BULL. AUSTRAL. MATH. SOC. 57R45, 58C27, 58C28
VOL. 38 (1988) [161–169]

ISOTROPIC VARIETIES IN THE SINGULAR SYMPLECTIC GEOMETRY

STANISŁAW JANECZKO AND ADAM KOWALCZYK

Maximal isotropic varieties of the singular symplectic structure $x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ are characterised in terms of generating families. The normal forms of the simplest singularities (of codimension 1) are obtained with the help of the theory of boundary singularities.

1. INTRODUCTION

Many of the regular properties of physical systems have been described successfully in the symplectic geometry framework (see [**1, 9, 16**]). However the singularities of wave front evolution [**3**], critical regions phenomona [**8**] and the low-temperature thermodynamics require another approach. As a first step towards a better modelling of these peculiar phenomona we investigate the geometry of maximal isotropic submanifolds in the phase space endowed with the simplest stable singular symplectic structure

$$\sigma = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_1$$

introduced in the theory of singularities of closed 2-forms (see [**11**]).

In section 2 we give the canonical representations of maximal isotropic submanifolds (σ-*germs*) in $(\mathbb{R}^{2n}, \sigma)$ by means of generating functions. Then we obtain the σ-germs as pull-backs of Langrangian submanifolds in $\left(\mathbb{R}^{2n}, \sum_i dx_i \wedge dy_i\right)$. In section 3 we generalise the σ-germs to σ-varieties. Then we obtain the initial classification list of normal forms of the σ-varieties in terms of generating families. These results are derived in the standard singularity theory fashion with an essential use of Arnold's classification of boundary singularities [**2**].

Received 15 October,1987
We would like to express our gratitude to Dr A.J. Pryde for his help in preparation of this paper. S.J. wishes to thank Monash University for a visiting appointment.

2. Local structure of maximal isotropic manifolds

Let us consider $\mathbb{R}^{2n}$ with fixed coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$ and a 2-form $\sigma = x_1 dx_1 \wedge y_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$. A *maximal isotropic manifold* (σ*-manifold*) is defined as an immersed n-dimensional submanifold $M = \iota(\mathbb{R}^n)$ of $\mathbb{R}^{2n}$, where $\iota\colon \mathbb{R}^n \to \mathbb{R}^{2n}$ is a smooth immersion such that $\iota^*\sigma = 0$. In this section we characterise germs at $0 \in \mathbb{R}^{2n}$ of σ-manifolds. We denote them by $(M,0)$ and call them σ*-germs.* A germ $(\iota,0)$ of the immersion $\iota\colon \mathbb{R}^n \to \mathbb{R}^{2n}$ can always be written in one of the following two forms:

(1)

$$\iota\colon (x_I,y_1,y_J) \in \mathbb{R}^n \mapsto (X_1(x_I,y_1,y_J), x_I, X_J(x_I,y_1,y_J), y_1, Y_I(x_I,y_1,y_J), y_J) \in \mathbb{R}^{2n}$$

or

(2)

$$\iota\colon (x_1,x_I,y_J) \in \mathbb{R}^n \mapsto (x_1, x_I, X_J(x_1,x_I,y_J), Y_1(x_1,x_I,y_J), Y_I(x_1,x_I,y_J), y_J) \in \mathbb{R}^{2n}$$

where $X\colon \mathbb{R}^n \to \mathbb{R}^{|J|}$, $Y_I\colon \mathbb{R}^n \to \mathbb{R}^{|I|}$ and Y_1, $X_1\colon \mathbb{R}^n \to \mathbb{R}$ are smooth germs $(I \cup J = \{2,\dots,n\},\quad I \cap J = \emptyset)$. Using the results of [**2, 16**], we obtain,

Proposition 2.1. *A σ-germ, $(M,0)$, can be represented by at least one of the following systems of equations:*

$$\begin{aligned} \frac{1}{2}x_1^2 &= \frac{\partial F}{\partial y_1}(y_1,x_I,y_J) \\ y_I &= \frac{\partial F}{\partial x_I}(y_1,x_I,y_J) \\ -x_J &= \frac{\partial F}{\partial y_J}(y_1,x_I,y_J) \end{aligned} \tag{3}$$

or

$$\begin{aligned} x_1 y_1 &= \frac{\partial F}{\partial x_1}(x_1,x_I,y_J) \\ y_I &= \frac{\partial F}{\partial x_I}(x_1,x_I,y_J) \\ -x_J &= \frac{\partial F}{\partial y_J}(x_1,x_I,y_J) \end{aligned} \tag{4}$$

where F is a germ of smooth function on $\mathbb{R}^n$ and $I \cup J = \{2,\dots,n\}$, $I \cap J = \emptyset$.

A σ-germ having representation (3) is called *regular*. A diffeomorphism $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserving the 2-form σ and the fibration $\pi\colon \mathbb{R}^{2n} \to \mathbb{R}^n$, $(x,y) \to x$ is called a σ-*equivalence*.

LEMMA 2.2. *Any σ-germ is σ-equivalent to a regular σ-germ.*

PROOF: If an immersion $\iota\colon \mathbb{R}^n \to \mathbb{R}^{2n}$ is not regular, it has representation (1) with

$$\frac{\partial Y_1}{\partial x_1}(0) = 0. \tag{5}$$

In this case its composition with the σ-equivalence $\phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $(x,y) \to (x, x+y)$ is regular (since it has a representation of the form (1) but not satisfying (5)). ∎

Let us now consider a symplectic form $\omega \overset{\text{def}}{=} \sum dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$. We recall some basic notions of the standard theory of Lagrangian singularities [**2**, **15**]. A symplectomorphism of $(\mathbb{R}^{2n}, \omega)$ preserving the fibration π is called a *Lagrangian equivalence (L-equivalence).* An L-equivalence preserving the hyperplane $\{x_1 = 0\}$ will be called *restricted (rL-equivalence).* An n-dimensional immersed submanifold $\iota\colon \mathbb{R}^n \to \mathbb{R}^{2n}$ such that $\iota^*\omega = 0$ is called *Lagrangian*; in such a case the germ $(L,0)$, $L \overset{\text{def}}{=} \iota(\mathbb{R}^n)$, will be called an *L-germ.*

The transformation

$$\rho\colon (x,y) \in \mathbb{R}^{2n} \mapsto \left(\frac{1}{2}x_1^2, x_2, \dots, x_n, y_1, \dots, y_n\right) \in \mathbb{R}^{2n} \tag{6}$$

preserves the fibration π and satisfies the condition

$$\rho^*\omega = \sigma. \tag{7}$$

Obviously ρ is not a unique transformation with these properties. For example its composition with any Lagrangian equivalence of $(\mathbb{R}^{2n}, \omega)$ has the same properties.

PROPOSITION 2.3.

(i) *For any rL-equivalence Φ of $(\mathbb{R}^{2n}, \omega)$ there exists a σ-equivalence ϕ making the following diagram commutative:*

$$\begin{array}{ccc} \mathbb{R}^{2n} & \xrightarrow{\Phi} & \mathbb{R}^{2n} \\ \uparrow{\rho} & & \uparrow{\rho} \\ \mathbb{R}^{2n} & \xrightarrow{\phi} & \mathbb{R}^{2n} \end{array} \tag{8}$$

(ii) *$(\rho(L), 0)$ is an L-germ for any regular σ-germ $(L,0)$.*

PROOF:

(i) For any *rL-equivalence* Φ we have $\Phi(x,y) = (X_i,(x), Y_i(x,y))$, where $X_1(x) = x_1(a + \alpha(x))$, $0 \neq a \in \mathbb{R}$ and $\alpha \in \mathbf{m}_x^2$. A diffeomorhpism ϕ

satisfying diagram (8) and preserving the fibration π, can be defined as follows:

$$\phi(x,y) \stackrel{\text{def}}{=} \left(x_1\sqrt{a+\alpha(\xi)}, X_2(\xi), \dots, X_n(\xi), Y_1(\xi,y), \dots, Y_n(\xi,y)\right)\big|_{\xi=\left(\frac{1}{2}x_1^2, x_2, \dots, x_n\right)}.$$

For such ϕ we have $\phi^\star\sigma = \phi^\star\rho^\star\omega = \rho^\star\Phi^\star\omega = \rho^\star\omega = \sigma$ (see 7).

(ii) follows directly from equation (3). ∎

Example 2.4. For a regular σ-germ $(M,0)$, $M = \{(t,t)\}$, the set $L \stackrel{\text{def}}{=} \rho(M)$ is the parabola $x = y^2$. Its pre-image is given by the equation $x^2 - y^2 = 0$. It contains M as one of two smooth branches. $\rho^{-1}(L)$ is a symmetrisation (with respect to reflection in the y-axis) of this branch. On the basis of Proposition 2.1 we can easily calculate the generating function for L: $F(y) = \frac{1}{3}y^3$.

3. Modified classification of Lagrangian varieties

It is well known [**2**, **15**] that an L-germ $(L,0)$ in $(\mathbf{R}^{2n},\omega)$ is generated by the germ $(F,0)$ of a *Morse family*, that is, it is given by the equations

$$y = \frac{\partial F}{\partial x}(\lambda,x), \qquad 0 = \frac{\partial F}{\partial \lambda}(\lambda,x), \tag{9}$$

where $F(\lambda,x) \in C^\infty(\mathbf{R}^k \times \mathbf{R}^n)$ and

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial \lambda^2}, \frac{\partial^2 F}{\partial\lambda\partial x}\right)\Big|_0 = \max = k. \tag{10}$$

By dropping requirement (10) we generalise the notion of Morse family to *generating family* [**9**, **7**]. By applying equations (9) to the generating family we obtain a Lagrangian variety (*L-variety*) which is not necessarily a smooth submanifold of $\mathbf{R}^{2n}$. (Such L-varieties appeared naturally in Arnold's theory of singularities of systems of rays [**3**].) In the generic case, when the generating family F is polynomial, the corresponding L-variety is stratifiable with all strata isotropic and maximal strata Lagrangian [**9**, **6**]. Two generating families $(F_i,0)$, $F_i(\lambda,x) \in C^\infty(\mathbf{R}^k \times \mathbf{R}^n)$, $i = 1,2$, are called *equivalent* if there exists a diffeomorphism

$$\Phi\colon (\mathbf{R}^k \times \mathbf{R}^n, 0) \to (\mathbf{R}^k \times \mathbf{R}^n, 0), \qquad (\lambda,x) \mapsto (\Lambda(\lambda,x), X(x))$$

and a smooth function $f \in C^\infty(\mathbf{R}^n)$ such that

$$F_2(\Lambda(\lambda,x), X(x)) = F_1(\lambda,x) + f(x) \tag{11}$$

near $0 \in \mathbb{R}^k \times \mathbb{R}^n$. The equivalence of generating families which preserves the hyperplane $\{x_1 = 0\}$ will be called *restricted (r-equivalence)*. For r-equivalences the first coordinate of X is divisible by x_1, that is

$$X_1(x) = x_1(\alpha + \phi(x)), \tag{12}$$

where $\alpha =$ const $\neq 0$ and $\phi \in \mathbf{m}(n)$. By straightforward calculation we obtain:

PROPOSITION 3.1. *Two L-varieties generated by r-equivalent generating families are rL-equivalent.*

Remark 3.2. For Morse familes and L-germs the converse is true. From [**16**, **2**] it follows that any two L-equivalent L-germs have equivalent minimal Mores families (that is Morse families $F_i(\lambda, x)$ such that $\partial^2 F_1/\partial\lambda^2|_0 = 0$).

We recall [**2**, **5**] that a generating family $(F(\lambda, x), 0)$, $(\lambda, x) \in \mathbb{R}^k \times \mathbb{R}^n$, is *versal* if any other generating family $(F'(\lambda, x'), 0)$, $(\lambda, x') \in \mathbb{R}^k \times \mathbb{R}^{n'}$ such that $F'|_{x'=0} = F|_{x=0}$ is induced from F, that is there exists a mapping

$$(\lambda, x') \in \mathbb{R}^k \times \mathbb{R}^{n'} \mapsto (\Lambda(\lambda.x'), X(x')) \in \mathbb{R}^k \times \mathbb{R}^n \tag{13}$$

and a function $f\colon \mathbb{R}^{n'} \to \mathbb{R}$ such that

$$F'(\lambda, x') = F(\Lambda(\lambda, x'), X(x')) + f(x').$$

(Classifications of versal families can be found in [**12**, **10**].)

For the purposes of this paper it seems natural to consider *restricted versality* by imposing on the inducing mappings (13) a requirement of preservation of distinguished hyperplanes, that is in the case of hyperplanes $\{x_1 = 0\}$ and $\{x_1' = 0\}$, by assuming $X(\{x_1' = 0\}) \subset \{x_1 = 0\}$. This requirement means that X_1, the first coordinate of X, is of the form (12). The following result reduces the restricted versality to ordinary versality.

PROPOSITION 3.3. *A family $(F(\lambda, x), 0)$ is restricted versal if and only if the family $(F(\lambda, x)|_{x_1=0}, 0)$ is versal.*

PROOF: $\Longleftarrow$. Assume $(F(\lambda, x)|_{x_1=0}, 0)$, $(\lambda, x) \in \mathbb{R}^k \times \mathbb{R}^n$ is a versal family and $(F'(\lambda, x'), 0)$, $(\lambda, x') \in \mathbb{R}^k \times \mathbb{R}^m$ is such that $F'(\lambda, 0) = F(\lambda, 0)$. Then $(\lambda, x') \mapsto (\Lambda(\lambda, x'), 0, X_2(\lambda, x'), \ldots, X_n(\lambda, x'))$ is the demanded morphism.

$\Longrightarrow$. Following the standard lines of versality theory [**4**, **13**] for restricted versality we obtain the following necessary condition:

$$\left\langle \frac{\partial F}{\partial \lambda} \right\rangle_{\mathcal{E}_{\lambda_x}} + \left\langle x_1 \frac{\partial f}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots, \frac{\partial F}{\partial x_n}, 1 \right\rangle_{\mathcal{E}_x} = \mathcal{E}_{\lambda_x}.$$

Factorising by $\mathbf{m}_x\mathcal{E}_{\lambda_x}$ we get the following condition of infinitesimal versality for $F|_{x_1=0}$:

$$\left\langle \frac{\partial F}{\partial \lambda}|_{x=0} \right\rangle_{\mathcal{E}_\lambda} + \left\langle \frac{\partial F}{\partial x_2}|_{x=0}, \dots, \frac{\partial F}{\partial x_n}|_{x=0}, 1 \right\rangle_{\mathbf{R}} = \mathcal{E}_\lambda.$$

As is well known this condition implies varsality of $F|_{x_1=0}$ [**2, 4, 11**]. ∎

In the case when the vector space $\mathcal{E}_\lambda / \langle \frac{\partial F}{\partial \lambda}(\lambda, x)|_{x=0} \rangle_{\mathcal{E}_\lambda}$ has a finite number of generators, say $\{e_1(\lambda), \dots, e_m(\lambda), 1\}$, we have the decomposition

$$F(\lambda, x) = F(\Lambda(\lambda, x), 0) + \sum_{i=1}^{m} e_i \circ \Lambda(\lambda, x) u_i(x) + f(x)$$

for some smooth $u = (u_1, \dots, u_m) \colon \mathbf{R}^n \to \mathbf{R}^m$ and $f \colon \mathbf{R}^n \to \mathbf{R}$ [**4, 14**], where $\Lambda \colon \mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}^k$, $\Lambda|_{\mathbf{R}^k \times \{0\}} = id_{\mathbf{R}^k}$. From Proposition 3.3 we find that any other r-equivalent family $(F', 0)$ has the form

$$F'(\lambda, x) = F(\Lambda(\lambda, x), 0) + \sum_{i=1}^{m} e_i(\Lambda(\lambda, x)) u'_i(x) + f(x),$$

where $\Lambda|_{\mathbf{R}^k \times \{0\}}$ is a diffeomorphism of $(\mathbf{R}^k, 0)$ and u' makes the following diagram commutative:

$$\text{(14)} \qquad \begin{array}{ccc} (\mathbf{R}^n, \{x_1 = 0\}, 0) & \xrightarrow{\;u\;} & (\mathbf{R}^m, 0) \\ \big\downarrow \phi & & \big\uparrow u' \\ (\mathbf{R}^n, \{x_1 = 0\}, 0) & = \!= \!= & (\mathbf{R}^n, \{x_1 = 0\}, 0) \end{array}$$

Here ϕ is a diffeomorphism preserving the hyperplane $\{x_1 = 0\}$. It is apparent that r-equivalence classes of generating families $(F(\lambda, x), 0)$ are parametrised by singularities of $F|_{x=0}$ and equivalence classes of mappings u in the sense of diagram (14) (we call them $\mathcal{A}_r$*-equivalences*). In this context it is natural to introduce the following characteristics of F: (i) *codimension of* $(F, 0)$, $\operatorname{codim} F \stackrel{\text{def}}{=} \dim \left(\mathcal{E}_\lambda / \langle \frac{\partial F}{\partial \lambda}(\lambda, x)|_{x=0} \rangle_{\mathcal{E}_\lambda}\right)$ and (ii) *corank of* $F = m - \operatorname{rank}\left(\frac{\partial \widetilde{u}}{\partial x}\right)|_{x=0}$, where $\widetilde{u} \colon \mathbf{R}^n \to \mathbf{R}^m$ is assumed to be such that F is induced via a pull-back $\left(\widetilde{\Lambda}, \widetilde{u}\right)$ from a universal unfolding $\widetilde{F}$ of $F|_{x=0}$. It is easily seen that these two characteristics are invariants of r-equivalences. Now using Arnold's classification methods [**3**] we obtain lists of normal forms for some simplest r-equivalence classes. We consider here the simplest case of codim $= 1$. The case of codim $= 2$ and 3 will be considered subsequently in the forthcoming paper.

PROPOSITION 3.4. *The list of simple normal forms of r-equivalence classes of generating families $F(\lambda, x)$, $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ of codimension 1 is the following:*

$$
\begin{array}{lll}
A_2A_0^0: & \lambda^3 + x_2\lambda; & \\
A_2A_k^0: & \lambda^3 + \left(\pm x_2^{k+1} \pm x_1 + q\right)\lambda, & k \geqslant 1; \\
A_2D_k^0: & \lambda^3 + \left(x_2x_3^2 \pm x_2^{k-1} \pm x_1 + q\right)\lambda, & k \geqslant 4; \\
A_2E_6^0: & \lambda^3 + \left(x_2^3 \pm x_3^4 \pm x_1 + q\right)\lambda; & \\
A_2E_7^0: & \lambda^3 + \left(x_2^3 + x_2x_3^3 \pm x_1 + q\right)\lambda; & \\
A_2E_8^0: & \lambda^3 + \left(x_2^3 + x_3^5 \pm x_1 + q\right)\lambda; & \\
A_2B_k^1: & \lambda^3 + \left(\pm x_1^k + x_2^2 + q\right)\lambda, & k \geqslant 2; \\
A_2C_k^1: & \lambda^3 + \left(x_1x_2 \pm x_2^k + q\right)\lambda, & k \geqslant 2; \\
A_2F_4^1: & \lambda^3 + \left(\pm x_1^2 + x_2^3 + q\right)\lambda; &
\end{array}
$$

where q is a non-degenerate quatratic form of the remaining variables.

PROOF: Up to an r-equivalence we have

$$F(\lambda, x) = \lambda^3 + \lambda u(x),$$

where $u: \mathbb{R}^n \to \mathbb{R}$. Using the list of simple normal forms of singularities of u on the manifold $\{x_1 \geqslant 0\} \subset \mathbb{R}^n$ with boundary $\{x_1 = 0\}$ [**2**, Sec. 17.4] we obtain the above classification. ∎

Remark 3.5.

(i) In the above list $A_2A_0^0$ is the only restricted versal family.

(ii) Families $A_2A_k^0$, $A_2D_k^0$ and $A_2E_i^0$ are Morse families while $A_2B_k^1$, $A_2C_k^1$ and $A_2F_4^1$ are not (and provide L-varieties which are not manifolds).

(iii) Generating families $\left(\widetilde{F}(\lambda, x), 0\right)$, $(\lambda, x) \in \mathbb{R}^k \times \mathbb{R}^n$, $k \geqslant 2$ with $\widetilde{F}|_{x=0}$ having singularity A_2 have simple normal forms $F(\lambda_1, x) + Q(\lambda_2, \ldots, \lambda_k)$, where F has one of the normal forms in Proposition 3.4 and Q is a non-degenerate quadratic form. Obviously $\widetilde{F}$ and F generate the same L-variety.

We define a *σ-variety* as a ρ pull-back (see [**6**]) of a L-variety in $\mathbb{R}^{2n}$. Having a generating family $(F(\lambda, x), 0)$, $(\lambda, x) \in \mathbb{R}^m \times \mathbb{R}^n$ for the L-variety, we obtain the following equations for the corresponding σ-variety V_F:

$$y_1 = \frac{\partial F}{\partial \xi_i}\left(\lambda, \frac{1}{2}x_1^2, x_2, \ldots, x_n\right),$$
$$0 = \frac{\partial F}{\partial \lambda}\left(\lambda, \frac{1}{2}x_1^2, x_2, \ldots, x_n\right).$$

Directly from Proposition 2.1 and the existence theorem for Morse familes (for example [**2**, **16**]) we obtain:

PROPOSITION 3.6. *For any regular σ-germ, $(\Sigma, 0)$, there exists a generating family $(F, 0)$ on $\mathbb{R}^m \times \mathbb{R}^n$ such that*

$$\Sigma^{sym} \stackrel{\text{def}}{=} \{(\pm x_1, x_2, \ldots, x_n, y)\ ;\ (x, y) \in \Sigma\} = V_F \text{ near } 0 \in \mathbb{R}^{2n}.$$

From Lemma 3.7 and Proposition 2.3 follows immediately:

PROPOSITION 3.7. *Two σ-varieties corresponding to r-equivalent generating families are σ-equivalent.*

The above results show that the local classification of σ-germs is subordinate to the classification of σ-varieties, and subsequently to the classification of generating families up to r-equivalences (described in Section 3).

THEOREM 3.8. *Initial classification of generic σ-varieties is provided by the classification list of generating families in Proposition 3.4.*

REFERENCES

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics* (Benjamin/Cummings, Reading, 1978).

[2] V.I. Arnold, S.M. Gusein-Zade, and A.N. Varchenko, *Singularities of Differentiable Maps* **1**, Engl. ed. (Birkhauser, Boston, 1985).

[3] V.I. Arnold, 'Singularities of systems of rays', *Russian Math. Surveys* **38** (1983), 87–176.

[4] Th. Bröcker and L. Lander, *Differentiable Germs and Catastrophes* (Cambridge University Press, Cambridge, 1975).

[5] J.J. Duistermaat, 'Oscilatory integrals, lagrange immersions and unfoldings of singularities', *Comm. Pure Appl. Math.* **27** (1974), 207–281.

[6] S. Janeczko, 'Constrained Lagrangian submanifolds over singular constraining varieties and discriminant varieties', *Ann. Inst. H. Poincaré, Sect A. (N.S.)* **46** (1987), 1–25.

[7] S. Janeczko, 'Generating families for images of lagrangian submanifolds and open swallowtails', *Math. Proc. Cambridge Phil. Soc.* **10** (1986), 91–107.

[8] S. Janeczko, 'Geometrical approach to phase transitions and singularities of Lagrangian submanifolds', *Demonstratio Math.* **16** (1983), 487–502.

[9] S. Janeczko, 'On singular lagrangian submanifolds and thermodynamics', *Ann. Soc. Sci. Bruxelles Ser. I.* **99** (1985), 49–83.

[10] J. Martinet, *Singularities of Smooth Functions and Maps* (Cambridge Univ. Press, Cambridge, 1982).

[11] J. Martinet, 'Sur les singularites des formes differentielles', *Ann. Inst. Fourier (Grenoble)* **20** (1970), 95–178.

[12] T. Poston and I. Stewart, *Catastrophe Theory and its Applications* (Pitman, San Francisco, 1978).

[13] R. Thom, *Structural Stability and Morphogenesis* (Benjamin, New York, 1975).

[14] C.T.C. Wall, 'Geometric properties of generic differentiable manifolds', in *Geometry and Topology: Lecture Notes in Math 597*, A. Dold and B. Eckmann, editors, pp. 707–774 (Springer-Verlag, 1977).

[15] A. Weinstein, *Lectures on Symplectic Manifolds* (CBMS Regional Conf. Ser. in Math., 1977).

[16] V.M. Zakalyukin, 'Lagrangian and Legrendrian singularities', *Functional Anal. Appl.* **10** (1976), 23–31.

Dr S. Janeczko
Department of Mathematics
Monash University
Clayton, Vic. 3168
Australia
and
Institute of Math
Technical University of Warsaw
Pl. Jednošci Robotniczej 1
00661 Warsaw
Poland

Dr A. Kowalczyk,
Telecom Australia,
Research Laboratories,
770 Blacburn Rd.,
Clayton, Vic. 3168