# Lagrangian submanifolds in product symplectic spaces

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We analyze the global structure of Lagrangian Grassmannian in the product symplectic space and investigate the local properties of generic symplectic relations. The cohomological symplectic invariant of discrete dynamical systems is generalized to the class of generalized canonical mappings. Lower bounds for the number of two-point and three-point symplectic invariants for billiard-type dynamical systems are found and several examples of symplectic correspondences encountered from physics are presented. © *2000 American Institute of Physics.*
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## I. INTRODUCTION

Let $(M,\omega)$ be a symplectic manifold. We consider the product $M\times M$ endowed with the symplectic structure $\Omega=\pi_2^*\omega-\pi_1^*\omega$, where $\pi_i$ are the corresponding projections onto the components of $M\times M$. The space of Lagrangian submanifolds of $(M\times M,\Omega)$ is a natural generalization of the group of symplectic transformations of $(M,\omega)$. We notice that if $\phi:(M,\omega)\rightarrow(M,\omega)$ is a symplectomorphism, then its graph, graph $\phi\subset M\times M$ is the Lagrangian submanifold, $\Omega|_{\mathrm{graph}\ \phi}=0$. There is an obvious motivation to study the global and local structure of such Lagrangian submanifolds, which are also called symplectic relations or symplectic correspondences (cf. Ref. 1). They are coming from various branches of mathematics in which the symplectic ideas and methods were succesfully applied (cf. Refs. 1–5).

The very elementary examples of symplectic relations, which are not the graphs of symplectomorphisms, play an important role in geometrical diffraction theory (Ref. 6). Consider the set $M$ of all oriented affine lines in $\mathbb{R}^3$. $M$ is a four-dimensional symplectic manifold, $M\equiv T^*S^2$, constructed by the symplectic reduction from the free particle Hamiltonian hypersurface (cf. Ref. 7). This is the space of rays of geometrical optics. One of the most classical systems that provide the nontrivial symplectic relation is a billiard system. If $V$ is a smooth compact, convex region in $\mathbb{R}^n$ and $X$ denotes its boundary hypersurface, then the symplectic relation joining the incoming (going through the interior of the region) and outgoing rays, by the reflection law $y=x-2(x|\mathbf{n})\mathbf{n}$ (where $\mathbf{n}$ is the unit outer normal to $X$, and $(\bullet|\bullet)$ denotes the scalar product in $\mathbb{R}^n$), is a graph of symplectomorphism called the billiard map (cf. Ref. 8). However, if $V$ is no longer convex, then the reflection law should be extended by the diffraction role (cf. Ref. 6), which prescribe to one incoming ray the family of outgoing rays gliding in the tangential point of an incoming ray. The billiard symplectic relation is no more the graph of a symplectomorphism. In the case of incoming ray, say $x$, going through the one-dimensional edge of an aperture in $\mathbb{R}^3$, the outgoing rays $y$ form a cone defined by the equations $(x-y|\gamma)=0$, $|x|=|y|$, where $\gamma$ is a vector tangent to an edge oriented according to the incoming ray orientation. Our aim in this paper is to provide the geometric framework for the action of generalized mechanical (like nonconvex billiard) and optical (like diffraction on apertures) systems.

An important object in investigation of geometry of Lagrangian submanifolds is the manifold $\Lambda_n$ of linear Lagrangian spaces in $2n$-dimensional symplectic space, called the Lagrangian Grassmannian (cf. Ref. 9). Its natural stratification, allowing us to indicate the global structure of

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Lagrangian submanifolds and their singularities, is constructed as follows.

Let us fix $\alpha \in \Lambda_n$. By $\Lambda_n^\alpha$, we denote the set of all Lagrangian subspaces in $\Lambda_n$ that are not transversal to $\alpha$. We have $\Lambda_n^\alpha = \cup_{k=1}^n \Lambda_{n,k}^\alpha$,

$$\Lambda_{n,k}^\alpha = \{\beta \in \Lambda_n ; \dim(\beta \cap \alpha) = k\}$$

and

$$\dim \Lambda_{n,k}^\alpha = \frac{n(n+1)}{2} - \frac{k(k+1)}{2} \geqslant 3, \tag{1}$$

if $k > 1$.

$\Lambda_n^\alpha$ may be oriented by choosing the vectors from $T_{\Lambda_n^\alpha}\Lambda_n$ transversal to $\Lambda_n^\alpha$, formed by symmetric bilinear quadratic forms on elements $\lambda \in \Lambda_n^\alpha$, which are positive definite on $\alpha \cap \lambda$. So $\Lambda_n^\alpha$ with this orientation represents a singular cycle that is Poincaré dual to the universal Maslov class (cf. Ref. 10).

Now we pose the following problem: *Does there exists a similar (to that of the standard classification of Lagrangian singularities in a cotangent bundle) classification of Lagrangian submanifolds and their singularities in the product symplectic space exploiting the canonical product structure?* Approaching the answer for this question we investigate the canonical stratification (i.e., a partition into smooth submanifolds as it was done for $\Lambda_n$ above) of the Lagrangian Grassmannian $\Lambda_{2n}$ in the product symplectic space, induced by the product structure. This stratification naturally appears in the theory of linear symplectic relations and is especially important for searching the geometric structure of the images by symplectic relations (cf. Refs. 7 and 11). In Sec. II we prove that any linear symplectic relation in the product space is a composition of reduction relation and a symplectomorphism. By this decomposition property $\Lambda_{2n}$ is stratified and the codimension formulas are calculated. In Sec. III, the first step into the theory of classification of germs of nonlinear symplectic relations is done, and the generic appearance of some singular points is proved. In the last section the action of $\Lambda_{2n}$ onto elements of $\Lambda_n$ is considered in the framework of unitary group $U(2n)$ and homogeneous space $U(2n)/O(2n)$ representing $\Lambda_{2n}$ (cf. Refs. 10 and 9). In the nonlinear case of symplectic relations the iterational cohomological symplectic invariant was introduced and its cohomological properties were described. By the Morse theory approach (cf. Refs. 12 and 13), the number of two-point and three-point (defined on two-point and three-point periodic orbit of symplectic relation) symplectic invariants were estimated from below for a possibly nonconvex billiard system and systems of equally charged particles on the surface. To conclude, we note that this paper had its origin in an attempt to find the possible complete classification of symplectic invariants by action of generalized symplectic mappings (cf. Ref. 7). The results here show that there is an open area for such invariants with applications to classical physical problems.

## II. LAGRANGIAN GRASSMANNIAN IN THE PRODUCT SYMPLECTIC SPACE

Now we consider the linear product symplectic space,

$$\mathcal{M} = (M \times M, \pi_2^* \omega - \pi_1^* \omega),$$

where $(M,\omega)$ is a $2n$-dimensional symplectic vector space. By $\Lambda_{2n}$ we denote the Lagrangian Grassmannian of linear subspaces in $\mathcal{M}$. By $M_1$ and $M_2$ we denote the symplectic spaces canonically placed in $\mathcal{M}$, $M_1 = M \times \{0\}$, $M_2 = \{0\} \times M$. Equivalently, we write

$$(M_1 \times M_2, \pi_2^* \omega_2 - \pi_1^* \omega_1),$$

for $\mathcal{M}$, where

$$\omega_2=\pi_2^*\omega-\pi_1^*\omega|_{\{0\}\times M}, \quad -\omega_1=\pi_2^*\omega-\pi_1^*\omega|_{M\times\{0\}}.$$

At first, we have the natural decomposition.

*Lemma II.1: If $L\in\Lambda_{2n}$, then $L$ is transversal to $M_1$ and to $M_2$ simultaneously, or $L$ is not transversal to $M_1$ and $L$ is not transversal to $M_2$.*

*Proof:* If $L$ is transversal to $M_2$ then it may be parametrized by $M_1$ so $L$ is a graph of a maximal rank symplectic mapping $M_1\to M_2$ and so has to be transversal to $M_1$ (one can replace $M_2$ by $M_1$ in this argument). If $L$ is not transversal to $M_1$, then assuming that $L$ is transversal to $M_2$ on the basis of the previous argument we get the transversality of $L$ to $M_1$, which contradicts to our assumption. □

By the critical subset of $\Lambda_{2n}$ we denote the set $C\Lambda_{2n}$ of those Lagrangian subspaces of $\mathcal{M}$, which are not transversal simultaneously to both subspaces $M_1$ and $M_2$:

$$C\Lambda_{2n}=\{L\in\Lambda_{2n}:L \text{ is not transversal to } M_1 \text{ and } L \text{ is not transversal to } M_2\}.$$

Elements of $C\Lambda_{2n}$ cannot be obtained as the graphs of linear symplectic transformations between $M_1$ and $M_2$.

By supercritical set of $\Lambda_{2n}$ we denote the Cartesian product,

$$S\Lambda_{2n}=\Lambda_n\times\Lambda_n.$$

The elements of this set are Lagrangian subspaces $L=(W_1,W_2)$, where $W_1$ and $W_2$ are Lagrangian subspaces in $(M_1,\omega_1)$ and $(M_2,\omega_2)$, respectively.

By the formula $\dim\Lambda_n=n(n+1)/2$, we find

$$\operatorname{codim} S\Lambda_{2n}=n^2.$$

If $R_1\subset(M_1\times M_2,\pi_2^*\omega_2-\pi_1^*\omega_1)$ is a Lagrangian subspace (linear symplectic relation), then we define the corresponding transpose Lagrangian subspace $R_1^t$ in $(M_2\times M_1,\pi_1^*\omega_1-\pi_2^*\omega_2)$,

$$R_1^t=\{(v_2,v_1)\in M_2\times M_1;(v_1,v_2)\in R\}.$$

If we have another Lagrangian subspace, say $R_2$ in the product space $(M_2\times M_3,\pi_3^*\omega_3-\pi_2^*\omega_2)$, then we define the composition of $R_1$ and $R_2$, $R_2\circ R_1$ as the following Lagrangian subspace:

$$R_2\circ R_1=\{(v_1,v_3)\in M_1\times M_3;\exists_{v_2\in M_2}(v_1,v_2)\in R_1,(v_2,v_3)\in R_2\},$$

in the product symplectic space $(M_1\times M_3,\pi_3^*\omega_3-\pi_1^*\omega_1)$.

*Proposition II.1: If $L\in C\Lambda_{2n}$ then $L$ has the following decomposition:*

$$L=R_2^t\circ\tilde{L}\circ R_1,$$

*where $\tilde{L}$, $R_1$, $R_2$ are linear Lagrangian subspaces:*

$$\tilde{L}\subset(\tilde{M}_1\times\tilde{M}_2,\pi_2^*\tilde{\omega}_2-\pi_1^*\tilde{\omega}_1),\quad R_1\subset(M_1\times\tilde{M}_1,\pi_2^*\tilde{\omega}_1-\pi_1^*\omega),$$

$$R_2\subset(M_2\times\tilde{M}_2,\pi_2^*\tilde{\omega}_2-\pi_1^*\omega),$$

*and $R_1$, $R_2$ are graphs of projections $\rho_1$ and $\rho_2$ onto $\tilde{M}_1$ and $\tilde{M}_2$, respectively,*

$$\rho_1^*\tilde{\omega}_1=\omega|_{\pi_1(L)},\quad \rho_2^*\tilde{\omega}_2=\omega|_{\pi_2(L)}.$$

*The symplectic forms $\tilde{\omega}_1$, $\tilde{\omega}_2$ are defined uniquely by the above formulas, and $\tilde{L}\in\Lambda_{2n-2k}-C\Lambda_{2n-2k}$, for some $k\in N$.*

*Proof:* If $L \in C\Lambda_{2n}$ then, by Lemma II.1 we have $\pi_1(L) \subseteq V_1$, $\pi_2(L) \subseteq V_2$, where $V_1$, $V_2$ are hypersurfaces in $M_1$ and $M_2$, respectively. If there is an equality above then $V_1$ and $V_2$ are coisotropic, so we have the natural projections $\rho_i$ along the symplectic polars $V_1^{\angle} \subset V_1$, $V_2^{\angle} \subset V_2$ onto the symplectic reduced spaces $\tilde{M}_1 = (V_1/V_1^{\angle}, \tilde{\omega}_1)$, $\tilde{M}_2 = (V_2/V_2^{\angle}, \tilde{\omega}_2)$. The symplectic polar to the subspace $V \subset (M, \omega)$ is defined to be the subspace $V^{\angle} = \{v \in M; \omega(v,u) = 0, \forall_{u \in V}\}$. So we represent $L$ uniquely by two hyperspaces $V_i$ and the Lagrangian subspace $\tilde{L} \in \Lambda_{2n-2}$ in $(\tilde{M}_1 \times \tilde{M}_2, \pi_2^* \tilde{\omega}_2 - \pi_1^* \tilde{\omega}_1)$. If $\tilde{L} \in C\Lambda_{2n-2}$, then we may proceed in an analogous way and obtain the noncritical representation for $\tilde{L}$. □

*Example II.1: If $n=2$ we have only two strata of the singular set $C\Lambda_4$: Elements of the first maximal stratum $C_1\Lambda_4$ are determined by the pairs of two coisotropic subspaces, $V_1$ in $M_1$ and $V_2$ in $M_2$ and the symplectic linear maps between the corresponding reduced symplectic spaces. It is easy to calculate* $\dim C_1\Lambda_4 = 9$. *The second stratum is the supercritical set $S\Lambda_4$, and its dimension* $\dim S\Lambda_4 = 6$.

In general, we have the following result concerning the structure of the singular set $C\Lambda_{2n}$.

**Theorem II.1:** *There is the following partition of the singular set $C\Lambda_{2n}$ into the smooth submanifolds,*

$$C\Lambda_{2n} = \bigcup_{k=1}^{n} C_k\Lambda_{2n},$$

*where the elements of $C_k\Lambda_{2n}$ are determined by the pairs of two coisotropic subspaces $V_1$ in $M_1$ and $V_2$ in $M_2$ of codimension $k$ and the symplectic linear automorphism of the $2n-2k$-dimensional symplectic space. In this partition $C_n\Lambda_{2n} = S\Lambda_{2n}$.*

*Proof:* In fact, it follows from the property that the projection of $L \in \Lambda_{2n}$ onto $M_1$ and $M_2$ is always coisotropic (or Lagrangian). Thus, starting from the hypersurfaces we see that the corresponding $\tilde{L} \in \Lambda_{2n-2}$, in the product of reduced symplectic spaces (as it was proved in the Proposition II.1), projects onto these spaces or onto their hypersurfaces in the more degenerated case. Repeating this argument for further representations of $L$, we get the natural decomposition by equally dimensional coisotropic subspaces and linear symplectic maps in, respectively, smaller dimensional symplectic space. □

*Corollary II.1:*

$$\operatorname{codim} C_k\Lambda_{2n} = k^2, \quad k = 1, \dots, n.$$

*Proof:* We calculate the dimension of the isotropic Grassmannian $I_k^{2n}$ of $k$-isotropic planes in $2n$-dimensional symplectic space $V$ (cf. Ref. 14),

$$\dim I_k^{2n} = 2nk - \tfrac{1}{2}k(3k-1).$$

This is the dimension of the corresponding space of $2n-k$-dimensional coisotropic subspaces in $V$. Since $\dim \Lambda_{2n} = 2n^2 + n$, we get

$$\begin{aligned} \operatorname{codim} C_k\Lambda_{2n} &= \dim \Lambda_{2n} - 2 \dim I_k^{2n} - \dim(\Lambda_{2n-2k}) \\ &= n(2n+1) - 2\left(2nk - \tfrac{1}{2}k(3k-1)\right) - (n-k)(2n-2k+1) = k^2. \end{aligned}$$

□

In comparison to the inequality (1), we have

$$\operatorname{codim} C_k\Lambda_{2n} \geqslant 4,$$

if $k > 1$.

## III. LOCAL CLASSIFICATION OF SYMPLECTIC RELATIONS

Let $(L,p)$ be germ of a symplectic relation (Lagrangian submanifold) in $\mathcal{M}$. Now we introduce the natural equivalence group acting on the space of such germs.

*Definition III.1: We say that the two germs* $(L_1,p_1)$, $(L_2,p_2)$ *of symplectic relations in* $\mathcal{M}$ *are equivalent if there exist two symplectomorphism germs* $B_1:(M_1,\pi_1(p_1))\to(M_1,\pi_1(p_2))$ *and* $B_2:(M_2,\pi_2(p_1))\to(M_2,\pi_2(p_2))$ *such that the symplectomorphism* $B_1\times B_2$ *of* $\mathcal{M}$ *sends* $L_1$ *into* $L_2$ *and* $p_1$ *into* $p_2$.

For the symplectic relation $L\subset\mathcal{M}$, we define the corresponding symplectic Gauss map,

$$G:L\ni p\to T_pL\in\Lambda_{2n}.$$

We call $L$ to be, in general, position [or generic (cf. Ref. 15)] if $G$ is transversal to $C\Lambda_{2n}=\cup_{k=1}^{n}C_k\Lambda_{2n}$. We say that $L$ has a *k-vertical position* at $p\in L$ if $G(p)\in C_k\Lambda_{2n}$. We call $k$ a rank of $k$-vertical position. A 0-vertical position corresponds to the case when $L$ is a graph of a local symplectomorphism at $p$, i.e., $G(p)\in\Lambda_{2n}-C\Lambda_{2n}$. For generic $L$ the isolated points of vertical position appear only if $n=2s^2$, for some $s\in N$. In this case they are isolated points in the $2s$-vertical position. In their neighborhood there are $k$-vertical positioned points with $k\leqslant 2s$. Following the standard representation of Lagrangian germs (cf. Ref. 16) we have the following preparatory lemma.

*Lemma III.1: For any germ of a Lagrangian submanifold* $(L,p)\subset\mathcal{M}$ *there are local cotangent bundle structures around* $\pi_1(p)$, *say* $T^*X$ *and around* $\pi_2(p)$, *say* $T^*Y$, *such that* $(L,p)$ *is generated in*

$$\mathcal{M}\cong(T^*X\times T^*Y,\pi_2^*\omega_Y-\pi_1^*\omega_X),$$

*by a germ of a generating function* $F:(X\times Y,\pi_{X\times Y}(p))\to\mathbb{R}$, *such that, in local coordinates on* $(X\times Y,\pi_{X\times Y}(p))$ *we have*

$$F(x,y)=\sum_{ij=1}^{n}x_iy_j\phi_{ij}(x,y), \tag{2}$$

*where* $\omega_X$ *and* $\omega_Y$ *are the corresponding Liouville symplectic structures on* $T^*X$ *and* $T^*Y$, *respectively.*

*Proof:* If $((p,q),(\tilde{p},\tilde{q}))$ are Darboux coordinates on $\mathcal{M}$, then we find the partition $I\cup J=\{1,\dots,n\}$, $I\cap J=\emptyset$, $\tilde{I}\cup\tilde{J}=\{1,\dots,n\}$, $\tilde{I}\cap\tilde{J}=\emptyset$, such that there exists a smooth function $S(p_I,q_J,\tilde{p}_{\tilde{I}},\tilde{q}_{\tilde{J}})$, which is a generating function for $(L,p)$ (cf. Refs. 5 and 16, Sec. III.19.3). By the symplectomorphism

$$\Phi(p,q;\tilde{p},\tilde{q})=(-q_I,p_J,p_I,q_J;-\tilde{q}_{\tilde{I}},\tilde{p}_{\tilde{J}},\tilde{p}_{\tilde{I}},\tilde{q}_{\tilde{J}})=(\xi,x;\eta,y),$$

which preserves the product structure of $\mathcal{M}$, we find the generating function $F(x,y)$ for $(L,p)$ in the canonical special symplectic structure $T^*X\times T^*Y$ on $\mathcal{M}$. The coordinates $(\xi,x)\in T^*X$, $(\eta,y)\in T^*Y$ are new coordinates on the cotangent bundles in which $(L,p)$ is generated by the generating function $F$. Then, further on, using the symplectomorphisms of $M_1$ and $M_2$ preserving the corresponding cotangent bundle structures, we obtain the reduced form (2) of function $F$. We recall that $L$ is described by the following equations:

$$\eta_i=\frac{\partial F}{\partial y_i}(x,y),\quad \xi_j=-\frac{\partial F}{\partial x_j}(x,y),\quad 1\leqslant i,\ j\leqslant n.$$

□

**Theorem III.1:** *Let* $p\in L$ *and we assume that the Lagrangian submanifold* $L$, *around* $p$, *is generated by the generating function in the normal form (2). Then we have the following.*

(1) *Rank of the vertical position of* $L$ *at* $p$ *is equal to the corank of the matrix* $(\phi_{ij} = \partial^2 F/\partial x_i\, \partial y_j)$ *at* $\pi_{X\times Y}(p)$.

(2) *At each* $p \in L$, *for a generic* $L$, *the family of mappings,*

$$\Phi : X\times Y \to \mathbb{R}^n, \quad \Phi(x,y) = \left( \sum_{j=1}^{n} y_j \phi_{1j}(x,y), \ldots, \sum_{j=1}^{n} y_j \phi_{nj}(x,y) \right),$$

*where* $\phi_{ij}(x,y)$ *are defined in Eq. (2), has a generic singularity at* $\pi_{X\times Y}(p)$.

*Proof:*

(1) Any linear relation $L$, by Lemma III.1, is equivalent to one generated by the quadratic form $\Sigma_{ij=1}^{n} x_i y_j a_{ij}$. So the dimension of the kernel of the matrix $(a_{ij})$ is exactly equal to the rank of verticality of $L$. This is a local symplectic invariant of $(L,p)$ that does not depend on the choice of the corresponding cotangent bundle structures.

(2) By Lemma III.1, any relation $L$ is locally generated by the generating function $F(x,y) = \Sigma_{ij} x_i y_j \phi_{ij}(x,y)$, and by the form of function $F$ uniquely represented by a smooth family of mappings,

$$\Phi(x,y) = (\hat{\phi}_1(x,y), \ldots, \hat{\phi}_n(x,y)),$$

such that $\Phi(x,0) \equiv 0$. We see that the Gauss map $G : L \to T_p L$ corresponds exactly to the one-jet extension $j^1\Phi(x,y)$ of the mapping $\Phi$, so the transversality of $G$ is equivalent to the corresponding transversality of $\Phi$ to the canonical stratification of smooth mappings of $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$ (cf. Refs. 15, 16). □

*Corollary III.1: At any point* $p \in L$ *of the 0-vertical position of* $L$, *symplectic relation* $L$ *is parallelizable, i.e., it is locally symplectically equivalent to its tangent space* $T_p L$ *with the following generating function:*

$$F(x,y) = \sum_{i=1}^{n} x_i y_i .$$

*Remark III.1: If* $n=2$ *then the supercritical points appear in generic* $L$ *as the isolated points, in fact* $codim\, C_2\Lambda_4 = 4$, *and* $G$ *is transversal to* $S\Lambda_4 = C_2\Lambda_4$ *(see Example II.1). If* $p \in L$ *is a supercritical transversal point then on the basis of Lemma III.1, on a neighborhood of* $p$, $L$ *is generated locally by the following generating function:*

$$F(x,y) = \sum_{ij=1}^{2} x_i y_j \phi_{ij}(x,y),$$

*where* $\phi_{ij}(0,0) = 0$, $ij = 1,2$, $p = 0$, *and the transversality condition is equivalent to*

$$\operatorname{rank} D\Phi(0) = 4,$$

*where* $\Phi(x,y) = (\phi_{ij}(x,y)) \in M_{2\times 2}$.

If we need to iterate a symplectic relation $L$ we have to use the symplectic equivalence group preserving the canonical product structure of $\mathcal{M} = (M\times M, \pi_2^*\omega - \pi_1^*\omega)$. We say that the two germs $(L_1,p_1)$, $(L_2,p_2) \subset \mathcal{M}$, where $\pi_1(p_i) = \pi_2(p_i) = \tilde{p}_i$, $i = 1,2$, are D-equivalent (diagonal equivalence) if there exists a symplectomorphism germ $B : (M,\tilde{p}_1) \to (M,\tilde{p}_2)$ such that $(B\times B) \times (L_1) = L_2$. Using the notation of the composition of symplectic relations, we can write

$$L_2 = \hat{B} \circ L_1 \circ \hat{B}^t.$$

Now slightly extending the proof of Lemma III.1, we have the following result.

*Lemma III.2: For any germ* $(L,p)\subset\mathcal{M}$ *there exists a local cotangent bundle structure* $T^*X$ *(D equivalence) around* $\pi_1(p)$, *such that* $(L,p)$ *is generated in*

$$\mathcal{M}\cong(T^*X\times T^*X,\pi_2^*\omega_X-\pi_1^*\omega_X),$$

*by a Morse Family germ* $F:(X\times X\times R^k,0)\to\mathbb{R}$ [*we assumed* $\pi_1(p)=0$],

$$F(x,y,\lambda)=\sum_{i=1}^{n} x_i\phi_i(x,y,\lambda), \tag{3}$$

*such that* $k\leqslant\dim X$. *If the integer k is minimal then it is an invariant of D-equivalence symplectic group action.*

We see that the linear symplectic relations in $\mathcal{M}$ are classified by the classes of linear mappings,

$$\Phi=(\phi_1,\dots,\phi_n):\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n,$$

extracted from the normal form (3), with the standard equivalence relation

$$\Xi:(x,y,\lambda)\to(A(x),Y(y),\Lambda(x,y,\lambda)),$$

where the equivalent mapping, say $\Phi'$, is given by

$$\Phi'(x,y,\lambda)=A^T\Phi(A(x),B(y),\Lambda(x,y,\lambda)).$$

*Remark III.2: An important source of the nontrivial symplectic relations is given by the KdV and MKdV hierarchies of nonlinear differential equations (cf. Ref. 2). The main example is the Darboux relation (correspondence). If we consider the Schrödinger operator,*

$$L=-D^2+u=-(D+f)(D-f)=-D^2+(f^2-f'),$$

*and its isospectral deformation,*

$$\tilde{L}=-D^2+v=-(D-f)(D+f)=-(D+g)(D-g)=-D^2+g'+g^2 \quad \left(D=\frac{d}{dx}\right),$$

*then we get the Darboux relation*

$$-f'+f^2=g'+g^2,$$

*in the product of two copies of function space* $\mathcal{F}$ *of variable* $x$, *endowed with the difference of the Poisson structures,*

$$\{F,G\}=\int\frac{\delta F}{\delta f}D\frac{\delta G}{\delta f}dx,$$

*where* $F[f],G[f]$ *are the functionals on* $\mathcal{F}$.

*For the Schrödinger operator* $-D^2+u$ *there is an infinite hierarchy of isospectral flows, with a corresponding set of integrals* $I_k[u]=\int L_k(u,u_x,u_{xx},\dots)dx$. *The first integrals expressed by* $f$*-variable can be written in the form*

$$H_k[f]=\int\mathcal{L}_k(f,f_x,f_{xx},\dots)dx;$$

$$H_0[f]=\int(f_x+f^2)dx,\quad H_1[f]=\frac{1}{2}\int(f_x^2+2f_xf^2+f^4)dx,$$

$$H_2[f]=\int\left(\frac{1}{2}f_{xx}^2+5f^2f_x^2+f^6+\left(ff_x^2+\frac{3}{5}f^5\right)_x\right)dx.$$

*The Darboux transformation preserves these integrals and gives the difference of the integrands that is the $x$ derivative, i.e.,*

$$\mathcal{L}_k[f]-\mathcal{L}_k[g]=\frac{d}{dx}\overline{F}_k[f,g], \tag{4}$$

*where $\overline{F}_k[f,g]$ are differential polynomials. Now we restrict the problem to $n$ jets of functions of $x$, so $\mathcal{L}=\mathcal{L}(f,f_x,\ldots,f^{(n)}_{x\ldots x})$ and provide the symplectic reduction to finite-dimensional symplectic space by reduction of functional parameter; $\delta H/\delta f=0$, which gives the finite-dimensional reduced space and indicates the symplectic structure from the formula*

$$d\mathcal{L}=\frac{\delta H}{\delta f}df+\frac{d}{dx}\sum_i p_i\,dq_i,$$

*where $q_1=f$, $q_2=f_x,\ldots,q_n=f^{(n-1)}$. Thus, after restriction of Darboux transformation $f\to g$ to the ''stationary hypersurface'' $\delta H/\delta f=0$, on the basis of (4), we get the symplectic relation*

$$\frac{d}{dx}\left(\sum_i p_i\,dq_i-\sum_j P_j\,dQ_j\right)=\frac{d}{dx}d\overline{F},$$

*generated by the Morse family S:*

$$\sum_i p_i\,dq_i-\sum_j P_j\,dQ_j=dS.$$

*As an example one can consider the integrand (cf. Ref. 2),*

$$\mathcal{L}=\tfrac{1}{2}(f_x^2+f^4)+\tfrac{1}{2}af^2+bf,$$

*where $a$, $b$ are real parameters. In the symplectic variables $q=f$, $p=f_x$, $Q=g$, $P=g_x$, the corresponding Darboux relation is generated by the function*

$$S(q,Q)=\tfrac{1}{3}(q^3+Q^3)+\tfrac{1}{2}a(Q+q)+b\log(q+Q).$$

## IV. ACTION AND ITERATION OF SYMPLECTIC RELATIONS

Let $L\subset\mathcal{M}$ be a symplectic relation and $S\subset(M,\omega)$ be a subset of $M$. Then we define the image of $S$ by $L$;

$$L(S)=\{p\in M:\exists_{p'\in S},(p',p)\in L\}.$$

Obviously this action of $L$ on subsets of $M$ preserves all their symplectic properties. Thus, if $S$ is Lagrangian, isotropic, or coisotropic, then $L(S)$ is also Lagrangian, isotropic, or coisotropic, respectively, unless it is singular (cf. Refs. 7, 17, and 18).

Now we consider the linear case, and assume $S$ is a Lagrangian subspace. As a canonical pair we define $(l,L)$, where $L\in\Lambda_{2n}$ and $l\in\Lambda_n$.

*Proposition IV.1: There is a natural mapping,*

$$H:\Lambda_n\times\Lambda_{2n}\to\Lambda_n,\quad H(l,L)=L(l).$$

*Proof:* We have to check that $L(l)$ is a Lagrangian subspace of $(M,\omega)$. Indeed, we can choose the cotangent bundle fibration on $\mathcal{M}$ and represent $l$ and $L$ by generating families, say $(\lambda,q_1)\rightarrow G(\lambda,q_1)$ for $l$ and $(\mu,q_1,q)\rightarrow F(\mu,q_1,q)$ for $L$. Then the generating family for the image $L(l)$ is defined by

$$H(\mu,\nu,\lambda,q)=F(\mu,\nu,q)+G(\lambda,\nu).$$

By the standard reduction of $(\mu,\nu,\lambda)$-Morse parameters, we get the generating family for $L(l)$ in the form (cf. Refs. 16 and 5)

$$H\cong\tilde{F}(\rho,q)=f(q)+\sum_{i=1}^{k}\rho_i g_i(q),$$

where $f$ is a quadratic form and $g(q)=(g_1(q),\ldots,g_k(q))$ is a linear mapping, $g(q)=Aq$. Now we easily see that the space

$$\left\{(p,q)\in M:\exists_{\rho\in\mathbb{R}^k}:p=\frac{\partial\tilde{F}}{\partial q}(\rho,q),0=\frac{\partial\tilde{F}}{\partial\rho}(\rho,q)\right\}$$

is an $n$-dimensional Lagrangian subspace of $(M,\omega)$ because $\dim(\operatorname{Ker}A)+\dim(\operatorname{Im}A)=n$. □

Another view on the image $L(l)$ is given through the unitary group reconstruction of Lagrangian subspaces (cf. Ref. 9). We write

$$L=\left\{\begin{pmatrix}A\\B\end{pmatrix}\begin{pmatrix}\alpha_1\\ \cdot\\ \alpha_{2n}\end{pmatrix},\quad \alpha_i\in\mathbb{R}\right\},$$

where

$$\begin{pmatrix}A\\B\end{pmatrix}\in U(2n).$$

The corresponding projections onto the first and the second component of $\mathcal{M}$ are given in the form

$$L^t(M)=\left\{A\begin{pmatrix}\alpha_1\\ \cdot\\ \alpha_{2n}\end{pmatrix},\alpha_i\in\mathbb{R}\right\}=\pi_1(L),$$

$$L(M)=\left\{B\begin{pmatrix}\alpha_1\\ \cdot\\ \alpha_{2n}\end{pmatrix},\alpha_i\in\mathbb{R}\right\}=\pi_2(L).$$

If $l$ is given in the form

$$l=\left\{G\begin{pmatrix}\beta_1\\ \cdot\\ \beta_n\end{pmatrix},\beta_i\in\mathbb{R}\right\},$$

where $G\in U(n)$. Then, at first, we define the subspace $V_{(A,G)}$,

$$\mathbb{R}^{2n}\supset V_{(A,G)}=\left\{\alpha:\exists_{\beta\in\mathbb{R}^n}A\begin{pmatrix}\alpha_1\\ \cdot\\ \alpha_{2n}\end{pmatrix}=G\begin{pmatrix}\beta_1\\ \cdot\\ \beta_n\end{pmatrix}\right\},$$

and finally $L(l)$ is defined in the following way:

$$L(l)=\left\{ B\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_{2n}\end{pmatrix};\alpha\in V_{(A,G)}\right\},$$

where we denote $\alpha=(\alpha_1,\ldots,\alpha_{2n})$, and $\beta=(\beta_1,\ldots,\beta_n)$.

The mapping

$$\rho:\Lambda_{2n}\ni L\rightarrow H(l_0,L)=L(l_0)\in\Lambda_n,$$

where $l_0$ is a fixed element of $\Lambda_n$, is a fiber bundle projection. Let $L$ be represented by $\binom{A}{B}\in U(2n)$, where we denote by $A=A_1+iA_2$, $B=B_1+iB_2$ the decomposition of $A$ and $B$ complex matrices.

*Proposition IV.2: The fiber of ρ is defined by* $\binom{A}{B}\in U(2n)$, *such that we have the following.*

(1) *rank* $\binom{A_2}{B_2}\leqslant n$.

(2) $A_1$ *and* $B_1$ *are surjective on* $V=Ker\binom{B_2}{A_2}$.

*Proof:* It is enough to find such $L\in\Lambda_{2n}$, that $L(l_0)=l_0$. Choose $l_0=\{(q+ip)\in M:p=0\}$, then we get the condition

$$\begin{pmatrix}A\\ B\end{pmatrix}\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_{2n}\end{pmatrix}\in\mathbb{R}^{2n},$$

and surjectivity of $A_1$ and $B_1$ on the Kernel of $\binom{A_2}{B_2}$ corresponds to the property that $l_0=\mathbb{R}^n\subset\mathbf{C}^n$ is mapped by $L$ onto $l_0$. □

Now using the Proposition II.1 we investigate the geometric structure of an image $L(l)$.

*Proposition IV.3: Any image of* $l\in\Lambda_n$ *by* $L\in\Lambda_{2n}$ *is a Lagrangian subspace* $l'$ *of a coisotropic space* $L(M)=\pi_2(L)\subset(M,\omega)$. *Thus, it is a counterimage, by a canonical reduction* $L(M)\xrightarrow{\pi}L(M)/L(M)^{\angle}$ *of some Lagrangian subspace in the reduced symplectic space* $(L(M)/L(M)^{\angle},\tilde{\omega})$, $\pi^*\tilde{\omega}=\omega|_{L(M)}$.

*Proof:* We define $L^t(M)=\pi_1(L)$, $L(M)=\pi_2(L)$, which are the coisotropic subspaces of the same codimension in both components of $\mathcal{M}$. $L(M)$ is a coisotropic space, so the Kernel of $\pi_1|_L$ projected onto $L(M)$, by $\pi_2$, is a symplectic polar of $L(M)$, and *vice versa* the Kernel of $\pi_2|_L$ projected onto $L^t(M)$, by $\pi_1$, is a symplectic polar of $L^t(M)$. Thus, for the pair $(l,L)$, in any common position, the image $L(l)$ is a Lagrangian subspace of the coisotropic space $L(M)$. □

*Remark IV.1:*

(1) *The mapping ρ is smooth on all strata of* $\Lambda_{2n}=GSp_{2n}\cup\cup_{k=1}^{n}C_k\Lambda_{2n}$, *where* $GSp_{2n}$ *denotes the graphs of symplectomorphisms. Only on these strata can we pull back the universal Maslov class* $[\rho^*\mu]$, $(\Lambda_{2n}\supset GSp_{2n}\xrightarrow{\rho}\Lambda_n\xrightarrow{det^2}S^1)$ *(cf. Refs. 10 and 9).*

(2) *In the smooth nonlinear case,* $L(l)$ *is always isotropic on his smooth strata. The corresponding local generating family is of the form (cf. Ref. 7)*

$$H(\lambda,\mu,\nu,q)=G(\lambda,\mu)+F(\nu,\mu,q),$$

*where* $G(\lambda,q_1)$ *is a generating Morse family for* $l$ *and* $F(\nu,q_1,q)$ *is a generating Morse family for* $L$. *We notice that* $H$ *is not necessary Morse family so the corresponding image* $L(l)$ *may not be smooth.*

There is an interesting symplectic invariant prescribed to the symplectic relation and based on the cohomological properties of the relation (cf. Refs. 19 and 20). Now we will assume that the Lagrangian submanifold $L\subset\mathcal{M}$ is compact (with boundary), and the first cohomology group

$H^1(L,\mathbb{R})$ is trivial. Instead of $(M_1,\omega_1)$, $(M_2,\omega_2)$ we take two copies of the same symplectic manifold $(M,\omega)$. For any choice of 1-form $\alpha$, such that $\omega = d\alpha$, the 1-form $\pi_2^*\alpha - \pi_1^*\alpha|_L$ is exact and

$$\pi_2^*\alpha - \pi_1^*\alpha|_L = dH, \tag{5}$$

for some smooth generating function $H{:}L\to\mathbb{R}$.

If $\alpha_1$ is another choice of a 1-form for which $d\alpha_1=\omega$, then $d(\alpha_1-\alpha)=0$ and $\alpha_1-\alpha = dG$, for some smooth function $G{:}M\to\mathbb{R}$ (where $M$ has a boundary or is not compact). For a new underlying 1-form $\alpha_1$, the Lagrangian submanifold $L$ has another generating function $H_1{:}L\to\mathbb{R}$, such that

$$\pi_2^*\alpha_1 - \pi_1^*\alpha_1|_L = dH_1. \tag{6}$$

*Proposition IV.4: The generating functions $H_1$ and $H$ defined by the formulas (6) and (5) are joined by the following relation:*

$$H_1 = H + (\pi_2^*G - \pi_1^*G)|_L. \tag{7}$$

*Proof:* Subtracting formula (5) from the formula (6), by sides, we get

$$\pi_2^*(\alpha_1-\alpha) + \pi_2^*(\alpha_1-\alpha) = d(H_1-H),$$

and because $\alpha_1-\alpha=dG$ we get finally

$$d(\pi_2^*G - \pi_2^*G) = d(H_1-H).$$

If we normalize the additive constants in the definitions of $H$, $H_1$, and $G$, we obtain the formula (7). □

Now we consider the multiple, iterated images by the relation $L$. Let $\sigma=\{(x_0,x_1)\in L,(x_1,x_2)\in L,\ldots,(x_{k-1},x_0)\in L,\}$. We will call $\sigma$ the periodic orbit of $L$. We will associate with $\sigma$ the following number:

$$N(\sigma)=\sum_{i=0}^{k-2} H(x_i,x_{i+1}) + H(x_{k-1},x_0).$$

Now using the formula (7) we have a natural property of $N(\sigma)$ (cf. Ref. 19).

*Corollary IV.1: $N(\sigma)$ is an invariant with respect to the action of the group of symplectomorphisms operating on $(M,\omega)$.*

If $L=\mathrm{graph}\,\Phi$, where $\Phi{:}(M,\omega)\to(M,\omega)$ is a symplectomorphism, then the set of numbers $\{N(\sigma)\}$, where $\sigma$ is any periodic orbit of $\Phi$ is called the spectrum of $\Phi$. This spectrum is extensively studied if $\Phi$ is a billiard mapping associated with the convex region in $\mathbb{R}^n$ (cf. Refs. 20 and 8). In the case of a graph, the formula (7) reduces to the following one:

$$H_1 = H + \Phi^*G - G,$$

where

$$\Phi^*\alpha - \alpha = dH,$$

and the iterational invariant to the periodic orbit of $\Phi$, $\sigma=\{x_0,x_1,\ldots,x_{k-1}\}$ is given in the form

$$N(\sigma)=\sum_{i=0}^{k-1} H(x_i).$$

As an example, we consider the billiard symplectic map. Let $\Omega$ be a smooth compact convex region in $\mathbb{R}^n$. Let $X$ be the boundary of $\Omega$ and $T^*X$ the cotangent bundle of $X$. The symplectic billiard map $B:T^*X\to T^*X$ is defined on the set $U=\{(x,\xi)\in T^*X:|\xi|<1\}$. To the point $(x,\xi)\in U$ we prescribe the point $(x',\xi')\in U$ in the following way. Let $n(x)$ be the outward unit normal vector to $X$ at $x$. There is a unique, unit element $\tilde{\eta}\in(\mathbb{R}^n)^*$ such that $\langle\tilde{\eta},n(x)\rangle<0$ and $\langle\tilde{\eta},v\rangle=\langle\xi,v\rangle$ for all $v\in T_xX$, where we identify $T_xX$ with the corresponding subspace in $\mathbb{R}^n$. To a given $(x,\xi)$ there exists $x'\in X$, which is the unique point of intersection of $X$ with the positive line segment; $x'=X\cap\{x+t\eta,t>0\}$, where $\eta$ is a unique vector in $\mathbb{R}^n$ corresponding to $\tilde{\eta}$, and $\xi'$ is defined as the unique element of $T^*_{x'}X$ for which we have $\langle\eta,v\rangle=\langle\xi',v\rangle$ for all $v\in T_{x'}X$. Obviously $B(x,\xi)=(x',\xi')$ is symplectic and the generating function for graph $B\in(T^*X\times T^*X,\pi_2^*\theta_X-\pi_1^*\theta_X)$ is defined as the perimeter,

$$\tilde{H}:X\times X\to\mathbb{R},\quad \tilde{H}(x,x')=|x'-x|,$$

where $\theta_X$ is the Liouville one-form on $T^*X$. By the projection $\pi_{X\times X}:T^*X\times T^*X\to X\times X$ and the smooth map

$$\rho:T^*X\to T^*X\times T^*X,\quad \rho(x,\xi)=(x,-\xi,x',\xi'),$$

we get the function $H:T^*X\to\mathbb{R}$, $H=\rho^*\pi^*_{X\times X}\tilde{H}$ such that $B^*\theta_X-\theta_X=dH$, which gives the symplectic invariant

$$N(\sigma)=\sum_{i=0}^{k-1}H(x_i,\xi_i),$$

for the periodic orbit $\sigma=\{(x_0,\xi_0),\ldots,(x_{k-1},\xi_{k-1})\}$, which is the length

$$|x_1-x_0|+\cdots+|x_0-x_{k-1}|$$

of the closed geodesic of the billiard system and defines, for all closed geodesics, the length spectrum of $\Omega$ (cf. Ref. 8).

In general, if we assume that

$$L\subset\mathcal{M}=(T^*X\times T^*X,\pi_2^*\omega_X-\pi_2^*\omega_X)$$

is generated by the smooth generating function $F:(x,y)\to F(x,y)$, then the invariant $N(\sigma)$ defined on the periodic orbit of $L$,

$$\pi_{X\times X}(\sigma)=\{(x_0,x_1),(x_1,x_2),\ldots,(x_{k-1},x_0)\}$$

is the critical value of the function

$$G:X\times\ldots\times X\to\mathbb{R},\quad G(\tilde{x}_0,\tilde{x}_1,\ldots,\tilde{x}_{k-1})=F(\tilde{x}_0,\tilde{x}_1)+F(\tilde{x}_1,\tilde{x}_2)+\cdots+F(\tilde{x}_{k-1},\tilde{x}_0),$$

for which $\pi_{X\times X}(\sigma)$ is a critical point.

*Definition IV.1: By $\{N(\sigma)\}^k$ we denote the set of symplectic invariants for k-point periodic orbits. We will call them k-point symplectic invariants.*

For the general billiard system the Lagrangian submanifold $L$ may not be the graph of any symplectomorphism. Let $\phi:X\to\mathbb{R}^n$ be an imbedding of a closed orientable surface and we assume that it is generic, i.e., the function

$$\tilde{H}_\phi(x,y)=|\phi(x)-\phi(y)|,$$

defined on $X\times X$ outside of the diagonal $\Delta$ has only nondegenerate critical points on $X\times X-\Delta$. We easily see that the critical points of $\tilde{H}_\phi$ are, in fact, the two-point orbits or double normals of the possibly nonconvex billiard system. The corresponding invariants are the critical values of $\tilde{H}_{\phi}$. Using the Morse theory methods, Morse inequalities, (cf. Refs. 13, 12, Theorem 1), we obtain an estimation for the number of k-point invariants for small k.

**Theorem IV.1:** *If $\phi$ is a generic imbedding of a surface of genus $g$, then we have the following lower bound for the number of two-point symplectic invariants:*

$$\#\{N(\sigma)\}^2 \geqslant 2g^2 + 3g + 3.$$

For the generic billiards on the plane we have at least two two-point symplectic invariants. In the case of an ellipsoid, $g=0$, with the three unequal axes $\#\{N(\sigma)\}^2=3$. If this is an imbedding of the torus we have at least eight two-point symplectic invariants of the toruslike billiard system. In three-dimensional billiard systems, the lower bound for $\#\{N(\sigma)\}^2$ is expressed by the first Betti number $d_1$ of $X$. In fact, if $\phi:X\to\mathbb{R}^n$ is generic, then the lower bound is given by the numbers:

$$2d_1^2 + 3d_1 + 4, \quad \text{if } d_1 \text{ is even},$$

or

$$2d_1^2 + 3d_1 + 5, \quad \text{if } d_1 \text{ is odd}.$$

If $X=S^{n-1}$ then we have $\#\{N(\sigma)\}^2 \geqslant n$.

Now we consider an imbedding $\phi:X\to\mathbb{R}^n$, which is generic with respect to the generating function

$$V_{\phi}(x,y) = \frac{1}{|\phi(x)-\phi(y)|},$$

defined outside of the diagonal $\Delta$. The corresponding symplectic relation $\tilde{L}\subset\mathcal{M}$, defined by $V_{\phi}$, provides the geometric setting for finding the equilibrium positions of equally charged particles on an imbedded surface in Euclidean space. The iterational symplectic invariants $N(\sigma)$ define the least potential energy of the number of charged particles in equilibrium on $X$. The two-point symplectic invariants for $\tilde{L}$ exactly correspond to the double normals—equilibrium positions of the two equally charged particles—and thus the corresponding lower bounds are analogous to that established for the billiard system in Proposition IV.1. The three-point invariants are defined by the critical points of the function,

$$V_{\phi}^{(3)}(x,y,z) = \frac{1}{|\phi(x)-\phi(y)|} + \frac{1}{|\phi(y)-\phi(z)|} + \frac{1}{|\phi(z)-\phi(x)|},$$

defined outside the total diagonal $\Delta$ in $X\times X\times X$ and generic. Now, following the further Morse-theory estimations obtained in Ref. 12 (Theorem 3), we get the following lower bound for the number of three-point symplectic invariants of $\tilde{L}$.

**Theorem IV.2:** *If $\phi:X\to\mathbb{R}^n$ is a generic imbedding of a surface of genus $g$, then we have the following lower bounds:*

$$\#\{N(\sigma)\}^3 \geqslant (4g^3 + 8g^2 + 6g + 12)/3, \quad \text{for } g \neq = 2(\mathrm{mod}\, 3),$$

*or*

$$\#\{N(\sigma)\}^3 \geqslant (4g^3 + 8g^2 + 6g + 14)/3, \quad \text{for } g = 2(\mathrm{mod}\, 3),$$

*if $g\neq=1$ or $\#\{N(\sigma)\}^3 \geqslant 11$ if $g=1$.*

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