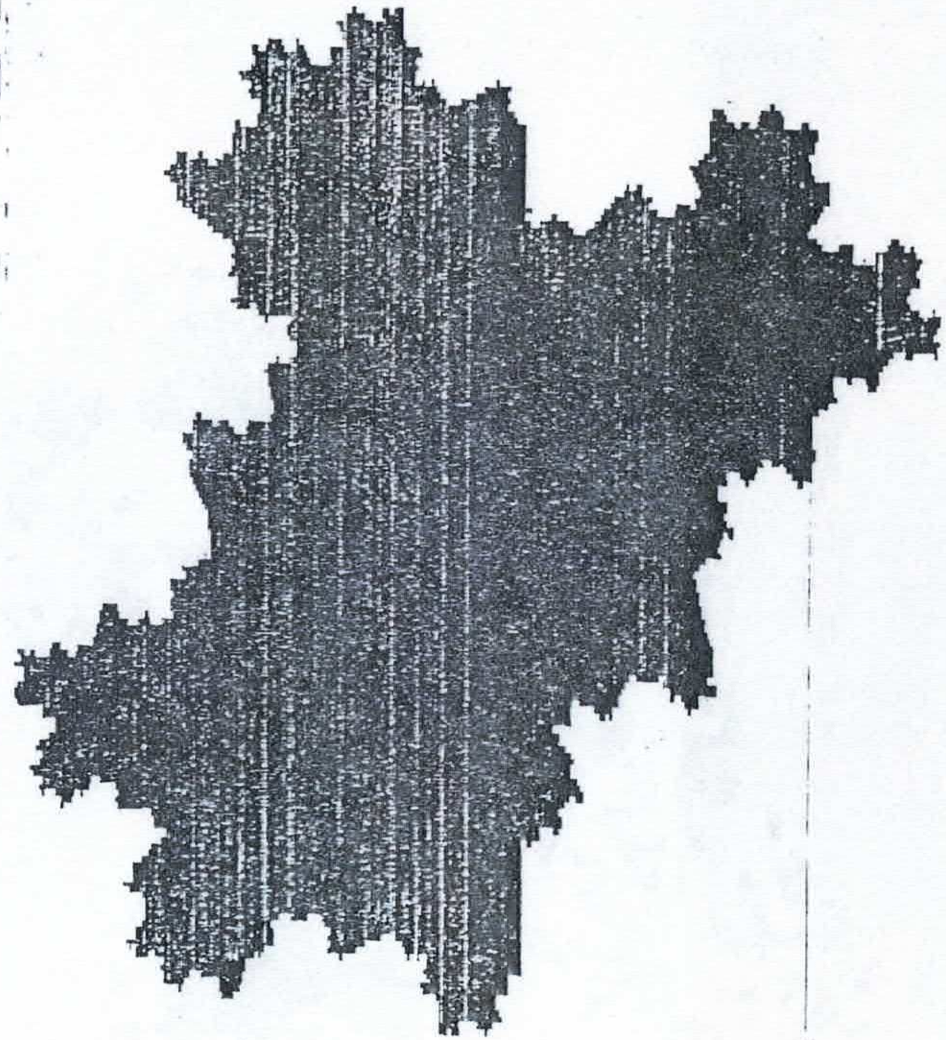

Figure 2d. $c_1 = 0.44 + i0.2$, $c_2 = -0.2 + i0.6$, $h = 0.25$

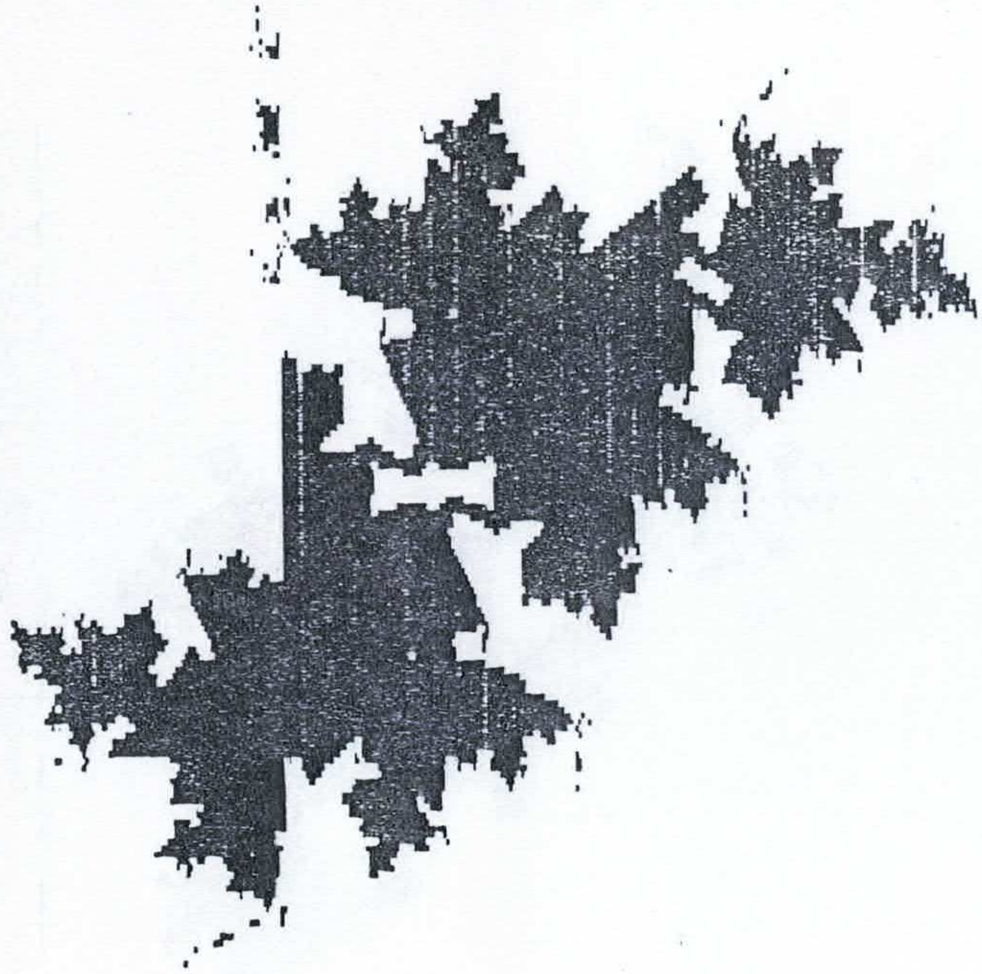

Figure 2e. $c_1 = 0.44 + i0.2$, $c_2 = -0.2 + i0.6$, $h = -0.35$

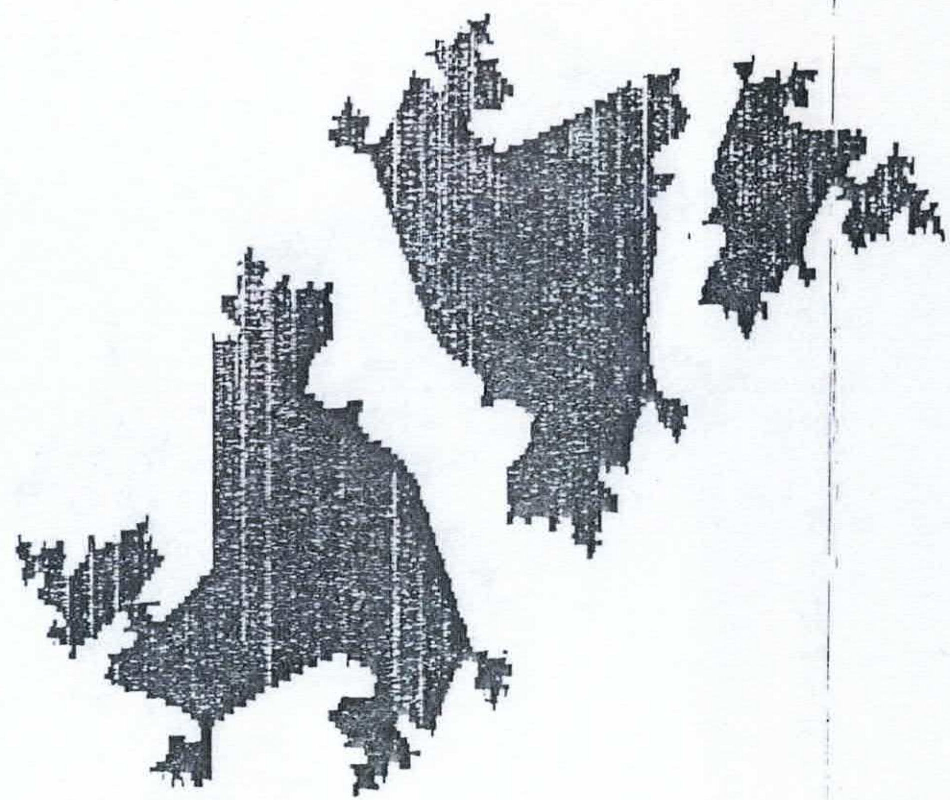

Figure 2f. $c_1 = 0.44 + i0.2$, $c_2 = -0.2 + i0.6$, $h = -0.5$

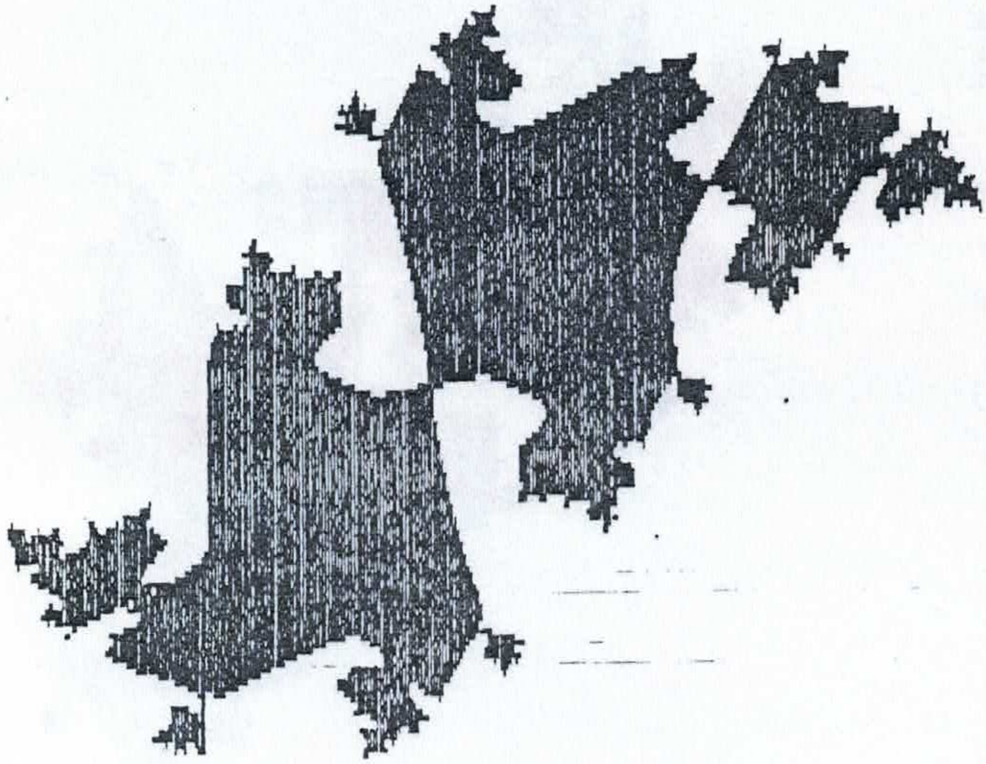

Figure 2g. $c_1 = 0.44 + i0.2$, $c_2 = -0.2 + i0.6$, $h = -0.5205$

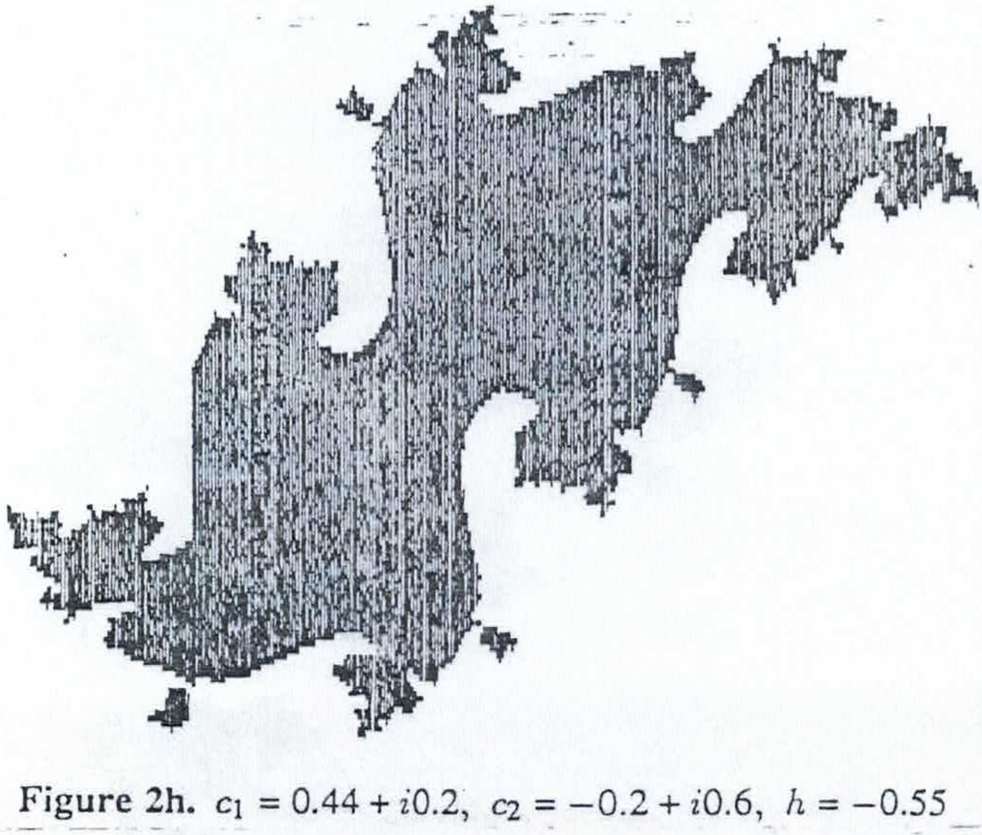

Figure 2h. $c_1 = 0.44 + i0.2$, $c_2 = -0.2 + i0.6$, $h = -0.55$

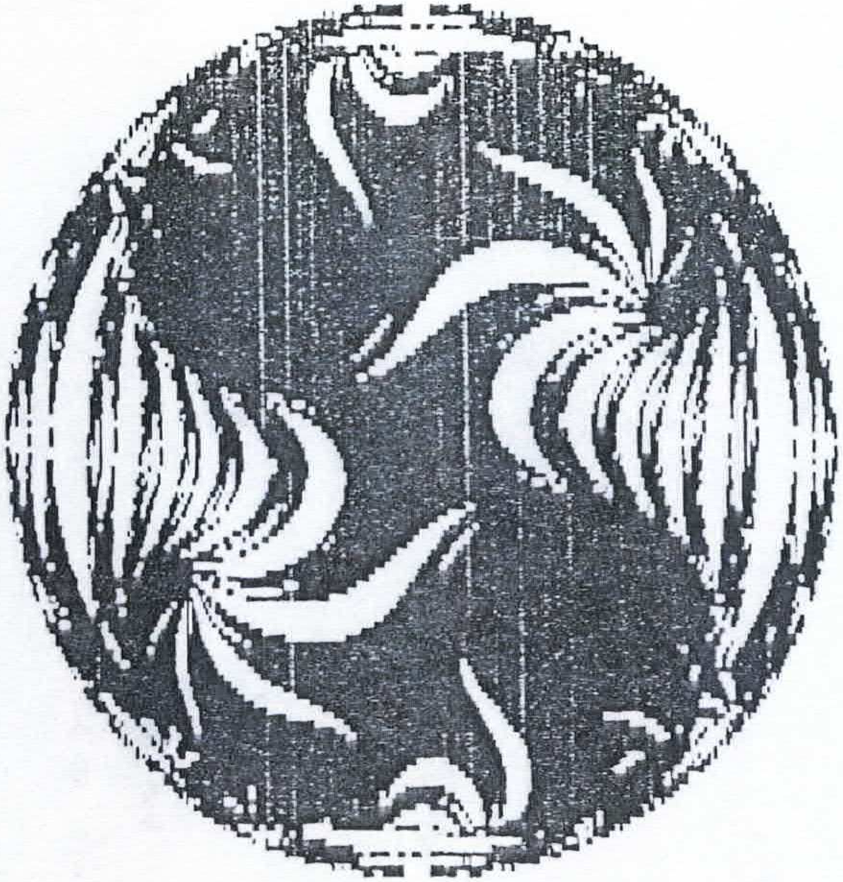

Figure 5e. $c_1 = 0.35 + i0.05$, $c_2 = 0.0 + i0.0$, $h = 0.89$

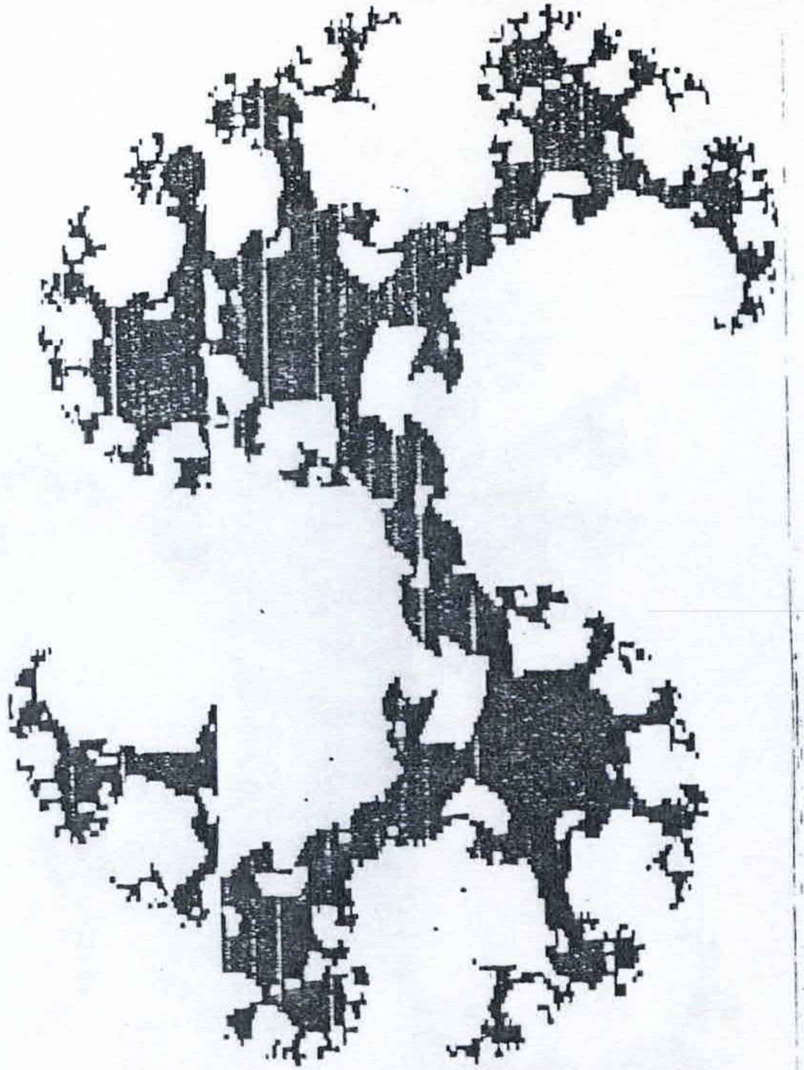

Figure 3a. $c_1 = 0.0 + i0.0$, $c_2 = 0.44 + i0.2$, $h = -0.5$

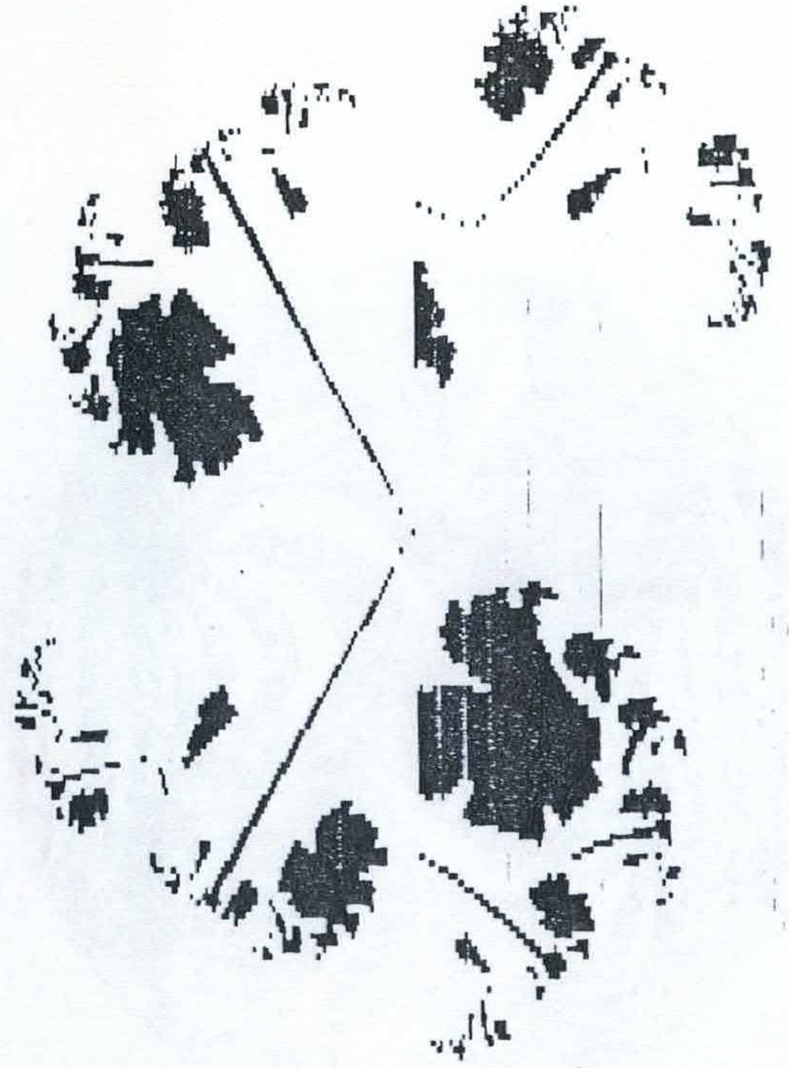

Figure 3b. $c_1 = 0.0 + i0.0,\ c_2 = 0.44 + i0.2,\ h = 0.0$

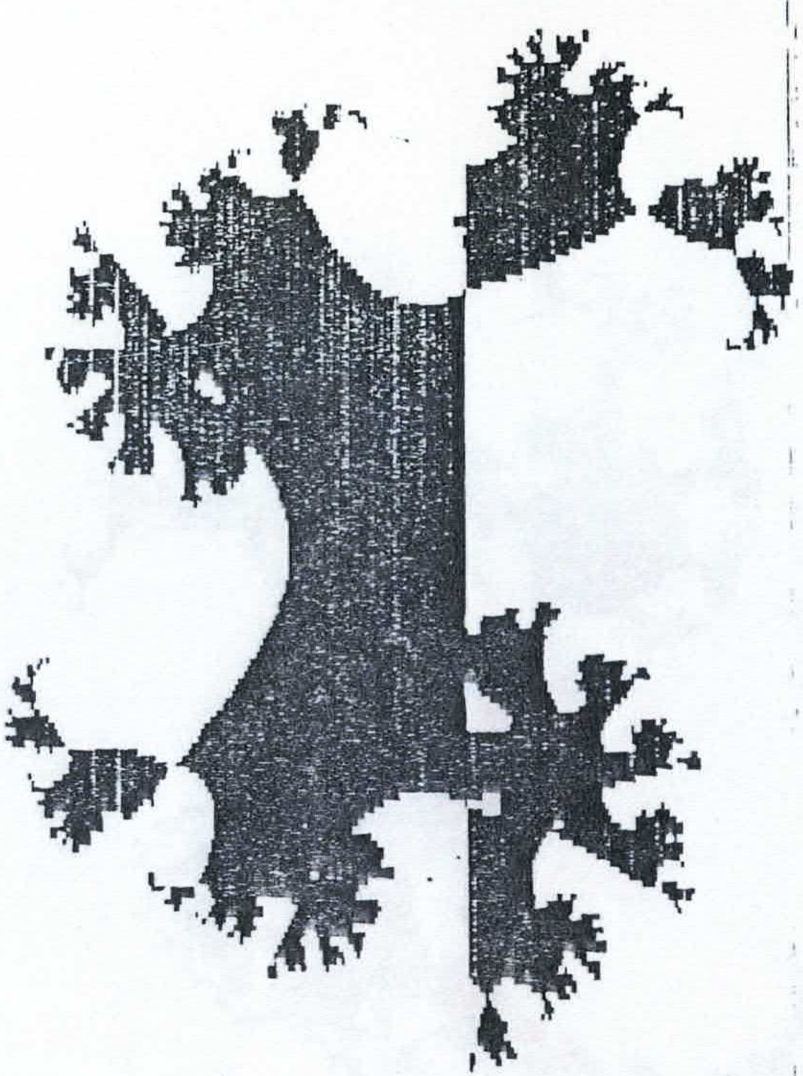

Figure 3c. $c_1 = 0.0 + i0.0,\ c_2 = 0.44 + i0.2,\ h = 0.1$

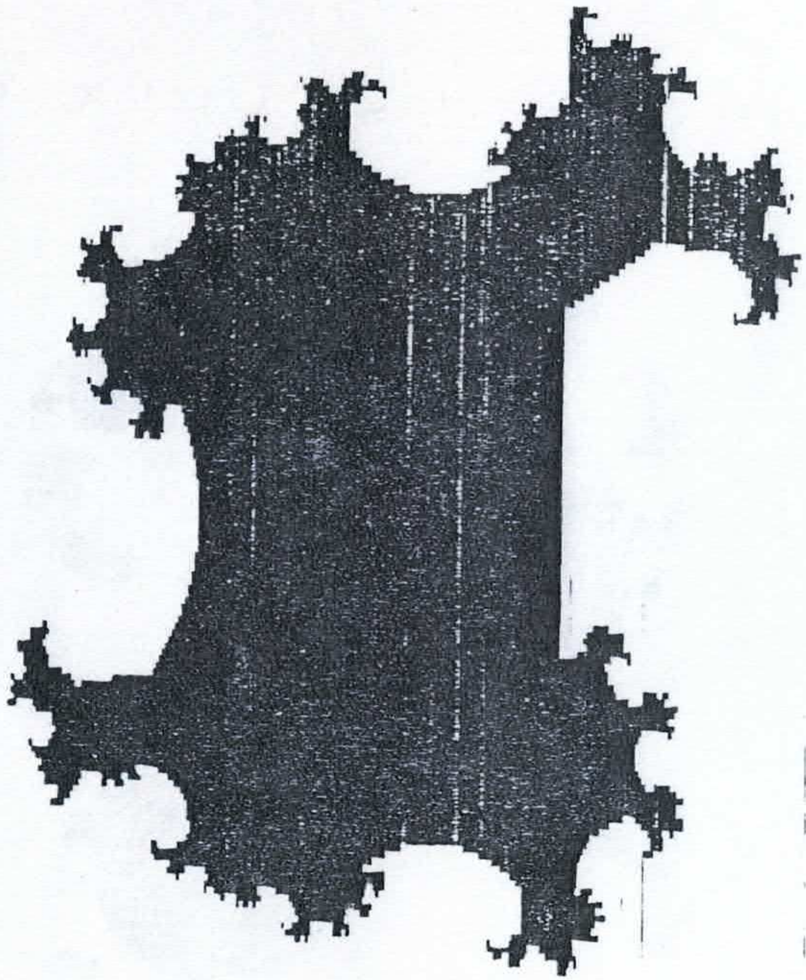

Figure 3d. $c_1 = 0.0 + i0.0$, $c_2 = 0.44 + i0.2$, $h = 0.3$

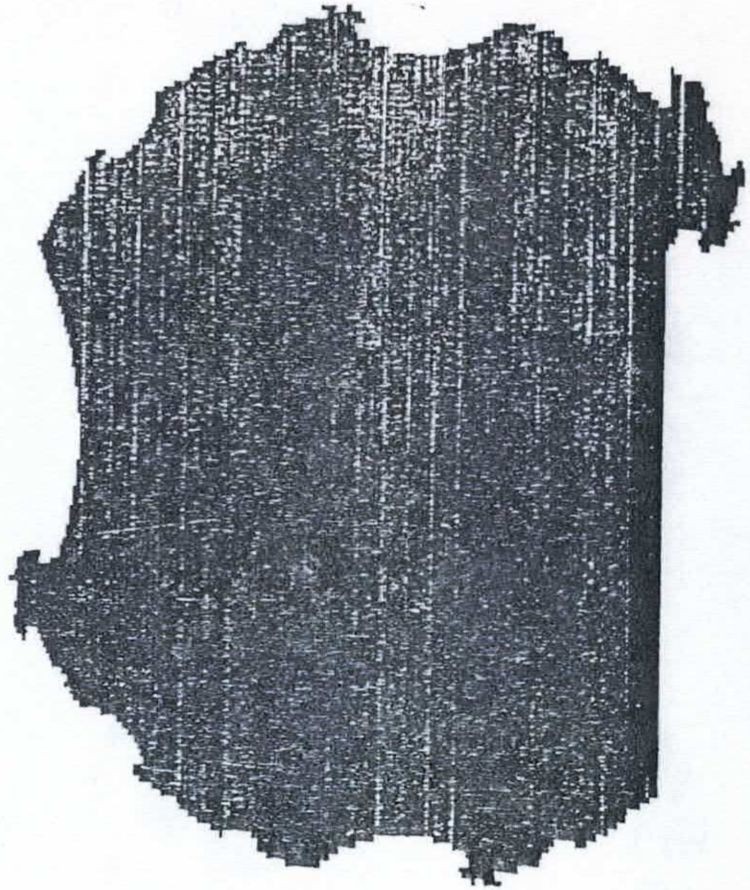

Figure 3e. $c_1 = 0.0 + i0.0$, $c_2 = 0.44 + i0.2$, $h = 0.63$

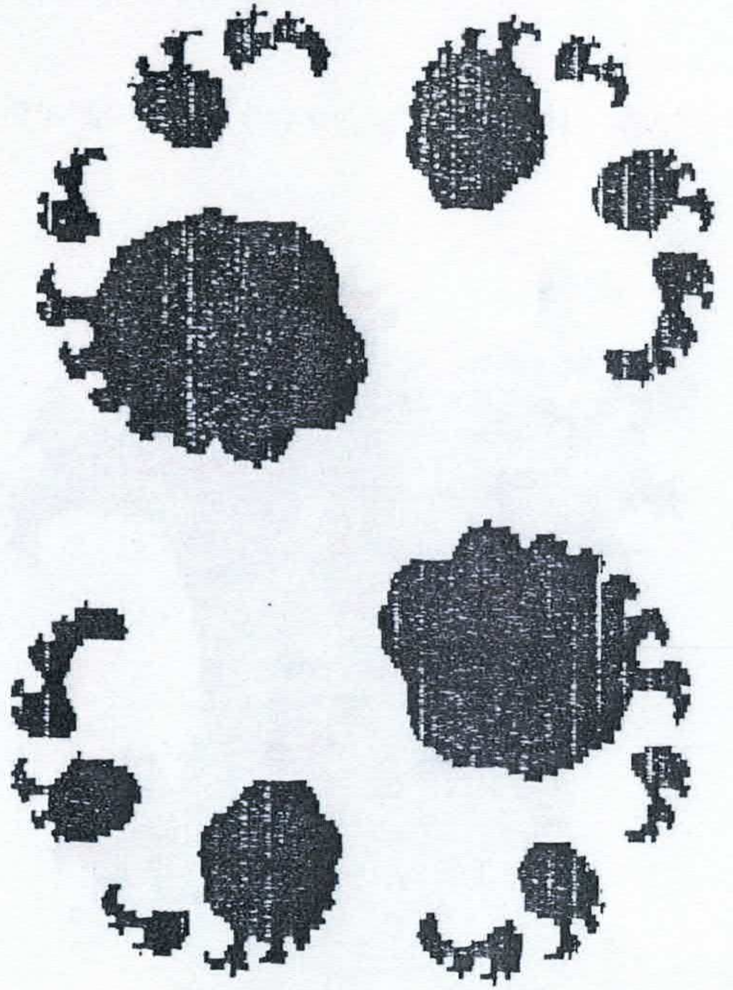

Figure 5a. $c_1 = 0.35 + i0.05$, $c_2 = 0.0 + i0.0$, $h = 100$

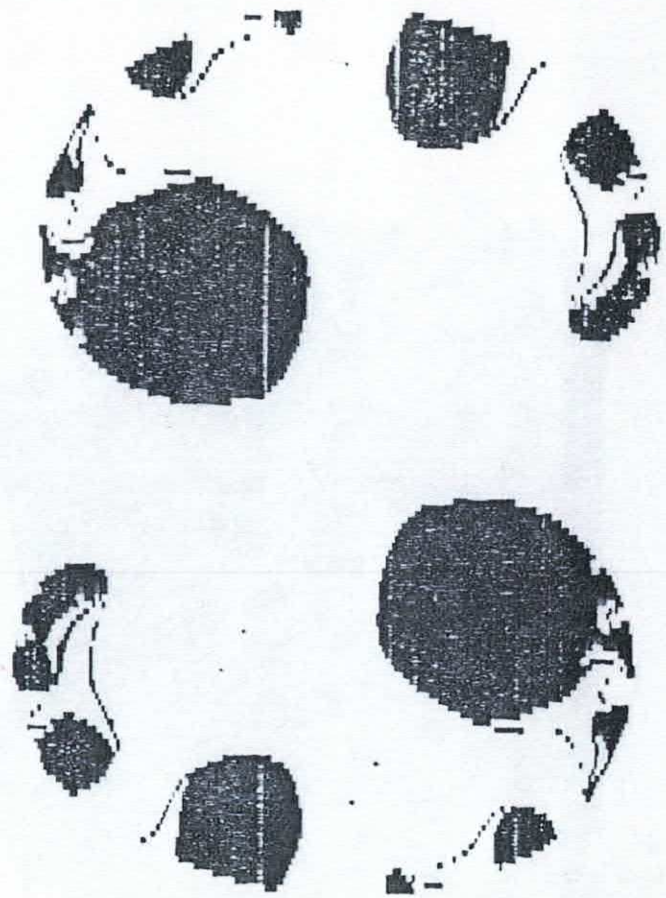

Figure 5b. $c_1 = 0.35 + i0.05$, $c_2 = 0.0 + i0.0$, $h = 1.02$

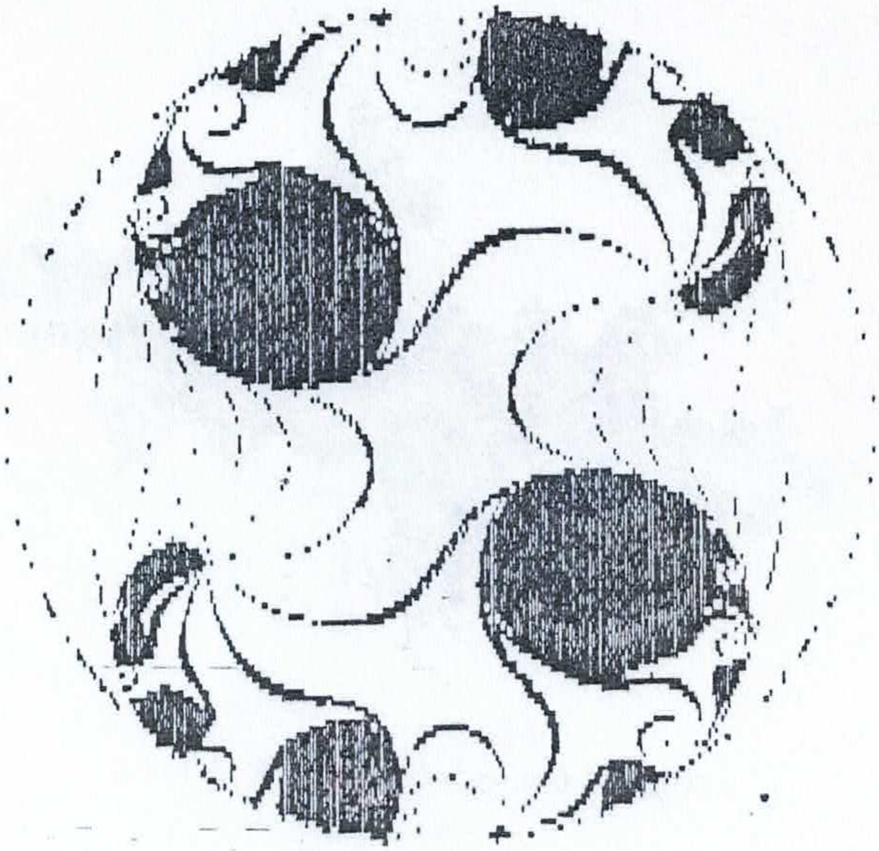

Figure 5c. $c_1 = 0.35 + i0.05$, $c_2 = 0.0 + i0.0$, $h = 0.99$

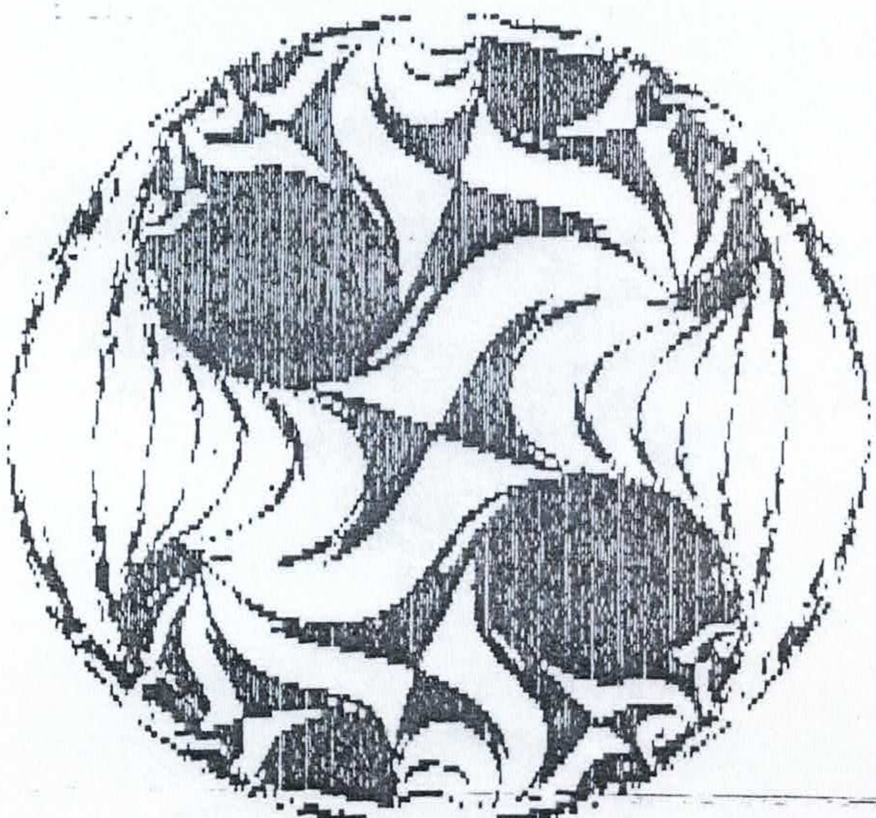

Figure 5d. $c_1 = 0.35 + i0.05$, $c_2 = 0.0 + i0.0$, $h = 0.94$

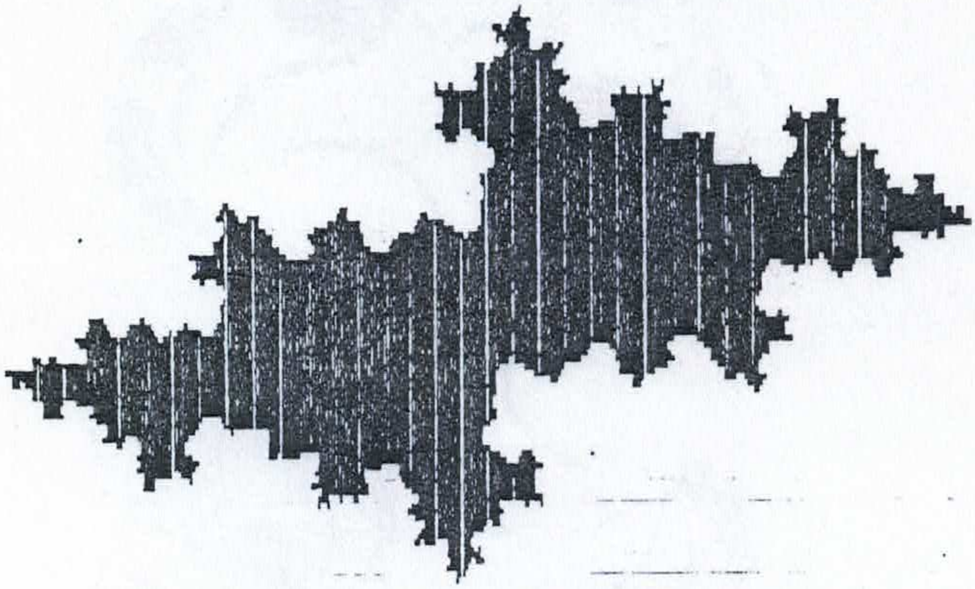

Figure 6a. $c_2 = 0.65 + i0.4$, $h = 0.0$

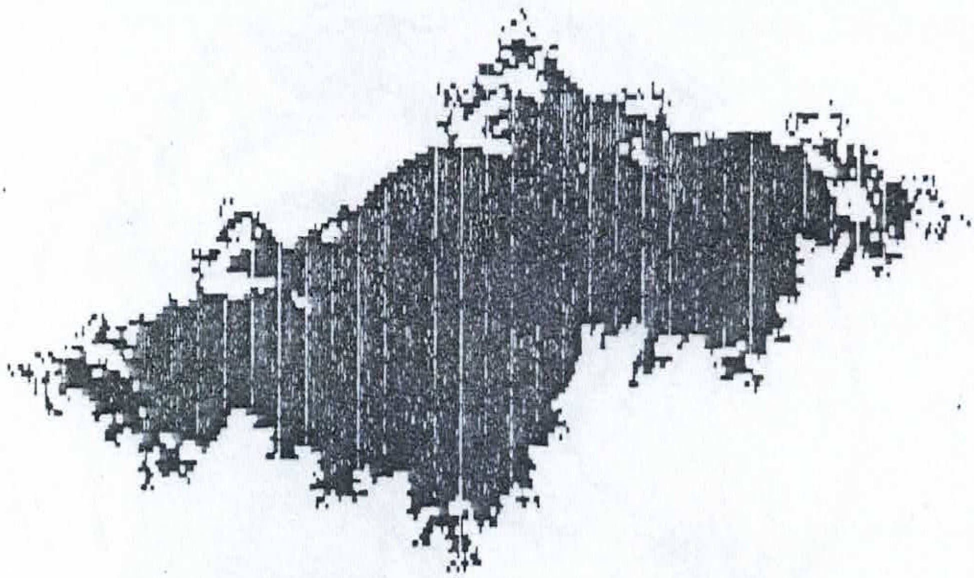

Figure 6b. $c_2 = 0.65 + i0.4$, $h = 0.5$

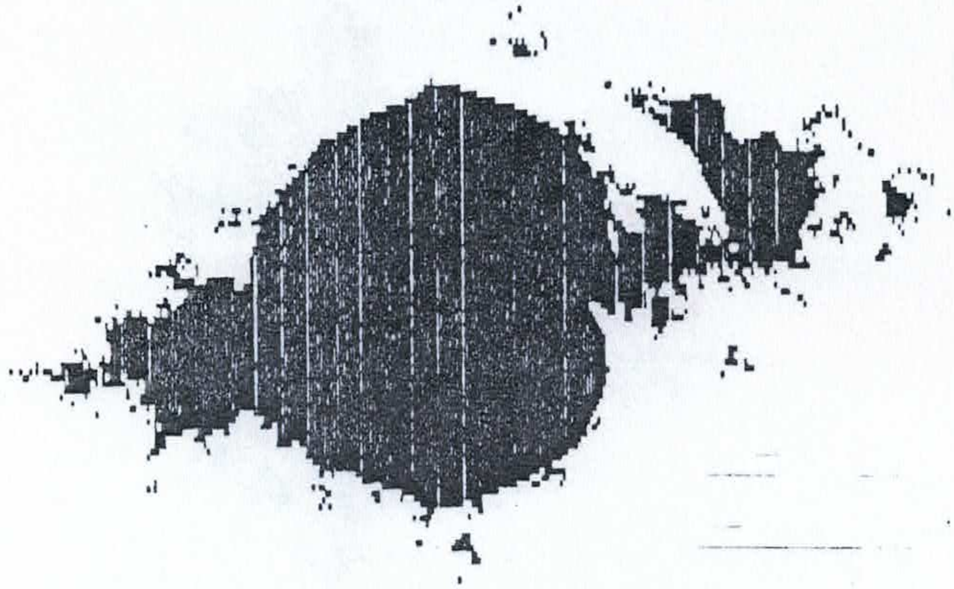

Figure 6c. $c_2 = 0.65 + i0.4$, $h = 0.75$

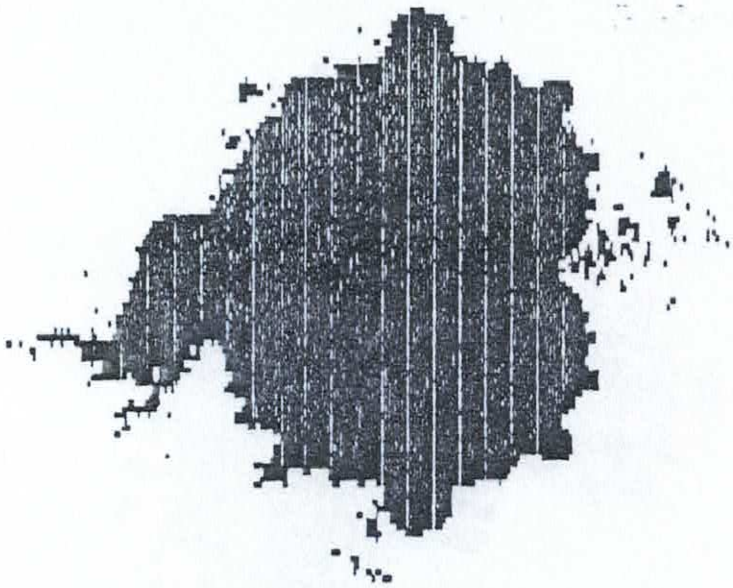

Figure 6d. $c_2 = 0.65 + i0.4$, $h = 1.0$

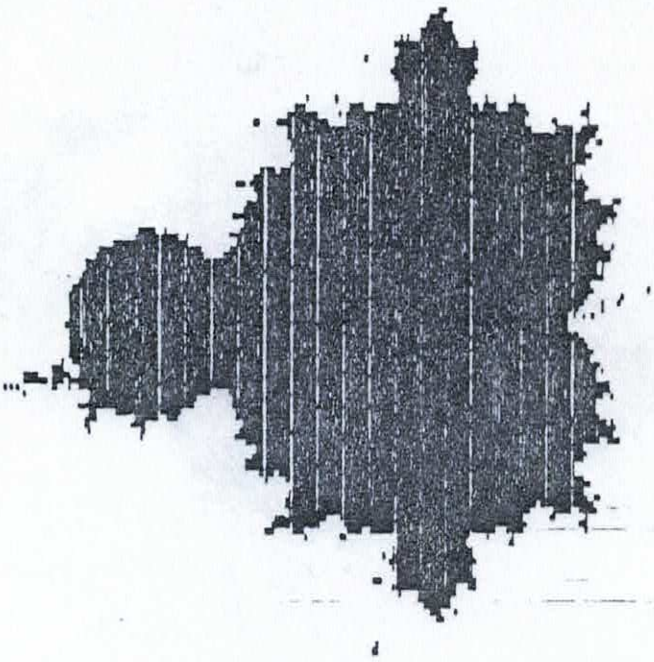

Figure 6e. $c_2 = 0.65 + i0.4,\ h = 1.25$

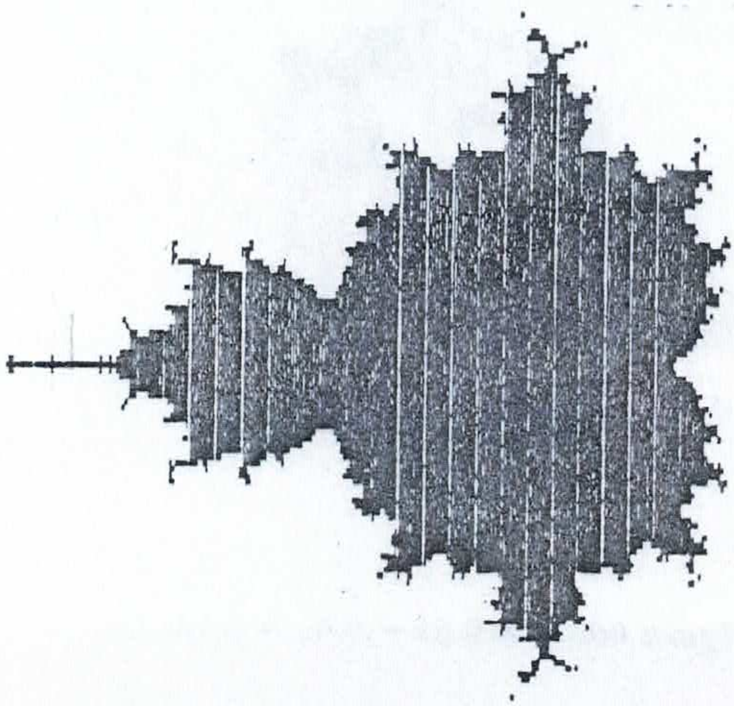

Figure 6f. $c_2 = 0.65 + i0.4,\ h = 3.0$

PREPRINTS

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