J. London Math. Soc. (2) 69 (2004) 503–517
DOI: 10.1112/S0024610703004952

LINEAR AUTOMORPHISMS THAT ARE SYMPLECTOMORPHISMS

STANISŁAW JANECZKO AND ZBIGNIEW JELONEK

Abstract

Let $\mathbf{K}$ be the field of real or complex numbers. Let $(X \cong \mathbf{K}^{2n}, \omega)$ be a symplectic vector space and take $0 < k < n, N = \binom{2n}{2k}$. Let $L_1, \dots, L_N \subset X$ be $2k$-dimensional linear subspaces which are in a sufficiently general position. It is shown that if $F : X \longrightarrow X$ is a linear automorphism which preserves the form ω^k on all subspaces $L_1, \dots, L_N$, then F is an ϵ_k-symplectomorphism (that is, $F^*\omega = \epsilon_k \omega$, where $\epsilon_k^k = 1$). In particular, if $\mathbf{K} = \mathbb{R}$ and k is odd then F must be a symplectomorphism. The unitary version of this theorem is proved as well. It is also observed that the set $\mathcal{A}_{l,2r}$ of all l-dimensional linear subspaces on which the form ω has rank $\leqslant 2r$ is linear in the Grassmannian $G(l, 2n)$, that is, there is a linear subspace L such that $\mathcal{A}_{l,2r} = L \cap G(l, 2n)$. In particular, the set $\mathcal{A}_{l,2r}$ can be computed effectively. Finally, the notion of symplectic volume is introduced and it is proved that it is another strong invariant.

1. *Introduction*

Let $\mathbf{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let (X, ω) be a symplectic vector space over $\mathbf{K}$, that is, $X \cong \mathbf{K}^{2n}$ is a vector space and ω is a bilinear, non-degenerate skew-symmetric form on X. The symplectic complement of a linear subspace $L \subset X$ is defined as the subspace

$$L^\omega = \{x \in X : \omega(x, y) = 0, \ \forall\, y \in L\}.$$

This space may not be transversal to L. A subspace $L \subset X$ is called isotropic if $L \subset L^\omega$, coisotropic if $L^\omega \subset L$, symplectic if $L \cap L^\omega = \{0\}$ and Lagrangian if $L^\omega = L$. L is symplectic if and only if $\omega\,|_L$ is a non-degenerate form. For any subspace L we have $\dim L + \dim L^\omega = \dim X$ and $(L^\omega)^\omega = L$. There exists a basis of X, called a symplectic basis, $u_1, \dots, u_n, v_1, \dots, v_n$, such that

$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0, \quad \omega(u_i, v_j) = \delta_{ij}.$$

If $L \subset X$ is a subspace, then there is a basis $u_1, \dots, u_k, v_1, \dots, v_k, w_1, \dots, w_l$ of L such that $\omega\,|_L(u_j, v_k) = \delta_{jk}$ and all other pairings $\omega\,|_L(\bullet, \bullet)$ vanish. This basis extends to a symplectic basis for (X, ω) and the integer $2k$ is the rank of $\omega\,|_L$.

We say that a linear automorphism $F : X \longrightarrow X$ is a symplectomorphism (or is symplectic on X) if $F^*\omega = \omega$, that is, $\omega(x, y) = \omega(F(x), F(y))$ for every $x, y \in X$. If $L \subset X$ is a linear subspace, then we say that F is symplectic on L if $\omega(x, y) = \omega(F(x), F(y))$ for every $x, y \in L$. The group of automorphisms of (X, ω) is called the symplectic group and is denoted by $\mathrm{Sp}(X, \omega)$. Via a symplectic basis, $\mathrm{Sp}(X, \omega)$ can be identified with the group $\mathrm{Sp}(2n, \mathbb{R})$ of real $2n \times 2n$ matrices A which satisfy $A^T J_0 A = J_0$, where J_0 is the $2n \times 2n$ matrix of ω (in a symplectic basis). If we identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ in the usual way: $\mathbb{R}^{2n} \ni (x, y) \longrightarrow x + iy$, then multiplication

Received 11 January 2002; revised 4 June 2003.

2000 *Mathematics Subject Classification* 51N10, 51N20, 15A04.

The second author was partially supported by KBN grant 2PO3A 01722.

by J_0 in $\mathbb{R}^{2n}$ corresponds to multiplication by i in $\mathbb{C}^n$. With this identification the complex linear group $\mathrm{GL}(n,\mathbb{C})$ is a subgroup of $\mathrm{GL}(2n,\mathbb{R})$ and the unitary group $U(n)$ is a maximal, compact subgroup of $\mathrm{Sp}(2n,\mathbb{R})$; $U(n)=\mathrm{Sp}(2n,\mathbb{R})\cap O(2n,\mathbb{R})$.

If L is a subspace of (X,ω), then obviously the form $\omega\mid_L$ and its exterior powers ω^k (volume form in particular) are symplectic invariants (cf. [**1**]). The following two natural questions are raised by this property.

(1) Does the assumption that such linear subspace data are preserved determine interesting transformations of the symplectic vector space?

(2) Is it possible to collect such partial 'information' from a finite family of subspaces and construct a complete system of invariants for the symplectic group?

In this paper, using the methods and ideas introduced in [**4**], we establish the structural data on subspaces and find the groups of automorphisms defined by them. The criteria for symplectomorphisms are confirmed by geometrical intuition coming from standard symplectic geometry. While symplectic structures naturally arise in diverse contexts such as Hamiltonian mechanics, field theory, differential and algebraic geometry, we focus on the motivation to extend the considerations including the time-reversible (including anti-symplectomorphisms) Hamiltonian systems (cf. [**6**]) and composite systems with weakened phase space structure (cf. [**7**]).

In Section 3, we discuss the case of a finite family of 2-dimensional subspaces of a symplectic vector space and describe the properties of linear automorphisms which are symplectic on all members of this family. The crucial property that the subspaces of the system are not co-planar, that is, they do not belong to any hyperplane in the appropriate Grassmannian, allows us to get symplectomorphisms or even more generally linear symplectic relations. The aim of the rest of this paper is to set up a method of generalizing this type of argument. In Section 4, the case of linear automorphisms preserving the symplectic data ω^k on $2k$-dimensional subspaces is studied and conditions under which these automorphisms become symplectomorphisms are obtained. We need this symplectic version to prove similar results for the Hermitian case in Section 5. Generically, elements of the Grassmannian of $2k$-dimensional linear subspaces of (X,ω) are symplectic subspaces. In Section 6, we show that the set of all linear subspaces of a given dimension on which the symplectic form is degenerate up to a fixed rank, is an irreducible algebraic subset of the Grassmannian. Using this type of strata of the Grassmannian, another, more general characterization theorem for symplectomorphisms is proved. The symplectic Gram matrix invariants connected with symplectic volume are studied in Section 7. We prove that the symplectic volume is a strong invariant on the linear subspaces of the symplectic vector space, which gives another useful characterization for automorphisms which are conformal symplectomorphisms.

2. *Notation and basic results*

Let X be a vector space of dimension $2n$. Let $L\subset X$ be an l-dimensional linear subspace ($0<l<2n$). If vectors $v_1,\dots,v_l$ form a basis of L, then the line $\mathbf{K}(v_1\wedge\dots\wedge v_l)$ is uniquely determined by L and it does not depend on the basis $v_1,\dots,v_l$. This line determines a unique point $\Psi(L)$ in the Grassmannian $G(l,2n)\subset\mathbb{P}^{N-1}$, where $N=\binom{2n}{l}$ and $\mathbb{P}^{N-1}$ denotes the $(N-1)$-dimensional projective space. We have the following notion of co-planar spaces.

DEFINITION 2.1. Let $L_1, \ldots, L_N$ be l-dimensional linear subspaces of X. We say that they are *co-planar* if the points $\Psi(L_1), \ldots, \Psi(L_N) \in G(l, 2n)$ are co-planar, that is, if there is a hyperplane $\Lambda \subset \mathbb{P}^{N-1}$ containing all points $\Psi(L_1), \ldots, \Psi(L_N)$.

REMARK 2.1. Note that subspaces $L_1, \ldots, L_N$ are not co-planar if the points $\Psi(L_1), \ldots, \Psi(L_N) \in G(l, 2n)$ span linearly the whole space $\mathbb{P}^{N-1}$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_{2n}$ be a basis of X. Since the Grassmannian $G(l, 2n)$ contains the subset $\{\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_l}\}_{0 < i_1 < \ldots < i_l \leqslant 2n}$ we see that the subspaces $L_1, \ldots, L_N$ are not co-planar if the points $\Psi(L_1), \ldots, \Psi(L_N)$ span linearly the Grassmannian $G(l, 2n)$.

If $L_1, \ldots, L_m$ are not co-planar, then we can always choose a subfamily $L_1, \ldots, L_k$ of non-co-planar subspaces with $k = N$, and conversely, it is easy to construct a collection $L_1, \ldots, L_N$ which is not co-planar. Hence we can always assume that $m = N$. We prove the following theorem.

THEOREM 2.1. *Let (X, ω) be a symplectic vector space of dimension $2n$ and let $F: X \longrightarrow X$ be a linear automorphism. Let $0 < k < n$ be a natural number. Assume that F preserves the form ω^k on a collection $L_1, \ldots, L_N$ of $2k$-dimensional subspaces which are not co-planar. Then F is an ϵ_k-symplectomorphism* (*that is, $F^*\omega = \epsilon\omega$, where $\epsilon^k = 1$*). *In particular, if $\mathbf{K} = \mathbb{R}$ and the number k is odd, then F is a symplectomorphism.*

COROLLARY 2.1. *Let (X, ω) be a symplectic vector space of dimension $2n$ and let $F: X \longrightarrow X$ be a linear automorphism. Let $0 < k < l \leqslant n$ be natural numbers such that $(k, l) = 1$. Assume that F preserves the forms ω^k and ω^l. Then F is a symplectomorphism.*

We also observe that the following is true.

THEOREM 2.2. *Let (X, ω) be a $2n$-dimensional symplectic vector space and let $0 < l < 2n$ be a natural number. The set $\mathcal{A}_l$ of all l-dimensional linear subspaces on which the form ω degenerates* (*that is, does not have maximal rank*) *is co-planar in the Grassmannian $G(l, 2n) \subset \mathbb{P}^{N-1}$. More precisely, if $\mathcal{A}_{l,2r}$ denotes the set of all l-dimensional linear subspaces on which the form ω has rank $\leqslant 2r$, then $\mathcal{A}_{l,2r-2}$ is co-planar in $\mathcal{A}_{l,2r}$, that is, there is a linear projective subspace $L \subset \mathbb{P}^{N-1}$ such that $\mathcal{A}_{l,2r} \cap L = \mathcal{A}_{l,2r-2}$. Moreover, $\mathcal{A}_{l,2r}$ is an irreducible algebraic variety with effectively computable equations.*

Finally, we introduce the $2k$-dimensional symplectic volume as $\mathrm{svol}_{2k}(v_1, \ldots, v_{2k}) = \det[\omega(v_i, v_j)]$ and we show the following theorem.

THEOREM 2.3. *Let (X, ω) be a symplectic vector space of dimension $2n$ and let $F: X \longrightarrow X$ be a linear automorphism. Let $0 < k < n$ and $M = 2\binom{2n}{2k}$. Assume that F preserves the $2k$-dimensional symplectic volume on a collection $L_1, \ldots, L_M$ of $2k$-dimensional subspaces which are sufficiently general* (*not two-co-planar, that is there are no hyperplanes $\Lambda_1, \Lambda_2 \subset \mathbb{P}^{N-1}$ such that $\Psi(L_1), \ldots, \Psi(L_M) \in \Lambda_1 \cup \Lambda_2$*). *Then F is an ϵ_{2k}-symplectomorphism.*

3. *The case of $k=1$*

Let (X,ω) be a symplectic vector space, that is, $X\cong\mathbf{K}^{2n}$ is a vector space and ω is a bilinear, non-degenerate skew-symmetric form.

THEOREM 3.1. *Let X be a symplectic vector space of dimension $2n$ and let $F:X\longrightarrow X$ be a linear automorphism. Assume that F is symplectic on a collection $L_1,\dots,L_N$ of 2-dimensional subspaces, which are not co-planar. Then F is a symplectomorphism.*

Proof. It is well known (see for example [**1**]) that we can always choose a vector basis $\mathbf{e}_1,\dots,\mathbf{e}_{2n}$ in X (symplectic basis) such that $\omega((\sum v_i\mathbf{e}_i),(\sum w_i\mathbf{e}_i))=\sum_{0<i\leqslant n}(v_iw_{i+n}-v_{i+n}w_i)$. We have $\omega=\sum_{i=1}^n\mathbf{e}_i^*\wedge\mathbf{e}_{i+n}^*$ in the dual basis $\mathbf{e}_1^*,\dots,\mathbf{e}_{2n}^*$.

Denote by $\overline{G(2,2n)}\subset\bigwedge^2X=:Y$ the set of all vectors $v\wedge w$, where $v,w\in X$. Let (v_1^i,v_2^i) be a basis of the linear subspace L_i, $i=1,\dots,N$. It is easy to see that the linear subspaces $L_1,\dots,L_N$ are co-planar if the vectors $\{v_1^i\wedge v_2^i\}_{i=1,\dots,N}\subset Y$ are co-planar in Y.

Set $u_i=v_1^i\wedge v_2^i$. Now, consider the mapping $R:=\bigwedge^2F:Y\longrightarrow Y$. In Y we have the basis $\mathbf{e}_{ij}=\mathbf{e}_i\wedge\mathbf{e}_j, 0<i<j\leqslant 2n$. For $y=\sum y_{ij}\mathbf{e}_{ij}$ let $\eta(y)=\sum_{i=1}^n y_{ii+n}$. Of course, η is a linear form on Y.

Observe that $\eta(v\wedge w)=\omega(v,w)$. Consequently, the mapping F is symplectic if and only if for every $v,w\in X$ we have $\eta(v\wedge w)=\eta(R(v\wedge w))$. However, the form $\eta(y)-\eta(R(y))$ is linear on Y and by the assumption it vanishes on the vectors $u_i, i=1,\dots,N$. Since the latter set is not co-planar in Y, the form $\eta(y)-\eta(R(y))$ vanishes identically on Y. This means that $\omega(v,w)=\omega(F(v),F(w))$ for every $v,w\in X$, that is, F is a symplectomorphism. □

From the proof we have the following.

COROLLARY 3.1. *Let (X,ω) be a $2n$-dimensional symplectic vector space. Then the set of all 2-dimensional linear subspaces on which the form ω degenerates is co-planar in the Grassmannian $G(2,2n)$.*

Proof. Indeed, this set is given by the equation $\{y\in G(2,2n):\eta(y)=0\}$. □

REMARK 3.1. The Lagrangian Grassmann manifold Λ_2 is isomorphic to a non-singular quadric in $\mathbb{P}^4$. Moreover, if $\mathbf{K}=\mathbb{R}$, then this quadric has the signature $(+++--)$. Indeed, the Grassmannian $G(2,4)$ is given by the equation $y_{12}y_{34}-y_{13}y_{24}+y_{14}y_{23}=0$ (see, for example, [**2**, p. 211]). The manifold Λ_2 is a section of the Grassmannian $G(2,4)$ by the hyperplane $y_{13}+y_{24}=0$ (cf. the proof of Theorem 2.1). Consequently, Λ_2 in homogeneous coordinates $y_{12},y_{34},y_{24},y_{14},y_{23}$ has the equation $y_{12}y_{34}+y_{24}^2+y_{14}y_{23}=0$.

We apply the above results to the case of more general symplectic transformations. Let (X,ω), (Y,μ) be two symplectic vector spaces, $X\cong\mathbf{K}^{2n}$, $Y\cong\mathbf{K}^{2m}$, $m\geqslant n$.

DEFINITION 3.1. Let F be a linear subspace of $X\oplus Y$, $\dim F=m+n$. We say that F is an *ϵ-symplectic relation* if F is a Lagrangian subspace of the product symplectic vector space $(X \oplus Y, \mu \ominus \epsilon\omega)$, that is $\mu \ominus \epsilon\omega\,|_F = 0$, where $\epsilon = \pm 1$, $\mu \ominus \epsilon\omega = \pi_Y^* \mu - \pi_X^* \epsilon\omega$ and $\pi_X(\pi_Y) : X \oplus Y \longrightarrow X(Y)$ are the canonical projections.

Remark 3.2. If F is an ϵ-symplectic relation of $(X \oplus Y,\ \mu \ominus \epsilon\omega)$, $m = n$, $\epsilon = +1(-1)$, and $\pi_X\,|_F : F \longrightarrow X$ is onto, then F is the graph of a symplectomorphism (anti-symplectomorphism). If $m > n$, $\epsilon = +1(-1)$, and the projection $\pi_X\,|_F$ is onto, then an ϵ-symplectic relation F is the graph of a symplectic relation called the symplectic reduction relation (anti-reduction) (cf. [**8**]). In this case $W = \pi_Y(F)$ is a coisotropic subspace of (Y, μ), that is the μ-complementary space W^μ is a subspace of W, and f is the graph of a linear projection of W onto X along W^μ.

Let F be a linear subspace of $X \oplus Y$. Let S be an affine subspace of X. We define the image $F(S)$ of S under F, by $F(S) = \{y \in Y : \exists_{x \in S}(x, y) \in F\}$.

Theorem 3.2. *Let $(X, \omega), (Y, \mu)$ be two symplectic vector spaces with $\dim X = 2n$, $\dim Y = 2m$, $m \geqslant n$. Let F be a linear subspace of the product symplectic vector space $(X \oplus Y, \mu \ominus \epsilon\omega)$, $\dim F = m + n$. Assume that $\pi_X\,|_F$ is onto and the form $\mu \ominus \epsilon\omega$ is isotropic on a collection of subspaces $L_1 \oplus F(L_1), \ldots, L_N \oplus F(L_N) \subset X \oplus Y$, where $L_1, \ldots, L_N \subset X$ are non-co-planar, 2-dimensional subspaces of X. Then the following hold.*

(1) *F is the graph of an ϵ-symplectomorphism if $m = n$.*

(2) *F is an ϵ-symplectic reduction relation if $m > n$.*

Proof. First we consider the case $m = n$. By assumption, F is the graph of a linear map $\phi : X \longrightarrow Y$. Take symplectic bases $\{e_i\}, \{h_i\}$ for X and Y. Consider the linear forms $\bar{\omega} : \bigwedge^2 X \longrightarrow \mathbf{K}$ and $\bar{\mu} : \bigwedge^2 Y \longrightarrow \mathbf{K}$ given by $\bar{\omega}(v \wedge w) = \omega(v, w)$, $\bar{\mu}(s \wedge u) = \mu(s, u)$. Then $\bar{\omega}(\sum x_{ij} e_{ij}) = \sum_{i=1}^n x_{ii+n}$, and $\bar{\mu}(\sum y_{ij} h_{ij}) = \sum_{i=1}^m y_{ii+m}$, where $e_{ij} = e_i \wedge e_j$, $h_{ij} = h_i \wedge h_j$. The forms $\bar{\omega}$ and $\bar{\mu}$ define a linear form on $\bigwedge^2 X$ by

$$\bar{\mu}(\Phi(x)) - \epsilon\bar{\omega}(x), \tag{3.1}$$

where $\Phi = \bigwedge^2 \phi : \bigwedge^2 X \longrightarrow \bigwedge^2 Y$. As in the proof of Theorem 3.1, let (v_1^i, v_2^i) be a basis of L_i, $i = 1, \ldots, N$, and $u_i = v_1^i \wedge v_2^i$. By assumption, the form $\bar{\mu}(\Phi(x)) - \epsilon\bar{\omega}(x)$ vanishes on $u_i, i = 1, \ldots, N$. Since $\{u_i\}_{i=1}^N$ is not co-planar in $\bigwedge^2 X$, the form (3.1) vanishes identically on $\bigwedge^2 X$, that is, F is Lagrangian, so $\mathrm{Ker}\phi = \{0\}$, since otherwise ω would be degenerate. Obviously ϕ is an ϵ-symplectomorphism. Indeed, $\mu(\phi(v), \phi(w)) - \epsilon\omega(v, w) = 0$ for every $v, w \in X$.

In the case $m > n$ we choose a linear section $\psi : X \longrightarrow Y$ of the canonical projection $\pi_X\,|_F : F \longrightarrow X$ and get the vanishing of the form

$$\mu(\psi(v), \psi(w)) - \epsilon\omega(v, w) = 0$$

for all sections ψ and vectors $v, w \in X$. From this, using the previous arguments, we deduce that F is Lagrangian and so an ϵ-symplectic reduction relation. $\square$

4. *The general case*

Definition 4.1. Let (X, ω) be a symplectic vector space and let $F : X \longrightarrow X$ be a linear automorphism. We say that F is an *ϵ_k-symplectomorphism* if $F^*\omega = \epsilon\omega$, where $\epsilon^k = 1$. Moreover, we say that F is an *anti-symplectomorphism* if $F^*\omega = -\omega$.

Remark 4.1. We can treat an ϵ_k-symplectomorphism as a root of a symplectomorphism. Indeed, if F is an ϵ_k-symplectomorphism, then F^k is a symplectomorphism.

Theorem 4.1. *Let (X,ω) be a symplectic vector space of dimension $2n$ and let $F: X \longrightarrow X$ be a linear automorphism. Let $0<k<n$. Assume that F preserves the form ω^k on a collection $L_1,\ldots,L_N$ of $2k$-dimensional subspaces which are not co-planar. Then F is an ϵ_k-symplectomorphism. In particular, if $\mathbf{K}=\mathbb{R}$ and the number k is odd, then F is a symplectomorphism.*

Proof. Denote by $\overline{G(2k,2n)} \subset \bigwedge^{2k} X =: Y$ the set of all vectors $v_1 \wedge \ldots \wedge v_{2k}$, where $v_1,\ldots,v_{2k} \in X$. Let $(v_1^i,\ldots,v_{2k}^i)$ be a basis of L_i, $i=1,\ldots,N$. It is easy to see that $L_1,\ldots,L_N$ are co-planar if the vectors $u_i := v_1^i \wedge \ldots \wedge v_{2k}^i$, $i=1,\ldots,N$ are contained in a hyperplane in Y.

In Y we have the basis $(\underline{\mathbf{e}_{i_1\ldots i_{2k}}} = \mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_{2k}}, 0<i_1<\ldots<i_{2k} \leqslant 2n)$. For a vector $y = x_1 \wedge \ldots \wedge x_{2k} \in \overline{G(2k,2n)}$ let $y = \sum y_{i_1\ldots i_{2k}} \mathbf{e}_{i_1\ldots i_{2k}}$. It is easy to see that the mappings $y_{i_1\ldots i_{2k}}$ treated as functions of the vectors $x_1,\ldots,x_{2k}$ form a basis of the space of all $2k$-linear skew-symmetric forms on X. In particular, there are numbers $a_{i_1\ldots i_{2k}}$ such that $\omega^k(x_1,\ldots,x_{2k}) = \sum a_{i_1\ldots i_{2k}} y_{i_1\ldots i_{2k}}$. Take $\eta(x_1 \wedge \ldots \wedge x_{2k}) = \omega^k(x_1,\ldots,x_{2k})$. We have $\eta(y) = \sum a_{i_1\ldots i_{2k}} y_{i_1\ldots i_{2k}}$ and consequently we can treat η as a linear form on the whole of Y.

Now, consider the mapping $R := \bigwedge^{2k} F : Y \longrightarrow Y$. Then F preserves the form ω^k on L_i if and only if $\eta(u_i) = \eta(R(u_i))$. However, the form $\eta(y) - \eta(R(y))$ is linear on Y and by assumption it vanishes on the set of vectors u_i, $i=1,\ldots,N$. Since the latter set is not co-planar in Y, the form $\eta(y) - \eta(R(y))$ vanishes identically on Y. This means that $\omega^k(v_1,\ldots,v_{2k}) = \omega^k(F(v_1),\ldots,F(v_{2k}))$ for every $v_1,\ldots,v_{2k} \in X$, that is, F preserves the form ω^k. Now we prove the following.

Lemma 4.1. *Let (X,ω) be a symplectic vector space. Let $W \subset X$ be a $2k$-dimensional symplectic subspace of X, that is $(W, \omega|_W)$ is a symplectic vector space. If*

$$\omega^k(v, w_1, \ldots, w_{2k-1}) = 0$$

for any $\{w_1,\ldots,w_{2k-1}\}$, $w_i \in W$, then v is complementary to W with respect to ω.

Proof. Because W is symplectic we can choose a symplectic basis $\{\mathbf{e}_i\}_{i=1}^{2n}$ in X such that W is generated by the vectors $\mathbf{e}_i$, $i \leqslant 2k$. In this basis we have $\omega|_W = \sum_{i=1}^k \mathbf{e}_i^* \wedge \mathbf{e}_{i+k}^*$. Moreover, we can assume that the subspace V ω-complementary to W is generated by the vectors $\mathbf{e}_i$, $i>2k$. Now let $v = \sum_{i=1}^{2n} a_i \mathbf{e}_i$. Then taking the $(2k-1)$-elements $\mathbf{e}_1,\ldots,$‘j’$,\ldots,\mathbf{e}_{2k}$, we get

$$\omega^k(v, \mathbf{e}_1, \ldots, \text{‘}j\text{’}, \ldots, \mathbf{e}_{2k}) = \pm a_j \omega^k(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{2k}) = \pm\, a_j k!,$$

so by assumption we have $a_j = 0$, $j=1,\ldots,2k$. This means that $v \in V$. □

Now consider a general vector subspace $V \subset X$ of dimension $2k+2$. Take a general subspace $L_V \subset V$ of dimension $2k$ and let $K_V \subset V$ be the subspace which is complementary to L_V with respect to ω. Since the form ω^{k+1} is proportional to the usual volume form Ω on V and any linear mapping changes the volume form by multiplying it by a suitable constant, there is a non-zero constant c such that

$$\Omega(F(v_1),\dots,F(v_{2k+2})) = c\Omega(v_1,\dots,v_{2k+2}) \tag{4.1}$$

for every $v_1,\dots,v_{2k+2} \in V$.

Now take in (4.1) $v_1,\dots,v_{2k} \in L_V$ and $v_{2k+1} = v$, $v_{2k+2} = w \in K_V$. Computing directly we have (note that v, w are ω-complementary to $v_1,\dots,v_{2k}$!)

$$\Omega(v_1,\dots,v_{2k},v,w) = a\omega(v,w)\omega^k(v_1,\dots,v_{2k}), \tag{4.2}$$

where a is a suitable non-zero constant. Since F preserves ω^k, using Lemma 4.1, we can deduce that the subspace $F(V_L)$ is complementary to $F(K_L)$ with respect to ω (in the space $F(V)$). In fact $\omega^k(F(v),F(v_1),\dots,F(v_{2k-1})) = 0$ for all $v_1,\dots,v_{2k-1} \in L_V$, so we can apply Lemma 4.1.

Hence as before we have

$$\Omega(F(v_1),\dots,F(v_{2k}),F(v),F(w)) = a\omega(F(v),F(w))\omega^k(F(v_1),\dots,F(v_{2k})). \tag{4.3}$$

Now from (4.2) and (4.3) we conclude that

$$\omega(F(v),F(w)) = c\omega(v,w).$$

Since the spaces K_V are not co-planar in $G(2,V)$ we deduce (as in the proof of Theorem 3.1) that $F^*\omega = c\omega$ on V. Finally, since $F^*\omega^k = \omega^k = c^k\omega^k$, we have $c^k = 1$. Thus there are only a finite number of possibilities for c. Let $\epsilon_1,\dots,\epsilon_k$ be all the kth roots of unity in $\mathbf{K}$ (if $\mathbf{K} = \mathbb{R}$, then we consider 1 and -1 only). Now in the Grassmannian $G(2,X)$ consider the subsets $S_i := \{y : \epsilon_i\eta(y) = \eta(\bigwedge^2 F(y))\}$. Since $G(2,X) = \bigcup_{i=1}^k S_i$, there is an i such that S_i is not co-planar in $G(2,X)$. Consequently, we must have $\epsilon_i\eta(y) = \eta(\bigwedge^2 F(y))$ identically, that is, $F^*\omega = \epsilon_i\omega$ on the whole of X. Moreover, if k is odd and $\mathbf{K} = \mathbb{R}$, we have $\epsilon_i = 1$. □

COROLLARY 4.1. *Let (X,ω) be a symplectic vector space and let $F : X \longrightarrow X$ be a linear automorphism. Let $0 < l < k \leqslant n$ be natural numbers such that $(k,l) = 1$. Assume that F preserves the forms ω^k and ω^l. Then F is a symplectomorphism.*

REMARK 4.2. The simplest example of an ϵ_k-symplectomorphism $F : X \longrightarrow X$ is the mapping

$$J_k\left(\sum_{i=1}^n (v_i\mathbf{e}_i + v_{i+n}\mathbf{e}_{i+n})\right) = \sum_{i=1}^n (v_i\mathbf{e}_i + \epsilon_k v_{i+n}\mathbf{e}_{i+n})$$

in symplectic coordinates, where ϵ_k is a primitive root of unity. Thus if the linear mapping F preserves the form ω^k then it has the form $F = J_l S$, where S is a symplectomorphism and l is a number which divides k.

From the proof of Theorem 4.1 we also have the following corollary.

COROLLARY 4.2. *Let (X,ω) be a $2n$-dimensional symplectic vector space. Then the set of all $2k$-dimensional linear subspaces on which the form ω degenerates (that is, does not have maximal rank) is co-planar in the Grassmannian $G(2k,2n)$.*

5. *The Hermitian case*

In this section, we give an application of our results to $\mathbb{C}$-linear mappings of n-dimensional complex space $X=\mathbb{C}^n$. In particular, we extend the main result of [**4**] to the case of $\mathbb{C}$-linear mappings. Let $\mathcal{G}(k,n)$ be the set of all k-dimensional complex linear subspaces of X. Of course $\mathcal{G}(k,n)$ is a subset of $G(2k,2n)$. We have the following variant of Definition 2.1.

Definition 5.1. Let $L_1,\dots,L_m$ be $\mathbb{C}$-linear k-dimensional subspaces of X. We say that they are *not co-planar* if the points $\Psi(L_1),\dots,\Psi(L_m)\in G(2k,2n)$ span linearly (over $\mathbb{R}$) the set $\mathcal{G}(k,n)$.

Remark 5.1. Let $N=\binom{2n}{2k}$. It is easy to see that a sufficiently general collection $\{L_1,\dots,L_N\}$ of k-dimensional $\mathbb{C}$-linear subspaces of X is not co-planar.

Let us recall that a Hermitian product H on a complex vector space X is a map

$$H: X\times X\longrightarrow \mathbb{C}$$

which is bilinear over $\mathbb{R}$ and satisfies the conditions

$$H(\mathbf{i}x,y)=\mathbf{i}H(x,y),\quad H(x,y)=\overline{H(y,x)},\qquad H(x,x)>0,\qquad \text{for } x\neq 0,$$

where $x,y\in X$, $\mathbf{i}^2=-1$ and $\bar{\cdot}$ is complex conjugation. In particular, on $X=\mathbb{C}^n$ we have the standard Hermitian product

$$H(z,w)=\sum_{j=1}^{n} z_j\overline{w_j},$$

which corresponds to the standard inner product of $\mathbb{R}^{2n}$. The space X with the standard Hermitian product H is called the Hermitian space. The metric induced by H is called the Hermitian metric, and the volume induced by it the Hermitian volume. This metric and volume coincide in fact with the usual Euclidean metric and volume induced on X by the standard inner product ξ. If $F: X\longrightarrow X$ is a $\mathbb{C}$-linear automorphism and $F^*H=H(F,F)=H$, then we call F a Hermitian isomorphism.

Now we can formulate the main result of this section.

Theorem 5.1. *Let (X,H) be a complex Hermitian space of complex dimension n and let $F: X\longrightarrow X$ be a $\mathbb{C}$-linear automorphism. Let $0<k<n$ and let $N=\binom{2n}{2k}$. Assume that F preserves the Hermitian volume on a collection $L_1,\dots,L_N$ of k-dimensional complex subspaces which are not co-planar. Then F is a Hermitian isomorphism. In particular, F is an isometry.*

Proof. We follow the lines of the proof of Theorem 4.1.

Expressing $H=\xi+\mathbf{i}\omega$ we find that ξ is the standard inner product and ω is a symplectic form on X. It is not difficult to check that on a k-dimensional complex linear subspace $L\subset X$ we have $\omega^k=(k!)v_{2k}$, where v_{2k} is the volume form. Hence by assumption the mapping F preserves the form ω^k on every such L.

We now recall some notation. By $\overline{G(2k,2n)}\subset\bigwedge^{2k}X=:Y$ (here we consider X as a real vector space) we denote the set of all vectors $v_1\wedge\ldots\wedge v_{2k}$, where

$v_1, \dots, v_{2k} \in X$. Let $(v_1^i, \dots, v_{2k}^i)$ be a basis of L_i, $i = 1, \dots, N$. It is easy to see that the linear subspaces $L_1, \dots, L_N$ are not co-planar if the vectors $u_i := v_1^i \wedge \dots \wedge v_{2k}^i$, $i = 1, \dots, N$ are not co-planar in Y, that is, if they linearly span the set $\overline{\mathcal{G}(k,n)}$, which is the affine cone over $\mathcal{G}(k,n)$. In Y we have the basis $(\mathbf{e}_{i_1 \dots i_{2k}} = \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_{2k}}, 0 < i_1 < i_2 < \dots < i_{2k} \leqslant 2n)$. For $y = x_1 \wedge \dots \wedge x_{2k} \in \overline{G(2k, 2n)}$ let $y = \sum y_{i_1 \dots i_{2k}} \mathbf{e}_{i_1 \dots i_{2k}}$. It is easy to see that the mappings $y_{i_1 \dots i_{2k}}$ treated as functions of the vectors $x_1, \dots, x_{2k}$ form a basis of the space of all $2k$-linear skew-symmetric forms on X. In particular, there are numbers $a_{i_1 \dots i_{2k}}$ such that $\omega^k(x_1, \dots, x_{2k}) = \sum a_{i_1 \dots i_{2k}} y_{i_1 \dots i_{2k}}$. Take $\eta(x_1 \wedge \dots \wedge x_{2k}) = \omega^k(x_1, \dots, x_{2k})$. We have $\eta(y) = \sum a_{i_1 \dots i_{2k}} y_{i_1 \dots i_{2k}}$ and consequently we can treat η as a linear form on the whole of Y.

Now, consider the mapping $R := \bigwedge^{2k} F : Y \longrightarrow Y$. Then F preserves the form ω^k on L_i if and only if $\eta(u_i) = \eta(R(u_i))$. However, the form $\eta(y) - \eta(R(y))$ is linear on Y and by assumption it vanishes on the set of vectors u_i, $i = 1, \dots, N$. Since the latter set is not co-planar in Y, the form $\eta(y) - \eta(R(y))$ vanishes identically on $\overline{\mathcal{G}(k,n)}$. This means that for every k-dimensional complex subspace L and vectors $v_1, \dots, v_{2k} \in L$ we have $\omega^k(v_1, \dots, v_{2k}) = \omega^k(F(v_1), \dots, F(v_{2k}))$. We can say that the mapping F preserves the form ω^k on all complex k-dimensional subspaces. We have the following variant of Lemma 4.1.

LEMMA 5.1. *Let (X, ω) be as above. Let $W \subset X$ be a k-dimensional complex subspace of X. If*

$$\omega^k(x, \mathbf{i}x, w_1, \mathbf{i}w_1, \dots, w_{k-1}, \mathbf{i}w_{k-1}) = 0$$

for any $w_i \in W$, then x is complementary to W with respect to ω.

Proof. Because W is a complex subspace we can choose a (complex) basis of X, $\{e_i\}_{i=1}^n$, such that ω is in the standard form on W, that is, $\omega\,|_W = (\mathbf{i}/2) \sum_{i=1}^k e_i^* \wedge \overline{e_i}^*$ and W is generated (over $\mathbb{C}$) by the vectors e_i, $i \leqslant k$. Moreover, we can assume that the subspace V orthogonal to W is generated by vectors e_i, $i > k$. Now let $v = \sum_{i=1}^{2n} a_i e_i$. Then taking the $(k-1)$-elements $e_1, \dots, `j', \dots, e_k$, we get

$$\omega^k(x, \mathbf{i}x, e_1, \mathbf{i}e_1, \dots, `j', \dots, e_k, \mathbf{i}e_k) = \pm a_j \omega^k(e_1, \mathbf{i}e_1, e_2, \mathbf{i}e_2, \dots, e_k, \mathbf{i}e_k) = \pm a_j k!,$$

so by assumption we have $a_j = 0, j = 1, \dots, k$. This means that $x \in V$. In particular, x is complementary to W with respect to ω. □

Now take a vector $x \in X$ and let V be a $(k+1)$-dimensional complex subspace of X which contains vector x. Let L be a complex subspace of V which is orthogonal to x.

Since the form ω^{k+1} is proportional to the usual volume form Ω on V and any linear mapping (here we consider F as a real mapping) changes the volume form by multiplying it by a suitable constant, there is a non-zero constant c such that

$$\Omega(F(v_1), F(\mathbf{i}v_1), \dots, F(v_{k+1}), F(\mathbf{i}v_{k+1})) = c\Omega(v_1, \mathbf{i}v_1, \dots, v_{k+1}, \mathbf{i}v_{k+1}) \qquad (5.1)$$

for every $v_1, \mathbf{i}v_1, \dots, v_{k+1}, \mathbf{i}v_{k+1} \in L$.

Now take in (5.1) $v_1, \dots, v_k \in L$ and $v_{k+1} = x$. Computing directly we have (note that x is complementary to $v_1, \dots, v_k$ with respect to ω!)

$$\Omega(v_1, \mathbf{i}v_1, \dots, v_k, \mathbf{i}v_k, x, \mathbf{i}x) = a\omega(x, \mathbf{i}x)\omega^k(v_1, \mathbf{i}v_1, \dots, v_k, \mathbf{i}v_k), \qquad (5.2)$$

where a is a suitable non-zero constant. Since F preserves ω^k on L, using Lemma 5.1, we can deduce that $F(L)$ is complementary to the subspace spanned by $\{F(x), iF(x)\}$ (in the space $F(V)$). In fact $\omega^k(F(x), \mathbf{i}F(x), F(v_1), \mathbf{i}F(v_1), \ldots, F(v_{k-1}), \mathbf{i}F(v_{k-1})) = 0$ for all $v_1, \ldots, v_{k-1} \in L_V$, so the lemma applies.

Hence as before we have

$$\begin{aligned}\Omega(F(v_1), F(\mathbf{i}v_1), \ldots, F(\mathbf{i}v_{k+1}))\\ = a\omega(F(x), \mathbf{i}F(x))\omega^k(F(v_1), F(\mathbf{i}v_1), \ldots, F(v_k), F(\mathbf{i}v_k)).\end{aligned} \tag{5.3}$$

Now from (5.2) and (5.3) we conclude that

$$\omega(F(x), \mathbf{i}F(x)) = c\omega(x, \mathbf{i}x).$$

This holds for all $x \in V$ and since $F^*\omega^k = \omega^k$ on V we have $c^k = 1$, but $H(F(x), F(x)) = \omega(\mathbf{i}F(x), F(x)) = c\omega(\mathbf{i}x, x) = cH(x, x)$ and consequently $c > 0$. Thus $c = 1$. Recall that $H = \xi + \mathbf{i}\omega$. We have

$$\xi(x, y) = (1/2)(H(x + y, x + y) - H(x, x) - H(y, y))$$

and so F preserves ξ. However, $\omega(x, y) = \xi(-\mathbf{i}x, y)$ and hence F also preserves ω. Finally, $F^*H = H$. This finishes the proof. □

Corollary 5.1. *Let X be an n-dimensional Hermitian space and let $F : X \longrightarrow X$ be a $\mathbb{C}$-linear automorphism. Let $0 < k < n$. Assume that F preserves the Hermitian volume for all k-dimensional complex linear subspaces of X. Then F is a Hermitian isomorphism, in particular F is an isometry.*

6. *Geometry of the set $\mathcal{A}_{l,2r}$*

Let $\mathcal{A}_{l,2r} \subset G(l, 2n)$ denote the set of all l-dimensional linear subspaces of X on which the form ω has rank $\leqslant 2r$. Of course $\mathcal{A}_{l,2r} \subset \mathcal{A}_{l,2r+2}$ if $2r + 2 \leqslant l$. We have the following.

Theorem 6.1. *Let (X, ω) be a symplectic vector space of dimension $2n$. Then the set $\mathcal{A}_{2k,2k-2} \subset G(2k, 2n) \subset \mathbb{P}^{N-1}$ is an irreducible algebraic subset of $G(2k, 2n)$ and it linearly spans a hyperplane in $\mathbb{P}^{N-1}$. More generally, for $r < k$ the set $\mathcal{A}_{2k,2r}$ is also irreducible and linear in $G(2k, 2n)$, that is, there is a linear projective subspace $L \subset \mathbb{P}^{N-1}$ such that $\mathcal{A}_{2k,2r} = G(2k, 2n) \cap L$. Moreover, the set $\mathcal{A}_{2k,2r}$ can be computed effectively.*

Proof. First assume that $\mathbf{K} = \mathbb{C}$. Let $\mathcal{A} = \mathcal{A}_{2k,2k-2}$ denote the set of all $2k$-dimensional subspaces on which the form ω has rank $< 2k$ (also called the set of subspaces of rank $< 2k$).

Now recall the notion of a projectively factorial variety. Let $X \subset \mathbb{P}^n$ be a complex algebraic subvariety of a complex projective space and let $C(X)$ be an affine cone over X. We consider the projective coordinate ring $R(X)$ of X as the ring $\mathbb{C}[C(X)] = \mathbb{C}[x_0, \ldots, x_n]/I(C(X))$, where $I(C(X)) = \{F \in \mathbb{C}[x_0, \ldots, x_n] : F|_{C(X)} = 0\}$. We say that X is projectively factorial if the ring $R(X)$ is factorial.

If X is a smooth projective variety, we can consider the Picard group $\mathrm{Pic}(X)$ of all algebraic line bundles on X (for details see, for example, [**2**, p. 133]). It is well known that if X is projectively factorial, then $\mathrm{Pic}(X) = \mathbb{Z}$ and $\mathrm{Pic}(X)$ is generated by the line bundle $\mathcal{O}(H)$, where $H \subset X$ is a hyperplane section.

By the Andreotti–Salmon theorem (see, for example, [**5**]) the embedded Grassmannian $G(2k,2n)\subset\mathbb{P}^{N-1}$ is projectively factorial. In particular, the Picard group of $G(2k,2n)$ is generated by a hyperplane section. It can be easily deduced from this that for every hyperplane $H\subset\mathbb{P}^{N-1}$ the set $H\cap G(2k,2n)$ is an irreducible variety which is not contained in a proper linear subspace of H (cf. [**3**, Lemma 3.18]). Let $H\subset\mathbb{P}^{N-1}$ be a hyperplane such that $\mathcal{A}=H\cap G(2k,2n)$ (cf. Corollary 4.2). Thus, by the above, $\mathcal{A}$ linearly spans the hyperplane H.

Now consider the set $\mathcal{A}_{2k,2r}$. It is irreducible since it contains an orbit of the symplectic group as a dense subset (in fact the orbit of any subspace l of dimension $2k$ on which the form ω^k has rank exactly $2r$ is dense in $\mathcal{A}_{2k,2r}$). We show that there is a linear subspace L such that $\mathcal{A}_{2k,2r}=L\cap G(2k,2n)$. Take a sequence of vectors $(x_1,\dots,x_{2k})\in X^{2k}$. Let $\mathbb{A}$ be the matrix which has (the coordinates of) the vectors $x_{i_1},\dots,x_{i_{2k-2r-2}}$ as rows. Let $\delta_{s_1,\dots,s_{2k-2r-2}}(x_{i_1},\dots,x_{i_{2k-2r-2}})$ denote the principal minor of $\mathbb{A}$ determined by the columns indexed by $s_1,\dots,s_{2k-2r-2}$. Consider all possible skew-symmetric forms of the type

$$\omega^{r+1}(x_{j_1},\dots,x_{j_{2r+2}})\delta_{s_1,\dots,s_{2k-2r-2}}(x_{i_1},\dots,x_{i_{2k-2r-2}}),$$

where $\{j_1,\dots,j_{2r+2}\}\cup\{i_1,\dots,i_{2k-2r-2}\}=\{1,\dots,2k\}$. It is not difficult to check that they simultaneously vanish only at the vectors $x_1\wedge\ldots\wedge x_{2k}$ which belong to $\mathcal{A}_{2k,2r}$. On the other hand, these skew-symmetric functions can be treated as linear forms on $\mathbb{P}^{N-1}$ (cf. the proof of Theorem 4.1). Of course, we can find these linear forms effectively (as functions of the variables $y_{i_1,\dots,i_{2k}}$, cf. the proof of Theorem 4.1). Since we know the equations of the Grassmannian $G(2k,2n)$ (see [**2**, p. 211]), we can compute the set $\mathcal{A}_{2k,2r}$ effectively. This finishes the proof in the case $\mathbf{K}=\mathbb{C}$.

Now we sketch the proof for the case $\mathbf{K}=\mathbb{R}$. As before, define $\mathcal{A}\subset G(2k,2n)$ as the set of all $2k$-dimensional linear subspaces of X of dimension $2n$ and of rank $<2k$. This set has a stratification into smooth subsets $\mathcal{A}_r=\{W\in\mathcal{A}:\operatorname{rank}\omega\mid_W=2r\}$, where $r=0,1,\dots,k-1$ and $\mathcal{A}_i\subset\operatorname{closure}(\mathcal{A}_{i+1})$. Moreover, every such subset is homogeneous with respect to the induced action of the group $\mathrm{Sp}(2n,\mathbf{K})$. Take a real subspace $L\in\mathcal{A}_{k-1}$. Let $H\subset\mathrm{Sp}(2n,\mathbb{R})$ be the stabilizer of L in the group $\mathrm{Sp}(2n,\mathbb{R})$. Thus

$$\dim\mathcal{A}=\dim\mathcal{A}_{k-1}=\dim\mathrm{Sp}(2n,\mathbb{R})-\dim H.$$

Now let us complexify X. Let $H'\subset\mathrm{Sp}(2n,\mathbb{C})$ be the stabilizer of $L\otimes\mathbb{C}$ in the group $\mathrm{Sp}(2n,\mathbb{C})$.

Let $\mathcal{A}'\subset G(2k,2n,\mathbb{C})$ be the set of all complex $2k$-dimensional linear subspaces of $X\otimes\mathbb{C}$ of dimension $2k$ and of rank $<2k$. Then $\mathcal{A}'$ has the same (complex) dimension as the orbit of $L\otimes\mathbb{C}$, and this dimension is equal to dim $\mathrm{Sp}(2n,\mathbb{C})-\dim H'$. However, H' contains the complexification of the subgroup H, thus $\dim_{\mathbb{C}}H\geqslant\dim_{\mathbb{R}}H$ and consequently $\dim_{\mathbb{C}}\mathcal{A}'\leqslant\dim_{\mathbb{R}}\mathcal{A}$, but in the complex case we have $\dim\mathcal{A}'=\dim G(2k,2n,\mathbb{C})-1$. From this we see immediately that $\dim\mathcal{A}=\dim G(2k,2n,\mathbb{R})-1$. This means that the complexification of $\mathcal{A}$ is $\mathcal{A}'$. Thus $\mathcal{A}$ spans linearly a (real) hyperplane if and only if $\mathcal{A}'$ spans a (complex) hyperplane. Now we can finish the proof as above. □

From the proof we see that part of Theorem 6.1 can be generalized.

Corollary 6.1. *Let (X,ω) be a symplectic vector space of dimension $2n$. Let l,r be integers such that $l\leqslant 2n$ and $2r+2\leqslant l$. Then there is a proper linear subspace*

$L\subset\mathbb{P}^{N-1}$ such that $\mathcal{A}_{l,2r}=L\cap G(l,2n)$. Moreover, we can compute the equations of $\mathcal{A}_{l,2r}$ effectively. In particular, this is true for the Lagrangian Grassmannian manifold $\Lambda_n=\mathcal{A}_{n,0}$.

DEFINITION 6.1. Let $L_1,\dots,L_{N-1}\in\mathcal{A}_{2k,2k-2}$ be $2k$-dimensional linear subspaces of X (of rank $<2k$). We say that they are *in general position* if they linearly span a hyperplane in $\mathbb{P}^{N-1}$.

REMARK 6.1. By Theorem 6.1, every sufficiently general subset $\{L_1,\dots,L_{N-1}\}\subset\mathcal{A}_{2k,2k-2}$ is in general position. Moreover, we can find such subspaces L_i with rank $L_i=2k-2$.

A slightly more general version of Theorem 4.1 is the following.

THEOREM 6.2. *Let (X,ω) be a symplectic vector space of dimension $2n$ and let $F:X\longrightarrow X$ be a linear automorphism. Let $0<k<n$. Let $L_1,\dots,L_{N-1}$ be $2k$-dimensional linear subspaces of X of rank $<2k$ which are in general position. Assume that F transforms $L_1,\dots,L_{N-1}$ onto subspaces which have rank $<2k$. Then there is a non-zero constant c such that $F^*\omega=c\omega$.*

Proof. We can assume that $\mathbf{K}=\mathbb{C}$. Let L be a $2k$-dimensional subspace of rank $2k$. There is a constant a such that $F^*\omega^k=a\omega^k$ on L. As in the proof of Theorem 4.1, denote by $\overline{G(2k,2n)}\subset\bigwedge^{2k}X:=Y$ the set of all vectors $v_1\wedge\dots\wedge v_{2k}$, where $v_1,\dots,v_{2k}\in X$. Let $\mathcal{A}=\{L_1,\dots,L_{N-1}\}$.

Let $H\subset\mathbb{P}^N$ be a hyperplane such that $\mathcal{A}_{2k,2k-2}=H\cap G(2k,2n)$. Thus, by assumption, $\mathcal{A}$ linearly spans the hyperplane H. Since $L\notin H$ we can easily deduce that the set $\mathscr{B}:=\{L\}\cup\mathcal{A}$ is not co-planar in Y. By assumption, for $W\in\mathcal{B}$ we have

$$a\eta(u_W)=\eta(R(u_W)),$$

where $u_W\in\overline{G(2k,2n)}\subset\bigwedge^{2k}X:=Y$ is a vector determined by W.

This implies that the linear form $a\eta(y)-\eta(R(y))$ vanishes on the vectors $u_W,W\in\mathcal{B}$. Since the set $\mathcal{B}$ is not co-planar in Y, the form $a\eta(y)-\eta(R(y))$ vanishes identically on Y. This means that $a\omega^k(v_1,\dots,v_{2k})=\omega^k(F(v_1),\dots,F(v_{2k}))$ for every $v_1,\dots,v_{2k}\in X$, that is, $F^*\omega^k=a\omega^k$.

Now we can repeat word for word the proof of Theorem 4.1 to find that there is a constant c such that $F^*(\omega)=c\omega$ and $c^k=a$. Since F is a linear automorphism, we have $c\neq 0$. □

COROLLARY 6.2. *Let (X,ω) be a symplectic vector space of dimension $2n$ and let $F:X\longrightarrow X$ be a linear automorphism. Let $0<k<n$. Assume that F transforms $2k$-dimensional subspaces of rank $2k-2$ onto subspaces which have rank $<2k$. Then there is a non-zero constant c such that $F^*\omega=c\omega$.*

COROLLARY 6.3. *Let (X,ω) be a symplectic vector space of dimension $2n$ and let $F:X\longrightarrow X$ be a linear automorphism. Let $0<l<2n$ and let $2r+2\leqslant l$. Assume that F transforms the set $\mathcal{A}_{l,2r}$ into the same set. Then there is a non-zero constant c such that $F^*\omega=c\omega$.*

Proof. Let $\mathcal{B}_{2r+2,2r}$ denote the set of $(2r+2)$-dimensional subspaces of rank $2r$. Since every subspace from $\mathcal{B}_{2r+2,2r}$ is contained in some subspaces from $\mathcal{A}_{l,2r}$, it is easy to see that F transforms the set $\mathcal{B}_{2r+2,2r}$ into the same set. Hence we are done by Corollary 6.2. $\square$

COROLLARY 6.4. *Let (X,ω) be a symplectic vector space of dimension $2n$ and let $F: X \longrightarrow X$ be a linear automorphism. Let $2 \leqslant l \leqslant n$ and assume that F transforms l-dimensional isotropic (for example, Lagrangian) subspaces onto subspaces of the same type. Then there is a non-zero constant c such that $F^*\omega = c\omega$.*

7. *Symplectic volume*

In this section we introduce and study the symplectic volume.

DEFINITION 7.1. Let (X,ω) be a $2n$-dimensional symplectic vector space. Let $v_1,\dots,v_{2k} \in X$. By the *symplectic $2k$-volume* of the collection $v_1,\dots,v_{2k}$ we mean the number

$$\mathrm{svol}_{2k}(v_1,\dots,v_{2k}) := \det([\omega(v_i,v_j)]_{1\leqslant i,j\leqslant 2k}).$$

The symplectic volume has the following nice property.

PROPOSITION 7.1. *Let (X,ω) be a symplectic vector space and let $v_1,\dots,$ $v_{2k} \in X$. Then $\mathrm{svol}_{2k}(v_1,\dots,v_{2k}) = 0$ if and only if either $v_1,\dots,v_{2k}$ are linearly dependent or the space $(W = \langle v_1,\dots,v_{2k}\rangle, \omega\,|_W)$ is not symplectic. Moreover, if $v_1,\dots,$ v_{2k} are linearly independent then $\mathrm{rank}[\omega(v_i,v_j)]_{1\leqslant i,j\leqslant 2k} = \mathrm{rank}\omega\,|_W$.*

Proof. If $v_1,\dots,v_{2k}$ are linearly dependent, then we can easily see (for example, by the Cramer rule) that $\det([\omega(v_i,v_j)]_{1\leqslant i,j\leqslant 2k}) = 0$.

Assume that $v_1,\dots,v_{2k}$ are linearly independent. There is a basis $\mathbf{e}_1,\dots,\mathbf{e}_{2k}$ of $W = \langle v_1,\dots,v_{2k}\rangle$ in which the form $\omega' = \omega\ |_W$ has the canonical form. In particular, rank $\omega' =$ rank $[\omega(\mathbf{e}_i,\mathbf{e}_j)]_{1\leqslant i,j\leqslant 2k}$. Also there is a non-singular $2k\times 2k$ matrix $\mathbb{A} = [a_{ij}]$ such that $v_i = \sum_{j=1}^{2k} a_{ij}\mathbf{e}_j$. Using bilinearity of ω, we have by direct computation

$$[\omega(v_i,v_j)] = \mathbb{A}[\omega(\mathbf{e}_i,\mathbf{e}_j)]\mathbb{A}^T.$$

This finishes the proof. $\square$

THEOREM 7.1. *Let (X,ω) be a symplectic vector space and let $v_1,\dots,v_{2k} \in X$. Then $(\omega^k(v_1,\dots,v_{2k}))^2 = (k!)^2\mathrm{svol}_{2k}(v_1,\dots,v_{2k})$.*

Proof. We can assume that $\mathbf{K} = \mathbb{C}$. Choose a basis $\{\mathbf{e}_1,\dots,\mathbf{e}_{2n}\}$ of X such that $\omega = \sum \mathbf{e}_i^* \wedge \mathbf{e}_{i+n}^*$. We can treat the determinant $\mathrm{svol}_{2k}(v_1,\dots,v_{2k}) =$ $\det([\omega(v_i,v_j)]_{1\leqslant i,j\leqslant 2k})$ as a polynomial function on X^{2k} of degree $2k$. Since the matrix $[\omega(v_i,v_j)]$ is skew-symmetric we have in fact $\mathrm{svol}_{2k}(v_1,\dots,v_{2k}) =$ $\mathrm{Pf}(v_1,\dots,v_{2k})^2$, where $\mathrm{Pf}(v_1,\dots,v_{2k})$ is also a polynomial (the Pfaffian of $[\omega(v_i,v_j)]$).

Note that $\mathrm{Pf}(v_1,\dots,v_{2k})^2 = \mathrm{svol}_{2k}(v_1,\dots,v_{2k}) = 0$ if and only if either $v_1,\dots,v_{2k}$ are linearly dependent or the space $(W = \langle v_1,\dots,v_{2k}\rangle, \omega\,|_W)$ is not symplectic (cf. Proposition 7.1). The polynomial function $\omega^k(v_1,\dots,v_{2k})$ has the same property. Moreover, it is not difficult to check that the latter polynomial is irreducible and of degree k. This means that the polynomials $\mathrm{Pf}(v_1,\dots,v_{2k})$ and $\omega^k(v_1,\dots,v_{2k})$ have the same degree and the same zero-sets. Since the latter is irreducible, by the Hilbert Nullstellensatz there is a constant C such that $\mathrm{Pf}(v_1,\dots,v_{2k}) = C\omega^k(v_1,\dots,v_{2k})$. In particular, $C^2(\omega^k(v_1,\dots,v_{2k}))^2 = \mathrm{svol}_{2k}(v_1,\dots,v_{2k})$. Moreover, by direct computations we have $(\omega^k(\mathbf{e}_1,\dots,\mathbf{e}_{2k}))^2 = (k!)^2$ and $\mathrm{svol}_{2k}(e_1,\dots,e_{2k}) = 1$. Thus $C^2 = (k!)^{-2}$ and $(\omega^k(v_1,\dots,v_{2k}))^2 = (k!)^2\mathrm{svol}_{2k}(v_1,\dots,v_{2k})$. □

To state the next result we need an additional notion. Let us recall that every $2k$-dimensional linear subspace $L \subset X$ determines a unique point $\Psi(L)$ in the Grassmannian $G(2k,2n) \subset \mathbb{P}^{N-1}$, where $N = \binom{2n}{2k}$.

Definition 7.2. Let $L_1,\dots,L_M$ be $2k$-dimensional linear subspaces of X. We say that they are two-co-planar if the points $\Psi(L_1),\dots,\Psi(L_M) \in G(2k,2n)$ are two-co-planar, that is, if there are hyperplanes $\Lambda_1,\Lambda_2 \subset \mathbb{P}^{N-1}$ such that $\Psi(L_1),\dots,\Psi(L_M) \in \Lambda_1 \cup \Lambda_2$.

Remark 7.1. Take $M = \binom{N}{2}$. Let $L_1,\dots,L_M$ be $2k$-dimensional linear subspaces of X. It is not difficult to check that if these subspaces are sufficiently general, then they are not two-co-planar.

Theorem 7.2. *Let (X,ω) be a symplectic vector space of dimension $2n$ and let $F : X \longrightarrow X$ be a linear automorphism. Let $0 < k < n$. Assume that F preserves $2k$-dimensional symplectic volume on a collection $L_1,\dots,L_M$ of $2k$-dimensional subspaces which are not two-co-planar. Then F is an ϵ_{2k}-symplectomorphism.*

Proof. By Theorem 7.1 on every L_i we have either $F^*(\omega^k) = \omega^k$ or $F^*(\omega^k) = -\omega^k$.

Denote by $\overline{G(2k,2n)} \subset \bigwedge^{2k} X := Y$ the set of all vectors $v_1 \wedge \dots \wedge v_{2k}$, where $v_1,\dots,v_{2k} \in X$. Let $(v_1^i,\dots,v_{2k}^i)$ be a basis of $L_i, i = 1,\dots,N$. It is easy to see that $L_1,\dots,L_N$ are co-planar if the vectors $u_i := v_1^i \wedge \dots \wedge v_{2k}^i, i = 1,\dots,N$ are co-planar in Y. We define a linear form η on the whole of Y as in the proof of Theorem 4.1.

Now, consider the mapping $R := \bigwedge^{2k} F : Y \longrightarrow Y$. Then $F^*\omega^k = \pm\omega^k$ on L_i if and only if $\pm\eta(u_i) = \eta(R(u_i))$. However, the forms $\eta(y) \pm \eta(R(y))$ are linear on Y and by assumption vectors $u_i, i = 1,\dots,M$ are in the union of the kernels of these forms. Since $u_i, i = 1,\dots,M$, are not two-co-planar in Y, either the form $\eta(y) - \eta(R(y))$ or the form $\eta(y) + \eta(R(y))$ vanishes identically on Y. This means that either $\omega^k(v_1,\dots,v_{2k}) = \omega^k(F(v_1),\dots,F(v_{2k}))$ for every $v_1,\dots,v_{2k} \in X$ or $-\omega^k(v_1,\dots,v_{2k}) = \omega^k(F(v_1),\dots,F(v_{2k}))$ for every $v_1,\dots,v_{2k} \in X$, that is, $F^*\omega^k = \pm\omega^k$.

Now we can repeat word for word the proof of Theorem 4.1 to see that there is a constant c such that $F^*(\omega) = c\omega$. Moreover, we have $c^k = 1$ or $c^k = -1$. In both cases $c^{2k} = 1$, hence F is an ϵ_{2k}-symplectomorphism. □

Corollary 7.1. *Let X be a symplectic vector space of dimension $2n$ and let $F : X \longrightarrow X$ be a linear automorphism. Let $0 < k < n$. Assume that F preserves $2k$-dimensional symplectic volume. Then F is an ϵ_{2k}-symplectomorphism.*

Corollary 7.2. *Let X be a real symplectic vector space of dimension $2n$ and let $F : X \longrightarrow X$ be a linear automorphism. Let $0 < k < n$. Assume that F preserves*

$2k$-dimensional symplectic volume. Then F is either a symplectomorphism or an anti-symplectomorphism.

Acknowledgements. This paper was written during the authors' stay at the Max Planck Institute of Mathematics in Bonn. The authors thank the MPI for financial support and warm hospitality. We also thank Jerzy Trzeciak for improving the readability of the paper.

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Stanisław Janeczko
Instytut Matematyczny PAN
ul. Śniadeckich 8
00-950 Warszawa
Poland

Wydział Matematyki i Nauk Informacyjnych,
Politechnika Warszawska
Pl. Politechniki 1
00-661 Warszawa
Poland

janeczko@ise.pw.edu.pl

Zbigniew Jelonek
Instytut Matematyczny PAN
Polska Akademia Nauk
Św. Tomasza 30
31-027 Kraków
Poland

Max Planck Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

najelone@cyf-kr.edu.pl