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Anti-orthotomics of frontals and their applications

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ABSTRACT

Let $f : N^n \to \mathbb{R}^{n+1}$ be a frontal with the Gauss mapping $\nu : N \to S^n$ and let $P \in \mathbb{R}^{n+1}$ be a point such that $(f(x) - P) \cdot \nu(x) \neq 0$ for any $x \in N$. In this study, for the mapping $\widetilde{f} : N \to \mathbb{R}^{n+1}$ defined by:

$$\widetilde{f}(x) = f(x) - \frac{||f(x) - P||^2}{2(f(x) - P) \cdot \nu(x)}\nu(x),$$

the following four statements hold. (1) $\widetilde{f}$ is a frontal with the Gauss mapping $\widetilde{\nu}(x) = \frac{f(x)-P}{||f(x)-P||}$ at $\widetilde{f}(x)$. (2) $\widetilde{f}$ is the unique anti-orthotomic of f relative to P. (3) The property that $(\widetilde{f}(x) - P) \cdot \widetilde{\nu}(x) \neq 0$ holds for any $x \in N$. (4) The equality of $||\widetilde{f}(x) - P|| = ||\widetilde{f}(x) - f(x)||$ holds for any $x \in N$. Moreover, three applications of the main result are given. A generalization of the Cahn–Hoffman vector formula is given as the first application. The second application involves clarifying an optical meaning of anti-orthotomics. The third application provides a criterion for determining a front for a given frontal.

1. Introduction

Throughout this study, let n, N be a positive integer and an n-dimensional C^∞ manifold without boundary, respectively. Moreover, all of the mappings in this study belong to the class C^∞ unless stated otherwise.

A mapping $f : N \to \mathbb{R}^{n+1}$ is called a *frontal* if a mapping $\nu : N \to \mathbb{R}^{n+1}$ exists such that the following two conditions are satisfied, where the dot in the center denotes the scalar product of two vectors in $\mathbb{R}^{n+1}$ and two vector spaces comprising $T_{f(x)}\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1}$ are identified.

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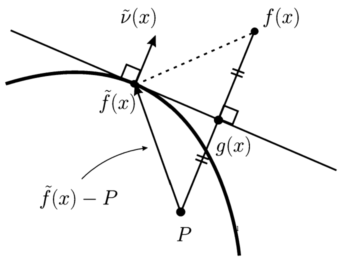

Fig. 1. Orthotomic f and pedal g of $(\widetilde{f}, \widetilde{\nu})$ relative to P.

(1) $\nu(x) \cdot \nu(x) = 1$, i.e., $\nu(x) \in S^n$ for any $x \in N$.
(2) $df_x(\mathbf{v}) \cdot \nu(x) = 0$ for any $x \in N$ and any $\mathbf{v} \in T_xN$.

By conditions (1) and (2) given above, it is natural to refer to $\nu : N \to S^n$ as the *Gauss mapping* of f. In this study, the mapping $(f, \nu) : N \to \mathbb{R}^{n+1} \times S^n$ is even called a frontal in some cases. The notion of a frontal was introduced in several independent studies (e.g., [6,12,23,27]) and it has been investigated widely (see [14]).

Definition 1. Let $\widetilde{f} : N \to \mathbb{R}^{n+1}$ be a frontal with its Gauss mapping $\widetilde{\nu} : N \to S^n$ and let P be a point of $\mathbb{R}^{n+1}$.

(1) A mapping $f : N \to \mathbb{R}^{n+1}$ is called the *orthotomic* of $\widetilde{f}$ relative to P if the following equality holds for any $x \in N$.

$$f(x) = 2\left(\left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x)\right) \widetilde{\nu}(x) + P.$$

(2) A mapping $g : N \to \mathbb{R}^{n+1}$ is called the *pedal* of $\widetilde{f}$ relative to P if the following equality holds for any $x \in N$.

$$g(x) = \left(\left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x)\right) \widetilde{\nu}(x) + P.$$

Fig. 1 clearly illustrates the relationship between the orthotomic f and pedal g of $\widetilde{f}$ relative to P. The following Proposition 1 guarantees that the orthotomic of a given frontal is a frontal.

Proposition 1. *Let $(\widetilde{f}, \widetilde{\nu}) : N \to \mathbb{R}^{n+1}$ be a frontal and let P be a point of $\mathbb{R}^{n+1}$ such that the following condition is satisfied for any $x \in N$.*

$$\left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x) \neq 0.$$

Then, the orthotomic of $\widetilde{f}$ relative to P defined by:

$$f(x) = 2\left((\widetilde{f}(x) - P) \cdot \widetilde{\nu}(x)\right) \widetilde{\nu}(x) + P$$

is a frontal with its Gauss mapping: $\nu(x) = \frac{f(x) - \widetilde{f}(x)}{||f(x) - \widetilde{f}(x)||}$. Moreover, the condition that:

$$(f(x) - P) \cdot \nu(x) \neq 0$$

holds for any $x \in N$.

Proposition 1 is proved in Section 2. By definition, it is clear that for the pedal g, $f = 2g - P$ is the orthotomic. Thus, it is clear that Proposition 1 yields the following corollary, which is a generalization of that given by [18].

Corollary 1. *Let $(\widetilde{f}, \widetilde{\nu}) : N \to \mathbb{R}^{n+1}$ be a frontal and let P be a point of $\mathbb{R}^{n+1}$ such that the following condition is satisfied for any $x \in N$.*

$$\left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x) \neq 0.$$

Then, the pedal of $\widetilde{f}$ relative to P defined by:

$$g(x) = \left((\widetilde{f}(x) - P) \cdot \widetilde{\nu}(x)\right) \widetilde{\nu}(x) + P$$

is a frontal with its Gauss mapping: $\nu(x) = \frac{2g(x)-P-\widetilde{f}(x)}{||2g(x)-P-\widetilde{f}(x)||}$. Moreover, the condition that:

$$(g(x) - P) \cdot \nu(x) \neq 0$$

holds for any $x \in N$.

It should be noted that in the case where $\widetilde{f}$ is a plane regular curve, it is well known that $\widetilde{f}(x) - f(x)$ is a normal vector to f at $f(x)$ (e.g., see [4]). Therefore, a part of Proposition 1 may be simply regarded as a generalization of the classical result to frontals of general dimension.

It should also be noted that even if $\widetilde{f} : N \to \mathbb{R}^{n+1}$ is non-singular, then the condition that "$(\widetilde{f}(x) - P) \cdot \widetilde{\nu}(x) \neq 0$ for any $x \in N$" seems not very mild. Thus, even when $\widetilde{f} : N \to \mathbb{R}^{n+1}$ is an embedding, if the image of the Gaussian curvature function of $\widetilde{f}(N)$ is a large interval containing zero as an interior point, then no points P satisfy the condition that "$(\widetilde{f}(x) - P) \cdot \widetilde{\nu}(x) \neq 0$ for any $x \in N$". In addition, if $\widetilde{f} : N \to \mathbb{R}^{n+1}$ is an embedding and the Gaussian curvature of $\widetilde{f}(N)$ is always positive, then the set $\{P \in \mathbb{R}^{n+1} \mid (\widetilde{f}(x) - P) \cdot \widetilde{\nu}(x) \neq 0 \text{ for any } x \in N\}$ is a non-empty open set. Moreover, the assumptions that "$\widetilde{f} : N \to \mathbb{R}^{n+1}$ is an embedding and that the Gaussian curvature of $\widetilde{f}(N)$ is always positive" appear to be common in the study of orthotomics and pedals. (e.g., see [1,5]). Therefore, the assumption given in Proposition 1 generalizes the common assumption to the study of orthotomics and pedals.

The same condition used in the assumption in Proposition 1 was introduced in the study by Bruce and Giblin [4] 7.14 in the case where $\widetilde{f} : I \to \mathbb{R}^2$ is regular, and also by the second author of the present study ([21] in the case where $\widetilde{f} : N \to S^{n+1}$ is an immersion, by [22] in the case where $\widetilde{f} : S^n \to \mathbb{R}^{n+1}$ is a Legendrian map, and by [16] in the case where $n = 1$ and $\widetilde{f} : S^1 \to \mathbb{R}^2$ is an embedding). In particular, in the study by [16], the following set was defined called the *no-silhouette* of $\widetilde{f}$ and denoted by $\mathcal{NS}_{\widetilde{f}}$:

$$\mathcal{NS}_{\widetilde{f}} = \left\{ P \in \mathbb{R}^2 \;\middle|\; \mathbb{R}^2 - \bigcup_{x \in S^1} \left(\widetilde{f}(x) + d\widetilde{f}_x(T_x S^1)\right) \right\}.$$

For a frontal $\widetilde{f} : N \to \mathbb{R}^{n+1}$ with its Gauss mapping $\widetilde{\nu}$, the notion of the no-silhouette $\mathcal{NS}_{\widetilde{f}}$ can be generalized naturally as follows. The optical meaning of the no-silhouette is illustrated in Fig. 2.

$$\mathcal{NS}_{\widetilde{f}} = \left\{ P \in \mathbb{R}^{n+1} \mid \left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x) \neq 0 \text{ for any } x \in N \right\}.$$

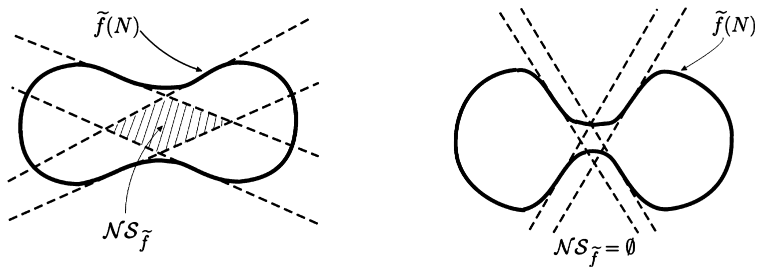

Fig. 2. Left: $\mathcal{NS}_{\widetilde{f}}$ is not empty. Right: $\mathcal{NS}_{\widetilde{f}}$ is empty.

Definition 2.

(1) Let $f : N \to \mathbb{R}^{n+1}$ be a C^∞ mapping and let P be a point of $\mathbb{R}^{n+1}$. A frontal $\widetilde{f} : N \to \mathbb{R}^{n+1}$ with its Gauss mapping $\widetilde{\nu} : N \to S^n$ is called the *anti-orthotomic* of f relative to P if the following equality holds for any $x \in N$.

$$f(x) = 2\left(\left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x)\right) \widetilde{\nu}(x) + P.$$

(2) Let $g : N \to \mathbb{R}^{n+1}$ be a C^∞ mapping and let P be a point of $\mathbb{R}^{n+1}$. A frontal $\widetilde{f} : N \to \mathbb{R}^{n+1}$ with its Gauss mapping $\widetilde{\nu} : N \to S^n$ is called the *negative pedal* of g relative to P if the following equality holds for any $x \in N$.

$$g(x) = \left(\left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x)\right) \widetilde{\nu}(x) + P.$$

By definition, if $f = 2g - P$, then the anti-orthotomic of f relative to P is exactly the same as the negative pedal of g relative to P. Depending on the situation, in some cases, the negative pedal is also called the *primitive* (e.g., see [2]) or the *Cahn–Hoffman map* (e.g., see [10,15,17]). In geometric optics, the notion of an anti-orthotomic is very important (e.g., see [1,4,5]) and the notion of a negative pedal is the core concept for Wulff shape studies (e.g., see [7,15,17,19,26]).

By definition, it is clear that an anti-orthotomic (resp., a negative pedal) is a frontal solution for a given orthotomic equation (resp., pedal equation). Therefore, obtaining anti-orthotomics or negative pedals may be considered inverse problems. Except in the case where the Gauss mapping $\widetilde{\nu}$ of $\widetilde{f}$ is non-singular (i.e., the case where $\widetilde{f} : N \to \mathbb{R}^{n+1}$ is an embedding and the Gaussian curvature of $\widetilde{f}(N)$ is always positive), these inverse problems have usually been investigated only by solving simultaneous function equations for the envelopes.

Example 1. Let $g : \mathbb{R} \to \mathbb{R}^2$ be the mapping defined by $g(\theta) = \left(\cos\left(\theta^3\right), \sin\left(\theta^3\right)\right)$. Define $\nu : \mathbb{R} \to \mathbb{R}^2$ by $\nu = g$. Then, $(g, \nu) : \mathbb{R} \to \mathbb{R}^2 \times S^1$ is a frontal. Set $P = (0, 0)$ and $\widetilde{f} = \widetilde{\nu} = g$. Then, clearly, $(\widetilde{f}, \widetilde{\nu}) : \mathbb{R} \to \mathbb{R}^2 \times S^1$ is the negative pedal of g relative to P and $\widetilde{f}(\mathbb{R}) = S^1$.

In addition, the function $\Phi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ given by:

$$\Phi(X, Y, \theta) = \nu(\theta) \cdot ((X, Y) - g(\theta))$$

defines the one-parameter family of affine tangent lines to $g(\mathbb{R}) = S^1$. The figure solution for the simultaneous equation $\Phi = 0$, $\frac{\partial \Phi}{\partial \theta} = 0$ is $S^1 \cup \{(1, Y) \mid Y \in \mathbb{R}\}$.

Example 2. Let $x : \mathbb{R} \to \mathbb{R}$ be a C^∞ periodic function of period 8 that satisfies the following condition for any integer n.

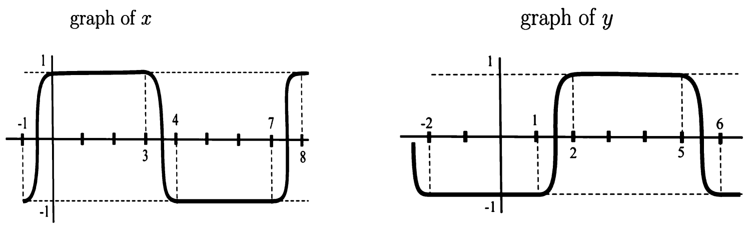

Fig. 3. Graphs for x and y.

$$\begin{cases} x(t) = 1 & (\text{if } 8n \le t \le 8n+3), \\ 1 \ge x(t) \ge -1 & (\text{if } 8n+3 \le t \le 8n+4), \\ x(t) = -1 & (\text{if } 8n+4 \le t \le 8n+7), \\ -1 \le x(t) \le 1 & (\text{if } 8n+7 \le t \le 8n+8). \end{cases}$$

Let $y : \mathbb{R} \to \mathbb{R}$ be a C^∞ periodic function of period 8 that satisfies the following condition for any integer n.

$$\begin{cases} y(t) = -1 & (\text{if } 8n-2 \le t \le 8n+1), \\ -1 \le y(t) \le 1 & (\text{if } 8n+1 \le t \le 8n+2), \\ y(t) = 1 & (\text{if } 8n+2 \le t \le 8n+5), \\ 1 \ge x(t) \ge -1 & (\text{if } 8n+5 \le t \le 8n+6). \end{cases}$$

Define $\widetilde{f} : \mathbb{R} \to \mathbb{R}^2$ by $\widetilde{f}(t) = (x(t), y(t))$. Then, $\widetilde{f}$ is a C^∞ periodic mapping of period 8 and the set of singular points of $\widetilde{f}$ contains infinitely many closed intervals:

$$\cdots, [-2,-1], [0,1], [2,3], [4,5], [6,7], [8,9], \cdots.$$

Fig. 3 shows clearly that even if $\widetilde{f}$ has other singular points, the image of $\widetilde{f}$ is always the square with the following 4 vertexes:

$$(1,-1) = \widetilde{f}([8n, 8n+1]), \quad (1,1) = \widetilde{f}([8n+2, 8n+3]),$$
$$(-1,1) = \widetilde{f}([8n+4, 8n+5]), (-1,-1) = \widetilde{f}([8n+6, 8n+7]).$$

Next, in order to show that $\widetilde{f}$ is a frontal, we construct a non-zero normal vector $(n_1(t), n_2(t))$ to $\widetilde{f}$ at $\widetilde{f}(t)$. Let $n_1 : \mathbb{R} \to \mathbb{R}$ be a C^∞ periodic function of period 8 that satisfies the following condition for any integer n.

$$\begin{cases} 0 \le n_1(t) \le 1 & (\text{if } 8n \le t \le 8n+1), \\ n_1(t) = 1 & (\text{if } 8n+1 \le t \le 8n+2), \\ 1 \ge n_1(t) \ge 0 & (\text{if } 8n+2 \le t \le 8n+3), \\ n_1(t) = 0 & (\text{if } 8n+3 \le t \le 8n+4), \\ 0 \ge n_1(t) \ge -1 & (\text{if } 8n+4 \le t \le 8n+5), \\ n_1(t) = -1 & (\text{if } 8n+5 \le t \le 8n+6), \\ -1 \le n_1(t) \le 0 & (\text{if } 8n+6 \le t \le 8n+7), \\ n_1(t) = 0 & (\text{if } 8n+7 \le t \le 8n+8). \end{cases}$$

Let $n_2 : \mathbb{R} \to \mathbb{R}$ be a C^∞ periodic function of period 8 that satisfies the following condition for any integer n.

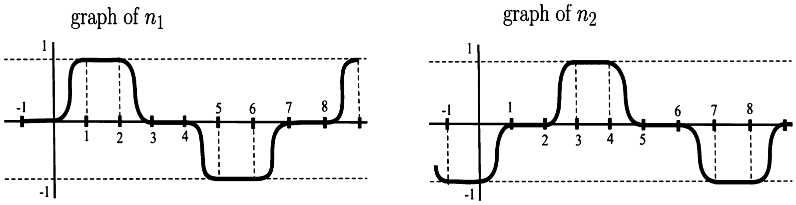

Fig. 4. Graphs for n_1 and n_2.

$$\begin{cases} -1 \leq n_2(t) \leq 0 & (\text{if } 8n \leq t \leq 8n+1), \\ n_2(t) = 0 & (\text{if } 8n+1 \leq t \leq 8n+2), \\ 0 \leq n_2(t) \leq 1 & (\text{if } 8n+2 \leq t \leq 8n+3), \\ n_2(t) = 1 & (\text{if } 8n+3 \leq t \leq 8n+4), \\ 1 \geq n_2(t) \geq 0 & (\text{if } 8n+4 \leq t \leq 8n+5), \\ n_2(t) = 0 & (\text{if } 8n+5 \leq t \leq 8n+6), \\ 0 \geq n_2(t) \geq -1 & (\text{if } 8n+6 \leq t \leq 8n+7), \\ n_2(t) = -1 & (\text{if } 8n+7 \leq t \leq 8n+8). \end{cases}$$

Fig. 3 and Fig. 4 clearly show that the following two properties hold for any $t \in \mathbb{R}$.

$$(n_1(t), n_2(t)) \neq (0,0) \quad \text{and} \quad \left(\frac{dx}{dt}(t), \frac{dy}{dt}(t)\right) \cdot (n_1(t), n_2(t)) = 0.$$

Set

$$\widetilde{\nu}(t) = \frac{1}{\sqrt{(n_1(t))^2 + (n_2(t))^2}}(n_1(t), n_2(t)).$$

Then, $(\widetilde{f}, \widetilde{\nu}) : \mathbb{R} \to \mathbb{R}^2 \times S^1$ is actually a frontal.

For the frontal $(\widetilde{f}, \widetilde{\nu})$, the envelope of the one-parameter family of lines ℓ_t perpendicular to the unit vector $\widetilde{\nu}(t)$ and that passes through the point $\widetilde{f}(t)$ does not restore the square $\widetilde{f}(\mathbb{R})$.

These two examples show that the classical notion of an envelope cannot generally restore the original figure for frontals. In order to eliminate the effects of singularities of frontals, Takahashi improved the notion of envelopes ([24,25]). The improvement made by Takahashi is suitable, and thus for Example 1, the original figure $g(\mathbb{R}) = S^1$ can actually be obtained as the envelope in the sense of Takahashi. However, the *variability condition* defined by [24,25] is unfortunately not satisfied for the frontal $(\widetilde{f}, \widetilde{\nu})$ given in Example 2. Thus, even Takahashi's envelope cannot restore the original square $\widetilde{f}(\mathbb{R})$ for Example 2. According to Ishikawa, a frontal that satisfies Takahashi's variability condition is called a *proper frontal* ([14], §6). The frontal $(\widetilde{f}, \widetilde{\nu}) : \mathbb{R} \to \mathbb{R}^2 \times S^1$ of Example 2 is not a proper frontal. To the best of our knowledge, except for Example 2.5 given by [14], all of the frontals investigated in previous detailed studies are proper frontals. We note that non-proper frontals are also useful, especially in surface science applications (see §5).

In the following, Example 3 shows that the uniqueness of an anti-orthotomic (resp., negative pedal) does not hold in general even when the given mapping f (resp., g) is a frontal.

Example 3. Let $f, g : \mathbb{R} \to \mathbb{R}^2$ be the constant mappings defined by $f(t) = (0, -1)$, $g(t) = (0, 0)$. Let $\nu : \mathbb{R} \to S^1$ be the constant mapping defined by $\nu(t) = (-1, 0)$. Set $P = (0, 1)$ and define the constant mapping $\widetilde{\nu} : \mathbb{R} \to S^1$ by $\widetilde{\nu}(t) = (0, 1)$. Then, for any C^∞ mapping $\widetilde{f} : \mathbb{R} \to \mathbb{R}^2$ with the form $\widetilde{f}(t) = (\widetilde{f}_1(t), 0)$, the frontal $(\widetilde{f}, \widetilde{\nu})$ is an anti-orthotomic of f relative to P and a negative pedal of g relative to P.

The main aim of this study is to obtain the unique solution of the inverse problem for a given orthotomic f relative to P such that $P \in \mathcal{NS}_f$.

Theorem 1. *Let* $(f, \nu) : N \to \mathbb{R}^{n+1} \times S^n$, P *be a frontal and a point of* $\mathbb{R}^{n+1}$ *such that* $P \in \mathcal{NS}_f$. *Let* $\widetilde{f} : N \to \mathbb{R}^{n+1}$ *be the mapping defined by:*

$$\widetilde{f}(x) = f(x) - \frac{||f(x) - P||^2}{2(f(x) - P) \cdot \nu(x)}\nu(x).$$

Then, the following four statements hold.

(1) *The mapping* $\widetilde{f}$ *is a frontal with its Gauss mapping* $\widetilde{\nu}(x) = \frac{f(x)-P}{||f(x)-P||}$.
(2) *The mapping* $\widetilde{f}$ *is the unique anti-orthotomic of* f *relative to* P.
(3) *The point* P *belongs to* $\mathcal{NS}_{\widetilde{f}}$.
(4) *The equality* $||\widetilde{f}(x) - P|| = ||\widetilde{f}(x) - f(x)||$ *holds for any* $x \in N$.

In the case where $P = O$ and $f : I \to \mathbb{R}^2$ is a regular curve such that $f(x) \cdot \nu(x) \neq 0$ for any $x \in I$, the same formula for $\widetilde{f}$ was given by [4] 7.14 to define the envelope of perpendicular bysectors of segments joining $f(x)$ and P. In addition, by Proposition 1, $\widetilde{f}(x)$ must be on the normal line $\{f(x) + a\nu(x) \mid a \in \mathbb{R}\}$. Therefore, in Theorem 1, simply by solving simultaneous linear equations, the same formula for $\widetilde{f}$ can easily be obtained as that for the intersections of the perpendicular bysectors and the normal lines, and the no-silhouette condition $P \in \mathcal{NS}_f$ guarantees that each simultaneous linear equation must have a unique solution.

Clearly, the following corollary follows from Theorem 1.

Corollary 2. *Let* $(g, \nu) : N \to \mathbb{R}^{n+1} \times S^n$, P *be a frontal and a point of* $\mathbb{R}^{n+1}$ *such that* $P \in \mathcal{NS}_g$. *Let* $\widetilde{f} : N \to \mathbb{R}^{n+1}$ *be the mapping defined by:*

$$\widetilde{f}(x) = 2g(x) - P - \frac{||g(x) - P||^2}{(g(x) - P) \cdot \nu(x)}\nu(x).$$

Then, the following four statements hold.

(1) *The mapping* $\widetilde{f}$ *is a frontal with its Gauss mapping* $\widetilde{\nu}(x) = \frac{g(x)-P}{||g(x)-P||}$.
(2) *The mapping* $\widetilde{f}$ *is the unique negative pedal of* g *relative to* P.
(3) *The point* P *belongs to* $\mathcal{NS}_{\widetilde{f}}$.
(4) *The equality* $||\widetilde{f}(x) - P|| = ||\widetilde{f}(x) - 2g(x) + P||$ *holds for any* $x \in N$.

The remainder of this paper is organized as follows. In Section 2, we prove Proposition 1. Theorem 1 is proved in Section 3. In the case where the Gauss mapping $\widetilde{\nu}$ of $\widetilde{f}$ is the identity mapping, we have the well-known Cahn–Hoffman vector formula for $\widetilde{f}$ ([9]). In Section 4, the Cahn–Hoffman formula is presented as the first application of Theorem 1. In Section 5, as the second application of Theorem 1, the optical meaning of the anti-orthotomic $\widetilde{f}$ is clarified even at a singular point of the Gauss mapping $\widetilde{\nu}$ of $\widetilde{f}$. Moreover, in order to demonstrate how the clarified optical meaning is useful, we employ it to construct the exact shape of the orthotomic f for the frontal $\widetilde{f}$ in Example 2 and a given point $P \in \mathcal{NS}_{\widetilde{f}}$. Finally, in Section 6, as the third application of Theorem 1, we give a criterion to ensure that a given frontal is actually a front.

2. Proof of Proposition 1

2.1. Proof that f *is a frontal with its Gauss mapping* $\nu(x) = \frac{f(x)-\widetilde{f}(x)}{||f(x)-\widetilde{f}(x)||}$

We recall that f is defined by $f(x) = 2\left((\widetilde{f}(x) - P) \cdot \widetilde{\nu}(x)\right)\widetilde{\nu}(x) + P$.

Lemma 2.1. *For any $x \in N$, $f(x) - \widetilde{f}(x)$ is a non-zero vector.*

Proof. Suppose that $f(x_0) = \widetilde{f}(x_0)$ for some $x_0 \in N$. Then, the following holds for x_0:

$$\widetilde{f}(x_0) - P = 2\left((\widetilde{f}(x_0) - P) \cdot \widetilde{\nu}(x_0)\right)\widetilde{\nu}(x_0),$$

which implies that $(\widetilde{f}(x_0) - P) \cdot \widetilde{\nu}(x_0) = 0$, and this contradicts the assumption that $P \in \mathcal{NS}_{\widetilde{f}}$. □

Set

$$\nu(x) = \frac{f(x) - \widetilde{f}(x)}{||f(x) - \widetilde{f}(x)||}.$$

Then, it is sufficient to show that $df_x(\mathbf{v}) \cdot \nu(x) = 0$ for any $x \in N$ and any $\mathbf{v} \in T_xN$, i.e., it is sufficient to show that:

$$(f \circ \xi)'(0) \cdot \nu(x) = 0$$

for any curve $\xi : (-\varepsilon, \varepsilon) \to N$ such that $\xi(0) = x$. The following lemma clearly holds.

Lemma 2.2.

(1) $(\widetilde{f} \circ \xi)'(0) \cdot \widetilde{\nu}(x) = 0$.
(2) $(\widetilde{\nu} \circ \xi)'(0) \cdot \widetilde{\nu}(x) = 0$.
(3) $(f \circ \xi)'(0) = 2\left(\left(\widetilde{f}(x) - P\right) \cdot (\widetilde{\nu} \circ \xi)'(0)\right)\widetilde{\nu}(x) + 2\left(\left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x)\right)(\widetilde{\nu} \circ \xi)'(0)$.

By using Lemma 2.2, we have the following:

$$\begin{aligned}
&||f(x) - \widetilde{f}(x)||\left((f \circ \xi)'(0) \cdot \nu(x)\right) \\
&= (f \circ \xi)'(0) \cdot \left(f(x) - \widetilde{f}(x)\right) \\
&= (f \circ \xi)'(0) \cdot \left(2\left(\left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x)\right)\widetilde{\nu}(x) - \left(\widetilde{f}(x) - P\right)\right) \\
&= 4\left((\widetilde{f}(x) - P) \cdot (\widetilde{\nu} \circ \xi)'(0)\right)\left((\widetilde{f}(x) - P) \cdot \widetilde{\nu}(x)\right) \\
&\qquad -2\left((\widetilde{f}(x) - P) \cdot (\widetilde{\nu} \circ \xi)'(0)\right)\left(\widetilde{\nu}(x) \cdot (\widetilde{f}(x) - P)\right) \\
&\qquad\qquad -2\left((\widetilde{f}(x) - P) \cdot \widetilde{\nu}(x)\right)\left((\widetilde{\nu} \circ \xi)'(0) \cdot (\widetilde{f}(x) - P)\right) \\
&= 0. \quad \square
\end{aligned}$$

2.2. Proof that $(f(x) - P) \cdot \nu(x) \neq 0$ for any $x \in N$

For any $x \in N$, we have the following:

$$\begin{aligned}
&||f(x) - \widetilde{f}(x)||\,(f(x) - P) \cdot \nu(x) \\
&= (f(x) - P) \cdot \left(f(x) - \widetilde{f}(x)\right) \\
&= 2\left(\left(\left(\widetilde{f}(x) - P\right) \cdot \widetilde{\nu}(x)\right)\widetilde{\nu}(x)\right) \cdot \left(f(x) - \widetilde{f}(x)\right)
\end{aligned}$$

$$= 2\left(\left(\left(\widetilde{f}(x) - P\right)\cdot\widetilde{\nu}(x)\right)\widetilde{\nu}(x)\right)\cdot\left(2\left(\left(\widetilde{f}(x) - P\right)\cdot\widetilde{\nu}(x)\right)\widetilde{\nu}(x) - \left(\widetilde{f}(x) - P\right)\right)$$
$$= 4\left(\left(\widetilde{f}(x) - P\right)\cdot\widetilde{\nu}(x)\right)^2 - 2\left(\left(\widetilde{f}(x) - P\right)\cdot\widetilde{\nu}(x)\right)\left(\widetilde{\nu}(x)\cdot\left(\widetilde{f}(x) - P\right)\right)$$
$$= 2\left(\left(\widetilde{f}(x) - P\right)\cdot\widetilde{\nu}(x)\right)^2.$$

By the assumption that $P \in \mathcal{NS}_{\widetilde{f}}$, it follows that $(f(x) - P)\cdot\nu(x) \neq 0$ for any $x \in N$. □

3. Proof of Theorem 1

3.1. Proof that $\widetilde{f}$ is a frontal with Gauss mapping $\widetilde{\nu}(x) = \frac{f(x)-P}{||f(x)-P||}$

From the assumption that $(f(x) - P)\cdot\nu(x) \neq 0$ for any $x \in N$, it follows that $f(x) \neq P$ for any $x \in N$. Thus,

$$\widetilde{\nu}(x) = \frac{f(x) - P}{||f(x) - P||},$$

is well defined. Then, it is sufficient to show that

$$(\widetilde{f}\circ\xi)'(0)\cdot\widetilde{\nu}(x) = 0$$

for any curve $\xi : (-\varepsilon, \varepsilon) \to N$ such that $\xi(0) = x$. $\widetilde{f}$ has the form

$$\widetilde{f}(x) = f(x) - \frac{||f(x) - P||^2}{2\,(f(x) - P)\cdot\nu(x)}\nu(x),$$

so we have the following:

$$\begin{aligned}
&||f(x) - P||\left(\left(\widetilde{f}\circ\xi\right)'(0)\cdot\widetilde{\nu}(x)\right)\\
&= (f\circ\xi)'(0)\cdot(f(x) - P) - \frac{(f\circ\xi)'(0)\cdot(f(x) - P)}{(f(x) - P)\cdot\nu(x)}\left(\nu(x)\cdot(f(x) - P)\right)\\
&\quad + \frac{(f(x) - P)\cdot(\nu\circ\xi)'(0)}{2\left((f(x) - P)\cdot\nu(x)\right)^2}||f(x) - P||^2\left(\nu(x)\cdot(f(x) - P)\right)\\
&\quad - \frac{||f(x) - P||^2}{2\left((f(x) - P)\cdot\nu(x)\right)}\left((\nu\circ\xi)'(0)\cdot(f(x) - P)\right)\\
&= (f\circ\xi)'(0)\cdot(f(x) - P) - (f\circ\xi)'(0)\cdot(f(x) - P)\\
&\quad + \frac{(f(x) - P)\cdot(\nu\circ\xi)'(0)}{2\left((f(x) - P)\cdot\nu(x)\right)}||f(x) - P||^2\\
&\quad - \frac{||f(x) - P||^2}{2\left((f(x) - P)\cdot\nu(x)\right)}\left((\nu\circ\xi)'(0)\cdot(f(x) - P)\right)\\
&= 0 + 0 = 0. \quad \square
\end{aligned}$$

3.2. Proof that $\widetilde{f}$ is the unique anti-orthotomic of f relative to P

The proof is essentially given in the paragraph next to Theorem 1. Thus, only a confirmation is given by definition in this subsection. We recall that $\widetilde{\nu}(x) = \frac{f(x)-P}{||f(x)-P||}$. We have the following:

$$
\begin{aligned}
& 2\left(\left(\widetilde{f}(x)-P\right)\cdot\widetilde{\nu}(x)\right)\widetilde{\nu}(x)+P \\
&= 2\left(\left((f(x)-P)-\frac{||f(x)-P||^2}{2\,(f(x)-P)\cdot\nu(x)}\nu(x)\right)\cdot\frac{f(x)-P}{||f(x)-P||}\right)\frac{f(x)-P}{||f(x)-P||}+P \\
&= 2\left(||f(x)-P||-\frac{||f(x)-P||}{2}\right)\frac{f(x)-P}{||f(x)-P||}+P \\
&= f(x)-P+P=f(x). \quad \square
\end{aligned}
$$

3.3. Proof that $\left(\widetilde{f}(x)-P\right)\cdot\widetilde{\nu}(x)\neq 0$ *for any* $x\in N$

For any $x\in N$, we have the following:

$$
\begin{aligned}
\left(\widetilde{f}(x)-P\right)\cdot\widetilde{\nu}(x) &= \left(\widetilde{f}(x)-P\right)\cdot\frac{f(x)-P}{||f(x)-P||} \\
&= \left(f(x)-\frac{||f(x)-P||^2}{2(f(x)-P)\cdot\nu(x)}\nu(x)-P\right)\cdot\frac{f(x)-P}{||f(x)-P||} \\
&= \left((f(x)-P)-\frac{||f(x)-P||^2}{2(f(x)-P)\cdot\nu(x)}\nu(x)\right)\cdot\frac{f(x)-P}{||f(x)-P||} \\
&= ||f(x)-P||-\frac{||f(x)-P||}{2} \\
&= \frac{||f(x)-P||}{2}\neq 0. \quad \square
\end{aligned}
$$

3.4. Proof that the equality $||\widetilde{f}(x)-P||=||\widetilde{f}(x)-f(x)||$ *holds for any* $x\in N$

$\widetilde{f}(x)-P=(f(x)-P)-\frac{||f(x)-P||^2}{2(f(x)-P)\cdot\nu(x)}\nu(x)$, so the following holds for any $x\in N$:

$$
\begin{aligned}
& \left\|\widetilde{f}(x)-P\right\|^2 \\
&= \left(\widetilde{f}(x)-P\right)\cdot\left(\widetilde{f}(x)-P\right) \\
&= \left((f(x)-P)-\frac{||f(x)-P||^2}{2(f(x)-P)\cdot\nu(x)}\nu(x)\right)\cdot\left((f(x)-P)-\frac{||f(x)-P||^2}{2(f(x)-P)\cdot\nu(x)}\nu(x)\right) \\
&= ||f(x)-P||^2-||f(x)-P||^2+\frac{||f(x)-P||^2}{4\left((f(x)-P)\cdot\nu(x)\right)^2} \\
&= \frac{||f(x)-P||^2}{4\left((f(x)-P)\cdot\nu(x)\right)^2} \\
&= \left\|\widetilde{f}(x)-f(x)\right\|^2. \quad \square
\end{aligned}
$$

4. Application 1: generalization of the Cahn–Hoffman vector formula

Let $(g,\nu): N\to\mathbb{R}^{n+1}\times S^n$ be a frontal. We assume that $\mathcal{NS}_g$ is not empty. Let P be a point of $\mathcal{NS}_g$. Then, by Corollary 2, the mapping $\left(\widetilde{f},\widetilde{\nu}\right): N\to\mathbb{R}^{n+1}\times S^n$ defined by

$$
\widetilde{f}(x)=2g(x)-P-\frac{||g(x)-P||^2}{(g(x)-P)\cdot\nu(x)}\nu(x),
$$

$$\widetilde{\nu}(x) = \frac{g(x) - P}{||g(x) - P||}$$

is a frontal and the unique negative pedal of g relative to P. Set $\gamma(x) = ||g(x) - P||$. Then, by using $\widetilde{\nu} : N \to S^n$ and $\gamma : N \to \mathbb{R}_+$, $g(x) - P$ can be expressed as follows:

$$g(x) - P = \gamma(x)\widetilde{\nu}(x).$$

Hoffman and Cahn showed the following [9] under the assumption that N is the unit sphere S^n and $\widetilde{\nu} : S^n \to S^n$ is the identity mapping, and under the identification $\mathbb{R}^{n+1} = T_{\widetilde{\nu}(x)}\mathbb{R}^{n+1} = T_{\widetilde{\nu}(x)}S^n \oplus \mathbb{R}\widetilde{\nu}(x)$.

Theorem 2 *(Cahn–Hoffman vector formula [9]). For any $x \in S^n$, the following equality holds:*

$$\widetilde{f}(x) - g(x) = \nabla\gamma(x) \oplus 0\widetilde{\nu}(x),$$

where $\nabla\gamma(x)$ is the gradient vector of γ at x with respect to the normal coordinate system of S^n around x. We generalize Theorem 2 as an application of Theorem 1 in the following.

Theorem 3. *Let $(g, \nu) : N \to \mathbb{R}^{n+1} \times S^n$ be a frontal and let P be a point of $\mathcal{NS}_g$. Suppose that $\widetilde{\nu}$ is non-singular at x. Then, the following equality holds:*

$$\widetilde{f}(x) - g(x) = \left((J\widetilde{\nu}(x))^{-1}\right)^t \nabla\gamma(x)\oplus 0\widetilde{\nu}(x),$$

where $\left((J\widetilde{\nu}(x))^{-1}\right)^t$ denotes the transposed matrix of the inverse of the Jacobian matrix of $\widetilde{\nu}$ with respect to an arbitrary local coordinate system around $x \in N$ and the normal coordinate system around $\widetilde{\nu}(x) \in S^n$. Theorem 3 yields Theorem 2 but also the following.

Corollary 3. *Let $(g, \nu) : N \to \mathbb{R}^{n+1} \times S^n$ be a frontal and let P be a point of $\mathcal{NS}_g$. Suppose that $\widetilde{\nu}$ is non-singular at x. Then, x is a singular point of γ if and only if $\widetilde{f}(x) = g(x)$ is satisfied.*

Proof of Theorem 3.

$$\widetilde{f}(x) = f(x) - \frac{||f(x) - P||^2}{2(f(x) - P) \cdot \nu(x)}\nu(x),$$

so it follows that:

$$\begin{aligned} f(x) - \widetilde{f}(x) &= \frac{||f(x) - P||^2}{2(f(x) - P) \cdot \nu(x)}\nu(x) \\ &= \frac{4\gamma^2(x)}{4\gamma(x)\left(\widetilde{\nu}(x) \cdot \nu(x)\right)}\nu(x) \\ &= \frac{\gamma(x)}{\widetilde{\nu}(x) \cdot \nu(x)}\nu(x) \\ &= \frac{\gamma(x)}{\widetilde{\nu}(x) \cdot \nu_2(x)}\left(\nu_1(x) \oplus \nu_2(x)\right), \end{aligned}$$

where $\nu(x) = \nu_1(x) \oplus \nu_2(x)$ and $\nu_1(x) \in T_{\widetilde{\nu}(x)}S^n, \nu_2(x) \in \mathbb{R}\widetilde{\nu}(x)$.

The technique employed by [20] is used in order to represent ν_1, ν_2 in terms of γ and $\widetilde{\nu}$. $f(x) - P = 2\gamma(x)\widetilde{\nu}(x)$, so for any $v \in T_xN$,

$$df_x(v) = 2\gamma(x)d\widetilde{\nu}_x(v) \oplus 2d\gamma_x(v)\widetilde{\nu}(x),$$

where $\gamma(x)d\widetilde{\nu}_x(v) \in T_{\widetilde{\nu}(x)}S^n$ and $d\gamma_x(v)\widetilde{\nu}(x) \in \mathbb{R}\widetilde{\nu}(x)$. Thus, the Jacobian matrix Jf of f at x with respect to an arbitrary local coordinate system around $x \in N$ and the direct product of the normal coordinate system and $\mathbb{R}$ around $f(x) = 2\gamma(x)\widetilde{\nu}(x)$ has the following form, where $J\widetilde{\nu}(x)$ denotes the Jacobian matrix of $\widetilde{\nu}$ at x and $(\nabla\gamma(t))^t$ is the transposed vector of the gradient of γ at x.

$$Jf(x) = \begin{pmatrix} 2\gamma(x)J\widetilde{\nu}(x) \\ 2\left(\nabla\gamma(x)\right)^t \end{pmatrix}.$$

Let $\widetilde{J\widetilde{\nu}}(x)$ and $|J\widetilde{\nu}|(x)$ be the cofactor matrix of the Jacobian matrix $J\widetilde{\nu}(x)$ and the Jacobian determinant of $\widetilde{\nu}$ at x, respectively. Moreover, let O be the $n \times 1$ zero vector. Multiplying the matrix:

$$\left(-\left(\nabla\gamma(x)\right)^t, 1\right)\begin{pmatrix} \widetilde{J\widetilde{\nu}}(x) & O \\ O^t & \gamma(x)|J\widetilde{\nu}|(x) \end{pmatrix}$$

with the Jacobian matrix $Jf(x)$ from the left-hand side yields the following, where E_n denotes the $n \times n$ unit matrix.

$$\begin{aligned} &\left(-\left(\nabla\gamma(x)\right)^t, 1\right)\begin{pmatrix} \widetilde{J\widetilde{\nu}}(x) & O \\ O^t & \gamma(x)|J\widetilde{\nu}|(x) \end{pmatrix}\begin{pmatrix} 2\gamma(x)J\widetilde{\nu}(x) \\ 2\left(\nabla\gamma(x)\right)^t \end{pmatrix} \\ &= \left(-\left(\nabla\gamma(x)\right)^t, 1\right)\begin{pmatrix} 2\gamma(x)|J\widetilde{\nu}|(x)E_n \\ 2\gamma(x)|J\widetilde{\nu}|(x)\left(\nabla\gamma(x)\right)^t \end{pmatrix} \\ &= (0, \ldots, 0). \end{aligned}$$

Hence we have the following.

Lemma 4.1. *Suppose that $|J\widetilde{\nu}|(x) \neq 0$. Then, we have the following:*

$$\begin{aligned} \nu_1(x) &= -\left(\widetilde{J\widetilde{\nu}}(x)\right)^t \nabla\gamma(x)/||\nu(x)|| \\ \nu_2(x) &= |J\widetilde{\nu}|(x)\gamma(x)\widetilde{\nu}(x)/||\nu(x)||. \end{aligned}$$

It should be noted that in order to show that $||\nu(x)|| \neq 0$, the assumption that "$|J\widetilde{\nu}|(x) \neq 0$" is used. Set $h(x) = 2g(x) - \widetilde{f}(x)$. Then, by elementary geometry, we have:

$$\begin{aligned} h(x) - P &= f(x) - \widetilde{f}(x) \\ &= \frac{\gamma(x)}{\widetilde{\nu}(x)\cdot\nu_2(x)}\left(\nu_1(x) \oplus \nu_2(x)\right) \\ &= \frac{1}{|J\widetilde{\nu}|(x)}\left(-\left(\widetilde{J\widetilde{\nu}}(x)\right)^t \nabla\gamma(x) \oplus |J\widetilde{\nu}|(x)\gamma(x)\widetilde{\nu}(x)\right) \\ &= -\frac{1}{|J\widetilde{\nu}|(x)}\left(\widetilde{J\widetilde{\nu}}(x)\right)^t \nabla\gamma(x) \oplus (g(x) - P). \end{aligned}$$

$\widetilde{f}(x) - g(x) = g(x) - h(x) = (g(x) - P) - (h(x) - P)$, so we have:

$$\widetilde{f}(x) - g(x) = \frac{1}{|J\widetilde{\nu}|(x)}\left(\widetilde{J\widetilde{\nu}}(x)\right)^t \nabla\gamma(x) \oplus 0\widetilde{\nu}(x) = \left(\left(J\widetilde{\nu}(x)\right)^{-1}\right)^t \nabla\gamma(x) \oplus 0\widetilde{\nu}(x). \quad \square$$

5. Application 2: opening the Gauss mapping of the anti-orthotomic

The application in Section 4 is a result obtained only at a non-singular point of $\widetilde{\nu}$. In this section, as a second application of Theorem 1, we investigate the results that can be obtained even at a singular point of $\widetilde{\nu}$. The following definitions are required for the investigation in this section.

Definition 3 *([13]).* Let $f = (f_1, \dots, f_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional C^∞ map-germ.

(1) Let Ω^1_n denote the $\mathcal{E}_n$-module of 1-forms on $(\mathbb{R}^n, 0)$. Then, the $\mathcal{E}_n$-module generated by df_i $(i = 1, \dots, n)$ in Ω^1_n is called the *Jacobi module* of f and denoted by $\mathcal{J}_f$, where dh denotes the exterior differential of h for a function-germ $h : (\mathbb{R}^n, 0) \to \mathbb{R}$.
(2) The *ramification module* of f (denoted by $\mathcal{R}_f$) is defined as the $f^*(\mathcal{E}_n)$-module comprising all function-germs γ such that $d\gamma$ belongs to $\mathcal{J}_f$.

Definition 4 *([13]).* Let $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be an equidimensional C^∞ map-germ and let $\delta : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a C^∞ function-germ. Then, the map-germ $(\varphi, \delta) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n \times \mathbb{R}, (0, 0))$ is called an *opening of* φ if $\delta \in \mathcal{R}_\varphi$.

Theorem 4. *Let $(f, \nu) : N \to \mathbb{R}^{n+1} \times S^n$ be a frontal. Let $x_0 \in N$ and $P \in \mathbb{R}^{n+1}$ satisfy $(f(x_0) - P) \cdot \nu(x_0) \neq 0$. Then, $(f - P) : (N, x_0) \to \mathbb{R}^{n+1}$ is an opening of the Gauss map-germ $\widetilde{\nu} = \frac{f-P}{||f-P||} : (N, x_0) \to S^n$ of its anti-orthotomic $\widetilde{f} : (N, x_0) \to \mathbb{R}^{n+1}$.*

The optical meaning of an anti-orthotomic is straightforward according to Theorem 4. By definition, we have the following corollary.

Corollary 4. *Let $(g, \nu) : N \to \mathbb{R}^{n+1} \times S^n$ be a frontal. Let $x_0 \in N$ and $P \in \mathbb{R}^{n+1}$ satisfy $(g(x_0) - P) \cdot \nu(x_0) \neq 0$. Then, $(g - P) : (N, x_0) \to \mathbb{R}^{n+1}$ is an opening of the Gauss map-germ $\widetilde{\nu} = \frac{g-P}{||g-P||} : (N, x_0) \to S^n$ of its negative pedal $\widetilde{f} : (N, x_0) \to \mathbb{R}^{n+1}$.*

Proof of Theorem 4. Let $V \subset S^n$ be a sufficiently small open neighborhood of $\widetilde{\nu}(x_0)$ and let $h = (h_1, \dots, h_n) : V \to T_{\widetilde{\nu}(x_0)} S^n$ be a normal coordinate system at $\widetilde{\nu}(x_0)$. Let U be a sufficiently small open neighborhood of x_0 such that $U \subset \widetilde{\nu}^{-1}(V)$ and set $\nu_1(x) = (\nu_{1,1}(x), \dots, \nu_{1,n}(x))$ for any $x \in U$. Moreover, for any i $(1 \leq i \leq n)$, set $\widetilde{\nu}_i = h_i \circ \widetilde{\nu}$. $\nu : N \to S^n$ is the Gauss mapping of $f : N \to \mathbb{R}^{n+1}$ and $f(x) - P = 2\gamma(x)\widetilde{\nu}(x)$, so we have:

$$\sum_{i=1}^{n} \nu_{1,i} \gamma d\widetilde{\nu}_i + ||\nu_2|| d\gamma = 0.$$

$(f(x_0) - P) \cdot \nu(x_0) \neq 0$, so it follows that $\nu_2(x_0) \neq 0$. Thus, we have

$$d\gamma = -\frac{1}{||\nu_2||} \sum_{i=1}^{n} \nu_{1,i} \gamma d\widetilde{\nu}_i \in \mathcal{J}_{\widetilde{\nu}}. \quad \square$$

Again, we consider the frontal $(\widetilde{f}, \widetilde{\nu}) : \mathbb{R} \to \mathbb{R}^2 \times S^1$ given in Example 2. We recall that the image $\widetilde{f}(\mathbb{R})$ is the square S with vertexes $(1, 1), (-1, 1), (1, -1), (-1, -1)$. Let $P = (p_1, p_2)$ be a point such that $-1 < p_1, p_2 < 1$. Then, P belongs to $\mathcal{NS}_{\widetilde{f}}$. Let $f : \mathbb{R} \to \mathbb{R}^2$ be the orthotomic of $\widetilde{f}$ relative to P. Theorems 1 and 4 reduce the construction of the image of f to elementary geometry, which is explained as follows (see Fig. 5). By the construction of $\widetilde{\nu}$, if t belongs to the union of closed intervals:

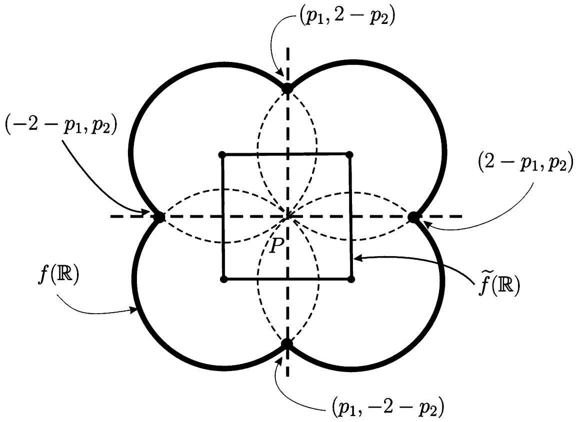

Fig. 5. Drawing $f(\mathbb{R})$ for the given square $\widetilde{f}(\mathbb{R})$.

$$\bigcup_{n\in\mathbb{Z}} \left([8n+1, 8n+2] \cup [8n+3, 8n+4] \cup [8n+5, 8n+6] \cup [8n+7, 8n+8]\right),$$

then t must be a singular point of $\widetilde{\nu}$. Thus, by Theorem 4, t must be a singular point of f, so each of $f([8n+1, 8n+2])$, $f([8n+3, 8n+4])$, $f([8n+5, 8n+6])$, and $f([8n+7, 8n+8])$ must be one point. By definition, the one point must be the mirror image of P, as follows:

Lemma 5.1.

$$f([8n+1, 8n+2]) = (2-p_1, p_2), \quad f([8n+3, 8n+4]) = (p_1, 2-p_2),$$
$$f([8n+5, 8n+6]) = (-2-p_1, p_2), \; f([8n+7, 8n+8]) = (p_1, -2-p_2).$$

By the construction of $\widetilde{f}$, we have:

$$\widetilde{f}([8n, 8n+1]) = (1, -1), \quad \widetilde{f}([8n+2, 8n+3]) = (1, 1),$$
$$\widetilde{f}([8n+4, 8n+5]) = (-1, 1), \; \widetilde{f}([8n+6, 8n+7]) = (-1, -1).$$

By assertion (4) in Theorem 1, the following holds.

Lemma 5.2.

$$||\widetilde{f}(t) - f(t)|| = \begin{cases} \sqrt{(p_1-1)^2 + (p_2+1)^2} & (\textit{if } 8n \le t \le 8n+1), \\ \sqrt{(p_1-1)^2 + (p_2-1)^2} & (\textit{if } 8n+2 \le t \le 8n+3), \\ \sqrt{(p_1+1)^2 + (p_2-1)^2} & (\textit{if } 8n+4 \le t \le 8n+5), \\ \sqrt{(p_1+1)^2 + (p_2+1)^2} & (\textit{if } 8n+6 \le t \le 8n+7). \end{cases}$$

By the construction of $\widetilde{\nu}$, we have the following.

Lemma 5.3.

(1) $f([8n, 8n+1])$ *is exactly the hemicircle centered at* $\widetilde{f}([8n, 8n+1]) = (1, -1)$ *with boundary* $f([8n+7, 8n+8]) = (p_1, -2-p_2)$ *and* $f([8n+1, 8n+2]) = (2-p_1, p_2)$ *that does not contain* P.
(2) $f([8n+2, 8n+3])$ *is exactly the hemicircle centered at* $\widetilde{f}([8n+2, 8n+3]) = (1, 1)$ *with boundary* $f([8n+1, 8n+2]) = (2-p_1, p_2)$ *and* $f([8n+3, 8n+4]) = (p_1, 2-p_2)$ *that does not contain* P.

(3) $f([8n+4,8n+5])$ *is exactly the hemicircle centered at* $\widetilde{f}([8n+4,8n+5]) = (-1,1)$ *with boundary* $f([8n+3,8n+4]) = (p_1, 2-p_2)$ *and* $f([8n+5,8n+6]) = (-2-p_1, p_2)$ *that does not contain* P.
(4) $f([8n+6,8n+7])$ *is exactly the hemicircle centered at* $\widetilde{f}([8n+6,8n+7]) = (-1,-1)$ *with boundary* $f([8n+5,8n+6]) = (-2-p_1, p_2)$ *and* $f([8n+7,8n+8]) = (p_1, -2-p_2)$ *that does not contain* P.

For the precise shape of the pedal $g : \mathbb{R} \to \mathbb{R}^2$ of $\widetilde{f}$ relative to P, we simply shrink $f(\mathbb{R})$ by 50 percent with respect to P.

It appears that the method proposed by Herring [8] is similar to our method, but his method seems to rely on a thermodynamical consideration of atoms, whereas our method requires no physical considerations. After the given shape is realized as the image of the frontal $\widetilde{f}(\mathbb{R})$, by applying Theorem 1 and Theorem 4, only elementary geometry is needed. Thus, in any physical situation, if the same square is given, then the γ-plot for the square must have the same shape.

6. Application 3: a criterion for fronts

Definition 5. A germ of the frontal $(f,\nu) : (N,x_0) \to \mathbb{R}^{n+1} \times S^n$ is called a *germ of front* (or *front-germ*) if (f,ν) is non-singular at x_0.

Given a frontal $(f,\nu) : N \to \mathbb{R}^{n+1} \times S^n$, if $(f,\nu) : (N,x) \to \mathbb{R}^{n+1} \times S^n$ is a germ of front for any $x \in N$, then (f,ν) is called a *front*. A front is also called a *wave-front*. For further details of fronts, please refer to [2,3].

Theorem 5. *Let* $(f,\nu) : N \to \mathbb{R}^{n+1} \times S^n$ *be a frontal and let* x_0 *be a point of* N*. Then, for any point* $P \in \mathbb{R}^{n+1}$ *such that* $(f(x_0) - P) \cdot \nu(x_0) \neq 0$*, the following are equivalent, where* $\left(\widetilde{f}, \widetilde{\nu}\right) : (N,x_0) \to \mathbb{R}^{n+1} \times S^n$ *is the anti-orthotomic germ of* f *relative to* P.

(1) $(f,\nu) : (N,x_0) \to \mathbb{R}^{n+1} \times S^n$ *is a front-germ.*
(2) $\left(\widetilde{f}, \widetilde{\nu}\right) : (N,x_0) \to \mathbb{R}^{n+1} \times S^n$ *is a front-germ.*
(3) $\left(f, \widetilde{f}\right) : (N,x_0) \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ *is non-singular.*

Theorem 5 answers the question posed by Honda and Teramoto ([11]). Theorem 5 yields the following corollaries.

Corollary 5. *Let* $(g,\nu) : N \to \mathbb{R}^{n+1} \times S^n$ *be a frontal and let* x_0 *be a point of* N*. Then, for any point* $P \in \mathbb{R}^{n+1}$ *such that* $(g(x_0) - P) \cdot \nu(x_0) \neq 0$*, the following are equivalent, where* $\left(\widetilde{f}, \widetilde{\nu}\right) : (N,x_0) \to \mathbb{R}^{n+1} \times S^n$ *is the negative pedal germ of* g *relative to* P.

(1) $(g,\nu) : (N,x_0) \to \mathbb{R}^{n+1} \times S^n$ *is a front-germ.*
(2) $\left(\widetilde{f}, \widetilde{\nu}\right) : (N,x_0) \to \mathbb{R}^{n+1} \times S^n$ *is a front-germ.*
(3) $\left(g, \widetilde{f}\right) : (N,x_0) \to \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ *is non-singular.*

Corollary 6. *Let* $(f,\nu) : N \to \mathbb{R}^{n+1} \times S^n$ *be a frontal. Let two points* $x_0 \in N$ *and* $P \in \mathbb{R}^{n+1}$ *satisfy* $(f(x_0) - P) \cdot \nu(x_0) \neq 0$*. Let* $(\widetilde{f}, \widetilde{\nu}) : N \to \mathbb{R}^{n+1} \times S^n$ *be the anti-orthotomic of* f *relative to* P*. If* x_0 *is not contained in* $Sing(\widetilde{f}) \cap Sing(\widetilde{\nu})$*, then the map-germ* $f : (N,x_0) \to \mathbb{R}^{n+1}$ *is a front-germ, where for a* C^∞ *mapping* $\varphi : X \to Y$*,* $Sing(\varphi)$ *denotes the singular set of* φ*. In particular, if* $\widetilde{f} : (N,x_0) \to \mathbb{R}^{n+1}$ *is non-singular, then* $f : (N,x_0) \to \mathbb{R}^{n+1}$ *must be a front-germ.*

Corollary 7. *Let* $(g, \nu) : N \to \mathbb{R}^{n+1} \times S^n$ *be a frontal. Let two points* $x_0 \in N$ *and* $P \in \mathbb{R}^{n+1}$ *satisfy* $(g(x_0) - P) \cdot \nu(x_0) \neq 0$*. Let* $(\widetilde{f}, \widetilde{\nu}) : N \to \mathbb{R}^{n+1} \times S^n$ *be the negative pedal of* g *relative to* P*. If* x_0 *is not contained in* $Sing(\widetilde{f}) \cap Sing(\widetilde{\nu})$*, then the map-germ* $g : (N, x_0) \to \mathbb{R}^{n+1}$ *is a front-germ. In particular, if* $\widetilde{f} : (N, x_0) \to \mathbb{R}^{n+1}$ *is non-singular, then* $g : (N, x_0) \to \mathbb{R}^{n+1}$ *must be a front-germ.*

Corollary 8. *Let* $(\widetilde{f}, \widetilde{\nu}) : N \to \mathbb{R}^{n+1} \times S^n$ *be a frontal. Let two points* $x_0 \in N$ *and* $P \in \mathbb{R}^{n+1}$ *satisfy* $(\widetilde{f}(x_0) - P) \cdot \widetilde{\nu}(x_0) \neq 0$*. Let* $(f, \nu) : N \to \mathbb{R}^{n+1} \times S^n$ *be the orthotomic of* $\widetilde{f}$ *relative to* P*. If* x_0 *is not contained in* $Sing(f) \cap Sing(\nu)$*, then the map-germ* $\widetilde{f} : (N, x_0) \to \mathbb{R}^{n+1}$ *is a front-germ.*

Corollary 9. *Let* $(\widetilde{f}, \widetilde{\nu}) : N \to \mathbb{R}^{n+1} \times S^n$ *be a frontal. Let two points* $x_0 \in N$ *and* $P \in \mathbb{R}^{n+1}$ *satisfy* $(\widetilde{f}(x_0) - P) \cdot \widetilde{\nu}(x_0) \neq 0$*. Let* $(g, \nu) : N \to \mathbb{R}^{n+1} \times S^n$ *be the pedal of* $\widetilde{f}$ *relative to* P*. If* x_0 *is not contained in* $Sing(g) \cap Sing(\nu)$*, then the map-germ* $\widetilde{f} : (N, x_0) \to \mathbb{R}^{n+1}$ *is a front-germ.*

Proof of Theorem 5. (2) $\Leftrightarrow$ (3) is simply a corollary of Theorem 4. (1) $\Rightarrow$ (3) is trivial. Thus, in order to complete the proof, it is sufficient to show that (3) $\Rightarrow$ (1). Suppose that $(f, \widetilde{f})$ is non-singular. Then, $(f, f - \widetilde{f})$ is non-singular. By the assumption that $(f(x_0) - P) \cdot \nu(x_0) \neq 0$, it follows that the projection $\pi : d_{x_0}(f - \widetilde{f})(\mathrm{Ker}(d_{x_0} f)) \to T_{\nu(x_0)} S^n$ is injective. Therefore, $(f, \nu) = \left(f, \frac{f - \widetilde{f}}{||f - \widetilde{f}||}\right) : (N, x_0) \to \mathbb{R}^{n+1} \times S^n$ is non-singular. □

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