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On geometric properties of Lagrangian submanifolds in product symplectic spaces

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Abstract. We study the generic properties of symplectic relations. Local models of symplectic relations are described and the corresponding local symplectic invariants are derived. A stratification of the Lagrangian Grassmannian in the product symplectic space $(N \times M,\, \pi_M^* \omega_M - \pi_N^* \omega_N)$ is constructed and global homological properties of the strata are investigated.

Key words: symplectic relation, Lagrangian Grassmannian, Maslov class.

1. Introduction

A symplectic structure on a manifold M is a 2-form ω which is closed and nondegenerate. For a symplectic manifold (M, ω), we can consider submanifolds L which are isotropic with respect to the symplectic form, $\omega\,|_L = 0$. If L has a maximal possible dimension equal to half of the dimension of the symplectic manifold, then L is called Lagrangian.

There are extensive local and global studies of Lagrangian submanifolds and their singularities (cf. [14, 3, 6, 7, 8]). An important object in global investigations of Lagrangian submanifolds is the Lagrangian Grassmannian Λ_n, the manifold of linear Lagrangian subspaces in a $2n$-dimensional linear symplectic space. The canonical stratification $\Lambda_n = \bigcup_{k=0}^n \Lambda_{n,k}^\alpha$, $\Lambda_{n,k}^\alpha = \{\beta \in \Lambda_n \colon \dim(\beta \cap \alpha) = k\}$, where α is a fixed element of Λ_n, allows us to describe the geometry of Lagrangian submanifolds and their singularities. The set $\Lambda_n^{(1)} = \bigcup_{k=1}^n \Lambda_{n,k}^\alpha$ is coorientable and its singular part has codimension strictly greater than 1 in $\Lambda_n^{(1)}$. Thus $\Lambda_n^{(1)}$ determines a singular cycle which is Poincaré dual to the universal Maslov class $\mu \in H^1(\Lambda_n, \mathbb{Z})$ (cf. [2]). Investigation of Lagrangian submanifolds in a product symplectic manifold $(M_1 \times M_2,\, \pi_2^* \omega_2 - \pi_1^* \omega_1)$, called symplectic relations, requires (as shown in [9]) the use of another natural stratification of the Lagrangian Grassmannian

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Λ_{n+m} ($\dim M_1 = 2n$, $\dim M_2 = 2m$) determined by the product structure properties. This stratification distinguishes the symplectically nonequivalent vertical positions of symplectic relations and measures their difference from canonical relations formed by graphs of symplectomorphisms.

To continue the investigation of symplectic relations, in Section 2 we construct a new stratification of the Lagrangian Grassmannian in a product linear symplectic space and investigate the generic properties of symplectic relations with respect to the canonical projections. In Section 3 we compute the first homology group of the strata and find a cycle whose homology class is dual to the universal Maslov class of the Grassmannian Λ_{n+m}.

2. Lagrangian submanifolds in product symplectic space

Let (N, ω_N), (M, ω_M) be two linear symplectic spaces, $\dim N = 2n$, $\dim M = 2m$, $m \leq n$. The product symplectic space is defined as

$$\mathcal{M} = (N \times M, \omega_M \ominus \omega_N), \tag{1}$$

where $\omega_M \ominus \omega_N = \pi_2^* \omega_M - \pi_1^* \omega_N$ and π_i, $i = 1, 2$, are the canonical projections.

By Λ_{n+m} we denote the Lagrangian Grassmannian of linear $(n+m)$-dimensional Lagrangian subspaces in $\mathcal{M}$. Let N and M be canonically placed in the product. If $L \in \Lambda_{n+m}$ then there are two possibilities:

1. L is transversal to N. Then $\mathrm{rank}(\pi_2 \mid_L)$ is maximal and L is called a linear reduction relation. The set of such L is called the regular subset of Λ_{n+m} and we will denote it by RSp_{n+m}.
2. L is not transversal to N. The set of such L will be denoted by $C\Lambda_{n+m}$ and called the critical subset of Λ_{n+m}.

We stratify Λ_{n+m} by the codimension of the image under the projection π_2. The most singular subset in Λ_{n+m} is denoted by $S\Lambda_{n+m}$, $S\Lambda_{n+m} = \Lambda_n \times \Lambda_m$, where Λ_n and Λ_m are the Lagrangian Grassmannians in N and M respectively; $\mathrm{codim}\, S\Lambda_{n+m} = nm$.

Now we introduce the composition of linear symplectic relations. Let R_1 and R_2 be Lagrangian subspaces in product spaces, say $(M_1 \times M_2, \omega_2 \ominus \omega_1)$ and $(M_2 \times M_3, \omega_3 \ominus \omega_2)$ respectively. Then we define the composition of R_1 and R_2, denoted by $R_2 \circ R_1$, as the Lagrangian subspace

$$R_2 \circ R_1 = \{(v_1, v_3) \in M_1 \times M_3 \colon \exists v_2 \in M_2,\ (v_1, v_2) \in R_1,\ (v_2, v_3) \in R_2\}$$

in the product symplectic space $(M_1 \times M_3, \omega_3 \ominus \omega_1)$.

Let V be a subspace of a linear symplectic space (N, ω). By $V^{\angle}$ we denote the symplectic-orthogonal subspace to V (symplectic polar), $V^{\angle} = \{v \in N \colon \omega(v, w) = 0 \text{ for every } w \in V\}$. If $V^{\angle} \subset V$ then V is called a coisotropic subspace of (N, ω). The quotient space $V/V^{\angle}$ endowed with the canonical reduced symplectic structure is called a reduced symplectic space.

Theorem 2.1 *The Lagrangian Grassmannian Λ_{n+m} has a canonical partition into smooth strata $C_k\Lambda_{n+m}$ $(k = 0, 1, \ldots, m)$ with $C_0\Lambda_{n+m} = RSp_{n+m}$ and $C_m\Lambda_{n+m} = S\Lambda_{n+m}$. Each stratum is characterized as follows: $L \in C_k\Lambda_{n+m}$ is uniquely represented as $L = (L_1 \circ R) \circ L_2$ with Lagrangian subspaces $L_1 \subset (N \times N_1, \omega_{N_1} \ominus \omega_N)$, $R \subset (N_1 \times N_2, \omega_{N_2} \ominus \omega_{N_1})$, $L_2 \subset (N_2 \times M, \omega_M \ominus \omega_{N_2})$, where L_i $(i = 1, 2)$ is the graph of the coisotropic projection (i.e. linear reduction relation) $\rho_i \colon \pi_i(L) \to N_i = \pi_i(L)/\pi_i(L)^{\angle}$ with $\dim N_1 = \dim N_2 = 2(m - k)$ and R is the graph of a symplectic isomorphism $N_1 \to N_2$.*

Proof. If $L \in RSp_{n+m}$, then $\pi_1(L)$ is a coisotropic subspace of N, i.e. the symplectic orthogonal $\pi_1(L)^{\angle}$ is contained in $\pi_1(L)$. And we can decompose L as $L = R \circ L_1$, where R and L_1 are symplectic relations, $R \subset (N_1 \times M, \omega_M \ominus \omega_{N_1})$ and $L_1 \subset (N \times N_1, \omega_{N_1} \ominus \omega_N)$. L_1 is the graph of the coisotropic projection $\rho_{N_1} \colon \pi_1(L) \to \pi_1(L)/\pi_1(L)^{\angle}$, where $\pi_1(L)/\pi_1(L)^{\angle}$ is endowed with the unique symplectic structure ω_{N_1} by the canonical formula $\rho_{N_1}^* \omega_{N_1} = \omega_N \mid_{\pi_1(L)}$ and R is the graph of a symplectic isomorphism $N_1 \to M$, $\dim N_1 = \dim M = 2m$.

A simple geometric argument shows that if $L \in C\Lambda_{n+m}$ then $\pi_1(L) \subset N$ and $\pi_2(L) \subset M$ are coisotropic subspaces with equally dimensional uniquely defined reduced symplectic spaces. Repeating the previous considerations for these subspaces we get the characterization of the corresponding strata of the partition. $\square$

Let $\mathcal{I}_k^{2n}$ denote the Grassmannian of isotropic k-dimensional subspaces in $2n-$ dimensional symplectic space.

As a consequence of the above theorem we obtain

Corollary 2.1 *The strata $C_k\Lambda_{n+m}$ are smooth submanifolds of Λ_{n+m} and*

$$\operatorname{codim} C_k\Lambda_{n+m} = k^2 + k(n - m). \tag{2}$$

Proof. It is just a matter of calculation:

$$\begin{aligned}\operatorname{codim} C_k\Lambda_{n-m} = &\dim \Lambda_{n+m} - \dim \mathcal{I}^{2n}_{n-m+k}\\ &- \dim \mathcal{I}^{2m}_{k} - \dim \Lambda_{2m-2k}.\end{aligned}$$

Using the standard formulas for the respective dimensions we get the result. □

Let (N, ω_N), (M, ω_M) be two symplectic manifolds and let the symplectic manifold $\mathcal{M}$ be defined as before in the linear case (cf. (1)). By (L, p) we denote a Lagrangian submanifold germ in $\mathcal{M}$. Now we introduce an equivalence relation in the space of germs of Lagrangian submanifolds.

Definition 2.1 Two Lagrangian germs (L_1, p_1), $(L_2, p_2) \subset (\mathcal{M}, \omega_M \ominus \omega_N)$ are called equivalent if there exist two symplectomorphism germs $B_1\colon (N, \pi_1(p_1)) \to (N, \pi_1(p_2))$ and $B_2\colon (M, \pi_2(p_1)) \to (M, \pi_2(p_2))$ such that the symplectomorphism $B_1 \times B_2\colon \mathcal{M} \to \mathcal{M}$ sends L_1 into L_2 and p_1 into p_2.

Now we have the preliminary

Lemma 2.1 *If (L, p) is a Lagrangian germ in $\mathcal{M}$, then there are local cotangent bundle structures around $\pi_1(p)$, say T^*X and around $\pi_2(p)$, say T^*Y, such that (L, p) is generated in the product space $\mathcal{M} \cong (T^*(X \times Y), \omega_{T^*Y} \ominus \omega_{T^*X})$ by the germ of a function $F\colon (X \times Y, \pi_{X\times Y}(p)) \to \mathbf{R}$ such that, in local coordinates on $(X \times Y, \pi_{X\times Y}(p))$,*

$$F(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \phi_{ij}(x, y), \tag{3}$$

for some smooth function-germs ϕ_{ij}, where $\pi_{X\times Y}\colon T^*(X \times Y) \to X \times Y$ is the canonical cotangent bundle projection, $\dim X = n$, $\dim Y = m$.

Proof. Let $((p, q), (\bar{p}, \bar{q}))$ be the Darboux coordinates on $T^*(X \times Y)$. Then

$$\omega_{T^*Y} \ominus \omega_{T^*X} = \sum_{i=1}^{m} d\bar{p}_i \wedge d\bar{q}_i - \sum_{i=1}^{n} dp_i \wedge dq_i.$$

By [3] (Section III, 19.3) we can find partitions $I \cup J = \{1, \ldots, n\}$, $I \cap J = \emptyset$, and $\bar{I} \cup \bar{J} = \{1, \ldots, m\}$, $\bar{I} \cap \bar{J} = \emptyset$, such that there exists a smooth function $(p_I, q_J, \bar{p}_{\bar{I}}, \bar{q}_{\bar{J}}) \mapsto S(p_I, q_J, \bar{p}_{\bar{I}}, \bar{q}_{\bar{J}})$ which is a generating function for (L, p) (cf. [2, 3]). By means of the symplectomorphism Φ of $\mathcal{M}$,

$$\Phi(p, q, \bar{p}, \bar{q}) = (-q_I, p_J, p_I, q_J, -\bar{q}_{\bar{I}}, \bar{p}_{\bar{J}}, \bar{p}_{\bar{I}}, \bar{q}_{\bar{J}}) = (\xi, x, \eta, y),$$

which preserves the product structure of $\mathcal{M}$, we get the generating function $(x, y) \mapsto F(x, y)$ for (L, p) in the canonical cotangent bundle structure $T^*X \times T^*Y$ on $\mathcal{M}$. We can write

$$F(x, y) = F_1(x) + \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \phi_{ij}(x, y) + F_2(y)$$

and then taking the equivalence $B_1 \times B_2$, $B_1(\xi, x) = (\xi - \operatorname{grad} F_1(x), x)$, $B_2(\eta, y) = (\eta - \operatorname{grad} F_2(y), y)$, we get the reduced form (3). □

Proposition 2.1 *If (L, p) projects onto $(M, \pi_2(p))$, $\dim M = 2m$, $m \leq n$, then (L, p) is equivalent to its tangent space T_pL with the generating function $F(x, y) = \sum_{i=1}^{m} x_i y_i$.*

Proof. By assumption, (L, p) can be parametrized by the mapping

$$\Phi(\bar{p}, \bar{q}, q_I) = (\phi_I(\bar{p}, \bar{q}, q_I), \phi_J(\bar{p}, \bar{q}, q_I), q_I, \psi_J(\bar{p}, \bar{q}, q_I); \bar{p}, \bar{q}),$$

i.e. $\Phi^*(\omega_M \ominus \omega_N) = 0$, where $I \cup J = \{1, \ldots, n\}$, $I \cap J = \emptyset$, and ϕ_I, ϕ_J, ψ_J are smooth map-germs. Consider the symplectomorphism $\Xi\colon N \to N$ given by $\Xi(\xi, x) = \big(\xi_I + \phi_I(\xi_J, x_J, x_I), \phi_J(\xi_J, x_J, x_I), x_I, \psi_J(\xi_J, x_J, x_I), \xi_J, x_J\big)$. It is really a symplectomorphism: in fact,

$$\begin{aligned}\Xi^*(dp_I \wedge dq_I + dp_J \wedge dq_J) = {}& d\xi_I \wedge dx_I + d\phi_I(\ldots) \wedge dx_I \\ & + d\phi_J(\ldots) \wedge d\psi_J(\ldots);\end{aligned}$$

but $d\bar{p} \wedge d\bar{q} - d\phi_I(\bar{p}, \bar{q}, q_I) \wedge dq_I - d\phi_J(\ldots) \wedge d\psi_J(\ldots) = 0$, so we get

$$\Xi^*(dp_I \wedge dq_I + dp_J \wedge dq_J) = d\xi_I \wedge dx_I + d\xi_J \wedge dx_J.$$

Then, via (Ξ^{-1}, id), the Lagrangian germ (L, p) is equivalent to the form $\xi_J = -y_J$, $\eta_J = x_J$, $\xi_I = 0$, where $J = \{1, \ldots, m\}$, and $\omega_M \ominus \omega_N = d\eta_J \wedge dy_J - d\xi \wedge dx$. But it is easy to check that this germ is generated by the generating function $F(x, y) = \sum_{i=1}^{m} x_i y_i$. □

Let $L \subset \mathcal{M}$ be a symplectic relation. We can associate with L the symplectic "Gauss" map, $G_L\colon L \ni p \mapsto T_pL \in \Lambda_{n+m}$, where the tangent space T_pL is identified with a linear subspace of $\mathcal{M}$.

Definition 2.2 We say that L is in general position (or that it is generic) if G_L is transversal to the stratification $C\Lambda_{n+m} = \bigcup_{k=1}^{m} C_k\Lambda_{n+m}$. We say that L has k-vertical position at $p \in L$ if $G_L(p) \in C_k\Lambda_{n+m}$. The index k (which is a symplectic invariant) is called the rank of the vertical position.

We see that the 0-vertical position at p corresponds to the case when $T_pL \in \Lambda_{n+m} - C\Lambda_{n+m}$. From Theorem 2.1 and Lemma 2.1 we obtain the following result.

Proposition 2.2 *If $p \in L$, and L has $k-$vertical position at p, then the germ (L, p) is equivalent to one in $(T^*X \times T^*Y, \omega_Y \ominus \omega_X)$ generated by the generating function*

$$F(x, y) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i y_j \phi_{ij}(x, y),$$

where the rank of the vertical position of L at p is equal to the corank of the matrix $((\partial^2 F/\partial x_i \partial y_j)(0, 0))$, $k = \mathrm{corank}(\phi_{ij}(0, 0))$ at $(0, 0) = \pi_{X\times Y}(p)$.

Let $M_{n\times m}$ $(n \geq m)$ denote the space of $n\times m$ matrices of real numbers. For each natural r, $0 \leq r \leq m$, let Σ_r denote the subset of $M_{n\times m}$ consisting of matrices of corank r. Then Σ_r is a submanifold of $M_{n\times m}$ of codimension $r^2 + r(n-m)$ (cf. [5]). Let $E_{n\times m}$ denote the set of $n\times m$ matrices of smooth function-germs at 0 on $X \times Y$, i.e. a representative of a germ is a smooth mapping of some open neighborhood of $0 \in X \times Y$ into $M_{n\times m}$. A germ $\Phi \in E_{n\times m}$ is called generic if it is transversal to all Σ_r, $r = 0, 1, \ldots, m$. By Lemma 2.1 and Proposition 2.2, to every neighborhood U of $p \in L$, for some choice of the cotangent bundle structure on $\mathcal{M}$, we associate the generating function

$$\pi_{X\times Y}(U) = V \ni (x, y) \mapsto F(x, y),$$

where L is transversal to the fibering. We can treat coordinates $(x, y) \in U$ as a parametrization of L. To each $(\bar{x}, \bar{y}) \in U$ we associate the two-jet

$$j^2_{(\bar{x}, \bar{y})}F = \sum_{i=1}^{n}\sum_{j=1}^{m}(x_i - \bar{x}_i)(y_j - \bar{y}_j)a_{ij}(\bar{x}, \bar{y}).$$

The smooth mapping $V \ni (x, y) \mapsto a_{ij} \in M_{n\times m}$ will be called the one-jet extension of the parametrization of L. Now we have

Proposition 2.3 *Let $L \subset \mathcal{M}$ be a symplectic relation in $\mathcal{M}$. Then the following conditions are equivalent*

1. *The mapping G_L is transversal to the stratification of the critical set $C\Lambda_{n+m} = \bigcup_{k=1}^{m} C_k\Lambda_{n+m}$.*
2. *For any germ of a symplectic relation (L, p) the corresponding one-jet extension of the local parametrization of L is generic.*

Proof. We know by Proposition 2.2 that the corresponding stratifications of Λ_{n+m} and $M_{n\times m}$ coincide. Thus the one-jet extension $(a_{ij}(x, y))$ reconstructs the Gauss map locally. So if $V \ni (x, y) \mapsto (a_{ij}(x, y))$ is generic, then G_L is transversal to the stratification of Λ_{n+m}, and conversely if G_L is transversal to the stratification of Λ_{n+m}, then by the extension and reduction method (see also [10, 14]) the corresponding local one-jet extensions are generic matrices. □

Remark 2.1

A. If $a_{ij}(\bar{x}, \bar{y})$ is generic then $\phi_{ij}(x, y)$ is generic in some open neighborhood of 0. And because of $a_{ij}(0, 0) = \phi_{ij}(0, 0)$, $\mathrm{rank}(a_{ij}(0, 0)) = \mathrm{rank}(\phi_{ij}(0, 0))$, we have the local equivalence of generic matrices $(a_{ij}(\bar{x}, \bar{y}))$ and $(\phi_{ij}(x, y))$ (cf. [10]). In fact

$$(a_{ij} \circ \psi)(x, y) = \sum_{kl} \alpha_{ik}(x, y)\phi_{kl}(x, y)\beta_{lj}(x, y),$$

where ψ is a local diffeomorphism, $(\phi_{kl}(0, 0)) = 0$, and α, β are local invertible matrices; $(\alpha_{ik}(0, 0)) = I_{n\times n}$, $(\beta_{lj}(0, 0)) = I_{m\times m}$.

B. By Proposition 2.3, the k-vertical points of generic $L \subset (N \times M, \omega_M \ominus \omega_N)$, $\dim N = 2n$, $\dim M = 2m$ cannot be removed by a small perturbation of L if

$$k^2 + k(n - m) \leq n + m \quad (n \geq m,\, k \leq m). \tag{4}$$

For generic L, the isolated points of k-vertical position appear only if $m \in \mathbb{N}$ satisfies the equation $m = (1/8)(4 + h^2 - s^2)$ for some $h \in \mathbb{N}$ and $s \in \mathbb{N}$, $s \geq 2$. In this case the relation L has an isolated k-vertical point, not removable by a small perturbation of L, for $k = (1/2)(2 + h - s)$. If $n = m$ the k-vertical points generically appearing in L are defined by the inequality $k^2 \leq 2n$. The isolated points of k-vertical position appear only if $n = 2h^2$ for some $h \in \mathbb{N}$, and these are $2h$-vertical points.

If $n = m = 2$, then the supercritical points (i.e. points $p \in L$ such that $G_L(p) \in C_2\Lambda_4$) appear in generic L as isolated points. At such points L is generated locally by a function $F(x,\, y) = \sum_{i,j=1}^{2} x_i y_j \phi_{ij}(x,\, y)$, where $\phi_{ij}(0,\, 0) = 0$, and the transversality of G_L to $C_2\Lambda_4$ is equivalent to the maximal rank property, $\operatorname{rank} D\Phi(0,\, 0) = 4$, where $\Phi(x,\, y) = (\phi_{ij}(x,\, y)) \in M_{2\times 2}$.

If $n = m = 1$, the supercritical points for generic L are not isolated. In this case the generating function has the form $F(x,\, y) = xyf(x,\, y)$, and the transversality condition means that f has no critical point at 0. Moreover the transversality condition ensures the infinitesimal symplectic stability of such supercritical points.

3. The geometry of Λ_{n+m}

We begin with the case $n = m$. Later we generalize the considerations to $n \neq m$, $n \geq m$.

We consider the partition of the critical subset $C\Lambda_{2n} \subset \Lambda_{2n}$ into the smooth submanifolds (cf. [9]), $C\Lambda_{2n} = \bigcup_{k=1}^{n} C_k\Lambda_{2n}$. Every stratum $C_k\Lambda_{2n}$, for $k = 1,\, \ldots,\, n-1$, is fibered in the following way:

$$Sp(2n-2k) \xrightarrow{r} C_k\Lambda_{2n} \xrightarrow{p} (\mathcal{I}_k^{2n})^2$$

where p is the canonical projection into symplectic polars (symplectic-orthogonal isotropic spaces) and r is the fiber inclusion. $\mathcal{I}_k^{2n}$ is the isotropic Grassmannian and $Sp(2n-2k)$ denotes the group of symplectic linear automorphisms of the $(2n-2k)$-dimensional linear symplectic space.

Proposition 3.1 *The first homology group of the set $C_k\Lambda_{2n}$, for $k = 1,\, \ldots,\, n-1$, with real coefficients is equal to $\mathbf{R}$.*

Proof. We take the exact homotopy sequence for the fibration

$$\cdots \to \pi_1(Sp(2n-2k)) \to \pi_1(C_k\Lambda_{2n}) \to \pi_1((\mathcal{I}_k^{2n})^2) \to \ldots$$

Since $\pi_1(Sp(2n-2k)) \simeq \mathbf{Z}$ and $\pi_1((\mathcal{I}_k^{2n})^2) \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$ ([11, 1]) is a torsion group, and $Sp(2n-2k)$ and $C_k\Lambda_{2n}$ are connected, we see that $H_1(C_k\Lambda_{2n},\, \mathbf{R})$ is equal to $\mathbf{R}$ or is trivial. We will examine the generator of the group $H_1(C_k\Lambda_{2n},\, \mathbf{R})$ coming from $Sp(2n-2k)$. We fix the point $I_0 = \mathbf{R}^k \times \mathbf{R}^k \in (\mathcal{I}_k^{2n})^2$ in the base, so over I_0 we have the inclusion of the fiber $r\colon Sp(2n-2k) \hookrightarrow C_k\Lambda_{2n}$. As a generator of $H_1(Sp(2n-2k),\, \mathbf{R})$ we take the class

$[\gamma_k(t)]$ of the matrix cycle in $Sp(2n-2k)$

$$\gamma_k(t) = \begin{pmatrix} e^{it} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

for $t \in [0,\, 2\pi]$. Let $(\varepsilon_1,\, \gamma_k(t)\varepsilon_1)$, $(\varepsilon_2,\, \gamma_k(t)\varepsilon_2)$, $\dots$, $(\varepsilon_{n-k},\, \gamma_k(t)\varepsilon_{n-k})$ be the complex basis of

$$\operatorname{graph} \gamma_k(t) \subset \mathbf{C}^{n-k} \oplus \mathbf{C}^{n-k},$$

where $\varepsilon_1, \dots, \varepsilon_{n-k}$ is the standard basis in $\mathbf{C}^{n-k}$.

We denote by $[\widetilde{\gamma_k(t)}]$ the image of the generator $[\gamma_k(t)]$ under the mapping

$$r_*\colon H_1(Sp(2n-2k),\, \mathbf{R}) \to H_1(C_k\Lambda_{2n},\, \mathbf{R}).$$

It is represented by the matrix $X \in Sp(2n)$, i.e.

$$X = \begin{pmatrix} X_A & 0 \\ 0 & Q_A \\ X_B & 0 \\ 0 & Q_B \end{pmatrix},$$

where:

1. $X_A = (\varepsilon_1,\, \overline{i\varepsilon_1},\, \dots,\, \varepsilon_{n-k},\, \overline{i\varepsilon_{n-k}})$ is an $(n-k) \times (2n-2k)$ matrix and the bar is the complex conjugation.
2. $Q_A = (E_k,\, 0)$ is a $k \times 2k$ matrix and consists of two parts: the identity matrix E_k and the zero matrix.
3. $X_B = (\gamma_k(t)\varepsilon_1,\, i\gamma_k(t)\varepsilon_1,\, \dots,\, \gamma_k(t)\varepsilon_{n-k},\, i\gamma_k(t)\varepsilon_{n-k})$, is a matrix of the same type as X_A.
4. $Q_B = (0,\, E_k)$.

We calculate

$$\det(\widetilde{\gamma_k(t)}) = \det \begin{pmatrix} A \\ B \end{pmatrix} = \pm e^{it}(2i)^{n-k}, \quad t \in [0,\, 2\pi].$$

Now we consider the mapping

$$\det{}^2\colon \Lambda_{2n} \to S^1, \quad [D] \mapsto (\det D)^2,$$

where the element $[D] \in \Lambda_{2n} \simeq U(2n)/O(2n)$ represents a Lagrangian linear subspace in $4n$-dimensional symplectic space. We recall that the universal Maslov class μ_{2n} of the Lagrangian Grassmannian is a generator of $H^1(\Lambda_{2n}, \mathbf{Z}) \simeq \mathbf{Z}$ and it is the image of a generator of $H^1(S^1, \mathbf{Z})$ under the mapping $\det^{2*}: H^1(S^1, \mathbf{Z}) \to H^1(\Lambda_{2n}, \mathbf{Z})$ ([2], [12], [7]). Dually the map of homology $\det^2_*: H_1(\Lambda_{2n}, \mathbf{Z}) \simeq \mathbf{Z} \to H_1(S^1, \mathbf{Z}) \simeq \mathbf{Z}$ is an isomorphism and the image of the cycle $[\widetilde{\gamma_k(t)}]$ under the mapping $\det^2_*$ is equal to $2g$, where g is a generator of $H_1(S^1, \mathbf{Z})$. We conclude that $[\widetilde{\gamma_k(t)}]$ is nonzero. The mapping r_* is an epimorphism, so r_* is an isomorphism and we have $H_1(C_k\Lambda_{2n}, \mathbf{R}) \simeq \mathbf{R}$. □

By μ^*_{2n} we denote the element of $H_1(\Lambda_{2n}, \mathbf{R}) \simeq \mathbf{R}$ dual to the universal Maslov class μ_{2n}. Thus we get

$$[\widetilde{\gamma_k(t)}] = [\deg \det^2(\widetilde{\gamma_k(t)})]\mu^*_{2n} = 2\mu^*_{2n}.$$

For the inclusion

$$j: C_k\Lambda_{2n} \hookrightarrow \Lambda_{2n}, \quad k = 1, \ldots, n-1,$$

we have $j_*([\widetilde{\gamma_k(t)}]) = 2\mu^*_{2n}$. Thus we proved the following theorem:

Theorem 3.1 *For every stratum $C_k\Lambda_{2n}$, $k = 1, \ldots, n-1$, we can find a cycle in $H_1(C_k\Lambda_{2n}, \mathbf{R}) \simeq \mathbf{R}$ realizing the class $\mu^*_{2n} \in H_1(\Lambda_{2n}, \mathbf{R})$ dual to the universal Maslov class for the Grassmannian Λ_{2n}.*

Now we are interested in two strata: the supercritical set $C_n\Lambda_{2n} = S\Lambda_{2n} \simeq \Lambda_n \times \Lambda_n$ and the biggest stratum $RSp_{2n} = Sp(2n)$ consisting of the graphs of the linear symplectomorphisms. The result for the strata RSp_{2n} is the same as for the strata $C_k\Lambda_{2n}$, where $k = 1, \ldots, n-1$.

Let l_0 be a fixed element of Λ_n and L be an arbitrary element of Λ_{2n}. We define the image of l_0 under L:

$$L(l_0) = \{p \in M : \exists p' \in l_0 \quad (p', p) \in L\},$$

which is obviously an element of Λ_n. We consider the mapping (cf. [9]) $\rho: \Lambda_{2n} \to \Lambda_n$, $\rho(L) = L(l_0)$. The mapping ρ restricted to the strata $C_k\Lambda_{2n}$ for $k = 0, \ldots, n-1$ is not continuous; in contrast, it is continuous on the supercritical stratum. Let $j: S\Lambda_{2n} \hookrightarrow \Lambda_{2n}$ be the inclusion map and let ρ_n be the map ρ restricted to the stratum $S\Lambda_{2n}$.

Corollary 3.1

1. *For the mapping* $j_*\colon H_1(S\Lambda_{2n},\,\mathbf{R}) \simeq \mathbf{R}\oplus\mathbf{R} \to H_1(\Lambda_{2n},\,\mathbf{R}) \simeq \mathbf{R}$ *we obtain*

$$j_*(a\mu_n^*,\, b\mu_n^*) = (-a+b)\mu_{2n}^*,$$

where the classes μ_n^* *and* μ_{2n}^* *are dual to the universal Maslov classes for the Grassmannians* Λ_n *and* Λ_{2n}*,* $a,\, b \in \mathbf{R}$*. In particular we have* $j_*(-\mu_n^*, \mu_n^*) = 2\mu_{2n}^*$*.*

2. *The map* $\rho_{n*}\colon H_1(S\Lambda_{2n},\,\mathbf{R}) \to H_1(\Lambda_n,\,\mathbf{R})$ *is the projection on the second factor, so* $\rho_{n*}(v,\,\mu_n^*) = \mu_n^*$ *for all* v*.*

The Lagrangian Grasssmannian is an orientable manifold if and only if n is an odd integer (cf. [4]). Thus in our case the Grassmannian Λ_{2n} is not orientable for any $n \in \mathbb{N}$.

Remark 3.1 The first singular stratum $C_1\Lambda_{2n}$ is coorientable in Λ_{2n}. The coorientability determines a cohomology class in $H^1(\Lambda_{2n},\,\mathbf{R})$.

The Grassmannian Λ_{n+m}, $n \geq m$ has the stratification (see Section 2):

$$\Lambda_{n+m} = RSp_{n+m} \cup \bigcup_{k=1}^{m-1} C_k\Lambda_{n+m} \cup C_m\Lambda_{n+m}.$$

The strata are fibered in the following ways:

1. the regular stratum RSp_{n+m}:

$$Sp(2m) \xrightarrow{r} RSp_{n+m} \xrightarrow{p} \mathcal{I}_{n-m}^{2n}$$

2. the critical strata $C_k\Lambda_{n+m}$ for $k = 1,\,\ldots,\,m-1$,

$$Sp(2m-2k) \xrightarrow{r} C_k\Lambda_{n+m} \xrightarrow{p} \mathcal{I}_{n-m+k}^{2n} \times \mathcal{I}_k^{2m}$$

3. the supercritical stratum

$$C_m\Lambda_{n+m} \simeq S\Lambda_{n+m} \simeq \Lambda_n \times \Lambda_m,$$

where p is the canonical projection into symplectic polars (isotropic spaces) and r is the fiber inclusion. We will examine the generators of the groups $H_1(RSp_{n+m},\,\mathbf{R})$ and $H_1(C_k\Lambda_{n+m},\,\mathbf{R})$, $k = 1,\,\ldots,\,m-1$, coming from $Sp(2m)$ and $Sp(2m-2k)$ respectively. Using analogous arguments to the first part of this section we get

Proposition 3.2

1. *The first homology groups of the sets* RSp_{n+m} *and* $C_k\Lambda_{n+m}$*, for* $k = 1, \ldots, m-1$*, with real coefficients are equal to* $\mathbf{R}$.
2. *For every stratum* RSp_{n+m} *and* $C_k\Lambda_{n+m}$*,* $k = 1, \ldots, m-1$*, we can find a cycle whose class in* $H_1(\Lambda_{n+m}, \mathbf{R}) \simeq \mathbf{R}$ *is the class* μ^*_{n+m} *dual to the universal Maslov class for the Grassmannian* Λ_{n+m}.
3. *For the inclusion* $j\colon S\Lambda_{n+m} \to \Lambda_{n+m}$ *we obtain*

$$j_*(a\mu^*_n, b\mu^*_m) = (-a+b)\mu^*_{n+m},$$

where the classes μ^*_n*,* μ^*_m *and* μ^*_{n+m} *are dual to the universal Maslov classes for the Grassmannians* Λ_n*,* Λ_m *and* Λ_{n+m} *(respectively),* $a, b \in \mathbf{R}$.
4. *The first singular set* $C_1\Lambda_{n+m}$ *is coorientable in* Λ_{n+m}.

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