Generalized Luneburg canonical varieties and vector fields on quasicaustics

S. Janeczko[a),b)]
Max-Planck-Institut für Mathematic, Gottfried-Claren-Strasse 26, 5300 Bonn, West Germany

(Received 2 December 1988; accepted for publication 22 November 1989)

Some aspects of a particular class of bifurcation varieties which are provided by simple and unimodal boundary singularities are studied. Their correspondence to diffraction theory is established. The generic caustics by diffraction on apertures are derived and their generating families for the corresponding Lagrangian varieties are calculated. It is proved that the quasicaustics associated to simple singularities are smooth hypersurfaces or Whitney's cross-caps. The procedure for calculating the modules of logarithmic vector fields is given, and the minimal sets of the corresponding generators are explicitly calculated. The general boundary singularities are constructed and the structure of quasicaustics defined by parabolic singularities is investigated.

I. INTRODUCTION

Let F: $(\mathbb{C}^{n+1}\times\mathbb{C}^p,0)\to(\mathbb{C},0)$ be a germ of a holomorphic function. By $(S,0)\subset(\mathbb{C}^{n+1},0)$ we denote a germ of a some hypersurface in $(\mathbb{C}^{n+1},0)$. The quasicaustic $Q(F)$ of F is defined as

$$Q(F)=\{a\in\mathbb{C}^p;\ F(\cdot,a)\ \text{has a critical point on } S\}.$$

Let F represent the distance function from the general wave front in the presence of an obstacle formed by an aperture (cf. Refs. 1 and 2) with boundary S. The corresponding quasicaustic $Q(F)$ is build up from the rays orthogonal to the given wave front and touching the boundary of the aperture (see the example of the quasicaustic illustrated in Fig. 1). The quasicaustic is a subvariety of the usual caustic (also called the bifurcation set[3,4])

$$\{a\in\mathbb{C}^p;\ F(\cdot,a)\ \text{or}\ F|_{s\times\mathbb{C}^p}(\cdot,a)\ \text{have a critical point}\},$$

and represents the structure of shadows formed by the common, pecular positions of aperture and incident wave front.

In this paper we investigate the structure of generic caustics and quasicaustics by diffraction on smooth obstacle curves and apertures (optical instruments). We use for this the classical phase space for general optical instruments, i.e., the space of pairs of rays $(l,\tilde{l})$, where l is an incident ray and $\tilde{l}$ is a transformed ray (produced by l and the optical instrument), endowed with the canonical symplectic structure. This space was first introduced by Luneburg[5] in his mathematical theory of optics and then revived by Guillemin and Sternberg[6] in their symplectic approach to various physical theories. To each optical instrument, in the mentioned phase space, there corresponds a Lagrangian subvariety, say A, defining all physical properties (from the point of view of the geometrical theory of optics[7]) of the system. So when A is fixed we can obtain all transformed wave fronts by taking the symplectic images $A(L)$ of all Lagrangian subvarieties L of incident rays (i.e., optical sources). (See, also, Ref. 8.)

The plan of the paper is as follows. In Sec. II we give preliminary results about the basic phase spaces and construct representative examples in the symplectic approach to general optical systems. The geometrical structure of caustics by diffraction on apertures, as well as their generic classification in the case of half-line aperture on the plane and half-plane aperture in Euclidean three-space, is investigated in Sec. III. We compute the normal forms for generating families of the generic canonical varieties in the case of diffraction on smooth curves in Sec. IV. When considering the caustics by diffraction on apertures, the quasicaustic component becomes important. In Sec. V we generalize the methods for ordinary caustics initiated by Bruce[9,10] to investigate the structure of logarithmic vector fields on quasicaustics. In Sec. VI we derive the generators for the modules of tangent vector fields to the quasicaustics corresponding to simple

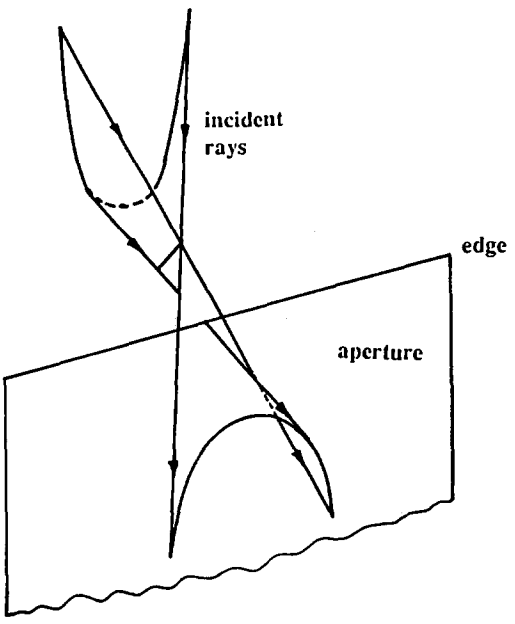

FIG. 1. Whitney's cross-cap quasicaustic.

[a)] On leave of absence from Institute of Mathematics, Technical University of Warsaw, Pl. Jednosci Robotniczej 1, 00-661 Warszawa, Poland.

[b)] This paper was partially written while the author was a SERC visitor to the Mathematics Institute, University of Warwick.

 0022-2488/90/040997-13$03.00

boundary singularities and prove that they are not free. Finally in Sec. VI we analyze the structure of quasicaustics and the reduction of functional moduli in normal forms of Lagrangian pairs.

II. SINGULARITIES IN ACTION OF OPTICAL INSTRUMENTS

Let (M, ω) be the symplectic manifold of all oriented lines in $V \cong \mathbb{R}^3$. We look on V as the configurational space of geometrical optics with refraction index n: $V \to \mathbb{R}$, $n \equiv 1$. Here (M, ω) is given by the standard symplectic reduction

$$\pi_M : H^{-1}(0) \to M \cong T^*S^2,$$

where the hypersurface $H^{-1}(0)$ is defined by the Hamiltonian

$$H : T^*V \to \mathbb{R}, \quad H(p,q) := \tfrac{1}{2}(\|p\|^2 - 1),$$

and π_M is the projection along characteristics of the associated Hamiltonian system.

Let (p,q) be coordinates on (T^*V, ω_V), where ω_V is an associated Liouville two-form. By (U,ω) we denote the local chart on (M,ω) described as an image $\pi_M(H^{-1}(0) \cap \{p_1 > 0\})$ with restricted symplectic form ω. The (p,q) form Darboux coordinates on (T^*V, ω_V). In corresponding Darboux coordinates (r,s) on (U,ω) we can write

$$(r,s) = \pi_M(p_2,p_3;q_1,q_2,q_3)$$

$$= \left(p_2,p_3;\ q_2 - \frac{q_1 p_2}{\sqrt{1-p_2^2-p_3^2}}, q_3 - \frac{q_1 p_3}{\sqrt{1-p_2^2-p_3^2}}\right),$$

where the unique reduced symplectic structure ω is given by the formula

$$\omega_V|_{H^{-1}(0)} = \pi_M^* \omega, \quad \omega|_U = \sum_{i=1}^{2} dr_i \wedge ds_i.$$

In the introduced coordinates on M, to each point $(r,s) \in U$ we can uniquely associate the corresponding ray (in parametric form):

$$(q_1,q_2,q_3)$$

$$= (0,s_1,s_2) + u\left(1, \frac{r_1}{\sqrt{1-r_1^2-r_2^2}}, \frac{r_2}{\sqrt{1-r_1^2-r_2^2}}\right), \quad u \in \mathbb{R}.$$

By the above formula one can translate the concrete optical problems into the language of the phase space (M,ω) and vice versa (cf. Refs. 5, 6, and 11).

Let (U,ω) and $(\tilde{U},\tilde{\omega})$ be two examples of the symplectic space of optical rays or its open subsets. Usually these manifolds denote the spaces of incident and transformed rays of an optical instrument.

Definition 2.1: The phase space of optical instruments is the following product symplectic manifold:

$$\Pi = (U \times \tilde{U};\ \pi_2^* \tilde{\omega} - \pi_1^* \omega),$$

where $\pi_{1,2} : U \times \tilde{U} \to U, \tilde{U}$ are canonical projections (this was first introduced by Luneburg[5]).

The process of optical transformation (say, reflection, refraction, or diffraction, etc., of the incident rays) is governed by the subvariety of Π, which is Lagrangian, i.e., it is stratified onto isotropic submanifolds of Π where maximal strata are Lagrangian (cf. Refs. 8, 12, and 13).

Definition 2.2: We define the general optical instrument to be a Lagrangian subvariety of Π (generalized symplectic relation[8,14]).

Remark 2.3: It is easily seen that reflecting or refracting optical instruments (cf. Ref. 15) correspond to graphs of symplectomorphisms between $(U;\omega)$ and $(\tilde{U},\tilde{\omega})$. But, for example, the diffraction process is described by a quite general Lagrangian subvariety of Π (cf. Ref. 1). In fact, let $(a,b,x,y,u,v,w) \to F(a,b,x,y,u,v,w)$ be the optical distance function (cf. Refs. 2 and 16) from the wave front

$$\{z = \varphi(x,y) = \lambda_1 x^2 + \lambda_2 xy + \lambda_3 y^2 + O_3(x,y)\}$$

in the presence of the aperture $\{a \geqslant 0,\ z = mb - 1\}$, where $m \geqslant 0$. If the incident ray goes from $(x,y) = (0,0)$ to $(a,b) = (0,0)$, then the transformed rays from $(a,b) = (0,0)$ to (u,v,w) are given by

$$\frac{\partial \overline{F}}{\partial b}(0,u,v,w) = 0,$$

$$\overline{F}(b,x,y,u,v,w) := F(0,b,x,y,u,v,w),$$

which, for the distance function

$$F = [(x-a)^2 + (y-b)^2 + (\varphi(x,y) - mb + 1)^2]^{1/2} + ((u-a)^2 + (v-b)^2 + (w - mb + 1)^2)^{1/2},$$

reads

$$m^2u^2 + v^2(m^2 - 1) - 2mv(1 + w) = 0$$

and

$$v + m(1 + w) \leqslant 0.$$

These conditions define the half-cone of diffracted rays (see Refs. 1 and 7).

Example 2.4: Reflection from the curve: Let the mirror be defined by $\{q_1 = 0\}$. Let (U,ω), the space of incident rays, be defined as $\pi_M(H^{-1}(0) \cap \{p_1 > 0\})$ and the corresponding space of reflected rays be defined as $\tilde{U} = \pi_M(H^{-1}(0) \cap \{p_1 < 0\})$. Then this reflecting optical instrument is equivalent to the Lagrangian subvariety of Π,

$$\Pi \supset \{((r,s),(\tilde{r},\tilde{s})) \in U \times \tilde{U};\ r = \tilde{r},\ s = \tilde{s}\} =: A,$$

and its corresponding generating family (cf. Refs. 17–19)

$$G(\lambda,s,\tilde{s}) = \lambda(s - \tilde{s}),$$

where $\lambda \in \mathbb{R}$, is a Morse parameter.

In our approach the sources of radiation produce rays in the space denoted by (U,ω). Thus we have the following definition.

Definition 2.5: We define the general source of light as a Lagrangian subvariety $L \subset (U,\omega)$ of the space of incident rays. If $A \subset \Pi$ is an optical instrument, then the transformed system of rays [or equivalently the transformed wave front (cf. Ref. 18)] is a symplectic image L' of L by means of A, i.e.,

$$L' := A(L) := \{\tilde{p} \in \tilde{U};\ \text{there exists } p \in L \text{ such that } (p,\tilde{p}) \in A\},$$

which is usually a Lagrangian subvariety of $(\tilde{U},\tilde{\omega})$ (cf. Ref. 8).

Example 2.6: Reflection of a parallel beam of rays: The

beam of parallel rays is given in (U,ω) by $L=\{r=0\}$ (a point source of light at infinity). By reflection in the mirror, $x\to(\varphi(x),x)\in\mathbb{R}^2, \varphi(0)=\varphi'(0)=0, \varphi''(0)\neq 0$, the canonical variety $A\subset\Pi$ (defining the reflection process) brings into L some focusing property and produces the well known caustic. The reflected beam of rays $A(L)$ has the form

$$(\tilde{r},\tilde{s})=\left(\frac{2\varphi'(x)}{\varphi'(x)^2+1},x-\frac{\varphi(x)\varphi'(x)(1+\varphi'(x)^2)^2}{\varphi'(x)^2-1}\right).$$

Remark 2.7: Local genericity of the wave front produced by $L\subset(U,\omega)$ is preserved during the process of reflection or refraction (cf. Ref. 15) because the corresponding canonical variety is a graph of symplectomorphism. Thus the caustics, produced by reflection or refraction, are classified by the simple singularities of type A_k, D_k, E_k.[20] It may not be so in a diffraction process, where $A\subset\Pi$ is no longer the graph of symplectomorphism. In this case the differentiable structure of L is drastically changed by A and $A(L)$ is no longer smooth. Its singular locus brings a completely new type of caustic responsible for the structure of shadows and half-shadows of an obstacle as well.

III. CAUSTICS AND QUASICAUSTICS BY DIFFRACTION

Let L be a source of light or transformed wave front in (M,ω). Now we recall the geometric construction that allows us to define caustic or wave front evolution in V, corresponding to L (cf. Refs. 12 and 13). Let Ξ be the product symplectic manifold

$$\Xi=(M\times T^*V,\pi_2^*\omega_V-\pi_1^*\omega),$$

where $\pi_{1,2}: M\times T^*V\to M, T^*V$ are the canonical projections. One can check that $\tilde{K}:=\text{graph}\,\pi_M\subset\Xi$ is a Lagrangian submanifold of Ξ. Thus there exists its local generating Morse family (cf. Ref. 17), say,

$$K:\mathbb{R}^k\times\tilde{X}\times V\to\mathbb{R}\ (\mu,\tilde{x},q)\to K(\mu,\tilde{x},q),$$

where $T^*\tilde{X}$ is an appropriate local cotangent bundle structure (special symplectic structure,[12-14] on (M,ω). The transformed system of rays forms a Lagrangian subvariety of (T^*V,ω_V) given as an image

$$\tilde{L}=(\tilde{K}\circ A)(L)\subset(T^*V,\omega_V),$$

where $\tilde{K}\circ A\subset\Xi$ is a composition of symplectic relations (cf. Refs. 12 and 17). If

$$G:\mathbb{R}^l\times X\times\tilde{X}\to\mathbb{R},\ (\nu,x,\tilde{x})\to G(\nu,x,\tilde{x}),\quad X,\tilde{X}\cong\mathbb{R}^n,$$

is a generating family for $A\subset\Pi$ and F: $\mathbb{R}^m\times X\to\mathbf{R}$, $(\lambda,x)\to F(\lambda,x)$ is a generating family for L, then the transformed Lagrangian subvariety $\tilde{L}\subset(T^*V,\omega_V)$ is generated by (not necessarily a Morse family)

$$\tilde{F}:\mathbb{R}^{k+l+m+2n}\times V\to\mathbb{R},$$
$$\tilde{F}(\lambda,\nu,\mu,x,\tilde{x};q):=G(\nu,x,\tilde{x})+K(\mu,\tilde{x},q)+F(\lambda,x),$$

where $\mathbb{R}^{k+l+m+2n}$ is a parameter space.

In optical arrangements the source of light is usually a smooth Lagrangian submanifold of (U,ω). Only after the transformation process through an optical instrument does it become singular.

Definition 3.1: Let $L\subset(U,\omega)$ be an initial source variety. We define the caustic by an optical instrument $A\subset\Pi$, to be a hypersurface of V formed by two components: (1) singular values of $\pi_V|_{\tilde{L}-\text{Sing}\tilde{L}}$; and (2) π_V (Sing $\tilde{L}$); where $\tilde{L}=(\tilde{K}\circ A)(L)$ and Sing $\tilde{L}$ denotes the singular locus of $\tilde{L}$.

Remark 3.2: In reflection or refraction we do not go beyond the smooth category of L (at least in this paper) so the associate caustics, in transformed wave fronts $\tilde{L}$, are those realizable by smooth generic sources (cf. Refs. 15 and 21). Thus in what follows we will be interested in caustics caused by diffraction, which will enrich substantially the list of optical events (cf. Ref. 22) and complete the correspondence between singularities of functions and groups generated by reflections.[16,23]

Diffracted rays are produced, for example, when an incident ray hits an edge of an impenetrable screen [i.e., an edge of a boundary or interface (cf. Ref. 1)]. In this case the incident ray produces infinitely many diffracted rays, which have the same angle with the edge as does the incident ray (see Remark 2.3.) This is so if both incident and diffracted rays lie in the same medium. Otherwise, the angles between the two rays and the plane normal to the edge are related by Snells law.[7] Furthermore, the diffracted ray lies on the opposite side of the normal plane from the incident ray; that is, all rules and laws of geometrical optics correspond exactly to the Lagrangian properties of the corresponding varieties $A\subset\Pi$.

Let I be the diagonal in Π. By Ω we denote the set of oriented lines in (U,ω) that do not intersect the screen. Thus we have the following proposition.

Proposition 3.3: In the edge diffraction in an arbitrary Euclidean space, the canonical variety $A\subset\Pi$ has two components

$$A=A^I\cup A^D,$$

where $A^I=\Omega\times\Omega\subset I$ and A^D is a pure diffraction of rays passing through the edge of an aperture, defined in Remark 2.3.

Corollary 3.4: Let $L\subset(U,\omega)$ be an incident system of rays. Then the edge diffracted system of rays,

$$\tilde{L}=(\tilde{K}\circ A)(L),$$

is a regular intersection (cf. Ref. 24) of two smooth components: $\tilde{L}_1=(\tilde{K}\circ A^I)(L)$ and $\tilde{L}_2=(\tilde{K}\circ A^D)(L)$, i.e.,

$$\tilde{L}=\tilde{L}_1\cup\tilde{L}_2,\quad \dim\tilde{L}_1\cap\tilde{L}_2=\dim\tilde{L}_1-1,$$
$$T_x(\tilde{L}_1\cap\tilde{L}_2)=T_x\tilde{L}_1\cap T_x\tilde{L}_2.$$

Thus we see that the caustic caused by the edge diffraction has three components: (1) the caustic of $\tilde{L}_1$, which is a part of the caustic in incident wave front L; (2) the caustic, purely by diffraction on the edge, i.e., the caustic of $\tilde{L}_2$; and (3) the image $\pi_V(\tilde{L}_1\cap\tilde{L}_2)$ of the rays passing exactly through an edge.

Definition 3.5: The set $\pi_V(\tilde{L}_1\cap\tilde{L}_2)\subset V$ is called the quasicaustic by diffraction on aperture. The rays belonging

to the quasicaustic that are contained in the aperture plane we will call the rays at infinity.

Usually the quasicaustics describe the structure of shadows and half-shadows in configurational space V (see Fig. 1).

Proposition 3.6: (1) Generic caustics by diffraction on the half-line aperture on the plane are diffeomorphic to the $\tilde{A}_2$, $\tilde{A}_3$, $B_2 \cong C_2$, B_3 boundary caustics. Normal forms for their generating families as images $A(L)$ [or pairs (A,L) in general position] are the following:

$\tilde{A}_2$: $-\frac{1}{3}\lambda^3 + \lambda(q_2 - a) - \frac{1}{2}q_1\lambda^2$, $a > 0$, and $A := \{q_1 = 0,\ q_2 \leqslant 0\}$;

$\tilde{A}_3$: $-\frac{1}{4}\lambda^4 + \lambda(q_2 - a) - \frac{1}{2}q_1\lambda_2$, $a > 0$, and $A := \{q_1 = 0,\ q_2 \leqslant 0\}$;

B_2: $-\frac{1}{2}\lambda^2 + q_2\lambda - \frac{1}{2}q_1\lambda^2$, $\{\lambda \geqslant 0\}$, and $A := \{q_1 = 0,\ q_2 \leqslant 0\}$;

B_3: $-\frac{1}{3}\lambda^3 - \frac{1}{2}q_1\lambda^2 + \lambda(q_2 - q_1 a)$, $\{\lambda \geqslant 0\}$, and $A := \{q_1 = 2a,\ q_2 \leqslant 2a^2\}$, $a > 0$;

where λ is a Morse parameter and a is the moduli of the common position.

(2) In generic one-parameter families of caustics by diffraction on the half-line aperture, which do not pass through infinity, the only possible configurations are those described in metamorphoses of optical caustics (see Ref. 21, p. 113, and Ref. 25) and the additional cases illustrated in Fig. 2.

Proof: It is easily seen that $\tilde{K} = \text{graph } \pi_M \subset \Xi$ is generated locally by

$$K(r,q_1,q_2) = q_2 r - \tfrac{1}{2}q_1 r^2.$$

The only stable systems of rays $\tilde{K}(L) \subset (T^*V, \omega_V)$ are generated in (M,ω) by $L := \{(r,s);\ s = -(\partial F/\partial r)(r)\}$, where

A_1: $F_1(r) = -\frac{1}{2}r^2$,

A_2: $F_2(r) = -\frac{1}{3}r^3$,

A_3: $F_3(r) = -\frac{1}{4}r^4$ (cf. Refs. 18 and 21).

Let the aperture be defined in its normal form by $q_1 = 0$, $q_2 \leqslant 0$ (so $A \subset \Pi$). Thus we have the boundary singularities (cf. Ref. 26) $A(L)$ defined in $(\tilde{M},\tilde{\omega})$ by the following generating functions:

$\tilde{A}_1$: $\tilde{F}_1(\tilde{r}) = -\frac{1}{2}\tilde{r}^2$, $\{\tilde{r} \geqslant 0\}$;

$\tilde{A}_2$: $\tilde{F}_2(\tilde{r}) = -\frac{1}{3}\tilde{r}^3$, $\tilde{r} \in \mathbb{R}$;

$\tilde{A}_3$: $\tilde{F}_3(\tilde{r}) = -\frac{1}{4}\tilde{r}^4$, $\{\tilde{r} \geqslant 0\}$.

Taking A_i in the general position with respect to A we obtain part (1) of Proposition 3.6. Part (2) follows by checking all the possible one-parameter evolutions (where the quasicaustic is not passing through infinity) of the stable caustic on the plane and in the presence of the half-line aperture. Two possible directions of intersection of the A_2 caustic by an edge of the aperture give us the cases (a) and (b) in Fig. 2. The evolution of an edge of the aperture passing through the ray tangent to the cusp caustic A_3 is illustrated in Fig. 2(c). Finally an evolution through the intersection point of the $A_2 + A_2$ caustic gives us the case of Fig. 2(d). This completes the proof of Proposition 3.6. □

Looking at the position of the quasicaustic in the diffraction problem with a half-plane aperture in $\mathbb{R}^3$ we can eliminate the C_4-boundary caustic. Thus we have the following proposition.

Proposition 3.7: Generic caustics by diffraction on the half-plane aperture in $\mathbb{R}^3$ are diffeomorphic to the $\tilde{A}_2$, $\tilde{A}_3$, $\tilde{A}_4$, B_2, B_3, B_4, F_4 boundary caustics.

Remark 3.8: (1) For the general linear hyperbolic system of first order (cf. Ref. 7),

$$\mathcal{L}u = u_t + \sum_{\nu=1}^{3} A^{\nu}\frac{\partial u}{\partial x_\nu} + Bu = 0,$$

where u represents, say, in the case of crystal optics, the pair of vectors (E,H), and $\mathcal{L}u = 0$ corresponds to Maxwell's equations. In the geometrical optics approximation, we obtain another characteristic equation (eikonal equation)

$$\det\left(\Phi_t + \sum_{\nu=1}^{3} A^{\nu}\frac{\partial \Phi}{\partial x_\nu}\right) = 0,$$

for the phase function $\Phi(x,t)$; $u \sim e^{i\omega\Phi(x,t)}a^0(x,t)$. In this case the conical refraction in crystal optics is an example of a

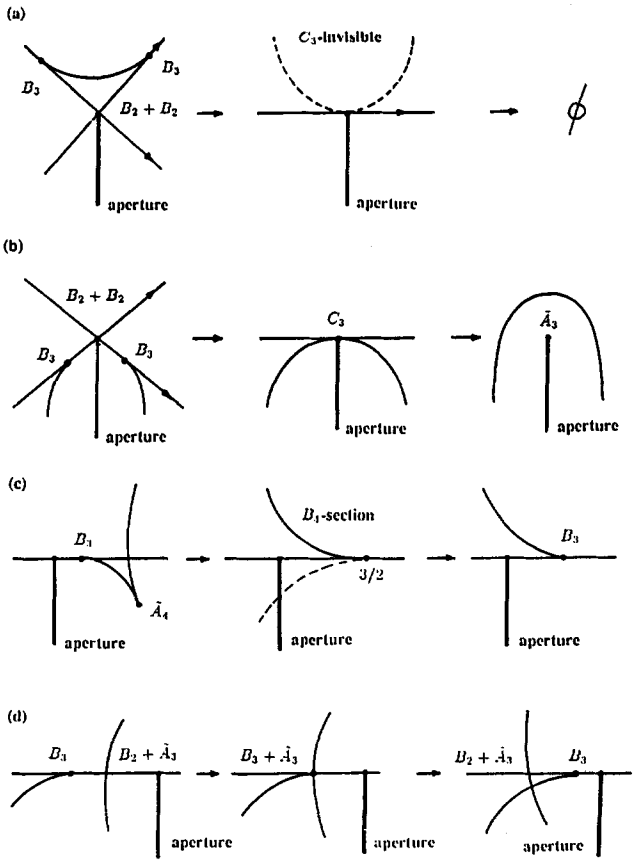

FIG. 2. Transformations of caustics in the presence of aperture.

Lagrangian variety quite generally situated in the associated phase space (cf. Refs. 6 and 7).

(2) In the edge diffraction on system of apertures (mentioned in Ref. 1) the singularities of the distance function are classified by the singularities on the many-dimensional corners.[27] In very constrained systems of apertures the classification is obtained using the methods of the theory of singularities of functions on singular varieties (cf. Refs. 9 and 16).

(3) The generic quasicaustic in the edge diffraction in $\mathbb{R}^3$, corresponding to the F_4 singularity of the distance function (cf. Ref. 25), is realized geometrically (see Fig. 1) when the curve of rays passing through the edge on the incident wave front is tangent to a constant curvature line on the wave front. This situation is generic (cf. Ref. 21).

IV. DIFFRACTION ON SMOOTH OBSTACLES

Now we can apply an introduced symplectic framework to describe the diffraction on smooth closed surfaces in $\mathbb{R}^3$. The problem is connected to the Riemannian obstacle problem (cf. Ref. 28), i.e., determination of geodesics on a Riemannian manifold with smooth boundary. Any geodesic on such a manifold is C^1 and consists of generically finitely many so-called switchpoints, where the geodesic has an initial or end point according to whether it lies in the interior part of the manifold or on the boundary. Cauchy uniqueness for manifolds with a boundary states that every boundary point (point of an obstacle) has a neighborhood in which, if two geodesic segments with the same initial point, initial tangent vector, and length do not coincide, then one of them has its right end point in the interior part of the manifold and is an involute of the other (in the planar case it lies on an appropriate involute of the obstacle curve). A geodesic γ' that has the same initial point, initial tangent vector, and length as γ is called an involute of a geodesic γ. The reformulation of the above obstacle problem in terms of geometrical optics of diffraction needs a definition of a surface diffracted ray. A surface diffracted ray is produced when a ray is incident tangentially on a smooth boundary or interface. It is a geodesic on the surface in the metric nds, where n is the refractive index of the medium on the side of the surface containing the incident ray. At every point it sheds a diffracted ray along its tangent (cf. Refs. 1 and 22). A surface diffracted ray is also produced on the second side of an interface by a ray incident from the first side at the critical angle [arcsin (n_1/n_2)]. In this case at every point it sheds rays back toward the first side at the critical angle. However, in what follows we will neglect these rays.

Let us consider an open subset S of an obstacle surface in $\mathbb{R}^3$. Let l_1 be the initial tangent line to the geodesic segment γ on S, and let l_2 be a tangent line to S. We say that l_2 is subordinate to l_1 with respect to an obstacle S if l_2 [or its piece in $(\mathbb{R}^3,S)$] belongs to the geodesic segment with the same initial point and the same tangent vector as γ has. By simple checking we have the following (cf. Ref. 18).

Proposition 4.1: Let γ be a geodesic flow on S. Then the set

$$A = \{(l,\tilde{l})\in\Pi;\ \tilde{l} \text{ is subordinate to } l \text{ with respect to } S \text{ and geodesic flow } \gamma\}$$

is a Lagrangian subvariety of Π defining the diffraction process on an obstacle S.

Now we look for the generic pairs (A,L). At first we consider the planar case.

Proposition 4.2: For the generic obstacle curve on the plane the only possible canonical varieties $A\subset\Pi$ have the following normal forms of generating families (or functions):

$$\tilde{A}_2\colon G(r,\tilde{r}) = -\tfrac{1}{12}(r^3+\tilde{r}^3),\ \text{(obstacle curve } q_2 = -q_1^2),$$

$$\tilde{H}_3\colon G(\lambda_1,\lambda_2,r,\tilde{r}) = \tfrac{9}{10}(\lambda_1^5+\lambda_2^5) - r\lambda_1^3 - \tilde{r}\lambda_2^3 + \tfrac{1}{2}r^2\lambda_1 + \tfrac{1}{2}\tilde{r}^2\lambda_2,$$

$$\text{(obstacle curve } q_2 = q_1^3),$$

$$\tilde{A}_{2,2}\colon G(r,\tilde{r}) = \tfrac{1}{2}(r|r| + \tilde{r}|\tilde{r}|),\ \text{(double tangent)}.$$

Proof: Let us take the noninflection point of the generic curve. Parametrically the curve is given as $(q_1,q_2) = (v,-v^2)$, $v\in\mathbb{R}$, and the corresponding family of tangent lines corresponding to the given incident ray has the form

$$(q_1,q_2) = (0,v^2) + u(1,-2v),\quad u\in\mathbb{R}.$$

By identification

$$s = v^2,\quad r = 4v/(1+4v^2),$$
$$\tilde{s} = \bar{v}^2,\quad \tilde{r} = 4\bar{v}/(1+4\bar{v}^2),$$

where $(v,\bar{v})\in\mathbb{R}^2$ parametrize the variety A, we obtain the case $\tilde{A}_2$ that corresponds to the Cartesian product of two ordinary folds. Taking the inflection point for an obstacle curve, we obtain, in the same way, the following parametrization for $A\subset\Pi$:

$$s = -2v^3,\quad r = \frac{3v^2}{\sqrt{1+9v^4}},\quad \tilde{s} = -2\bar{v}^3,\quad \tilde{r} = \frac{3\bar{v}^2}{\sqrt{1+9\bar{v}^4}}.$$

After straightforward calculations we obtain the generating family for it, denoted by $\tilde{H}_3$. Analogously we obtain the $\tilde{A}_{2,2}$ case. □

Corollary 4.3: For (A,L) in the general position we have the possible stable images $A(L)\subset(\tilde{M},\tilde{\omega})$,

$$A_2\colon \tilde{F}_1(\tilde{r}) = -\tfrac{1}{12}\tilde{r}^3,$$
$$H_3\colon \tilde{F}_2(\lambda,\tilde{r}) = \tfrac{9}{10}\lambda^5 - \tilde{r}\lambda^3 + \tfrac{1}{2}\tilde{r}^2\lambda,$$
$$A_{2,2}\colon \tilde{F}_3(\tilde{r}) = \tfrac{1}{2}|\tilde{r}|\tilde{r},$$

and the generating families for their corresponding configurational images,

$$K(A_2)\colon F_1(\lambda,q_1,q_2) = -\tfrac{1}{12}\lambda^3 + q_2\lambda - \tfrac{1}{2}q_1\lambda^2,$$
$$K(H_3)\colon F_2(\lambda_1,\lambda_2,q_1,q_2) = \tfrac{9}{10}\lambda_1^5 - \lambda_2\lambda_1^3 + \tfrac{1}{2}\lambda_2^2\lambda_1 + q_2\lambda_2 - \tfrac{1}{2}q_1\lambda_2^2,$$
$$K(A_{2,2})\colon F_3(\lambda,q_1,q_2) = \tfrac{1}{2}\lambda|\lambda| + q_2\lambda - \tfrac{1}{2}q_1\lambda^2$$

[see Figs. 3(a)–3(c) and also the figures in Ref. 22].

Proof: In the general position of A and L, only one point

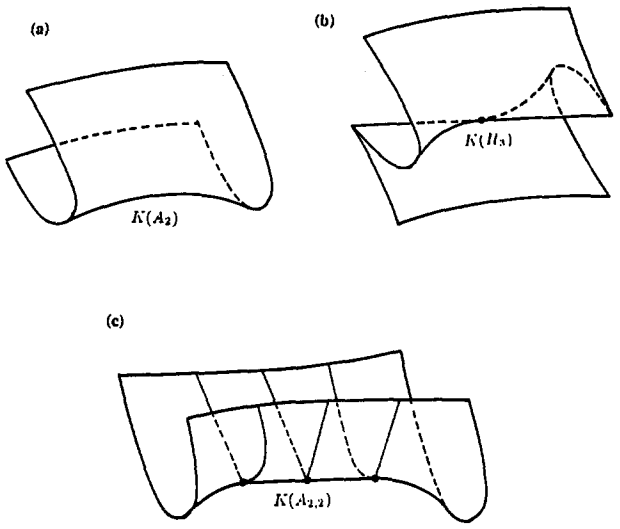

FIG. 3. Canonical varieties for the generic obstacle curve on the plane.

of L is tangent to an obstacle curve in the neighborhood of the considered point of this curve. Hence in the calculation of $(\tilde{K}\circ A)(L)$ in all the cases ($\tilde{A}_2$, $\tilde{H}_3$, and $\tilde{A}_{2,2}$) it is necessary to put $r=$ const in generating families of Proposition 4.2. □

Remark 4.4: (A) The first, most important, results in obstacle geometry and its correspondence to the structure of singular orbits of H_3 and H_4 group actions were discovered by Shcherbak.[16] The aim of the present paper is to show how singular wave front evolutions appear in the general setting of the mathematical theory of optics (cf. Refs. 5, 6, and 18) and to complete the investigations of the caustics and quasicaustics that appear there. As we see, the planar obstacle problem is connected to the studies of tangent developables. More degenerated singularities there can be described using the blowing-up construction (cf. Ref. 29).

(B) The $K(A_{2,2})$ singularity appeared as an adjacent to the higher singular one (see Fig. 4) in a generic one-parameter family of obstacles

$$q_2 = -\tfrac{1}{4}q_1^4 + \tfrac{1}{2}aq_1^2 - \tfrac{1}{4}a^2, \quad a\in\mathbb{R}_+,$$

i.e.,

$$r = -2av_\epsilon - 3\epsilon\sqrt{a}v_\epsilon^2 + (4a^3-1)v_\epsilon^3 + O(v_\epsilon^4),$$
$$s = 2\epsilon a^{3/2}v_\epsilon + 4av_\epsilon^2 + 3\epsilon\sqrt{a}v_\epsilon^3 + \tfrac{3}{4}v_\epsilon^4,$$

$\epsilon = \pm 1$, $v_+\geqslant 0$, $v_-\leqslant 0$.

(C) We can see that by choosing the special symplectic

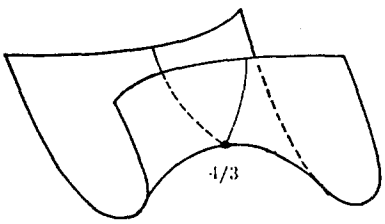

FIG. 4. The higher-order singularity of a canonical variety.

structure fibered over (p_1,p_2) in the H_3 case, we can investigate only a cuspidal edge of $A(L)$. In fact, with its generating family

$$F_2'(\lambda,\mu,p) = F_2(\lambda,\mu) - \mu_1 p_1 - \mu_2 p_2,$$

after reduction of the μ_1, μ_2, and λ_2 parameters, we obtain the generating family for the H_2 singularity,

$$F_2'(\lambda,p) = \tfrac{2}{10}\lambda^5 - p_2\lambda^3 + \tfrac{1}{2}p_2^2\lambda,$$

and its level sets (wave fronts) as in Table 2 of Ref. 16. This observation is connected with the much more general feature of obstacle singular wave front evolutions; namely, all singularities in obstacle geometry as indicated in Table 2 of Ref. 16 are generated by the generalized open swallowtails [in $(\tilde{M},\tilde{\omega})$ space] with generating family (see Ref. 8, p. 106)

$$\tilde{A}_{2(k+1)}: \int_0^\lambda \left(x^{k+1} + \sum_{i=2}^{k+1} \bar{s}_{i-1}x_{k-i+1}\right)^2 dx.$$

The Ξ_l ($l\geqslant 1$), Δ_l ($l\geqslant 2$) (cf. Ref. 16) singular wave front evolutions are reconstructed from $\tilde{A}_{2(k+1)}$ singularities by specifying appropriate common generic positions of $A\subset\Pi$ and $\tilde{A}_{2(k+1)}\subset(\tilde{M},\tilde{\omega})$.

V. VECTOR FIELDS ON CAUSTICS AND QUASICAUSTICS

As we can see from the preceding sections, caustics in the wave front evolution, or in a diffracted wave front on the aperture, are defined as bifurcation sets for the corresponding generating family (Morse family[2,17]) of functions or the family of functions on the manifold with boundary, respectively (cf. Refs. 20 and 26). To investigate the structure of these sets and modules of tangent vector fields on them, in what follows we shall consider the real analytic or holomorphic functions (germs). For the ordinary caustics, defined as the critical values of the Lagrange projections (cf. Ref. 20) from the Lagrangian submanifolds, which are not necessarily fibered by optical rays, the procedure is the following.[3,4]

Let f: $(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a holomorphic function of finite codimension, i.e., the dimension of the quotient $\mathcal{O}_{(x)}/J(f)$ as a complex vector space is finite, where $\mathcal{O}_{(x)}$ denotes the ring of holomorphic functions h: $(\mathbb{C}^n,0)\to(\mathbb{C},0)$ and $J(f)$ is the ideal in $\mathcal{O}_{(x)}$ generated by the partial derivatives $\partial f/\partial x_1,\ldots,\partial f/\partial x_n$. Let $\mathcal{M}_{(x)}$ denote the maximal ideal in $\mathcal{O}_{(x)}$. If $g_1,\ldots,g_p$ is a basis for $\mathcal{M}_{(x)}/J(f)$, then

$$F\colon (\mathbb{C}^n\times\mathbb{C}^p,0)\to(\mathbb{C},0),$$
$$F(x,a) = f(x) + \sum_{i=1}^{p} a_i g_i(x)$$

is a miniversal unfolding of f (cf. Ref. 30).

The caustic of F [or bifurcation set of F (see Refs. 4 and 9)] is the following set (germ):

$$B(F) = \{a\in\mathbb{C}^p;\ F_a \text{ has a degenerate critical point}\}.$$

The set of critical values of π: $(\Sigma F,0)\to(\mathbb{C}^p,0)$ (π is a canonical projection on the second factor), where

$$\Sigma F = \left\{(x,a)\in\mathbb{C}^n\times\mathbb{C}^p;\ \frac{\partial F}{\partial x_1} = \cdots = \frac{\partial F}{\partial x_n} = 0\right\},$$

is the caustic. It appears to be important to know the mod-

ules of tangent vector fields to caustics (as well as to wave fronts,[9,20,31] which is easier). They are useful in the reduction of functional moduli in the classification of generic symmetric and nonsymmetric Lagrangian submanifolds (cf. Ref. 20, p. 344, and Ref. 32). We recall some necessary definitions from Refs. 3 and 33. The set of germs of holomorphic vector fields on $\mathbb{C}^p$, at 0, tangent to the nonsingular part of $B(F)$, is called the set of logarithmic vector fields of $B(F)$ at 0. It is denoted also by Derlog $B(F)$. In Refs. 4, 9, and 10 (see, also, Refs. 31 and 33) a general method for computing these vector fields was given. It was shown that A_k singularities are the only ones whose module of tangent vector fields to $B(F)$ is free (i.e., caustic is a free divisor[33]). Applying the method used in these papers we investigate the modules of vector fields tangent to the quasicaustics in diffraction on apertures (this is a first step in the investigation of the structure of caustics by diffraction).

Let $\mathcal{O}_{(y,x)}$ denote the ring of holomorphic functions h: $(\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C},0)$. The hypersurface $S=\{y=0\}$ corresponds to the boundary of an aperture. Following the general scheme used in Ref. 20 for boundary singularities, we shall consider holomorphic functions f: $(\mathbb{C}\times\mathbb{C}^n,0)\to(\mathbb{C},0)$ of finite codimension, i.e.,

$$\dim_{\mathbb{C}} \mathcal{O}_{(y,x)}/\Delta(f)<\infty,$$

where

$$\Delta(f)=\left\langle y\frac{\partial f}{\partial y},\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right\rangle$$

denotes the ideal in $\mathcal{O}_{y,x}$ generated by the partial derivatives $\partial f/\partial x_1,\ldots,\partial f/\partial x_n$ and $y\,\partial f/\partial y$ (cf. Refs. 20 and 34). Let $g_0,\ldots,g_{\mu-1}$ form a basis for $\mathcal{O}_{(y,x)}/\Delta(f)$ with $g_0=1$ and $g_i\in\mathcal{M}_{(y,x)}$. Then the miniversal deformation, in the category of deformations of functions on the manifold with a boundary, as a Morse family for the corresponding diffracted Lagrangian variety (cf. Refs. 13 and 24) is defined as follows:

$$F\colon (\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^{\mu-1},0)\to(\mathbb{C},0),$$
$$F(y,x,a)=f(y,x)+\sum_{i=1}^{\mu-1}a_ig_i(y,x).$$

Proposition 5.1: The caustic (or bifurcation set) from diffraction on the aperture, having the generating family F: $(\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^p,0)\to(\mathbb{C},0)$ (p is not necessarily minimal) of functions on the manifold with boundary (extended edge) has three components

(1) $B_1(F)=\{a\in\mathbb{C}^p;\ F(\cdot,\cdot,a)$ has a degenerate critical point$\}$,

(2) $B_2(F)=\{a\in\mathbb{C}^p;\ F(0,\cdot,a)$ has a degenerate critical point$\}$,

(3) $Q(F)=\{a\in\mathbb{C}^p;\ F(\cdot,\cdot,a)$ has a critical point on $S=\{y=0\}\}$.

Proof: By Corollary 3.4, we have the three isotropic submanifolds defining the system of diffracted rays $\tilde{L}_1$, $\tilde{L}_2$, and $\tilde{L}_1\cap\tilde{L}_2$. It is easily seen that in terms of the generating family/distance function F, the corresponding caustics can be written in forms (1)–(3) of Proposition 5.1. □

The set (germ)

$$(\Sigma_r F,0)=\left(\left\{(x,a)\in\mathbb{C}^n\times C^p;\ \frac{\partial F}{\partial y}\Big|_{s\times\mathbb{C}^p}=\frac{\partial F}{\partial x_1}\Big|_{s\times\mathbb{C}^p}=\cdots=\frac{\partial F}{\partial x_n}\Big|_{s\times\mathbb{C}^p}=0\right\},0\right)$$

is called the restricted critical set.

Using the Splitting Lemma[30] and the versality property of F, we have the following proposition.

Proposition 5.2: (A) The restricted critical set $(\Sigma_r F,0)$ is the germ of a smooth manifold of dimension $p-1$.

(B) The quasicaustic of F, $(Q(F),0)$, is an image of $(\Sigma_r F,0)$ by the natural projection π: $\Sigma_r F,0\to\mathbb{C}^p,0$ to the second factor.

The set of logarithmic vector fields of $Q(F)$ at 0 is defined (cf. Refs. 3 and 33) to be the set of germs of holomorphic vector fields on $\mathbb{C}^p$ at 0, tangent to the nonsingular part of $Q(F)$; it is an $\mathcal{O}_{(a)}$ module.

Proposition 5.3: Let $\xi\in$Derlog $Q(F)$, then it is π liftable, i.e., for some germ of a vector field $\tilde{\xi}$, on $\mathbb{C}^n\times\mathbb{C}^p$, tangent to $\Sigma_r F$ at 0, we have

$$\xi\circ\pi=d\pi\circ\tilde{\xi}.$$

Proof: ξ lifts uniquely by π at every point $a\in\mathbb{C}^p-\Gamma(\pi|_{\Sigma_r F})$. Hence ξ lifts to a holomorphic vector field $\tilde{\xi}_1$ on $\mathbb{C}^n\times\mathbb{C}^p$, tangent to $\Sigma_r F$ and defined off a set of codimension 2 in $\mathbb{C}^n\times\mathbb{C}^p$. By Hartog's theorem, $\tilde{\xi}_1$ extends to a holomorphic vector field $\tilde{\xi}$ tangent to $\Sigma_r F$.

Now using the π-lowerable vector fields $\tilde{\xi}$ tangent to $\Sigma_r F$ we will construct the module Derlog $Q(F)$. Let F be as above. We define the ideal

$$I(F)=\left\langle\psi(x,a),\frac{\partial\overline{F}}{\partial x_1}(x,a),\ldots,\frac{\partial\overline{F}}{\partial x_n}(x,a)\right\rangle\mathcal{O}_{(x,a)},$$

where ψ and $\overline{F}$ are given by decomposition:

$$F(y,x,a)=F(0,x,a)+y\psi(x,a)+y^2g(y,x,a),$$
$$\overline{F}(x,a):=F(0,x,a).$$

Let

$$\tilde{\xi}=\sum_{i=1}^{n}\beta_i\frac{\partial}{\partial x_i}+\sum_{i=1}^{p}\gamma_i\frac{\partial}{\partial a_i},\quad \beta_i,\gamma_i\in\mathcal{O}_{(x,a)},$$

be the germ of a vector field at $0\in\mathbb{C}^n\times\mathbb{C}^p$, tangent to $\Sigma_r F$. Then we have

$$\tilde{\xi}\left(\frac{\partial F}{\partial y}(0,x,a)\right)\in I(F)$$

and

$$\tilde{\xi}\left(\frac{\partial F}{\partial x_i}(0,x,a)\right)\in I(F),\quad i=1,\ldots,n.$$

For our

$$F(y,x,a)=f(y,x)+\sum_{i=1}^{\mu-1}a_ig_i(y,x),$$

we have

$$\psi(x,a)=\frac{\partial f}{\partial y}(0,x)+\sum_{i=1}^{\mu-1}a_i\frac{\partial g_i}{\partial y}(0,x).$$

So we need

$$\sum_{i=1}^{n}\beta_i\frac{\partial\psi}{\partial x_i}+\sum_{i=1}^{\mu-1}\gamma_i\frac{\partial g_i}{\partial y}\Big|_{0\times\mathbb{C}^n}\in I(F)$$

and

$$\sum_{i=1}^{n}\beta_i\frac{\partial^2\overline{F}}{\partial x_i\,\partial x_j}+\sum_{i=1}^{\mu-1}\gamma_i\frac{\partial\overline{g}_i}{\partial x_j}\in I(F),\quad 1\leqslant j\leqslant n,$$

where $\overline{g}(x):=g(0,x)$. Thus we obtain the following lemma.

Lemma 5.4: $\tilde{\xi}$ is a lifting of $\xi\in\mathrm{Derlog}\,Q(F)$,

$$\xi=\sum_{i=1}^{p}\alpha_i(a)\frac{\partial}{\partial a_i},$$

if and only if, for some $\beta_i\in\mathcal{O}_{(x,a)}$ $(i=1,\ldots,n)$, we have

$$\begin{aligned}&\sum_{i=1}^{n}\beta_i\frac{\partial\psi}{\partial x_i}+\sum_{i=1}^{\mu-1}\alpha_i\frac{\partial g^i}{\partial y}\Big|_{0\times\mathbb{C}^n}\in I(F),\\&\sum_{i=1}^{n}\beta_i\frac{\partial^2\overline{F}}{\partial x_i\,\partial x_j}+\sum_{i=1}^{\mu-1}\alpha_i\frac{\partial\overline{g}_i}{\partial x_j}\in I(F).\end{aligned}\tag{5.1}$$

We have chosen the normal form for F in such a way that the variables $a_\mu,\ldots,a_p$ $(p\geqslant\mu-1)$ do not appear in F. Now, following the general scheme used in Refs. 3 and 10 for ordinary bifurcation sets, we can propose a procedure for constructing the tangent vector fields to quasicaustics.

By the Preparation Theorem,[20,30] the module

$$\mathcal{O}_{(y,x,a)}/\overline{\Delta}(F),$$

where

$$\overline{\Delta}(F)=\left\langle y\frac{\partial F}{\partial y},\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n}\right\rangle\mathcal{O}_{(y,x,a)},$$

is a free $\mathcal{O}_{(a)}$ module[20] generated by $1,g_1,\ldots,g_{\mu-1}$. So, for any $h\in\mathcal{O}_{(y,x,a)}$, we can write

$$\begin{aligned}h(y,x,a)=&\beta(y,x,a)y\frac{\partial F}{\partial y}(y,x,a)\\&+\sum_{i=1}^{n}\beta_i(y,x,a)\frac{\partial F}{\partial x_i}(y,x,a)\\&+\sum_{j=1}^{\mu-1}a_j(a)g_j(y,x)+\alpha(a),\end{aligned}\tag{5.2}$$

for some $\beta_i\in\mathcal{O}_{(y,x,a)}$, $\alpha_j\in\mathcal{O}_{(a)}$, $\alpha\in\mathcal{O}_{(a)}$.

By straightforward checking we obtain the following proposition.

Proposition 5.5: Let $h\in\mathcal{O}_{(y,x,a)}$ satisfy

$$\frac{\partial h}{\partial y}\Big|_{0\times\mathbb{C}^n\times\mathbb{C}^p}\in I(F),\quad\frac{\partial h}{\partial x_i}\Big|_{0\times\mathbb{C}^n\times\mathbb{C}^p}\in I(F),\quad i=1,\ldots,n.$$

Then the vector field

$$\xi=\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial a_i},$$

where α_i, $i=1,\ldots,\mu-1$, are defined in (5.2) and α_i, $i=\mu,\ldots,p$, are arbitrary holomorphic functions from $\mathcal{O}_{(a)}$, is tangent to the quasicaustic $Q(F)=\pi(\Sigma,F)$. Conversely, suppose

$$\xi=\sum_{i=1}^{p}\alpha_i\frac{\partial}{\partial a_i}$$

is tangent to $Q(F)$. Then there is some $h\in\mathcal{O}_{(y,x,a)}$ as above with

$$h=\sum_{i=1}^{n}\beta_i\frac{\partial F}{\partial x_i}+\beta y\frac{\partial F}{\partial y}+\sum_{i=1}^{\mu-1}\alpha_i g_i+\alpha,$$

and

$$\frac{\partial h}{\partial x_i}\Big|_{0\times\mathbb{C}^n\times\mathbb{C}^p}\in I(F),\quad\frac{\partial h}{\partial y}\Big|_{0\times\mathbb{C}^n\times\mathbb{C}^p}\in I(F).\qquad\square$$

We see that the set of all such h with $(\partial h/\partial y)|_{\overline{s}}\in I(F)$, $(\partial h/\partial x_i)|_{\overline{s}}\in I(F)$, $1\leqslant i\leqslant n$, form an $\mathcal{O}_{(a)}$ module: in fact, it is the kernel of the $\mathcal{O}_{(a)}$ module homomorphism,

$$\Phi:\ \mathcal{O}_{(y,x,a)}\ni h\to\left(\frac{\partial h}{\partial y},\frac{\partial h}{\partial x_1},\ldots,\frac{\partial h}{\partial x_n}\right)\in\left(\frac{\mathcal{O}_{(y,x,a)}}{I(F)+\langle y\rangle\mathcal{M}_{(y,x,a)}}\right)^{n+1}.$$

Here, $\overline{\Delta}(F)\subset I(F)+\langle y\rangle\mathcal{M}_{(y,x,a)}$ and clearly the set of tangent vector fields to $Q(F)$ is a finitely generated $\mathcal{O}_{(a)}$ module. We denote $\overline{s}=\mathcal{O}\times\mathbb{C}^n\times\mathbb{C}^p$.

VI. QUASICAUSTICS OF SIMPLE AND UNIMODAL BOUNDARY SINGULARITIES

The simple singularities of functions on the boundary $\{y=0\}$ of a manifold with a boundary were classified in Ref. 20, p. 281. Their miniversal unfoldings are

$$\tilde{A}_\mu:\ \pm y\pm x^{\mu+1}+\sum_{i=1}^{\mu-1}a_ix^i,\quad\mu\geqslant1;$$

$$B_\mu:\ \pm y^\mu\pm x^2+\sum_{i=1}^{\mu-1}a_i\,y^{\mu-i},\quad\mu\geqslant2;$$

$$C_\mu:\ yx\pm x^\mu+\sum_{i=1}^{\mu-1}a_ix^{\mu-i},\quad\mu\geqslant2;$$

$$\tilde{D}_\mu:\ \pm y+x_1^2x_2\pm x_2^{\mu-1}+\sum_{i=1}^{\mu-2}a_ix_2^i+a_{\mu-1}x_1,\quad\mu\geqslant4;$$

$$\begin{aligned}\tilde{E}_6:&\ \pm y+x_1^3\pm x_2^4+a_1x_1+a_2x_2+a_3x_2^2\\&+a_4x_1x_2+a_5x_1x_2^2;\end{aligned}$$

$$\begin{aligned}\tilde{E}_7:&\ \pm y+x_1^3+x_1x_2^3+a_1x_1+a_2x_2+a_3x_2^2+a_4x_1x_2\\&+a_5x_2^3+a_6x_2^4;\end{aligned}$$

$$\begin{aligned}\tilde{E}_8:&\ \pm y+x_1^3+x_2^5+a_1x_1+a_2x_2+a_3x_2^2+a_4x_1x_2+a_5x_2^3\\&+a_6x_1x_2^2+a_7x_1x_2^3;\end{aligned}$$

$$F_4:\ \pm y^2+x^3+a_2y+a_3x+a_1xy.$$

Thus we have, after direct checking, the following proposition.

Proposition 6.1: The quasicaustics for simple boundary singularities are

$$\tilde{A}_\mu,\ \tilde{D}_\mu,\ \tilde{E}_k:\ Q(F)=\varnothing,$$

$$B_\mu:\ Q(F)=\{a\in\mathbb{C}^{\mu-1};\ a_{\mu-1}=0\},$$

$$C_\mu:\ Q(F)=\{a\in\mathbb{C}^{\mu-1};\ a_{\mu-1}=0\},$$

$$F_4:\ Q(F)=\{a\in\mathbb{C}^3;\ a_2^2+\tfrac{1}{3}a_1^2a_3=0\}$$

(i.e., Whitney's cross-cap, see Fig. 1).

Thus we need to calculate only the module of vector fields tangent to $Q(F_4)$. Let us define the germ, at zero, of the variety $X:=Q(F_4)\cup\{a_1=0\}$. We see that the vector fields tangent to $(X,0)$ lie in Derlog $Q(F_4)$.

Proposition 6.2: The vector fields

$$V_1=-\frac{1}{6}a_1^2\frac{\partial}{\partial a_2}+a_2\frac{\partial}{\partial a_3},$$

$$V_2=a_1\frac{\partial}{\partial a_1}+a_2\frac{\partial}{\partial a_2},$$

$$V_3=-\frac{1}{3}a_1\frac{\partial}{\partial a_1}+a_3\frac{2}{3}\frac{\partial}{\partial a_3}$$

form a free basis for the $\mathcal{O}_{(a)}$ module Derlog X.

Before we prove this theorem we need the following proposition.

Proposition 6.3: For corank-2 boundary singularities F: $(\mathbb{C}\times\mathbb{C}\times\mathbb{C}^p,0)\to(\mathbb{C},0)$, the space of functions $h\in\mathcal{O}_{(y,x,a)}$ reconstructing the $\mathcal{O}_{(a)}$ module of vector fields tangent to the quasicaustic $Q(F)$ has the form

$$h(y,x,a)=\int_0^x\left(\frac{\partial F}{\partial y}(0,s,a)\psi_1(s,a)+\frac{\partial F}{\partial x}(0,s,a)\psi_2(s,a)\right)ds+y^2\xi(y,x,a),$$

where $\psi_i\in\mathcal{O}_{(x,a)}$ $(i=1,2)$, $\xi\in\mathcal{O}_{(y,x,a)}$.

Proof: Every function $h\in\mathcal{O}_{(y,x,a)}$ can be written in the form

$$h(y,x,a)=\eta_2(x,a)+y\eta_1(x,a)+y^2\eta(y,x,a),$$

and thus

$$\frac{\partial h}{\partial y}(0,x,a)=\eta_1(x,a),\quad \frac{\partial h}{\partial x}(0,x,a)=\frac{\partial\eta_2}{\partial x}(x,a).$$

By Proposition 5.5, we can take

$$\eta_1(x,a)\in I(F)\ \text{ and }\ \eta_2(x,a)=\int_0^x g(s,a)ds,\quad g\in I(F),$$

obtaining all functions

$$\eta_2(x,a)+y\eta_1(x,a)+y^2\eta(v,x,a)(\mathrm{mod}\,\overline{\Delta}(F)),$$

defining the $\mathcal{O}_{(a)}$ module of vector fields tangent to $Q(F)$. Now we see that

$$\eta_2(x,a)+y\eta_1(x,a)+y^2\eta(y,x,a)=\eta_2(x,a)+y^2\xi(y,x,a)\left(\mathrm{mod}\left\langle y\frac{\partial F}{\partial y},y\frac{\partial F}{\partial x}\right\rangle\mathcal{O}_{(y,x,a)}\right),$$

where $\xi\in\mathcal{O}_{(y,x,a)}$. Adding an element of $\langle y\rangle\overline{J}(F)$ [$\overline{J}(F)$ is an ideal of $\mathcal{O}_{(y,x,a)}$ generated by $\partial F/\partial y,\partial F/\partial x_1,\ldots,\partial F/\partial x_n$] does preserve the space of functions and does not affect the resulting vector field.

Proof of Proposition 6.2: $I(F_4)=\langle a_1x+a_2,3x^2+a_3\rangle\mathcal{O}_{(x,a)}$. By Proposition 6.3, taking $\psi_1,\psi_2,\xi\equiv1$, we have

$$h_1(x,a)=\tfrac{1}{2}a_1x^2+a_2x=-\tfrac{1}{8}a_1^2y+a_2x-\tfrac{1}{8}a_1a_3(\mathrm{mod}\,\overline{\Delta}(F_4)),$$

$$h_2(x,a)=y^2=-a_1xy-a_2y(\mathrm{mod}\,\overline{\Delta}(F_4)),$$

$$h_3(x,a)=x^3+xa_3=-\tfrac{1}{3}a_1xy+\tfrac{2}{3}a_3x(\mathrm{mod}\,\overline{\Delta}(F_4)).$$

Then the corresponding V_i belongs to Derlog $Q(F_4)$ $(i=1,2,3)$. By simple computation we obtain

$$V_1(a_1)=0,\quad V_2(a_1)=-a_1,\quad V_3(a_1)=-\tfrac{1}{3}a_1;$$

Thus $V_i\in$Derlog X as well. We also have that

$$\det(V_1(a),V_2(a),V_3(a))=-\tfrac{1}{3}a_1(a_2^2+\tfrac{1}{3}a_3a_1^2)$$

is a reduced equation for $(X,0)$; thus, by the results of Saito[33] (see, also, Ref. 3), we find that $(X,0)$ is a free divisor. □

We define the following ideals of $\mathcal{O}_{(y,x)}$ and $\mathcal{O}_{(y,x,a)}$, respectively:

$$\Theta(f)=\langle y\rangle J(f)+\left\langle\frac{\partial f}{\partial x_1},\ldots,\frac{\partial f}{\partial x_n}\right\rangle^2\mathcal{O}_{(y,x)}$$

and

$$\overline{\Theta}(F)=\langle y\rangle\overline{J}(F)+\left\langle\frac{\partial F}{\partial x_1},\ldots,\frac{\partial F}{\partial x_n}\right\rangle^2\mathcal{O}_{(y,x,a)}.$$

For determining all fields tangent to the quasicaustic we need the following lemma.

Lemma 6.4: The space $\mathcal{O}_{(y,x)}/\Theta(f)$ is finite dimensional. Its $\mathbb{C}$ basis also generates the quotient space $\mathcal{O}_{(y,x,a)}/\overline{\Theta}(F)$ as an $\mathcal{O}_{(a)}$ module.

Proof: $\Theta(f)\supset\Delta(f)$ and f is finitely determined as a boundary singularity. Thus $\mathcal{O}_{(y,x)}/\Theta(f)$ is $\mathbb{C}$-finite dimensional with the basis $\{g_1,\ldots,g_N\}$. Let us define the mapping

$$\Psi\colon(\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^p,0)\to(\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^{n(n+1)/2}\times\mathbb{C}^p,0),$$

$$\Psi(y,x,a)=\left(y\frac{\partial F}{\partial y}(y,x,a),y\frac{\partial F}{\partial x_1}(y,x,a),\ldots,y\frac{\partial F}{\partial x_n}(y,x,a),\frac{\partial F}{\partial x_i}(y,x,a)\frac{\partial F}{\partial x_j}(y,x,a),a\right),$$

with $1\leqslant i,j\leqslant n$, $i\leqslant j$, and ordered set of pairs (i,j). Thus we have

$$\mathcal{O}_{(y,x,a)}/\Psi^*(\mathcal{M}_{(y,x,a)})\mathcal{O}_{(y,x,a)}\cong\mathcal{O}_{(y,x)}/\Theta(f)\mathcal{O}_{(y,x)}.$$

By the Preparation Theorem,[30] every element h of $\mathcal{O}_{(y,x,a)}$ has the form

$$h(y,x,a)=\sum_{i=1}^N\phi_i\left(y\frac{\partial F}{\partial y}(y,x,a),y\frac{\partial F}{\partial x_1}(y,x,a),\ldots,y\frac{\partial F}{\partial x_n}(y,x,a),\frac{\partial F}{\partial x_i}(y,x,a)\frac{\partial F}{\partial x_j}(y,x,a),a\right)\times g_i(y,x).$$

Thus

$$\mathcal{O}_{(y,x,a)}/\overline{\Theta}(F)\cong\left\{\sum_{i=1}^N\psi_i(a)g_i(y,x)\right\},\quad\psi_i\in\mathcal{O}_{(a)},$$

which completes the proof of Lemma 6.4.

Let $\{g_1,\ldots,g_N\}$ be a $\mathbb{C}$ basis for $\mathcal{O}_{(y,x)}/\Theta(f)$. In general we have the following proposition.

Proposition 6.5: Functions $h\in\mathcal{O}_{(y,x,a)}$, which reconstruct the $\mathcal{O}_{(a)}$ module of vector fields tangent to $Q(F)$, can be written as

$$h(y,x,a)=\sum_{i=1}^N\alpha_i(a)g_i(y,x),$$

where

$$\sum_{i=1}^{N} \alpha_i(a) \frac{\partial g_i}{\partial y}(0,x) \in I(F),$$

$$\sum_{i=1}^{N} \alpha_i(a) \frac{\partial g_i}{\partial x_j}(0,x) \in I(F),$$

$1 \leqslant j \leqslant n$.

Proof: By Lemma 6.4, any $h \in \mathcal{O}_{(y,x,a)}$ can be written as

$$h(y,x,a) = \sum_{i=1}^{N} \alpha_i(a) g_i(y,x) + \beta(y,x,a) y \frac{\partial F}{\partial y}(y,x,a) + \sum_{j=1}^{n} \beta_j(y,x,a) y \frac{\partial F}{\partial x_j}(y,x,a) + \sum_{k,l=1}^{n} \beta_{k,l}(y,x,a) \frac{\partial F}{\partial x_k}(y,x,a) \frac{\partial F}{\partial x_l}(y,x,a),$$

where $\alpha_i \in \mathcal{O}_{(a)}$, $\beta, \beta_j, \beta_{kl} \in \mathcal{O}_{(y,x,a)}$. By simply checking the assumption of Proposition 5.5, we see that the three last terms in the above formula do not affect on the resulting vector field belonging to Derlog $Q(F)$. This proves Proposition 6.5. □

Proposition 6.6: The $\mathcal{O}_{(a)}$ module Derlog $Q(F_4)$, i.e., the module of holomorphic vector fields tangent to Whitnery's cross-cap, is generated by the vector fields

$$\begin{aligned} V_1 &= -\frac{1}{6} a_1^2 \frac{\partial}{\partial a_2} + a_2 \frac{\partial}{\partial a_3}, \\ V_2 &= a_1 \frac{\partial}{\partial a_1} + a_2 \frac{\partial}{\partial a_2}, \\ V_3 &= -\frac{1}{3} a_1 \frac{\partial}{\partial a_1} + \frac{2}{3} a_3 \frac{\partial}{\partial a_3}, \\ V_4 &= a_2 \frac{\partial}{\partial a_1} - \frac{1}{3} a_1 a_3 \frac{\partial}{\partial a_2}, \end{aligned} \tag{6.1}$$

which satisfy the relation

$$a_1 V_4 - 2a_3 V_1 + 3a_2 V_3 = 0.$$

Proof: We have

$$\Theta(f) = \langle y^2, x^2 y, x^4 \rangle \mathcal{O}_{(y,x)}$$

and

$$\mathcal{O}_{(y,x)} / \Theta(f) \cong [1, x, y, x^2, x^3, xy]_{\mathbb{C}}.$$

By Proposition 6.5 all functions $h \in \mathcal{O}_{(y,x,a)}$ leading to the construction of Derlog $Q(F_4)$ can be written in the form

$$h(y,x,a) = \alpha_1(a) + \alpha_2(a)x + \alpha_3(a)x^2 + \alpha_4(a)x^3 + \alpha_5(a)y + \alpha_6(a)xy,$$

where $\alpha_i \in \mathcal{O}(a)$, $i = 1,\ldots,6$, are such that

$$\begin{aligned} &\alpha_5(a) + \alpha_6(a)x \in I(F_4), \\ &\alpha_2(a) + 2\alpha_3(a)x + 3\alpha_4(a)x^2 \in I(F_4) \quad \text{(see Sec. V)}. \end{aligned} \tag{6.2}$$

By simple calculations we check that V_i, $i = 1,\ldots,4$, are tangent to Whitney's cross-cap. Calculations using power series or a homogeneous filtration show that these are the only vector fields generating Derlog $Q(F_4)$. In fact,

$$h = \alpha_1 - \tfrac{1}{3}\alpha_3 a_3 + (\alpha_2 - \tfrac{1}{3}\alpha_4 a_3)x + (\alpha_5 - \tfrac{1}{3}\alpha_3 a_1)y + (\alpha_6 - \tfrac{1}{3}\alpha_4 a_1)xy (\text{mod } \Delta(F_4)).$$

Hence all vector fields belonging to Derlog $Q(F_4)$ can be written in the form

$$V = \alpha_6 \frac{\partial}{\partial a_1} + \alpha_5 \frac{\partial}{\partial a_2} + \alpha_5' \frac{\partial}{\partial a_3} - \frac{1}{6} a_1 \alpha_6' \frac{\partial}{\partial a_2} + \alpha_4 V_3, \tag{6.3}$$

where $\alpha_4, \alpha_5, \alpha_6, \alpha_5', \alpha_6' \in \mathcal{O}(a)$ satisfy

$$\alpha_5 + \alpha_6 x \in I(F_4), \quad \alpha_5' + \alpha_6 x \in I(F_4), \tag{6.4}$$

which are a simple rewritten version of (6.2). Here we use the formula $x^2 = -\frac{1}{3} a_3 (\text{mod } I(F_4))$. Solving (6.4) using a power series, we obtain an expression for (6.3) that involves only V_i, $i = 1,2,3,4$. □

Proposition 6.5 gives an algorithm for calculating all vector fields tangent to quasicaustics corresponding to boundary singularities. Now we restrict our attention to quasicaustics corresponding to the unimodal boundary singularities.

Let us consider the miniversal deformations for parabolic boundary singularities[20]:

$$\begin{aligned} &F_{1,0}\colon\ y^3 + x^3 + a_1 y^2 x + a_2 xy + a_3 y^2 + a_4 y + a_5 x, \\ &K_{4,2}\colon\ y^2 + x^4 + a_1 yx^2 + a_2 xy + a_3 x^2 + a_4 x + a_5 y, \\ &D_{4,1}\ (= L_6)\colon\ \tfrac{1}{2}x_1^2 x_2 + \tfrac{1}{3}x_2^3 + yx_1 + a_1 yx_2 + \tfrac{1}{2}a_2 x_2^2 + a_3 y + a_4 x_1 + a_5 x_2, \end{aligned}$$

where a_1 is a modulus parameter. The Milnor number of these deformations is 6 and the boundary is $\{y = 0\}$. We treat these three cases separately, starting with $F_{1,0}$.

Proposition 6.7: The module Derlog $Q(F_{1,0})$ is not free and is generated by the following vector fields:

$$\begin{aligned} V_1 &= -\frac{1}{6} a_2^2 \frac{\partial}{\partial a_4} + a_4 \frac{\partial}{\partial a_5}, \\ V_2 &= a_2 \frac{\partial}{\partial a_2} + a_4 \frac{\partial}{\partial a_4}, \\ V_3 &= -a_2 \frac{\partial}{\partial a_2} + 2a_5 \frac{\partial}{\partial a_5}, \\ V_4 &= a_4 \frac{\partial}{\partial a_2} - \frac{1}{3} a_2 a_5 \frac{\partial}{\partial a_4}, \\ V_5 &= \frac{\partial}{\partial a_1}, \quad V_6 = \frac{\partial}{\partial a_3}. \end{aligned} \tag{6.5}$$

Proof: We have

$$I(F) = \langle a_2 x + a_4, 3x^2 + a_5 \rangle \mathcal{O}_{(x,a)}$$

and

$$\mathcal{O}_{(y,x)} / \Theta(f) = [1, x, y, x^2, x^3, xy, y^2, xy^2]_{\mathbb{C}}.$$

Thus

$$h = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 y + \alpha_6 xy + \alpha_7 y^2 + \alpha_8 xy^2.$$

The equations

$$\begin{aligned} \frac{\partial h}{\partial y}\bigg|_{y=0} &= \alpha_5 + \alpha_6 x \in I(F), \\ \frac{\partial h}{\partial x}\bigg|_{y=0} &= \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2 \in I(F) \end{aligned}$$

reduce the calculations to those in the proof of Proposition 6.6. □

In remaining cases we only need to calculate the one-jets of vector fields generating the module. We now treat the case $K_{4,2}$.

Proposition 6.8: All vector fields belonging to Derlog $Q(K_{4,2})$ have the following form:

$$\begin{aligned}&\left(\alpha_9-\frac{1}{2}a_1\alpha_5+U\alpha_6\right)\frac{\partial}{\partial a_1}+\left(\alpha_8-\frac{1}{6}a_1\alpha_4'\right.\\&\quad\left.-\frac{1}{4}a_2\alpha_5+\alpha_6\left(V-\frac{5}{12}a_1a_3\right)\right)\frac{\partial}{\partial a_2}\\&\quad+\left(\frac{1}{2}\alpha_3'+\frac{1}{2}\alpha_5a_3+\frac{3}{8}a_4\alpha_6\right)\frac{\partial}{\partial a_3}\\&\quad+\left(\alpha_2'-\frac{1}{6}a_3\alpha_4'+\frac{3}{4}\alpha_5a_4-\frac{1}{6}a_3^2\alpha_6\right)\frac{\partial}{\partial a_4}\\&\quad+\left(\alpha_7-\frac{1}{12}a_2\alpha_4'+\alpha_6\left(W-\frac{5}{24}a_2a_3\right)\right)\frac{\partial}{\partial a_5},\end{aligned}\tag{6.6}$$

where

$\alpha_5,\alpha_6\in\mathcal{O}(a)$,

$U=-\frac{1}{16}a_1^2a_2((8+a_1^2)/(4-a_1^2))-\frac{1}{16}a_1^2a_2-\frac{1}{4}a_2$,

$V=-\frac{1}{2}a_1^2((a_5+\frac{1}{8}a_1a_2^2-\frac{1}{2}a_1a_3)/(4-a_1^2))$

$\quad-\frac{1}{16}a_1a_2^2+\frac{1}{2}a_1a_3$,

$W=-\frac{1}{16}a_1^3((a_2a_5-2a_4)/(4-a_1^2))-\frac{1}{16}a_1a_2a_5$

$\quad+\frac{1}{8}a_1a_4+\frac{1}{8}a_3a_2$,

$\alpha_7+\alpha_8x+\alpha_9x^2=A(x,a)(a_1x^2+a_2x+a_5)$

$\quad\times\mathrm{mod}(\langle 4x^3+2a_3x+a_4\rangle\mathcal{O}_{(x,a)})$,

$\alpha_2'+\alpha_3'x+\alpha_4'x^2=B(x,a)(a_1x^2+a_2x+a_5)$

$\quad\times\mathrm{mod}(\langle 4x^3+2a_3x+a_4\rangle\mathcal{O}_{(x,a)})$,

$A,B\in\mathcal{O}_{(x,a)}$.

Proof: We easily calculate

$$I(F)=\langle a_1x^2+a_2x+a_5,\ 4x^3+2a_3x+a_4\rangle\mathcal{O}_{(x,a)},$$

$$\Theta(f)=\langle y^2,yx^3,x^6\rangle\mathcal{O}_{(y,x)},$$

so we can write

$$\begin{aligned}h&=\alpha_1+\alpha_2x+\alpha_3x^2+\alpha_4x^3+\alpha_5x^4\\&\quad+\alpha_6x^5+\alpha_7y+\alpha_8xy+\alpha_9x^2y,\\\left.\frac{\partial h}{\partial y}\right|_{y=0}&=\alpha_7+\alpha_8x+\alpha_9x^2\in I(F),\\\left.\frac{\partial h}{\partial x}\right|_{y=0}&=\alpha_2+2\alpha_3x+3\alpha_4x^2+4\alpha_5x^3+5\alpha_6x^4\in I(F).\end{aligned}$$

Introducing the functions

$$\alpha_2'=\alpha_2-\alpha_5a_4,$$

$$\alpha_3'=2\alpha_3-2\alpha_5a_3-\tfrac{5}{4}\alpha_6a_4,$$

$$\alpha_4'=3\alpha_4-\tfrac{5}{2}a_3\alpha_6,$$

and using the Malgrange preparation theorem

$$\mathcal{O}_{(x,a)}/\langle 4x^3+2a_3x+a_4\rangle\mathcal{O}_{(x,a)}\cong[1,x,x^2]_{\mathcal{O}_{(a)}},$$

we obtain the respective equations for α_7, α_8, α_9 and α_2', α_3', α_4'. □

Now, taking (6.5) into account, we can calculate the one-jets of the corresponding module generators:

$$\begin{aligned}j^1V_1&=a_1\frac{\partial}{\partial a_1}+a_2\frac{\partial}{\partial a_2}+a_5\frac{\partial}{\partial a_5},\\j^1V_2&=a_2\frac{\partial}{\partial a_1}+a_5\frac{\partial}{\partial a_2},\\j^1V_3&=a_5\frac{\partial}{\partial a_1},\\j^1V_4&=\frac{1}{2}a_2\frac{\partial}{\partial a_3}+a_5\frac{\partial}{\partial a_4},\\j^1V_5&=\frac{1}{2}a_5\frac{\partial}{\partial a_3},\\j^1V_6&=-2a_1\frac{\partial}{\partial a_1}-a_2\frac{\partial}{\partial a_2}+2a_3\frac{\partial}{\partial a_3}+3a_4\frac{\partial}{\partial a_4},\\j^1V_7&=-\frac{1}{4}a_2\frac{\partial}{\partial a_1}+\frac{3}{8}a_4\frac{\partial}{\partial a_3}.\end{aligned}\tag{6.7}$$

We now treat the last case $D_{4,1}$.

Proposition 6.9: The module of the logarithmic vector fields Derlog $Q(D_{4,1})$ has seven generators. Their one-jets are

$$\begin{aligned}j^1V_1&=\frac{\partial}{\partial a_1}+a_2\frac{\partial}{\partial a_3}-a_5\frac{\partial}{\partial a_4}+a_4\frac{\partial}{\partial a_5},\\j^1V_2&=j^1V_1,\\j^1V_3&=-2\frac{\partial}{\partial a_2}-2a_1\frac{\partial}{\partial a_3}+a_3\frac{\partial}{\partial a_4}-a_2\frac{\partial}{\partial a_5},\\j^1V_4&=-3a_1\frac{\partial}{\partial a_1}+2a_2\frac{\partial}{\partial a_2}-a_3\frac{\partial}{\partial a_3}+a_4\frac{\partial}{\partial a_4}+4a_5\frac{\partial}{\partial a_5},\\j^1V_5&=-\frac{1}{2}a_3\frac{\partial}{\partial a_1}+2a_5\frac{\partial}{\partial a_2}-\frac{1}{2}a_4\frac{\partial}{\partial a_3},\\j^1V_6&=a_2\frac{\partial}{\partial a_1}+a_4\frac{\partial}{\partial a_2},\\j^1V_7&=a_2\frac{\partial}{\partial a_2}+a_3\frac{\partial}{\partial a_3}+2a_4\frac{\partial}{\partial a_4}+2a_5\frac{\partial}{\partial a_5}.\end{aligned}$$

Proof: As in the preceding cases we follow the standard procedure:

$$\begin{aligned}I(F)&=\langle x_1+a_1x_2+a_3,x_1x_2+a_4,\tfrac{1}{2}x_1^2+x_2^2\\&\quad+a_2x_2+a_5\rangle\mathcal{O}_{(x,a)},\\\mathcal{O}_{(y,x)}/\Theta(f)&\cong[1,x_1,x_2,y,x_1^2,x_1x_2,x_2^2,x_2y,x_1^3,\\&\quad x_1^2x_2,x_1x_2^2,x_2^3,x_2^4,x_1x_2^3]_{\mathbb{C}}.\end{aligned}$$

Thus by Proposition 6.5 we have

$$\begin{aligned}h&=\alpha_0+\alpha_1x_1+\alpha_2x_2+\alpha_3y+\alpha_4x_1^2+\alpha_5x_1x_2+\alpha_6x_2^2\\&\quad+\alpha_7x_2y+\alpha_8x_1^3+\alpha_9F(0,x,a)+\alpha_{10}x_1x_2^2\\&\quad+\alpha_{11}x_2^3+\alpha_{12}x_2^4+\alpha_{13}x_1x_2^3\end{aligned}$$

and

$$\left.\frac{\partial h}{\partial y}\right|_{y=0}=\alpha_3+\alpha_7x_2\in I(F),$$

$$\left.\frac{\partial h}{\partial x_1}\right|_{y=0}$$
$$= \alpha_1 - 2\alpha_4 a_3 + 3(a_1 a_4 + a_3^2)\alpha_8 - \alpha_{10}(a_5 + \tfrac{1}{2}a_1 a_4 + \tfrac{1}{2}a_3^2) + \alpha_{13}\nu + x_2(-2a_1\alpha_4 + 3a_1 a_3\alpha_8 + \alpha_5 - \alpha_{10}(a_2 + \tfrac{1}{2}a_1 a_3) + \alpha_{13}\mu) \in I(F),$$

$$\left.\frac{\partial h}{\partial x_2}\right|_{y=0}$$
$$= \alpha_2 - \alpha_5 a_3 - 2a_4\alpha_{10} - 3\alpha_{11}(a_5 + \tfrac{1}{2}a_1 a_4 + \tfrac{1}{2}a_3^2) + 4\alpha_{12}\nu + x_2(2\alpha_6 - a_1\alpha_5 + 4\mu\alpha_{12} - 3\alpha_{11}(a_2 + \tfrac{1}{2}a_1 a_3) - 3\alpha_{13}a_4) \in I(F).$$

Because $1, x_2$ form a free basis for $\mathcal{O}_{(x,a)}/I(F)$, we immediately obtain

$$\alpha_3 = 0, \quad \alpha_7 = 0,$$
$$\alpha_1 = 2\alpha_4 a_3 - 3(a_1 a_4 + a_3^2)\alpha_8 + \alpha_{10}(a_5 + \tfrac{1}{2}a_1 a_4 + \tfrac{1}{2}a_3^2) - \alpha_{13}\nu,$$
$$\alpha_5 = 2a_1\alpha_4 - 3a_1 a_3\alpha_8 + \alpha_{10}(a_2 + \tfrac{1}{2}a_1 a_3) - \alpha_{13}\mu,$$
$$\alpha_2 = a_3(2a_1\alpha_4 - 3a_1 a_3\alpha_8 + \alpha_{10}(a_2 + \tfrac{1}{2}a_1 a_3) - \alpha_{13}\mu) + 2a_4\alpha_{10} + 3\alpha_{11}(a_5 + \tfrac{1}{2}a_1 a_4 + \tfrac{1}{2}a_3^2) - 4\alpha_{12}\nu,$$
$$\alpha_6 = \tfrac{1}{2}a_1(2a_1\alpha_4 - 3a_1 a_3\alpha_8 + \alpha_{10}(a_2 + \tfrac{1}{2}a_1 a_3) - \alpha_{13}\mu) - 2\mu\alpha_{12} - \tfrac{3}{2}\alpha_{11}(a_2 + \tfrac{1}{2}a_1 a_3) + \tfrac{3}{2}\alpha_{13}a_4,$$

where

$$\mu = (a_2 + \tfrac{1}{2}a_1 a_3)^2 - a_5 - \tfrac{1}{2}a_1 a_4 - \tfrac{1}{2}a_3^2,$$
$$\nu = (a_2 + \tfrac{1}{2}a_1 a_3)(a_5 + \tfrac{1}{2}a_1 a_4 + \tfrac{1}{2}a_3^2).$$

Inserting these into h we obtain the formula for

$$h(\mathrm{mod}\ \overline{\Delta}(F)) - h(\mathrm{mod}\ \overline{\Delta}(F))|_{x=0,\,y=0}.$$

From this formula we can read off not only the one-jets but also all the generators of the module. □

Let $\rho: \mathbb{C}^p \to \mathbb{C}^k$ be a projection on $\mathbb{C}^k \subset \mathbb{C}^p$. We say that $Q(F) \subset \mathbb{C}^p$ is locally equisingular along $\mathbb{C}^k$ near $p_0 \in \mathbb{C}^k$ if, for all $p \in \mathbb{C}^k$ near p_0, the pairs $(\rho^{-1}(p),0)$ and $(\rho^{-1}(p) \cap Q(F),0)$ are all diffeomorphic. Checking the vector fields listed in Propositions 6.7 and 6.9, we have the following corollary.

Corollary 6.10: (1) The quasicaustic $Q(F_{1,0})$ is equisingular along the two-dimensional singular locus, parametrized by $\{a_1,a_3\}$.

(2) The quasicaustic $Q(D_{4,1})$ is equisingular along the two-dimensional singular locus, parametrized by $\{a_1,a_2\}$.

In both cases the fiber $(\rho^{-1}(p) \cap Q(F),0)$ is diffeomorphic to Whitney's cross-cap.

The logarithmic vector fields can also be used for the classification of the generic Lagrangian pairs (L_1,L_2) up to quasicaustic equivalence (cf. Refs. 24 and 32). The singular Lagrangian variety $L_1 \cup L_2$ is provided by generic families of functions on the manifold with boundary. In this sense, to determine the germ of the Lagrangian pair means to define the generating family of functions on a manifold with a boundary (cf. Sec. III).

Let $f: (\mathbb{C}\times\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a finitely determined boundary singularity. Let $F: (\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^{\mu-1},0) \to (\mathbb{C},0)$ be its miniversal unfolding. If $G: (\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^p,0) \to (\mathbb{C},0)$ is a generating family for a Lagrangian pair, then generically G is a pullback from the miniversal unfolding F of the finitely determined germ $f(y,x) = G(y,x,0)$, i.e.,

$$G(y,x,a) = F(\Phi(y,x,a),\phi(a)) + h(a),$$

where $\Phi: (\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^p,0) \to (\mathbb{C}\times\mathbb{C}^n,0)$ is a family of biholomorphisms, germs preserving the hypersurface $\{y=0\}$. The pullback $\phi: (\mathbb{C}^p,0) \to (\mathbb{C}^{\mu-1},0)$, $\phi \in \mathcal{O}_{(a)}^{\mu-1}$ and $h \in \mathcal{O}_{(a)}$. Thus analogously to the classification of generic Lagrangian submanifolds (see Ref. 20, p. 337), the classification of generic Lagrangian pairs is done by specifying the miniversal unfoldings of finitely determined boundary singularities and their generic pullbacks $\phi \in \mathcal{O}_{(a)}^{\mu-1}$.

Let us assume that Lagrangian pairs are modeled on unimodal singularities $f: (\mathbb{C}\times\mathbb{C}^n,0) \to (\mathbb{C},0)$, i.e., the generic generating family with such f has the following prenormal form:

$$G: (\mathbb{C}\times\mathbb{C}^n\times\mathbb{C}^p,0) \to (\mathbb{C},0), \quad p \geqslant \mu - 2,$$
$$G(y,x,a) = f(y,x) + \sum_{i=1}^{\mu-2} g_i(y,x)a_i + g_{\mu-1}(y,x)\lambda(a),$$

where $g_{\mu-1}(y,x)$ defines the modulus direction.

Generically, the pullback ϕ is transversal to this direction, so

$$\bar{\lambda} := \lambda|_{\{a_1 = \cdots = a_{\mu-2} = 0\}}: (\mathbb{C}^{p-\mu+2},0) \to (\mathbb{C},0)$$

is a Morse function. Thus there are possible two generic normal forms for the generating families of Lagrangian pairs of unimodal type:

(1) $\lambda(a) = a_{\mu-1}$, when $p > \mu - 2$ and $D\bar{\lambda}(0) \neq 0$;

(2) $\lambda(a) = \eta(a_1,\ldots,a_{\mu-2}) \pm a_{\mu-1}^2 \pm \cdots \pm a_p^2$, when $D\bar{\lambda}(0) = 0$;

where $\eta \in \mathcal{O}_{(\bar{a})}$ $[\bar{a} = (a_1\ldots,a_{\mu-2})]$ is a functional modulus.

To obtain more information about classifying quasicaustics, we need to introduce a weaker equivalence relation in Lagrangian pairs (cf. Refs. 20 and 32 in the case of functional moduli in the standard classification of Lagrangian submanifolds). Let

$$G_1(y,x,a) = F(y,x,\phi_1(a)) + f_1(a),$$
$$G_2(y,x,a) = F(y,x,\phi_2(a)) + f_2(a)$$

be two generating families for the corresponding Lagrangian pairs $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. We say that $\mathcal{L}_1$, $\mathcal{L}_2$ are quasicaustic equivalent if ϕ_1, ϕ_2 are right–left equivalent, i.e.,

$$\phi_1(a) = (\psi \circ \phi_2 \circ \xi)(a),$$

for some biholomorphism $\xi: (\mathbb{C}^p,0) \to (\mathbb{C}^p,0)$, and some biholomorphism $\psi: (\mathbb{C}^{\mu-1},0) \to (\mathbb{C}^{\mu-1},0)$ preserving the quasicaustic $(Q(F),0)$.

Proposition 6.11: For unimodal boundary singularities $F_{1,0}, D_{4,1}$, by quasicaustic equivalence, the functional modulus λ can be reduced to zero.

Proof: On the basis of Ref. 20, p. 343, we need to check only that

$$\mathcal{M}_{(a)} \subseteq \langle A_1^1(a),\ldots,A_1^r(a)\rangle \mathcal{O}_{(a)}, \qquad (*)$$

which implies that

$$\mathcal{M}_{(a)}\mathscr{E}(\phi) \subseteq \phi^*\mathscr{E}(\mu-1) + T\phi(\mathscr{E}(p)),$$

for ϕ: $(\mathbb{C}^p,0)\to(\mathbb{C}^{\mu-1},0)$ being in the general position to the modulus direction. Here by $\mathscr{E}(\phi)$ we denote the vector fields along ϕ (cf. Ref. 30). Let $\mathscr{E}(\mu-1)$ and $\mathscr{E}(p)$ be the spaces of vector fields on $(\mathbb{C}^{\mu-1},0)$ and $(\mathbb{C}^p,0)$, respectively. This enables us to apply the ordinary homotopic method to eliminate the functional modulus λ. Taking into account the vector fields listed in the Propositions 6.7 and 6.9,

$$V_l=\sum_{i=1}^{5}A_i^l\frac{\partial}{\partial a_i},$$

we immediately have fulfilled $(*)$ for the parabolic singularities $F_{1,0}$ and $D_{4,1}$. □

ACKNOWLEDGMENTS

I am very grateful to Mark Roberts and Anthony Fletcher for useful discussions and their help in preparing this manuscript. I would also like to thank the referee for very helpful comments improving Sec. VI of the paper. I am also very grateful to Professor F. Hirzebruch and his Institute for their invitation and exceptional hospitality.

[1]J. B. Keller, "Rays, waves and asymptotics," Bull. Am. Math. Soc. **84**, 727 (1978).
[2]J. W. Bruce and S. Janeczko, "Classification of caustics by diffraction" (in preparation).
[3]J. W. Bruce, "Vector fields on discriminants and bifurcation varieties," Bull. London Math. Soc. **17**, 257 (1985).
[4]H. Terao, "The bifurcation set and logarithmic vector fields," Math. Ann. **263**, 313 (1983).
[5]R. K. Luneburg, *Mathematical Theory of Optics* (Univ. California Press, Berkeley, 1964).
[6]V. W. Guillemin and S. Sternberg, *Symplectic Techniques in Physics* (Cambridge U. P., London, 1984).
[7]M. Kline and I. W. Kay, *Electromagnetic Theory and Geometrical Optics* (Interscience, New York, 1965).
[8]S. Janeczko, "Generating families for images of Lagrangian submanifolds and open swallowtails," Math. Proc. Cambridge Philos. Soc. **100**, 91 (1986).
[9]J. W. Bruce, "Functions on discriminants," J. London Math. Soc. (2) **30**, 551 (1984).
[10]J. W. Bruce and D. L. Fidal, "Vector fields on caustics" (unpublished).
[11]T. Poston and I. Stewart, *Catastrophe Theory and its Applications* (Pitman, San Francisco, 1978).
[12]R. Abraham and J. E. Marsden, *Foundations of Mechanics* (Benjamin/Cummings, Reading, MA, 1978).
[13]S. Janeczko, "Covariant symplectic geometry of binary forms and singularities of systems of rays," Max-Planck-Institut für Mathematik, preprint, MPI, 1986–40.
[14]W. M. Tulczyjew, "The Legendre transformation," Ann. Inst. H. Poincaré **27**, 101 (1977).
[15]J. W. Bruce, P. J. Giblin, and C. G. Gibson, "On caustics by reflection," Topology **21**, 179 (1982).
[16]O. P. Shcherbak, "Wave fronts and groups generated by reflections," Usp. Mat. Nauk **43**, 125 (1988).
[17]A. Weinstein, "Lectures on symplectic manifolds," C. B. M. S. Conf. Ser. Am. Math. Soc. **29** (1977).
[18]S. Janeczko, "Singularities in the geometry of an obstacle," Suppl. Rend. Circolo Mat. Palermo, Ser. II **16**, 71 (1987).
[19]C. T. C. Wall, "Geometric properties of generic differentiable manifolds," in *Geometry and Topology, Lecture Notes in Mathematics*, Vol. **597**, edited by A. Dold and B. Eckmann (Springer, Berlin, 1977), pp. 707–774.
[20]V. I. Arnold, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of Differentiable Maps* (Birkhauser, Boston, 1985), Vol. 1.
[21]V. I. Arnold and A. B. Givental, "Symplectic geometry," Itogi Nauki Contemp. Probl. Math. Fund. Direct. **4**, 5 (1985).
[22]V. I. Arnold, "Singularities in variational calculus," Itogi Nauki Contemp. Probl. Math. **22**, 3 (1983).
[23]I. G. Shcherbak, "Focal set of a surface with boundary, and caustics of groups generated by reflections B_k, C_k, and F_4," Funct. Anal. Appl. **18**, 84 (1984).
[24]Nguyên hûu Dùc, Nguyên tiên Dai, "Stabilite de l'interaction geometrique entre deux composantes holonomes simples," C. R. Acad. Sci. Paris A **291**, 113 (1980).
[25]Yu. V. Chekanov, "Caustics in geometrical optics," Funct. Anal. Appl. **20**, 223 (1986).
[26]V. I. Matov, "Unimodal and bimodal germs of functions on manifold with boundary," Proc. Petrovski Sem. **7**, 174 (1981).
[27]D. Siersma, "Singularities of functions on boundaries, corners, etc.," Quart. J. Math. Oxford (2) **32**, 119 (1981).
[28]S. B. Alexander, I. D. Berg, and R. I. Bishop, "The Riemannian obstacle problem," Ill. J. Math. **31**, 167 (1987).
[29]D. Mond, "On the tangent developable of a space curve," Math. Proc. Camb. Philos. Soc. **91**, 351 (1982).
[30]J. Martinet, *Singularities of Smooth Functions and Maps* (Cambridge U. P., Cambridge, 1982).
[31]V. M. Zakalyukin, "Bifurcations of waveforms depending on one parameter," Funct. Anal. Appl. **10**, 139 (1976).
[32]S. Janeczko and M. Roberts, "Classification of symmetric caustics," preprint, University of Warwick, October (1983).
[33]K. Saito, "Theory of logarithmic differential forms and logarithmic vector fields," J. Fac. Sci. Univ. Tokyo Sec. I A **27**, 265 (1980).
[34]E. J. N. Looijenga, *Isolated Singular Points on Complete Intersections* (Cambridge U.P., London, 1984).