CLASSIFICATION OF SYMMETRIC CAUSTICS II: CAUSTIC EQUIVALENCE

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ABSTRACT

Symplectic equivalence of Lagrangian projections is too strong to yield a useful classification of projections which commute with a symmetry group action. A weaker equivalence relation, caustic equivalence, is introduced and used to classify the caustics of Lagrangian submanifolds that are invariant under symplectic involutions.

1. *Introduction*

Symmetric caustics arise naturally in a number of contexts including, for example, phase transitions [**4**] and phonon focusing [**12**] in crystals. In this paper and its predecessor [**6**] we describe some general singularity theory machinery that can be used to classify these caustics.

Let X be a smooth manifold with a smooth action of the compact Lie group G. This action extends to an action on the cotangent bundle T^*X which leaves invariant the natural symplectic form. If L is a G-invariant Lagrange submanifold of T^*X then the Lagrange projection $\pi_L: L \to X$ is G-equivariant and its discriminant, the caustic C_L of L, is a G-invariant subvariety of X. In [**6**] we considered the classification of the pairs (T^*X, L) up to symplectic equivalence, that is, G-equivariant symplectomorphisms of T^*X which preserve its natural fibration. Some of the main ideas are reviewed in §2 of this paper. However it appears that this equivalence relation is too strong to be really useful; the classification of corank 1 $\mathbb{Z}_2$-invariant caustics described in [**6**] showed that there can exist generic projections which have infinite codimension if $\dim X \geqslant 3$.

In this paper we introduce a weaker equivalence relation, caustic equivalence, which however still preserves the equivariant diffeomorphism type of the caustic. The main idea is to describe the caustics as sections through larger 'G-versal' caustics and to classify these using singularity theory machinery developed by Damon [**3**] for rather different purposes. The tools we need are described in §3.

In §4 we apply this machinery to the classification of corank 1 $\mathbb{Z}_2$-invariant caustics, giving a complete classification of generic caustics that can occur if $\dim X \leqslant 6$. In higher dimensions we again have the problem of generic caustics with infinite codimension, but these results represent a considerable improvement on the classification of [**6**].

We would like to apply the techniques described in this paper to higher corank singularities which are invariant under other symmetry groups, such as the corank 2 caustics with square symmetry investigated by Nye [**8**]. The main difficulty appears to be the computational problem of finding enough information about vector fields

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tangent to bifurcation sets of versal unfoldings to be able to prove that pull back mappings are infinitesimally stable. This could perhaps be overcome by using symbolic computation software.

A variation on the ideas developed in this paper is given by Montaldi [7]. He has used caustic equivalence, and the assistance of the software Macaulay, to classify certain corank 3 Lagrange singularities with an *antisymplectic* symmetry that occur in time reversible Hamiltonian systems.

The classification of symmetric Legendrian projections (that is, singularities of symmetric wavefronts) turns out to be much more straightforward than that of Lagrangian projections. This is because the distinction between the analogues of symplectic equivalence and caustic equivalence does not exist. A discussion with examples is given in [13], which also contains alternative proofs of Theorem 4.1 of this paper and the main classification theorem of [7].

Finally we note that the ideas described in this paper can be interpreted in the context of multi-parameter gradient bifurcation problems which are invariant under the action of a symmetry group which acts on the parameter space as well as on the 'state space'. In such an interpretation the basic objects of study are taken to be parametrized potential functions. These can be identified with the Morse families of the Lagrangian theory. However in the case of gradient bifurcation problems there seems to be no reason to require that the nondegeneracy condition (2.2) in §2 should hold.

2. *Symmetric caustics*

We first review some of the results of [5, 6]. We identify the manifold (X, x_0) with $(\mathbb{R}^n, 0)$ and assume that the action of G on $(\mathbb{R}^n, 0)$ is linear and orthogonal. We denote $\mathbb{R}^n$ with this action of G by V. We also identify T^*V with $V \oplus V^*$, where V^* is the dual of V. The orthogonality of the action of G on V implies that V^* is isomorphic to V. If $(L, 0) \subset (T^*V, 0)$ is a G-invariant Lagrange submanifold germ and $\pi_L : (L, 0) \to (V, 0)$ its associated G-equivariant Lagrange projection, then $\ker D\pi_L(0) = T_0 L \cap V^*$ is a G-invariant subspace of V^* which we denote by W^*. Let $W^{*\perp}$ denote a G-invariant complement to W^* in V^* and define $W = (W^*)^*$ and $W^\perp = (W^{*\perp})^*$. We can identify V with $W \oplus W^\perp$. Let $q_1 \dots q_k$ denote coordinates for W, $q_{k+1}, \dots, q_n$ for $W^\perp$, $p_1, \dots, p_k$ for W^* and $p_{k+1}, \dots, p_n$ for $W^{*\perp}$. Then the submanifold germ $(L, 0)$ is given by a set of equations:

$$\left.\begin{aligned} \frac{\partial F}{\partial p_i}(p_1, \dots, p_k, q_1, \dots, q_n) &= 0, \quad i = 1, \dots, k, \\ -\frac{\partial F}{\partial q_j}(p_1, \dots, p_k, q_1, \dots, q_n) &= p_j, \quad j = k+1, \dots, n, \end{aligned}\right\} \tag{2.1}$$

where $F(p_1, \dots, p_k, q_1, \dots, q_n)$ is a G-invariant smooth function germ satisfying:

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial p_a \partial p_b}, \frac{\partial^2 F}{\partial p_c \partial q_d}\right)_{\substack{a,b,c=1,\dots,k \\ d=1,\dots,n}} = k \tag{2.2}$$

and

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial p_a \partial p_b}\right)_{a,b=1,\dots,k} = 0 \tag{2.3}$$

at $p = q = 0$. Condition (2.2) means that the equations (2.1) do indeed define a smooth submanifold germ while (2.3) ensures that $\ker D\pi_L(0) = W^*$. We call a

G-invariant function germ satisfying (2.2) and (2.3) a (G-invariant) Morse family. If V' is a representation of G which has a G-invariant subspace isomorphic to V, then an invariant Morse family $F: W^* \oplus V \to \mathbb{R}$ also defines an invariant Lagrange submanifold L' of T^*V'. Let $q_1, \ldots, q_n$ be coordinates on the subspace isomorphic to V and extend these to a system $q_1, \ldots, q_{n'}$ on V'. Then the equations for L' are obtained by supplementing (2.1) by $q_j = 0$ for $j = n+1, \ldots, n'$. We say that the Lagrange submanifold L' is a trivial extension of L.

The discriminant of π_L (the caustic of L) is given by:

$$C_F = \left\{ q \in V : \exists p \in W^* \text{ such that } \frac{\partial F}{\partial p}(p,q) = 0 \text{ and } \det \frac{\partial^2 F}{\partial p^2}(p,q) = 0 \right\}.$$

If $F: W^* \oplus V \to \mathbb{R}$ is regarded as an unfolding, that is a family of functions on W^* parametrized by V, then C_F is the set of parameter values q for which $F(\cdot, q)$ has non-Morse critical points—the local bifurcation set of F. The caustic of a trivial extension of a Lagrange submanifold L is the product of the caustic of L with a smooth G-manifold. Two G-invariant Lagrange submanifold germs $(L_j, 0) \subset (T^*V, 0)$, $(j = 1, 2)$ are symplectically equivalent if there exist germs of a G-equivariant symplectomorphism $\Phi: (T^*V, 0) \to (T^*V, 0)$ and a G-equivariant diffeomorphism $\phi: (V, 0) \to (V, 0)$ such that $\pi \circ \Phi = \phi \circ \pi$ and $\Phi(L_1) \subset L_2$. Two G-invariant Morse families $F_j: W^* \oplus V \to \mathbb{R}$ $(j = 1, 2)$ generate symplectically equivalent Lagrange submanifold germs if and only if they are R_G^+-equivalent. This means that there exists a G-equivariant diffeomorphism germ $\Psi: (W^* \oplus V, 0) \to (W^* \oplus V, 0)$, a G-equivariant diffeomorphism germ $\psi: (V, 0) \to (V, 0)$ and a G-invariant function germ $\alpha: (V, 0) \to \mathbb{R}$ such that $\pi_2 \circ \Psi = \psi \circ \pi_2$ and $F_1(p, q) = F_2(\Psi(p, q)) + \alpha(q)$, where π_2 is the natural projection from $W^* \oplus V$ to V. The 'tangent space' for this equivalence relation, and some simple results concerning infinitesimally R_G^+-stable Morse families, are described in [**6**].

We call the G-invariant function germ $f = F(\cdot, 0)$ on W^* the *organizing centre* of F. Let R_G denote the group of germs of G-equivariant diffeomorphisms of $(W^*, 0)$. Recall from [**9**, §4] that finite R_G-determinacy holds in general in $\mathscr{E}^G(W^*)$, the space of G-invariant function germs on $(W^*, 0)$. It follows that for generic Morse families the organizing centres f are finitely R_G-determined. If G is finite it follows from [**10**, Prop. 5.1] that f is actually R-finitely determined, and so by [**11**, Theorem 2.1] has an unfolding which is 'G-versal'.

PROPOSITION 2.1. *Suppose G is finite and let $f \in \mathscr{E}^G(W^*)$ be R-finitely determined. Set $U = m(W^*)/J(f)$ with its induced action of G. Then there exists a G-invariant unfolding $\mathscr{F}: W^* \oplus U \to \mathbb{R}$ such that for any representation V of G and G-invariant unfolding $F: W^* \oplus V \to \mathbb{R}$ with an organizing centre $F(\cdot, 0)$ which is R_G-equivalent to f, there exists a G-equivariant map germ $\phi: (V, 0) \to (U, 0)$ such that $F(p, q)$ is R_G^+-equivalent to $\mathscr{F}(p, \phi(q))$.*

This gives a 'prenormal form' for generic G-invariant Morse families when G is finite. In [**6**] we showed that the infinitesimal R_G^+-stability of such a Morse family is equivalent to the infinitesimal R_G-stability of the 'pull-back' mapping $\phi: (V, 0) \to (U, 0)$. Here R_G is the group of germs of G-equivariant diffeomorphisms of $(V, 0)$.

For a general compact group G the organizing centre f of a generic G-invariant Morse family must be a germ that can appear in a generic family of G-invariant germs

parametrized by V^G. If f is R_G-simple [2, §17.3], this means that the codimension of the R_G-orbit of f in the space of G-invariant germs which vanish at 0 will be less than or equal to the dimension of V^G. Moreover the restriction $F|_{W^*\oplus V^G}$ will be an R_G-versal unfolding of f. If G acts trivially on W^* then an R_G-versal family $F|_{W^*\oplus V^G}$ is R-versal in the non-equivariant category. It follows from [11] that F must be R_G^+-equivalent to a trivial extension of $F|_{W^*\oplus V^G}$.

3. *Caustic equivalence*

In this section we introduce an equivalence relation between G-invariant Morse families that is weaker than R_G^+-equivalence, but still preserves the equivariant diffeomorphism type of caustics. Let G be a compact Lie group acting orthogonally on $V = \mathbb{R}^n$ and $W = \mathbb{R}^k$ as before. Let $C_F \subset V$ denote the caustic of the G-invariant Morse family $F: W^* \oplus V \to \mathbb{R}$.

DEFINITION 3.1. Two G-invariant Morse families $F_j: W^* \oplus V \to \mathbb{R}$ $(j = 1, 2)$ are *caustic equivalent* if the following conditions hold.

(1) There exists a representation U of G, a G-invariant Morse family $\mathscr{F}: W^* \oplus U \to \mathbb{R}$ and G-equivariant map germs $\phi_j: (V, 0) \to (U, 0)$ such that $F_j(p, q)$ is R_G^+-equivalent to $\mathscr{F}(p, \phi_j(q))$.

(2) There exists a pair of G-equivariant diffeomorphism germs (H, h) with $H: (V \times U, 0) \to (V \times U, 0)$ and $h: (V, 0) \to (V, 0)$ satisfying:

(a) $H(x, y) = (h(x), \tilde{H}(x, y))$, where $\tilde{H}: V \times U \to U$ is a map germ satisfying $\tilde{H}(x, 0) = 0$,

(b) $H(V \times C_{\mathscr{F}}) \subseteq V \times C_{\mathscr{F}}$,

(c) $H(x, \phi_1(x)) = (h(x), \phi_2(h(x)))$.

The equivalence between ϕ_1 and ϕ_2 defined by condition (2) is an equivariant version of the K_V-equivalence of Damon [3] with $V = C_{\mathscr{F}}$. We shall also say that ϕ_1 and ϕ_2 are $K_{C_{\mathscr{F}}}^G$-equivalent. Clearly h maps $\phi_1^{-1}(C_{\mathscr{F}})$ diffeomorphically to $\phi_2^{-1}(C_{\mathscr{F}})$, and hence C_{F_1} diffeomorphically to C_{F_2}.

If F_1, F_2 and $\mathscr{F}$ satisfy the conditions of the definition then their organizing centres f_1, f_2 and f (respectively) are all R_G-equivalent to each other. If they are finitely R-determined then $\mathscr{F}$ can be taken to be a G-versal unfolding of f. As we remarked in §2 this holds generically if G is finite. In this case $\mathscr{F}$ and $C_{\mathscr{F}}$ can be taken to be polynomial and hence analytic. This is essential in order to be able to apply the machinery of [3].

Let $\mathscr{F}: W^* \oplus U \to \mathbb{R}$ be an analytic G-invariant Morse family and $C = C_{\mathscr{F}}$ its caustic. The tangent space for K_C^G equivalence of equivariant map germs $\phi: (V, 0) \to (U, 0)$ is defined as follows (compare with [3, §1]). Let $\mathscr{E}^G(V, U)$ denote the $\mathscr{E}^G(V)$-module of terms of smooth G-equivariant mappings $(V, 0) \to U$. Let $\mathscr{E}_a^G(U)$ denote the ring of germs of G-invariant analytic functions on $(U, 0)$ and $\Theta^G(U)$ ($\Theta_a^G(U)$, respectively) the $\mathscr{E}^G(U)$- ($\mathscr{E}_a^G(U)$-respectively) module of germs of smooth (analytic, respectively) G-equivariant vector fields on $(U, 0)$. Similar definitions are made for vector fields on $(V, 0)$. Define

$$\Theta_a^G(C) = \{\xi \in \Theta_a^G(U) : \xi \cdot g \in I_a(C)\ \forall g \in I_a(C)\},$$

where $I_a(C)$ is the ideal in $\mathscr{E}_a^G(U)$ consisting of analytic function germs which vanish on C. Then $\Theta_a^G(C)$ is a finitely generated $\mathscr{E}_a^G(U)$-module with generators $\{\xi_j\}_{j=1}^m$,

say. Let $\{\zeta_i\}_{i=1}^{l}$ generate $\Theta_a^G(V)$ as an $\mathscr{E}_a^G(V)$-module. For any G-equivariant germ $\phi:(V,0)\to(U,0)$ let $\Theta^G(\phi)$ denote the $\mathscr{E}^G(V)$-module of germs of smooth G-equivariant vector fields along ϕ. We identify $\Theta^G(\phi)$ with $\mathscr{E}^G(V,U)$ in the usual way. We also identify $\Theta^G(U)$ with $\mathscr{E}^G(U,U)$, and so the vector fields ξ_j with G-equivariant germs $\xi_j: U\to U$. Finally we define the extended K_C^G-tangent space of ϕ to be the finitely generated submodule of $\mathscr{E}^G(V,U)$ given by:

$$T_e K_C^G(\phi) = \mathscr{E}^G(v)\cdot\{\zeta_i\cdot\phi, \xi_j\circ\phi\}_{\substack{i=1,\dots,l\\ j=1,\dots,m}}.$$

DEFINITION 3.2. A G-equivariant map germ $\phi:(V,0)\to(U,0)$ is *infinitesimally K_C^G-stable* if $T_e K_C^G(\phi) = \mathscr{E}^G(V,U)$.

Note that the group R_G of germs of G-equivariant diffeomorphisms of $(V,0)$ is contained in K_C^G and so, since infinitesimal R_G-stability of ϕ is equivalent to the infinitesimal R_G^+-stability of $\mathscr{F}(p,\phi(q))$, we see that infinitesimal stability with respect to caustic equivalence is a weaker condition than that with respect to symplectic equivalence. The group K_C^G lies between R_G and K_G, the group of all G-equivariant 'contact equivalences' of the mappings ϕ. Infinitesimal K_G-stability is a generic property for all pairs of representations V and U [9]. We shall see in §4 that the same is not true for K_C^G, though infinitesimal K_C^G-stability is generic more often than infinitesimal R_G-stability is.

One property that K_C^G does share with R_G is that infinitesimal stability implies that the associated Morse family is R_G-versal when restricted to $W^*\oplus V^G$, as long as $\mathscr{F}$ is a G-versal unfolding and the vector fields tangent to $C_{\mathscr{F}}$ all vanish at 0.

PROPOSITION 3.3. *Let $\mathscr{F}: W^*\oplus U\to\mathbb{R}$ be an analytic G-versal unfolding such that*

$$\Theta_a^G(C)\subset(m(U)\Theta_a(U))^G,$$

where $C = C_{\mathscr{F}}$. Then if the G-equivariant map germ $\phi:(V,0)\to(U,0)$ is infinitesimally K_C^G-stable, the Morse family $\mathscr{F}(p,\phi(q))$ restricted to $W^*\oplus V^G$ is R_G-versal.

Proof. The germ ϕ is infinitesimally K_C^G-stable if and only if the map

$$\psi:\Theta^G(V)\to\frac{\mathscr{E}^G(V,U)}{\mathscr{E}^G(V)\cdot\{\xi_j\circ\phi\}_{j=1,\dots,m}}$$

induced by $\zeta\mapsto\zeta\cdot\phi$ is surjective. This implies that

$$\bar{\psi}:\frac{\Theta^G(V)}{(m(V)\Theta(V))^G}\longrightarrow\frac{\mathscr{E}^G(V,U)}{\mathscr{E}^G(V)\cdot\{\xi_j\circ\phi\}_{j=1,\dots,m}+(m(V)\mathscr{E}(V,U))^G}$$

is surjective. The source of $\bar{\psi}$ is a real vector space with basis given by

$$\frac{\partial}{\partial q_j}\quad(j=1,\dots,a),$$

where $q_1,\dots,q_a$ are coordinates for V^G. The target of $\bar{\psi}$ is $\mathscr{E}^G(V,U)/(m(V)\mathscr{E}(V,U))^G$, since the vector fields ξ_j all vanish at 0. This in turn is isomorphic to U^G and hence to $m^G(W^*)/J(f)^G$ where f is the organizing centre of $\mathscr{F}$. Thus the surjectivity of $\bar{\psi}$ is equivalent to the R_G-versality of the restriction of $\mathscr{F}(p,\phi(q))$ to $W^*\oplus V^G$.

Our main tool in the classification of $\mathbb{Z}_2$-invariant caustics in the next section is the following 'reduction lemma', which is proved by standard techniques.

PROPOSITION 3.4. *Let C be an analytic subvariety of U. Let $\phi_t: V \to U$ be an analytic 1-parameter family of G-equivariant maps such that for each t the germ of ϕ_t at 0 is infinitesimally K_C^G-stable. Then there exists an analytic family of pairs of G-equivariant analytic diffeomorphisms (H_t, h_t) with $H_t: V \times U \to V \times U$ and $h_t: V \to V$, satisfying:*

(a) $H_t(x,y) = (h_t(x), \tilde{H}_t(x,y))$, where $\tilde{H}_t: V \times U \to U$,

(b) $H_t(V \times C) \subseteq V \times C$,

(c) *the germs of $H_t(x, \phi_0(x))$ and $(h_t(x), \phi_t(h_t(x)))$ at 0 are equal.*

REMARKS 3.5. (1) In general the K_C^G equivalences constructed by the proposition do not preserve the origin in V, that is $h_t(0) \neq 0$. However, if all analytic vector fields tangent to C vanish at 0 it follows that $\tilde{H}_t(x,0) = 0$ for all x and t. Conditions (a) and (c) of the proposition imply that $\tilde{H}_t(0, \phi_0(0)) = \phi_t(h_t(0))$ and so if $\phi_0(0) = 0$ it follows that $h_t(0) \in \phi_t^{-1}(0)$. In particular, if 0 is an isolated solution of $\phi_t(x) = 0$ then $h_t(0) = 0$.

(2) In addition $\tilde{H}_t(x,0) = 0$ for all x implies that $D_x \tilde{H}_t(0,0) = 0$ and so, if $\phi_0(0) = 0$, we have that

$$D\tilde{H}_t(0, \phi_0(0)) \cdot D_x \phi_0(0) = D_x \phi_t(h_t(0)),$$

and so the rank of $D_x \phi_t$ at $h_t(0)$ must be equal to that of $D_x \phi_0$ at 0. This can also be used to prove that $h_t(0) = 0$. See the proof of Theorem 4.1 (b) in the next section.

4. $\mathbb{Z}_2$-*Symmetry*

We now apply the results of the preceding sections to give a classification of the caustics of corank 1 Lagrange projections from $\mathbb{Z}_2$-invariant Lagrange submanifolds of $T^*\mathbb{R}^n$, where $\mathbb{Z}_2 = \{1, \kappa\}$ acts on $V = \mathbb{R}^n$ by:

$$\kappa \cdot (x_1, \ldots, x_r, y_1, \ldots, y_s) = (-x_1, \ldots, -x_r, y_1, \ldots, y_s), \quad r+s = n. \tag{4.1}$$

We assume that $\mathbb{Z}_2$ acts non-trivially on $W^* = \mathbb{R}$: if λ is a coordinate on W^* then $\kappa \cdot \lambda = -\lambda$. The R_G-finitely determined $\mathbb{Z}_2$-invariant germs on W^* are R_G-equivalent to one of the $f_k(\lambda) = \lambda^{2(k+1)}$, with $\mathbb{Z}_2$-versal unfoldings $\mathscr{F}_k: W^* \oplus U \to \mathbb{R}$ given by:

$$\mathscr{F}_k(\lambda, u_1, \ldots, u_k, v_1, \ldots, v_k) = \lambda^{2(k+1)} + \sum_{i=1}^{k} v_i \lambda^{2i} + \sum_{i=1}^{k} u_i \lambda^{2i-1}, \tag{4.2}$$

where $u_1, \ldots, u_k, v_1, \ldots, v_k$ are coordinates on $U = \mathbb{R}^{2k}$, with $\mathbb{Z}_2$-action:

$$\kappa(u_1, \ldots, u_k, v_1, \ldots, v_k) = (-u_1, \ldots, -u_k, v_1, \ldots, v_k). \tag{4.3}$$

A $\mathbb{Z}_2$-invariant Morse family with organizing centre R_G-equivalent to f_k is therefore R_G^+-equivalent to one of the form:

$$\lambda^{2(k+1)} + \sum_{i=1}^{k} \psi_i(x,y) \lambda^{2i} + \sum_{i=1}^{k} \phi_i(x,y) \lambda^{2i-1}, \tag{4.4}$$

where the ψ_i are $\mathbb{Z}_2$-invariant, $\psi_i(-x,y) = \psi_i(x,y)$ and the ϕ_i satisfy

$$\phi_i(-x,y) = -\phi_i(x,y).$$

Some easy invariant theory implies that $\phi_i(x,y) = \sum_{j=1}^{k} \phi_{i,j}(x,y)\, x_j$ where the $\phi_{i,j}$ are $\mathbb{Z}_2$-invariant, and that the ψ_i and the $\phi_{i,j}$ are functions of the y_i and the products $x_i x_j$. Thus the pull-back mapping $\phi: V \to U$ is given by:

$$\begin{aligned}\phi(x,y) &= (\phi_1(x,y), \ldots, \phi_k(x,y), \psi_1(x,y), \ldots, \psi_k(x,y)) \\ &= \left(\sum_{j=1}^{r} \phi_{1,j} x_j, \ldots, \sum_{j=1}^{r} \phi_{k,j} x_j, \psi_1, \ldots, \psi_k\right). \end{aligned} \tag{4.5}$$

The restriction of $\mathscr{F}(\lambda, \phi(x,y))$ to $V^{\mathbb{Z}_2}$ is $\lambda^{2(k+1)} + \sum_{i=1}^{k} \psi_i(0,y)\lambda^{2i}$. If this is R_G-versal then a change of y coordinates (an R_G^+-equivalence) takes $\mathscr{F}(\lambda, \phi(x,y))$ to a Morse family of the form:

$$\lambda^{2(k+1)} + \sum_{i=1}^{k} y_i \lambda^{2i} + \sum_{i=1}^{k} \phi_i(x,y)\lambda^{2i-1}. \tag{4.6}$$

Generic Morse families, including Morse families given by infinitesimally K_C^G-stable pull-back mappings ϕ, will have this form. We shall say that a caustic is *infinitesimally stable* if it is the caustic of a Morse family given by an infinitesimally K_C^G-stable pull-pack mapping. The main results of this section are summarized in the following theorem.

THEOREM 4.1. (a) *If $r = 1$ then generic corank 1 $\mathbb{Z}_2$-invariant caustics in $\mathbb{R}^n$ are infinitesimally stable and are trivial extensions of the caustics of the Morse families:*

$$\lambda^{2(k+1)} + \sum_{j=1}^{k} y_j \lambda^{2j} + x_1 \lambda, \quad k = 0, \ldots, n-1.$$

(b) *If $n \leqslant 6$ then generic corank 1 $\mathbb{Z}_2$-invariant caustics in $\mathbb{R}^n$ are infinitesimally stable and are trivial extensions of the caustics of the Morse families listed in Table* 4.1

(c) *If $r \geqslant 2$ and $s > r^2/(r-1)$ (for example, $r = 2$, $s = 5$ and so $n = 7$) then there exist generic corank 1 $\mathbb{Z}_2$-invariant caustics in $\mathbb{R}^n$ which are not infinitesimally stable.*

REMARKS. (i) The normal forms I, II, V and XII in Table 4.1 are the standard versal unfoldings, with induced $\mathbb{Z}_2$-actions, of the A_l singularities where $l = 1, 3, 5$ and 7, respectively. Note that only those with l odd appear, despite the apparent $\mathbb{Z}_2$ symmetry of, for example, the swallowtail caustic ($l = 4$). However the versal unfoldings with l even are the Morse families of $\mathbb{Z}_2$ equivariant Lagrange projections for which the $\mathbb{Z}_2$ action on T^*X is antisymplectic, see [7, 13].

(ii) This classification should be compared with that up to symplectic equivalence given in [6]. It is shown there, for example, that if $s \geqslant r = 1$ then infinitesimally symplectically stable Morse families are equivalent to trivial extensions of the families

$$\lambda^{2(k+1)} + \sum_{i=1}^{k} y_i \lambda^{2i} + \sum_{i=1}^{k-1} (\alpha_i + y_{k+i}) x_1 \lambda^{2i+1} + x_1 \lambda$$

with $k \leqslant \frac{1}{2}(s+1)$. Here the parameters α_i are moduli. It follows from Theorem 4.1 (a) that these families are all caustic equivalent to trivial extensions of the families

$$\lambda^{2(k+1)} + \sum_{i=1}^{k} y_i \lambda^{2i} + x_1 \lambda.$$

TABLE 4.1. *Morse families of generic corank* 1 $\mathbb{Z}_2$*-invariant caustics in* $\mathbb{R}^n$, $n \leqslant 6$.

	n	r	Morse family
I	1	1	$\lambda^2+x_1\lambda$
II	2	1	$\lambda^4+y_1\lambda^2+x_1\lambda$
III	3	1	$\lambda^6+\sum_{j=1}^{2} y_j\lambda^{2j}+x_1\lambda$
IV	4	1	$\lambda^8+\sum_{j=1}^{3} y_j\lambda^{2j}+x_1\lambda$
V	4	2	$\lambda^6+\sum_{j=1}^{2} y_j\lambda^{2j}+x_2\lambda^3+x_1\lambda$
VI	5	1	$\lambda^{10}+\sum_{j=1}^{4} y_j\lambda^{2j}+x_1\lambda$
VII	5	2	$\lambda^8+\sum_{j=1}^{3} y_j\lambda^{2j}+x_2\lambda^3+x_1\lambda$
VIII	5	2	$\lambda^6+\sum_{j=1}^{2} y_j\lambda^{2j}+y_3x_2\lambda^3+x_1\lambda$
IX	6	1	$\lambda^{12}+\sum_{j=1}^{5} y_j\lambda^{2j}+x_1\lambda$
X$^{\pm}$	6	2	$\lambda^{10}+\sum_{j=1}^{4} y_j\lambda^{2j}+x_2(\lambda^7\pm\lambda^3)+x_1\lambda$
XI	6	2	$\lambda^8+\sum_{j=1}^{3} y_j\lambda^{2j}+x_2\lambda^5+y_4x_2\lambda^3+x_1\lambda$
XII	6	3	$\lambda^8+\sum_{j=1}^{3} y_j\lambda^{2j}+x_3\lambda^5+x_2\lambda^3+x_1\lambda$

In particular the moduli do not change the smooth equivariant diffeomorphism types of the caustics. Similar remarks apply to other normal forms in the two classifications.

(iii) The caustic of normal form III, the 'symmetric butterfly' is shown in Figure 4.1. In Figure 4.2 we show two views of the caustic of normal form IV.

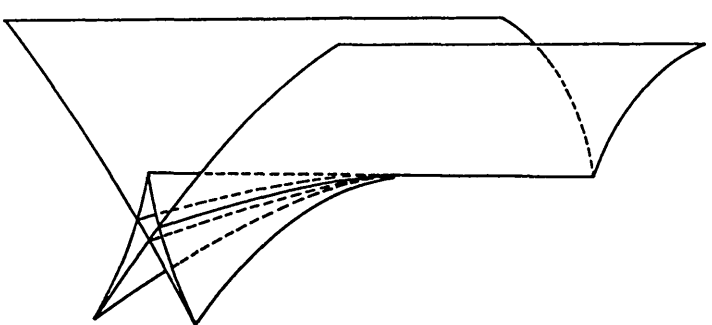

FIG. 4.1. The caustic of normal form III, the symmetric butterfly.

For the proof of the Theorem 4.1 we need a set of generators for the $\mathbb{Z}_2$-equivariant vector fields tangent to the caustic, C_k, of the $\mathbb{Z}_2$-versal family $\mathscr{F}_k$

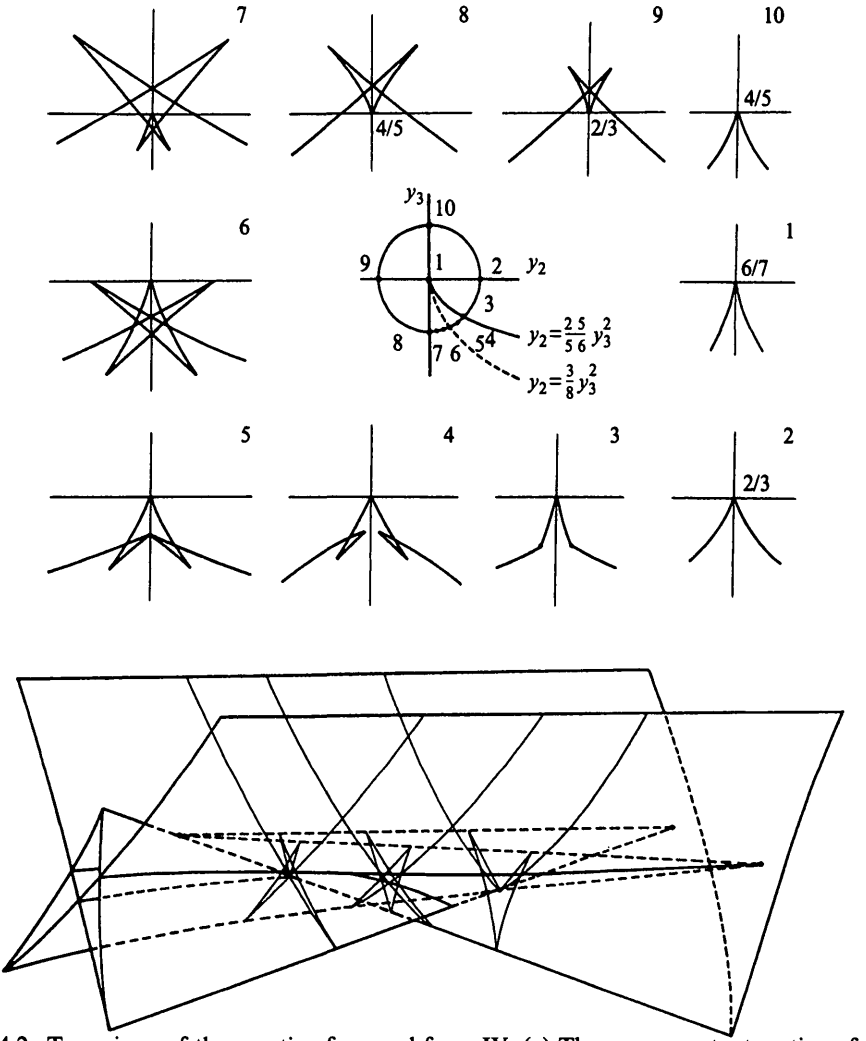

FIG. 4.2. Two views of the caustic of normal form IV. (a) The y_2, y_3 constant sections for a number of values of y_2 and y_3. (b) A y_3 constant section with y_3 negative.

given by (4.2). This is the same as the discriminant of $\partial\mathscr{F}_k/\partial\lambda$, and hence the discriminant of:

$$\lambda^{2k+1}+\sum_{j=1}^{k}\frac{2j}{2(k+1)}v_j\lambda^{2j-1}+\sum_{j=1}^{k}\frac{2j-1}{2(k+1)}u_j\lambda^{2j-2}. \tag{4.7}$$

Arnold [1] showed that the set of all vector fields tangent to the discriminant of

$$\lambda^{l+1}+\sum_{j=1}^{l}\mu_j\lambda^{l-j}$$

is a free module generated by the vector fields:

$$\xi_i=\sum_{j=1}^{l}\xi_{ij}\frac{\partial}{\partial\mu_j}\quad i=1,\ldots,l,$$

where

$$\xi_{ij}=\frac{i(l+1-j)}{l+1}\mu_{i-1}\mu_{j-1}-\sum_{\alpha=0}^{i-1}(i+j-2\alpha)\mu_{\alpha-1}\mu_{i+j-\alpha-1} \tag{4.8}$$

with $\mu_{-1} = 1$ and $\mu_j = 0$ for $j < -1$, $j = 0$ and $j > l$. If l is even, then with respect to the $\mathbb{Z}_2$-action

$$\kappa \cdot (\mu_1, \ldots, \mu_l) = (\mu_1, -\mu_2, \mu_3, \ldots, -\mu_l),$$

ξ_i is equivariant if i is odd and 'anti-equivariant' if i is even. By anti-equivariant we mean that $(\xi_i \cdot f)(\kappa \cdot \mu) = -\kappa \cdot (\xi_i \cdot f)(\mu)$ for any $\mathbb{Z}_2$-invariant function germ f on $\mathbb{R}^l$. Relabel vector fields ξ_i by $\xi_i^+ = \xi_{2i-1}$ and $\xi_i^- = \xi_{2i}$ for $i = 1, \ldots, \frac{1}{2}l$. It follows easily that the $\mathbb{Z}_2$-equivariant vector fields tangent to C_k are generated over the ring of invariant functions by $\{\xi_i^+, \mu_{2j}\xi_i^-\}_{i,j=1,\ldots,k}$.

To get back to our original coordinates (u, v) we set $l = 2k$ and make the substitutions:

$$\mu_{2j-1} = \frac{(k+1-j)}{(k+1)} v_{k+1-j}, \quad \mu_{2j} = \frac{(k+1-j)}{(k+1)} u_{k+1-j}.$$

We do this explicitly for the 1-jets of the vector fields ξ_i^+. Since

$$j^1\xi_i = -\sum_{j=1}^{2k+1-i} (i+j)\mu_{i+j-1} \frac{\partial}{\partial \mu_j}$$

we obtain

$$j^1\xi_i^+ = \sum_{j=i}^{k} a_{ij} u_{j+1-i} \frac{\partial}{\partial u_j} + \sum_{j=i}^{k} b_{ij} v_{j+1-i} \frac{\partial}{\partial v_j} \quad i = 1, \ldots, k, \tag{4.9}$$

where

$$a_{ij} = \frac{[2(k+i-j)+1][j-i+1]}{j}, \quad b_{ij} = \frac{2[k+i-j][j-i+1]}{j}. \tag{4.10}$$

Similar expressions can be obtained for $j^1\xi_i^-$, but these are not needed below.

We use these formulae to obtain a characterization of infinitesimally $K_{C_k}^{\mathbb{Z}_2}$-stable pull-back mappings.

PROPOSITION 4.2. *Let $k \leqslant s$. The map germ $\phi: \mathbb{R}^{r+s} \to \mathbb{R}^{2k}$ given by* (4.5) *with $\psi_j = y_j$ for $j = 1, \ldots, k$ is infinitesimally $K_{C_k}^{\mathbb{Z}_2}$-stable if and only if the following $k \times r$ matrices, evaluated at $x = y = 0$, span the vector space of all $k \times r$ matrices:*

$$\begin{pmatrix} 0 & \ldots & 0 \\ 0 & \ldots & 0 \\ \phi_{1,b} & \ldots & \phi_{k,b} \\ 0 & \ldots & 0 \\ 0 & \ldots & 0 \end{pmatrix} \leftarrow a\text{th row} \qquad a, b = 1, \ldots, r$$

$$\begin{pmatrix} 0 & \ldots & 0 & a_{\alpha\alpha}\phi_{1,1} & \ldots & a_{\alpha k}\phi_{k-\alpha+1,1} \\ \vdots & & & & & \vdots \\ \underbrace{0 \quad \ldots \quad 0}_{\alpha-1} & & & a_{\alpha\alpha}\phi_{1,r} & \ldots & a_{\alpha k}\phi_{k-\alpha+1,r} \end{pmatrix} \qquad \alpha = 1, \ldots, k$$

$$\begin{pmatrix} \dfrac{\partial \phi_{1,1}}{\partial y_c} & \ldots & \dfrac{\partial \phi_{k,1}}{\partial y_c} \\ \vdots & & \vdots \\ \dfrac{\partial \phi_{1,r}}{\partial y_c} & \ldots & \dfrac{\partial \phi_{k,r}}{\partial y_c} \end{pmatrix} \qquad c = k+1, \ldots, s.$$

Proof. The $K_{C_k^{\mathbf{Z}_2}}$-tangent space of ϕ is given by

$$TK_{C_k^{\mathbf{Z}_2}}(\phi) = \mathscr{E}^{\mathbf{Z}_2}(V)\cdot\left\{x_a\frac{\partial\phi}{\partial x_b}, \frac{\partial\phi}{\partial y_c}, \xi_\alpha^+\circ\phi, x_\beta\xi_\alpha^-\circ\phi\right\}$$

where $a, b = 1, \ldots, r$, $c = 1, \ldots, s$, and $\alpha, \beta = 1, \ldots, k$. By Nakayama's lemma $TK_{C_k^{\mathbf{Z}_2}}(\phi) = \mathscr{E}^{\mathbf{Z}_2}(V, U)$ if and only if the same equality holds modulo $m^{\mathbf{Z}_2}(V)\cdot\mathscr{E}^{\mathbf{Z}_2}(V, U)$. The space $\mathscr{E}^{\mathbf{Z}_2}(V, U)/m^{\mathbf{Z}_2}(V)\cdot\mathscr{E}^{\mathbf{Z}_2}(V, U)$ is isomorphic to $M(k, r)\times\mathbb{R}^k$, where $M(k, r)$ is the space of $k\times r$ matrices. Using (4.9) we see that modulo $m^{\mathbf{Z}_2}(V)\mathscr{E}^{\mathbf{Z}_2}(V, U)$ we have

$$x_a\frac{\partial\phi}{\partial x_b} = (\phi_{1,b}x_a, \ldots, \phi_{k,b}x_a, 0, \ldots, 0),$$

$$\frac{\partial\phi}{\partial y_c} = \begin{cases}\left(\sum_{j=1}^r \frac{\partial\phi_{1,j}}{\partial y_c}x_j, \ldots, \sum_{j=1}^r \frac{\partial\phi_{k,j}}{\partial y_c}x_j, 0, \ldots, 0, 1, 0, \ldots, 0\right), & c = 1, \ldots, k\\ \left(\sum_{j=1}^r \frac{\partial\phi_{1,j}}{\partial y_c}x_j, \ldots, \sum_{j=1}^r \frac{\partial\phi_{k,j}}{\partial y_c}x_j, 0, \ldots, 0\right), & c = k+1, \ldots, s\end{cases}$$

$$\xi_\alpha^+\circ\phi = \left(0, \ldots, 0, a_{\alpha\alpha}\sum_{j=1}^r \phi_{1,j}x_j, \ldots, a_{\alpha k}\sum_{j=1}^r \phi_{k+1-\alpha,j}x_j, 0, \ldots, 0\right),$$

and

$$x_\beta\xi_\alpha^-\circ\phi = 0,$$

where the $\phi_{i,j}$ and $\partial\phi_{i,j}/\partial y_c$ are all evaluated at $x = y = 0$.

The projections onto the component $\mathbb{R}^k$ of $\mathscr{E}^{\mathbf{Z}_2}(V, U)/m^{\mathbf{Z}_2}(V)\cdot\mathscr{E}^{\mathbf{Z}_2}(V, U)$ of the $\partial\phi/\partial y_c$ $(c = 1, \ldots, k)$ span that component while the other contributions to the tangent space project to 0. It follows that ϕ is infinitesimally stable if and only if $x_a\partial\phi/\partial x_b$ $(a, b = 1, \ldots, r)$, $\partial\phi/\partial y_c$ $(c = k+1, \ldots, s)$ and $\xi_\alpha^+\circ\phi$ $(\alpha = 1, \ldots, k)$ span the component $M(k, r)$. Translated into matrix notation this proves the lemma.

Proof of Theorem 4.1(c). There can exist infinitesimally $K_{C_k^{\mathbf{Z}_2}}$-stable germs $\phi:\mathbb{R}^{r+s}, 0\to\mathbb{R}^{r+s}, 0$ only if the number of matrices in Proposition 4.2 is greater than the dimension of $M(k, r)$ that is,

$$r^2+k+(s-k) = r^2+s \geqslant kr. \tag{4.11}$$

For any given r and s there exist generic caustics obtained by pulling back from $\mathscr{F}_s$. These can be stable only if (4.11) holds with $k = s$, that is $r = 1$ or $s \leqslant r^2/(r-1)$.

Proof of Theorem 4.1(a). If $r = 1$ then generic Morse families are equivalent to those given by pull-back mappings of the form:

$$\phi(x, y_1, \ldots, y_s) = (\phi_1 x, \ldots, \phi_k x, y_1, \ldots, y_k), \tag{4.12}$$

where the ϕ_i are functions of $y_1, \ldots, y_s$ and x^2. We may also assume that $\phi_1 = 1$. By Proposition 4.2 these mappings are infinitesimally stable if the vectors:

$$(1, \phi_2, \ldots, \phi_k),$$
$$(a_{11}, a_{12}\phi_2, \ldots, a_{1k}\phi_k),$$
$$(0, \ldots, 0, a_{jj}, a_{jj+1}\phi_2, \ldots, a_{jk}\phi_{k+1-j}),$$
$$(0, \ldots, 0, a_{kk}),$$

and

$$\left(0, \frac{\partial\phi_2}{\partial y_c}, \ldots, \frac{\partial\phi_k}{\partial y_c}\right) \quad \text{for} \quad c = k+1, \ldots, s$$

span $\mathbb{R}^k$. Since $a_{jj} \neq 0$ $(j = 1, \ldots, k)$ this is always true. Thus generic Morse families are given by infinitesimally stable pull-back mappings.

Consider now the one-parameter family of Morse families given by:

$$\phi_t(x, y_1, \ldots, y_s) = (x, (1-t)\,\phi_2 x, \ldots, (1-t)\,\phi_k x, y_1, \ldots, y_k).$$

These are all infinitesimally stable at $x = y = 0$ and so by Proposition 3.4 the germ of ϕ_0 at 0 is equivalent to the germ of ϕ_1 at some point $h_1(0)$. Since $\phi_t(x, y_1, \ldots, y_s) = 0$ implies $x = y = 0$, Remark 3.5 implies $h_1(0) = 0$. This proves the result.

Proof of Theorem 4.1 (b). Generic caustics are given by Morse families which are R_G-versal unfoldings when restricted to V^G. If $n \leqslant 6$ then $\dim V^G \leqslant 5$ and so for generic caustics $k \leqslant 5$. We treat each case $k = 0, 1, \ldots, 5$ in turn.

$k = 0$. This corresponds to nonsingular Lagrange projections, with Morse families equivalent to the normal form I in Table 4.1.

$k = 1$. The prenormal form (4.6) with $\phi_1(x, y) = x_1$ immediately gives normal form II in Table 4.1.

$k = 2$. We start with the prenormal form:

$$\lambda^6 + y_2\lambda^4 + y_1\lambda^2 + \left(\sum_{j=1}^{r} \phi_{2j} x_j\right)\lambda^3 + x_1\lambda.$$

By Proposition 4.2 this is stable if and only if the following matrix has rank $2r$.

$$\begin{bmatrix}
1 & 0 & \ldots & \ldots & 0 & \phi_{21} & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & \phi_{21} & 0 & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 1 & 0 & \ldots & \ldots & \ldots & \ldots & \phi_{21} \\
0 & \ldots & \ldots & \ldots & 0 & \phi_{22} & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & \phi_{2r} & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 0 & \phi_{22} & 0 & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 0 & \phi_{2r} & 0 & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & 0 & \ldots & \ldots & \ldots & 0 & \phi_{22} \\
0 & \ldots & \ldots & \ldots & 0 & 0 & \ldots & \ldots & \ldots & 0 & \phi_{2r} \\
a_{11} & 0 & \ldots & \ldots & 0 & a_{12}\phi_{21} & a_{12}\phi_{22} & \ldots & \ldots & \ldots & a_{12}\phi_2 \\
0 & \ldots & \ldots & \ldots & 0 & a_{22} & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & 0 & \dfrac{\partial\phi_{21}}{\partial y_3} & \ldots & \ldots & \ldots & \ldots & \dfrac{\partial\phi_{2r}}{\partial y_3} \\
0 & \ldots & \ldots & \ldots & 0 & \dfrac{\partial\phi_{21}}{\partial y_s} & \ldots & \ldots & \ldots & \ldots & \dfrac{\partial\phi_{2r}}{\partial y_s}
\end{bmatrix}$$

It follows that the Morse family is stable if and only if one of the following conditions holds:

(i) $r = 1$,

(ii) $r \geqslant 2$ and $\phi_{2j}(0) \neq 0$ for some $j \geqslant 2$,

(iii) $r \geqslant 2$, $\phi_{2j}(0) = 0$ for all j, and the matrix $\dfrac{\partial(\phi_{22}, \ldots, \phi_{2r})}{\partial(y_3, \ldots, y_s)}$ has rank $r-1$.

For $r > 2$ conditions (ii) and (iii) are equivalent to the statement that $(\phi_{22}, \ldots, \phi_{2r})$ restricted to $x = y_1 = y_2 = 0$ is transversal to 0, which implies that stability is generic. We obtain normal forms for each of three cases in turn.

(i) This is dealt with by Theorem 4.1(a). The normal form is III in Table 4.1.

(ii) In this case the normal form is G-versal and so is equivalent to normal form V.

(iii) Since $\dfrac{\partial(\phi_{22}, \ldots, \phi_{2r})}{\partial(y_3, \ldots, y_s)}$ has rank $r-1$ we must have $s-2 \geqslant r-1$ and so $s \geqslant r-1$.

Combined with $r \geqslant 2$ and $n \leqslant 6$ this implies $r = 2$ and hence $\dfrac{\partial \phi_{22}}{\partial y_j}(0) \neq 0$ for some $j \geqslant 3$. Without loss of generality we can take $\phi_{22} = y_3 + \psi$, where $D\psi(0) = 0$, and the prenormal form becomes

$$\lambda^6 + y_2 \lambda^4 + y_1 \lambda^2 + ((y_3 + \psi) x_2 + \phi_{21} x_1) \lambda^3 + x_1 \lambda.$$

Define

$$\phi_t = (x_1, (t(\phi_{21} x_1 + \psi x_2) + y_3 x_2), y_1, y_2).$$

For all t this is infinitesimally stable at 0 and so Proposition 3.4 says that there is a diffeomorphism h_t of V such that the germ of ϕ_t at $h_t(0)$ is equivalent to that of ϕ_0 at 0. It remains to show that $h_t(0) = 0$. By Remark 3.5 $h_t(0)$ lies in $\phi_t^{-1}(0)$ and hence in the set $\{(x, y) : x_1 = y_1 = y_2 = 0\}$. Moreover h_t is $\mathbb{Z}_2$-equivariant, so $h_t(0)$ must also lie in $V^G = \{(x, y) : x_1 = x_2 = 0\}$. From Remark 3.5(2) the rank of $D\phi_t$ at $h_t(0)$ must equal that of $D\phi_0$ at 0 and so $h_t(0)$ must lie in $\{(x, y) : x_1 = x_2 = y_1 = y_2 = 0 = y_3 + t\psi\}$. It follows that $h_t(0) = 0$.

$k = 3$. The initial prenormal form is

$$\lambda^8 + y_3 \lambda^6 + y_2 \lambda^4 + y_1 \lambda^2 + \left(\sum_{j=1}^{r} \phi_{3j} x_j\right) \lambda^5 + \left(\sum_{j=1}^{r} \phi_{2j} x_j\right) \lambda^3 + x_1 \lambda.$$

Since $n \leqslant 6$ we must have $r \leqslant 3$.

If $r = 1$ then the Morse family is equivalent to normal form IV by Theorem 4.1(a).

If $r = 2$ then $s = 3$ or 4 and by Proposition 4.2 the prenormal form is stable if and only if the following matrix has rank 6.

$$\begin{bmatrix} 1 & 0 & \phi_{21} & 0 & \phi_{31} & 0 \\ 0 & 1 & 0 & \phi_{21} & 0 & \phi_{21} \\ 0 & 0 & \phi_{22} & 0 & \phi_{32} & 0 \\ 0 & 0 & 0 & \phi_{21} & 0 & \phi_{21} \\ a_{11} & 0 & a_{12}\phi_{21} & a_{12}\phi_{22} & a_{13}\phi_{31} & a_{13}\phi_{32} \\ 0 & 0 & a_{22} & 0 & a_{23}\phi_{21} & a_{23}\phi_{22} \\ 0 & 0 & 0 & 0 & a_{33} & 0 \\ 0 & 0 & \dfrac{\partial \phi_{21}}{\partial y_4} & \dfrac{\partial \phi_{22}}{\partial y_4} & \dfrac{\partial \phi_{31}}{\partial y_4} & \dfrac{\partial \phi_{32}}{\partial y_4} \end{bmatrix}$$

The last row of this matrix is omitted if $s = 3$. A straightforward calculation shows that the matrix has rank 6 if and only if either of the following conditions holds:

(i) $\phi_{22}(0) \neq 0$,

(ii) $\phi_{22}(0) = 0$, $\phi_{32}(0) \neq 0$ and $\dfrac{\partial \phi_{22}}{\partial y_4}(0) \neq 0$.

Only the first of these is possible if $s = 3$. Note that generically one or the other of these conditions will hold and so stability is generic. In case (i) an argument similar to those above, using Remark 3.5(1), shows that the Morse families are equivalent to normal form VII. In case (ii) Remark 3.5(2) is used to show that the prenormal form is equivalent to XI.

If $r = 3$ then $s = 3$ and the prenormal form is stable if and only if the following matrix has rank 9.

$$\begin{bmatrix} 1 & 0 & 0 & \phi_{21} & 0 & 0 & \phi_{31} & 0 & 0 \\ 0 & 1 & 0 & 0 & \phi_{21} & 0 & 0 & \phi_{31} & 0 \\ 0 & 0 & 1 & 0 & 0 & \phi_{21} & 0 & 0 & \phi_{31} \\ 0 & 0 & 0 & \phi_{22} & 0 & 0 & \phi_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{22} & 0 & 0 & \phi_{32} & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{22} & 0 & 0 & \phi_{32} \\ 0 & 0 & 0 & \phi_{23} & 0 & 0 & \phi_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi_{23} & 0 & 0 & \phi_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & \phi_{23} & 0 & 0 & \phi_{33} \\ a_{11} & 0 & 0 & a_{12}\phi_{21} & a_{12}\phi_{22} & a_{12}\phi_{23} & a_{13}\phi_{31} & a_{13}\phi_{32} & a_{13}\phi_{33} \\ 0 & 0 & 0 & a_{22} & 0 & 0 & a_{23}\phi_{21} & a_{23}\phi_{22} & a_{23}\phi_{23} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{33} & 0 & 0 \end{bmatrix}$$

This matrix has rank 9 if and only if $\phi_{22}\phi_{33} - \phi_{32}\phi_{23} \neq 0$, from which it follows that the generic Morse families are G-versal and so are equivalent to the normal form XII.

$k = 4$. If $n \leqslant 6$ then generic Morse families with $k = 4$ can only occur if $r = 1$ or 2.

If $r = 1$ the Morse family is equivalent to VI by Theorem 4.1(a).

If $r = 2$ then $s = 4$ and the prenormal form is

$$\lambda^{10} + y_4\lambda^8 + y_3\lambda^6 + y_2\lambda^4 + y_1\lambda^2 + \left(\sum_{\substack{i=2,3,4 \\ j=1,2}} \phi_{ij} x_j \lambda^{2i-1}\right) + x_1\lambda.$$

This is stable if and only if the following matrix has rank equal to 8.

$$\begin{bmatrix} 1 & 0 & \phi_{21} & 0 & \phi_{31} & 0 & \phi_{41} & 0 \\ 0 & 1 & 0 & \phi_{21} & 0 & \phi_{21} & 0 & \phi_{21} \\ 0 & 0 & \phi_{21} & 0 & \phi_{21} & 0 & \phi_{21} & 0 \\ 0 & 0 & 0 & \phi_{21} & 0 & \phi_{21} & 0 & \phi_{21} \\ a_{11} & 0 & a_{12}\phi_{21} & a_{12}\phi_{22} & a_{13}\phi_{31} & a_{13}\phi_{32} & a_{14}\phi_{41} & a_{14}\phi_{42} \\ 0 & 0 & a_{22} & 0 & a_{23}\phi_{21} & a_{23}\phi_{22} & a_{24}\phi_{31} & a_{24}\phi_{32} \\ 0 & 0 & 0 & 0 & a_{33} & 0 & a_{34}\phi_{21} & a_{34}\phi_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{44} & 0 \end{bmatrix}$$

Taking the determinant of this matrix shows that its rank is 8 if and only if:

$$\phi_{22} \neq 0,$$

and

$$\{a_{23}a_{34}(a_{13}-a_{11})(\phi_{31}\phi_{22}-\phi_{32}\phi_{21}) + a_{23}a_{33}(a_{12}-a_{14})\phi_{42}\}\phi_{22} + (a_{34}a_{22}+a_{33}a_{24})(a_{13}-a_{12})\phi_{32}^2 \neq 0.$$

These conditions divide the space of stable prenormal forms into four connected regions with representative points $\phi_{22} = \pm 1$, $\phi_{42} = \pm 1$, $\phi_{21} = \phi_{31} = \phi_{32} = \phi_{41} = 0$. Since for all the prenormal forms $\phi^{-1}(0) = 0$ it follows from Proposition 3.4 and Remark 3.5(1) that two Morse families in the same connected component are equivalent to each other, and so all the Morse families are equivalent to one of the normal forms

$$\lambda^{10} + \sum_{j=1}^{4} y_j \lambda^{2j} \pm x_2(\lambda^7 \pm \lambda^3) + x_1 \lambda.$$

The change of coordinates $x_2 \to -x_2$ reduces these four normal forms to the two $X^{\pm}$.

$k = 5$. If $n \leqslant 6$ then generic Morse families with $k = 5$ can occur only if $r = 1$. By Theorem 4.1(a) we obtain normal form IX.

This completes the proof of Theorem 4.1.

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