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EQUIVARIANT SINGULARITIES OF LAGRANGIAN MANIFOLDS AND UNIAXIAL FERROMAGNET*

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Abstract. A classification of typical 3-dimensional Lagrangian singularities with $Z_2 \oplus Z_2$ symmetry (independent changes of sign in two coordinates) is presented. This provides the finite classification of typical and structurally stable local forms of a class of 4-dimensional internal energies with the uniaxial ferromagnet symmetry.

The equivalence relation in the latest classification preserves the basic thermodynamic features of internal energies: the symmetry, the internal stability regions and the inequalities of chemical potentials for states of the system with respectively equal remaining thermodynamic forces.

As an example of applications a phenomenological model of the Curie point for an uniaxial ferromagnet is presented. This demonstrates an alternative to classical "ad hoc" approaches in phenomenological modelling of critical phenomena.

Key words. uniaxial ferromagnet, Lagrangian singularities, equivariant singularities, critical point, phase transition

AMS(MOS) subject classifications. Primary 57R45, 58C27, 58C28, 80A10, 82A25, 82A60; secondary 53C57

Introduction. A geometrical approach to phase transitions as singularities of projections of stable Lagrangian submanifolds (modelling the set of equilibrium states) onto the space of intensive parameters was discussed by Janeczko [11]. This approach was designed to satisfy the laws of thermodynamics and thus to overcome shortcomings of the earlier applications of elementary catastrophe theory to phase transitions (see Fowler [7] and its critique by Lavis and Bell [19]). In particular, the equivalence relations introduced in [11] to classify internal energies do preserve equations of state unlike the previous approach [7].

In the present work we extend the theory to the case of internal energies with symmetries of the uniaxial ferromagnet and obtain the classification of these potentials. The paper is subdivided as follows. In § 1 we consider the space of invariant internal energies in which we introduce an equivalence relation preserving the basic thermodynamic features, i.e., the symmetry, the equality of thermodynamic forces in coexisting phases and the internal stability regions. The results of the local classification of the structurally stable and typical forms of the thermodynamic potentials are stated in Theorems 1.1 and 1.2. In § 2 we present an example of an application of the formalism. The set of isothermal sections of the stability region boundary (bifurcation set) for the potentials of the above classification are compared to the family of isothermal boundaries of the hysteresis region for the uniaxial ferromagnet [5], [29], [30]. This leads, with the help of the second law of thermodynamics, to the selection of the form ("singularity") of internal energy suitable for the description of the Curie region Fig. 1, cf. [28, p. 67]. The phenomenon of spontaneous magnetization is obtained as a feature of one of several typical forms of internal energy. Moreover, this internal energy is locally structurally by stable [23], [28], in the sense that any other "sufficiently close" form is equivalent to it, i.e., displays the same features. It is established which terms of the Taylor's expansion of the internal energy should be considered and which could

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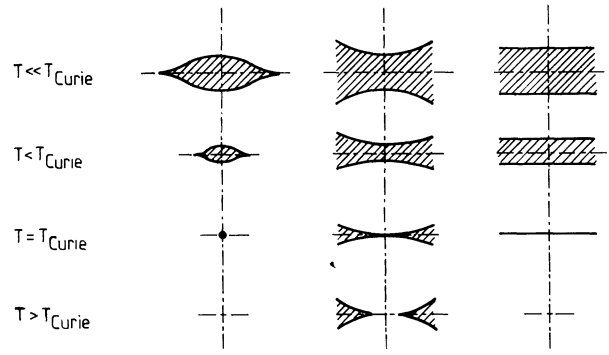

FIG. 1. *Typical cases of isothermal sections of the bifurcation sets for* (A_3) *forms of internal energy. The shaded regions consist of points of a* 3*-fold covering, the other points have at most* 1*-fold coverings.*

be neglected without losing any qualitative features of the model (see Example 1.1). Note that this result could be viewed as an additional justification for Landau's choice of the form of thermodynamic potential for an uniaxial ferromagnet [17].

Section 3 is devoted to the proof of the classification theorems of § 1. The whole problem is translated into the language of symplectic geometry and Lagrangian singularities. The main advantage of this formulation is that we may express the equivalence between densities of internal energies in terms of a group of transformations (Lagrangian equivalences) of the phase space. Then an equivariant modification of Arnold's [2] approach to classification of nonsymmetric Lagrangian mappings can be applied.

In relation to some other works in this area it is worth stressing the following aspects:

(i) In the nonsymmetric case the main step in the classification of Lagrangian singularities was to show its equivalence to the previously known classifications of singularities of functions and their versal unfoldings (elementary catastrophes) [2], [10], [33], [3]. In the symmetric case a similar procedure leads to a classification of unfoldings of symmetric functions with additional symmetry in the unfolding parameters (see [13]). Thus, unlike in the nonsymmetric case, the classifications like that of Wasserman [32], which do not assume any symmetry in the unfolding parameters, cannot be utilized for the classification of symmetric Lagrangian singularities.

(ii) In the space of Lagrangian mappings the equivariant ones form a subset of infinite co-dimension. For this reason the results of the nonsymmetric classification [2], [3] cannot be used directly in the symmetric case. Using the results of Slodowy [25], one may show [12] that some of nonsymmetric versal unfoldings are also versal unfoldings in the symmetric case. But, in general, this does not lead to a complete classification of symmetric unfoldings and still leaves the question of structural stability open [13], [16].

1. Classification of thermodynamic potentials. On the phenomenological level the equilibrium state of an uniaxial ferromagnet with x-axis as the direction of easy magnetization is described by four extensive parameters $(\tilde{S}, \tilde{M}_x, \tilde{M}_y, N) = (\text{entropy}, \text{magnetizations in the } x\text{- and } y\text{-directions, mole number})$ and four conjugated intensive parameters or thermodynamic forces $(T, H_x, H_y, \mu) = (y_1, \cdots, y_4)$ (temperature magnetic fields in the x- and y-direction, chemical potential). All thermodynamic information concerning the ferromagnet could be extracted from a (homogeneous of the first

order) function $U = U(\tilde{S}, \tilde{M}_x, \tilde{M}_y, N)$ describing the internal energy of the system. In the region of interest the mole number N is positive, so the internal energy could be written in the form

$$U = N \cdot u\left(\frac{\tilde{S}}{N}, \frac{\tilde{M}_x}{N}, \frac{\tilde{M}_y}{N}\right) = N \cdot u(x_1, x_2, x_3)$$

where u is called mole internal energy and x_1, x_2, x_3 (denoted also by S, M_x, M_y) are usually called mole entropy and respective mole magnetizations. Because of the limitations of the mathematical formalism (see § 3) we are forced to assume that mole internal energy u is a C^∞ function. The function u satisfies the following symmetry condition

$$u(x_1, x_2, x_3) = u(x_1, \varepsilon x_2, \delta x_3) \tag{1.1}$$

for any $\varepsilon, \delta = \pm 1$ and $(x_1, x_2, x_3) \in \mathbb{R}^3$. This condition expresses the symmetry properties of the uniaxial ferromagnet with respect to the choice of the sign of the x- or y-axis [13]. Thus the function u is an invariant of the group $G := Z_2 \oplus Z_2 \simeq \{(\varepsilon, \delta);\ \varepsilon, \delta = \pm 1\}$ of transformations $\mathbb{R}^3$, (ε, δ): $(x_1, x_2, x_3) \to (x_1, \varepsilon x_2, \delta x_3)$. We denote by $C^\infty_G(3)$ the set of all such functions.

In equilibrium the following equations of state are satisfied [5], [27]:

$$y_1 = \frac{\partial U}{\partial S} = \frac{\partial u}{\partial x_1}, \tag{1.2}$$

$$y_2 = \frac{\partial U}{\partial \tilde{M}_x} = \frac{\partial u}{\partial x_2}, \tag{1.3}$$

$$y_3 = \frac{\partial U}{\partial \tilde{M}_y} = \frac{\partial u}{\partial x_3}, \tag{1.4}$$

$$y_4 = \frac{\partial U}{\partial N} = u - \sum_{i=1}^{3} x_i \frac{\partial u}{\partial x_i}, \tag{1.5}$$

and the condition of *internal stability* also must be fulfilled [5], [27]:

(1.6) The matrix $d^2u = \left[\frac{\partial^2 u}{\partial x_i \, \partial x_j}\right]$ is positive definite.

For a mole internal energy u denote by ∇u the transformation

$$x \in \mathbb{R}^3 \to \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right) \in \mathbb{R}^3.$$

Note that ∇u is G-equivariant, i.e., $\nabla u \circ g = g \circ \nabla u$ for any $g \in G$. Let us denote by Γ_u the mole Gibbs potential:

$$\Gamma_u := u - \sum x_i \frac{\partial u}{\partial x_i} \tag{1.7}$$

for $x = (x_i) \in \mathbb{R}^3$.

DEFINITION 1.1. We say two mole internal energies, u, $\tilde{u} \in C^\infty_G(3)$, are *equivalent* if there exists a pair, say ϕ and ψ, of G-equivariant diffeomorphisms of $\mathbb{R}^3$ such that the following conditions are satisfied:

(i) $\phi \circ \nabla u = \nabla \tilde{u} \circ \psi$,

(ii) if $\nabla u(x) = \nabla u(x')$ then $\Gamma_u(x) - \Gamma_u(x') = \Gamma_{\tilde{u}}(\psi x) - \Gamma_{\tilde{u}}(\psi x')$,

(iii) the matrix d^2u is positive definite at x if and only if $d^2\tilde{u}$ is positive definite at $\psi(x)$.

We say that u and $\tilde{u}$ are *weakly equivalent* if (i), (ii) and the weaker version of (iii) with "semidefinite" (i.e., $\det(d^2u)=0$) substituting "positive definite" holds true.

Similarly we define a *local weak equivalence* of mole internal energies or, in other words, of their germs, (u, x_0) and $(\tilde{u}, \tilde{x}_0)$. In this case we assume the $\tilde{x}_0=\psi(x_0)$ and that the above requirements are satisfied in an open neighborhood of x_0. In our considerations below we always assume x_0 to be a G-invariant point, i.e., of the forms $x_0=(x_{01}, 0, 0)$.

Let us introduce a transformation,

$$z:\mathbb{R}^3\to\mathbb{R}^3,\ (x_1, x_2, x_3)\to(x_1, x_2^2, x_3^2). \tag{1.8}$$

Obviously $F\circ z\in C^\infty_G(3)$ if $F\in C^\infty(3)$. From Schwarz's Theorem [24] it results that any functions belonging to C^∞_G is of that form.

It is natural to introduce topology of uniform convergence or C^∞ Whitney's topology on $C^\infty_G(3)$. All our results presented below could be easily modified to be valid with respect to any of these topologies. But to avoid unnecessary complications in proofs we use neither of them. Instead we follow a simple definition characterizing magnitude of perturbation of a function in $C^\infty_G(3)$.

DEFINITION 1.2. Let $\delta:\mathbb{R}^3\to\mathbb{R}^+$ be a continuous, positive valued function and $u\in C^\infty_G(3)$. A function $u'\in C^\infty_G(3)$ is called *δ-close* to u (or just *close*) if there exist functions $F, F'\in C^\infty(3)$ such that $u=F\circ z$ and $u'=F'\circ z$ and the difference $(F-F')$ and all its partial derivatives up to order 3 are bounded by δ on $\mathbb{R}^3$.

Now we are prepared to introduce a fundamental notion of local structural stability.

DEFINITION 1.3. A germ of mole internal energy, (u, x_0), $u\in C^\infty_G(3)$, is said to be *structurally stable* (*weakly structurally stable*) if for any $\tilde{u}\in C^\infty_G(3)$ sufficiently close to u there exists a point $\tilde{x}_0$ such that $(\tilde{u}, \tilde{x}_0)$ is equivalent (weakly equivalent, respectively) to (u, x_0).

The following three theorems summarize the results of the classification of *structurally stable* and *typical* local forms of the mole internal energy.

First, given a G-invariant point $x_0\in\mathbb{R}^3$ we introduce three disjoint subsets $A_1(x_0)$, $A_2(x_0)$ and $A_3(x_0)$ in the space of the germs (u, x_0) mole internal energies $u\in C^\infty_G(3)$. Each of these sets is defined by one of the following conditions:

(A_1) $F_{,11}F_{,2}F_{,3}\neq 0$ at x_0, or

(A_2) $F_{,11}=0$ and $F_{,111}F_{,2}F_{,3}\neq 0$ at x_0, or

(A_3) $F_{,i}=0$ and $F_{,111}F_{,j}F_{,1i}(F^2_{,1i}-F_{,11}F_{,ii})\neq 0$ at x_0 for $i=2$, $j=3$ or vice versa,

where $F\in C^\infty(3)$ is a function such that $u=F\circ z$ with z given by (8) and $F_{,j}$, $F_{,ij}$, etc., denote the partial derivatives $\partial F/\partial x_j$, $\partial^2F/\partial x_i\partial x_j$, etc.

THEOREM 1.1. *A germ (u, x_0) of mole internal energy $u\in C^\infty_G(3)$ at a G-invariant point is structurally stable if and only if it satisfies one of the above conditions* (A_1)-(A_3).

THEOREM 1.2. *Any structurally stable germ of mole internal energy from $C^\infty_G(3)$ at a G-invariant point is equivalent to a germ $(u, 0)$ of one of the following normal forms*:

$$(A_1)\qquad u(x)=\pm(x_1^2+x_2^2+x_3^2),$$

$$(A_2)\qquad u(x)=x_1^3\pm(x_2^2+x_3^2)),$$

$$(A_3)\qquad u(x)=\pm x_2^4\pm((x_1+x_2^2)^2+x_3^2).$$

THEOREM 1.3. *Let $\delta:\mathbb{R}^3\to\mathbb{R}^+$ be a continuous positive valued function. For any $u\in C^\infty_G(3)$ there exists $\tilde{u}\in C^\infty_G(3)$ which is δ-close to u and such that any germ $(\tilde{u}, x_0)$ at a G-invariant point $x_0\in\mathbb{R}^3$ is structurally stable.*

In other words, any germs of a typical function (or mole internal energy) from $C^{\infty}_{G}(3)$ at a G-invariant point is equivalent to one of the normal forms of Theorem 1.2.

The results of this section can be summarized as follows. We consider internal energies describing a thermodynamic system with symmetries of the uniaxial ferromagnet. We treat all energies giving the "same thermodynamics" (in the sense of Definition 1.1) as equivalent and then we consider those energies which are insensitive to small variations (structurally stable in the sense of Definition 1.3). We are then able to show that near fixed points of the symmetry ($x_2 = x_3 = 0$—axis) such energies can take only three distinct forms, A_1, A_2 and A_3 (given, up to an equivalence, by Theorem 1.2). Moreover, any internal energy structurally *unstable* at a fixed point can be reduced, using an arbitrary small perturbation, to a form belonging to one of the structurally stable classes (A_1)-(A_3).

Remark 1.1. For points of $\mathbb{R}^3$ which are not G-invariant the list of stable forms is longer than that given in Theorem 1.2 [16]. Thus each stable germ $(\tilde{u}, \tilde{x})$ mole of internal energy $u \in C^{\infty}_{G}(3)$ at $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, 0)$, or $x = (\tilde{x}_1, 0, \tilde{x}_3)$ where $\tilde{x} \neq 0$, $\tilde{x}_2 \neq 0$ is equivalent to the germs of u at $(0, 1, 0)$ of one of the following four normal forms:

$$(A_1) \qquad u(x) = \pm(x_1^2 + (x_2 - 1)^2 + x_3^2),$$

$$(A_2) \qquad u(x) = x_1^3 \pm ((x_2 - 1)^2 + x_3^2),$$

$$(A_3) \qquad u(x) = \pm(x_2 - 1)^4 \pm (x_1 + (x_2 - 1)^2)^2 + x_3^2,$$

$$(D_4) \qquad u(x) = \pm x_1^3 \pm x_1(x_2 - 1)^2 + (x_3 + x_1^2)^2.$$

Similarly, the following five germs at $(0, 1, 1)$ are normal forms of stable mole internal energy at points $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, $\tilde{x}_2 \tilde{x}_3 \neq 0$:

$$(A_1) \qquad u(x) = \pm(x_1^2 + (x_2 - 1)^2 + (x_3 - 1)^2),$$

$$(A_2) \qquad u(x) = \pm(x_1^3 \pm ((x_2 - 1)^2 + (x_3 - 1)^2)),$$

$$(A_3) \qquad u(x) = \pm(x_2 - 1)^4 \pm (x_1 + (x_2 - 1)^2)^2 + (x_3 - 1)^2,$$

$$(A_4) \qquad u(x) = \pm x_1^5 + (x_1^3 + x_2 - 1)^2 + (x_1^2 + x_2 - 1)^2,$$

$$(D_4) \qquad u(x) = \pm x_1^3 \pm x_1(x_2 - 1)^2 + (x_1^2 + x_3 - 1)^2.$$

Below we give an example showing how small "higher order" perturbation can change the properties of a structurally unstable mole internal energy dramatically.

EXAMPLE 1.1. Consider a family of mole internal energies:

$$u_{\varepsilon\delta}(s, M) = M^2 s^3 + \varepsilon M^6 + \delta s^7.$$

First note that for $\varepsilon = \delta = 0$ there is no stable states of the system (i.e., $d^2 u_{00}$ is nowhere positive definite). Now consider $u_{\varepsilon 0}$ for $\varepsilon > 0$. Figures 2(a), (b) show distribution of eigenvalues of $d^2 u_{\varepsilon 0}$ near $M = 0$, $s = 0$ and the image of the transformation $\nabla u_{\varepsilon 0}$ near $H = 0$, $t = 0$. Now for every point (H, t) belonging to the shaded regions in Fig. 2(b) there exists exactly one stable state of the system. Note that for smaller ε the regions become bigger. In this sense, the smaller the perturbation, (i.e., ε) the more dramatic the change of the properties of $u_{\varepsilon 0}$ in comparison to u_{00}.

For $u_{\varepsilon\delta}$, ε, $\delta > 0$ we observe even bigger changes. The internal stability regions are shown as shaded areas in Fig. 2(c). In Fig. 2(d) the image of $\nabla u_{\varepsilon\delta}$ is sketched. We observe that for any (H, t), $t > 0$ there exists a stable state of a system and that for every (H, t) in the cross-hatched regions even two such states are present.

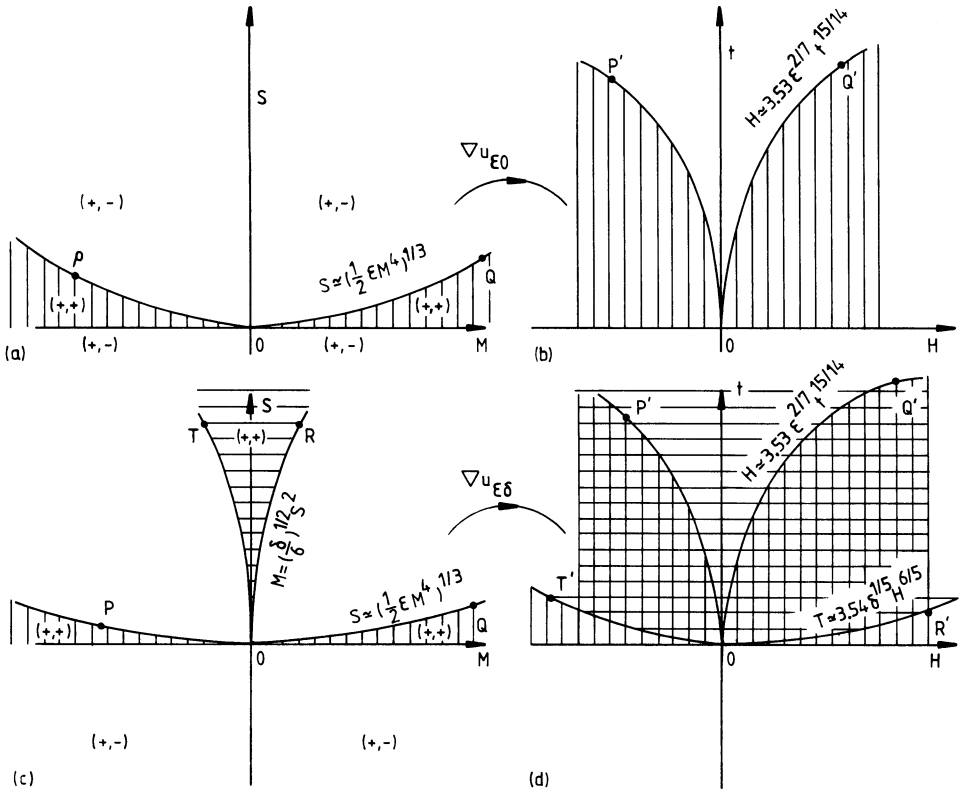

FIG. 2. *Distributions of eigenvalues of* $d^2u_{\varepsilon\delta}$ *near* $M=0$, $s=0$ *and the image of internal stability regions near* $H=0$, $t=0$ *given by* $\nabla u_{\varepsilon\delta}$ *for different values of* ε *and* δ: *in Figs.* 2(a) *and* 2(b) *for* $\varepsilon>0$ *and* $\delta=0$ *and in Figs.* 2(c) *and* 2(d) *for* ε, $\delta>0$, *respectively.*

2. Application to uniaxial ferromagnet. In experiments with a ferromagnet the parameters that are controlled are the three intensive ones: the temperature $y_1=T$, both magnetic fields $y_2=H_x$, $y_3=H_y$ and one extensive parameter, the mole number N (we consider here a small single domain crystal). The values of the remaining extensive parameters, S, M_x, H_y, are established by the response of the system and they could be derived with the help of the principle of minimum energy [5].

Let us now consider the equivalence described in Definition 1.1. *Condition* (i) assures the following natural requirement is satisfied: two states differing by the values of the extensive parameters only always correspond to a pair of states with the same property. In the presence of sufficient fluctuations for a given control point (T, H_x, H_y) the system chooses the state with the lowest chemical potential μ or, in other words, the lowest density of Gibbs energy Γ_u [5], [19], [28]. In particular, necessary condition for phase transition of the first order is equality of the Gibbs energies in two phases for a given (T, H_x, H_y) point. In this respect the differences of chemical potentials between different phases of the system over the same point in the control space are of major interest. *Condition* (ii) of Definition 1.1 assures that these differences are preserved by the structural equivalences. *Condition* (iii) assures the preservation of internal stability regions. Finally, the equivariance of ψ and ϕ preserves the symmetry of the system.

In order to describe the explicit form of the internal energy density, given by

Theorems 1.1–1.3, it is convenient to introduce the following notation. We denote by $\mathfrak{m}^r$, $r=1, 2, \cdots$, the set of all C^∞-functions $\alpha(x_1, x_2, x_3)$ vanishing at $0\in\mathbb{R}^3$ together with all their partial derivatives of order $1, 2, \cdots, r-1$. It is customary to denote $\mathfrak{m}^1$ by $\mathfrak{m}$.

From Theorem 1.1 we immediately obtain the next corollary.

COROLLARY 2.1. *The form of the structurally stable G-invariant mole internal energy* $u(S, M_x, M_y)$ *in a neighborhood of the point* $(S_0, 0, 0, T_0, 0, 0)$ *is as follows*:

$$\begin{aligned} u(S, M_x, M_y) &= T_0(S-S_0)+a_2M_x^2+a_3M_y^2+a_{11}(S-S_0)^2 \\ &\quad +a_{12}(S-S_0)M_x^2+a_{22}M_x^4+a_{13}(S-S_0)M_y^2 \\ &\quad +a_{23}M_x^2M_y^2+a_{33}M_y^4+\alpha(S-S_0, M_x^2, M_y^2)+\text{const}, \end{aligned} \tag{2.1}$$

where $\alpha\in\mathfrak{m}^3$ *and the constants* a_i, a_{ij} *satisfy one of the following conditions*:

(A_1) $\quad a_2a_3a_{11}\neq 0,$

(A_2) $\quad a_{11}=0 \quad \textit{and} \quad a_2a_3\,\partial^3\alpha/\partial S^3\neq 0 \quad \textit{at } (S_0, 0, 0),$

$(A_{3'})$ $\quad a_2=0 \quad \textit{and} \quad a_3a_{11}a_{12}(a_{12}^2-4a_{11}a_{22})\neq 0,$

$(A_{3''})$ $\quad a_3=0 \quad \textit{and} \quad a_2a_{11}a_{13}(a_{13}^2-4a_{11}a_{33})\neq 0.$

To study the bifurcation set and other properties of mole of internal energy $u(s, M_x, M_y)$ it is convenient to consider the associated equations of state as the (Lagrangian [2]) mapping ∇u, which has in our case the following form:

$$\nabla u\colon \mathbb{R}^3\to\mathbb{R}^3,\ (S, M_x, M_y)\to\left(\frac{\partial u}{\partial S}, \frac{\partial u}{\partial M_x}, \frac{\partial u}{\partial M_y}\right).$$

The boundary of the stability region of u [27] or the set of critical points of ∇u is given by the condition

$$|d^2u| := \det\left[\frac{\partial^2 u}{\partial^2(S, M_x, M_y)}\right]=\frac{\partial(\nabla u)}{\partial(S, M_x, M_y)}=0. \tag{2.2}$$

The image of the above set given by the transformation ∇u will be called the *bifurcation set* of u.

We recall now experimental data in order to characterize the bifurcation set of the uniaxial ferromagnet and then obtain a structurally stable local form of its internal energy that would reproduce the experimentally observed behavior. The experimental bifurcation set (or, the critical surface, according to Thomas [30]) is scheduled in Fig. 3 (see [18], [29]). This surface with cusp-like edges divides the space of intensive parameters into two regions marked by "1^x" and "2^x". The number ("1" or "2") indicates the number of different stable states of the system that can exist for a given point (T, H_x, H_y) in the respective region.

For u of the form (A_1), the bifurcation set is empty while in the case of (A_2) it forms a regular surface. Hence, neither of these two forms is suitable as a mole internal energy in the Curie region.

Now let us consider u of the form $(A_{3'})$. First note that the Hessian d^2u at $(S_0, 0, 0)$ is in this case diagonal with 0, a_3 and a_{11} on the main diagonal. Thus the necessary condition for the existence of stable states in the neighborhood of $(T_0, 0, 0)$ is

$$a_{11}>0, \qquad a_3>0. \tag{2.3}$$

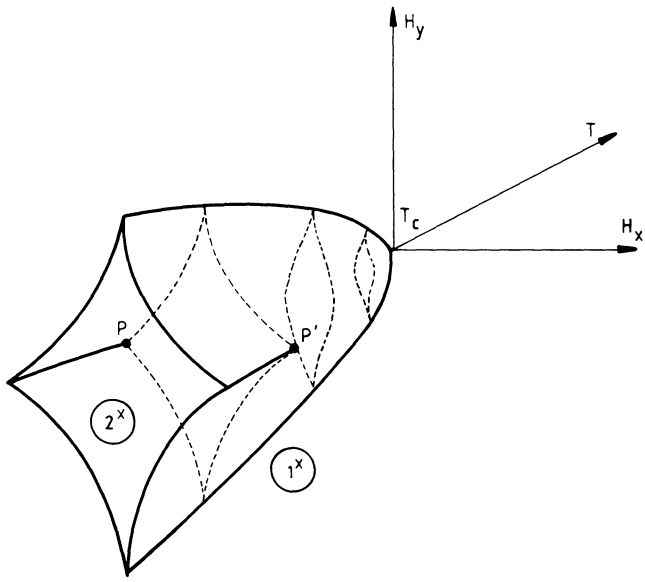

FIG. 3. *The experimental bifurcation set (critical surface) for an uniaxial ferromagnet. Here T_C is the Curie temperature.*

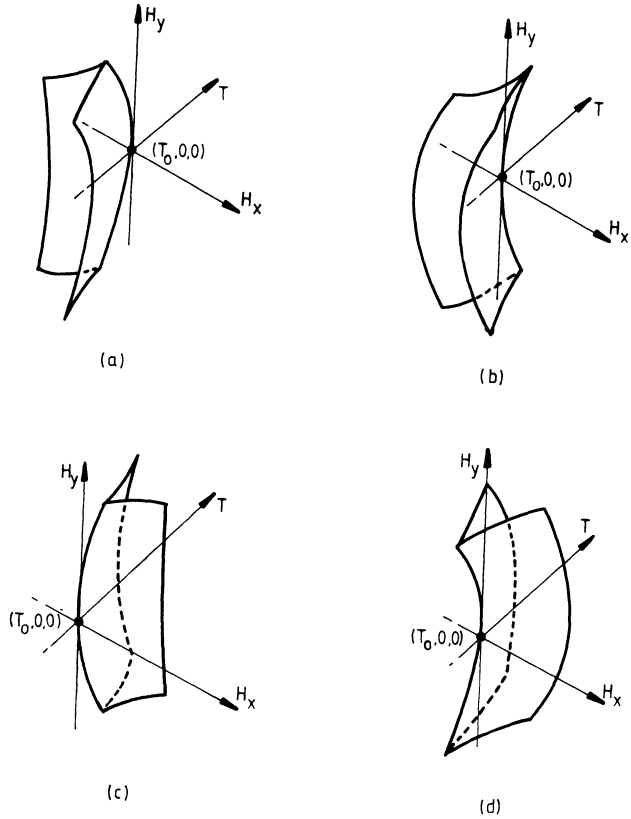

FIG. 4. *Bifurcation sets for four different cases of u of the form* (A_3): (a) $\operatorname{sgn}(a_{12}\delta) = -1$, $\operatorname{sgn}(a_{12}\Delta) = -1$; (b) $\operatorname{sgn}(a_{12}\delta) = -1$, $\operatorname{sgn}(a_{12}\Delta) = +1$; (c) $\operatorname{sgn}(a_{12}\delta) = 1$, $\operatorname{sgn}(a_{12}\Delta) = 1$; (d) $\operatorname{sgn}(a_{12}\delta) = 1$, $\operatorname{sgn}(a_{12}\Delta) = -1$.

In Fig. 4, we sketch the bifurcation sets for the mole internal energy ($A_{3'}$) satisfying conditions (2.3) and for four different combinations of sgn $(a_{12}\delta)$ and sgn $(a_{12}\Delta)$ where

$$\delta := a_{12}^2 - 4a_{11}a_{22}, \tag{2.4}$$

$$\Delta := a_{13}a_{12} - 2a_{11}a_{23}. \tag{2.5}$$

In the remaining case ($A_{3''}$) the bifurcation set is as in Fig. 4(a), (b) but with axis H_x and H_y interchanged. By inspection we find that only the bifurcation set shown on Fig. 4(a) corresponds to the experimental one. Thus u must be of the form ($A_{3'}$) with the conditions (2.3) and

$$T_0 = T_c, \quad a_{12}\delta < 0 \quad \text{and} \quad a_{12}\Delta < 0. \tag{2.6}$$

First we observe that the boundary of the stability region ($|d^2u| = 0$) divides the space of extensive parameters into two domains "S_c^+" and "S_c^-", with "S_c^+ containing the semi-axis $M_y = M_x = 0$, $S > S_c$ (see Fig. 5). If $a_{12}\delta < 0$, for each point (T, H_x, H_y) from "2^x" there exist two points in "S_c^+" and one point in "S_c^-" which are translated by ∇u onto (T, H_x, H_y). Similarly ∇u transforms exactly one point belonging to the region "S_c^+" onto each point in "1^x" (note how the circle in the plane $M_y = 0$ in Fig. 5(a) is mapped by ∇u). Thus, to have two stable states of the systems for each (T, H_x, H_y) belonging to the regions "2^x" and one such state for each (T, H_x, H_y) "1^x" the matrix d^2u must have all positive eigenvalues in the region "S_c^+" near $(S_c, 0, 0)$. Since condition (2.3) assures that at least two eigenvalues are positive (near $(S_c, 0, 0)$), all three eigenvalues will be positive if

$$\begin{aligned} |d^2u| = 8a_{11}[&a_{12}a_3(S-S_c) + a_{12}a_{13}(S-S_c)^2 + 6a_3a_{22}M_x^2 + a_{23}^2M_y^2] \\ &+ a_3a_{12}^2M_x^2 + O^3(S-S_c, M_x, M_y) \end{aligned} \tag{2.7}$$

is positive in "S_c^+". Using (2.3) we see immediately that $|d^2u| > 0$ in "S_c^+" if

$$a_{12} > 0. \tag{2.8}$$

The following corollary summarizes the results of the above discussion.

COROLLARY 2.2. *The structurally stable mole internal energy $u(S, M_x, M_y)$ for uniaxial ferromagnet near the Curie point, with x-easy axis, must be of the form* ($A_{3'}$) *of Corollary* 2.1 (*with T_0 and S_0 being the temperature and entropy at Curie point*) *and must satisfy the following conditions*:

$$a_{11} > 0, \quad a_{12} > 0, \quad a_3 > 0, \tag{2.9}$$

$$\delta = a_{12}^2 - 4a_{11}a_{22} < 0, \tag{2.10}$$

$$\Delta = a_{13}a_{12} - 2a_{11}a_{23} < 0. \tag{2.11}$$

2.1. Spontaneous magnetization $H = 0$. If we set $s = S - S_c$, $t = T - T_c$, with S_c and T_c being entropy and temperature at the Curie point, respectively, then the fundamental equation of a uniaxial ferromagnet takes the form:

$$\begin{aligned} u = u(s, M_x, M_y) = &a_{11}s^2 + a_3M_y^2 + a_{12}sM_x^2 + a_{13}sM_y^2 + a_{22}M_x^4 \\ &+ a_{23}M_x^2M_y^2 + a_{33}M_y^4 + \alpha(s, M_x^2, M_y^2) \end{aligned} \tag{2.12}$$

with $\alpha \in \mathfrak{m}^3$ and a_{ij} satisfying conditions (2.9)-(2.11).

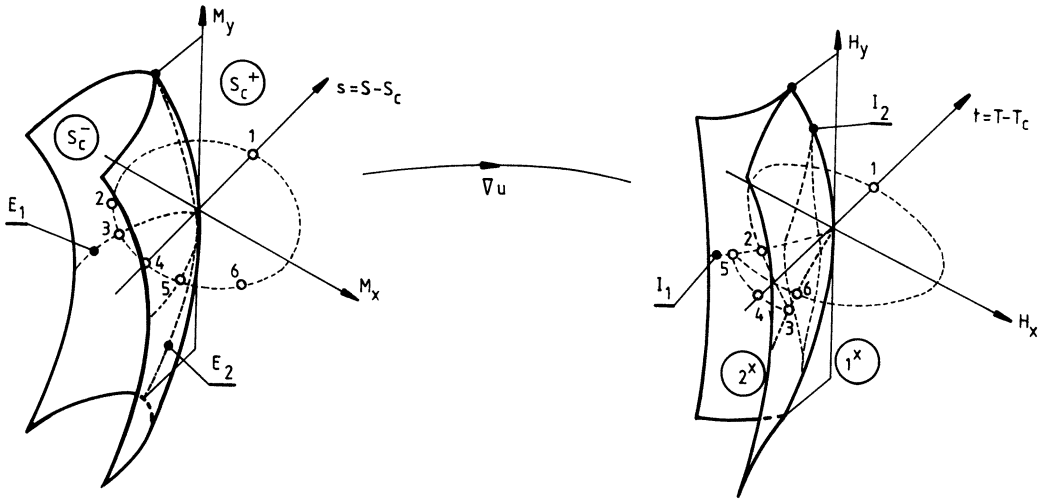

FIG. 5. *Transformation* ∇u *for* u *of the form* $(A_{3'})$ *and such that* $a_{12}\delta < 0$ *and* $a_{12}\Delta > 0$. *Up to the terms of higher order the equations of curves* E_i, I_i *are as follows*:

$$I_1: \quad H_y = 0, \quad t = \text{const} \cdot \operatorname{sgn}(\delta a_{12}) |H_x|^{2/3} \text{ with const} > 0,$$

$$I_2: \quad H_x = 0, \quad t = \frac{\Delta}{(2a_3)^2 a_{12}} H_y^2,$$

$$E_1: \quad M_y = 0, \quad s = \left(\frac{\delta}{a_{11}a_{12}} - 2\right) M_x^2,$$

$$E_2: \quad M_x = 0, \quad s = -\frac{a_{23}}{a_{12}} M_y^2.$$

The equations of state are

$$t = \frac{\partial u}{\partial s} = 2a_{11}s + a_{12}M_x^2 + a_{13}M_y^2 + \alpha_{,1}(s, M_x^2, M_y^2),$$

$$H_x = \frac{\partial u}{\partial M_x} = 2a_{12}sM_x + 4a_{22}M_x^3 + 2a_{23}M_xM_y^2 + 2M_x\alpha_{,2}(s, M_x^2, M_y^2), \tag{2.13}$$

$$H_y = \frac{\partial u}{\partial M_y} = 2a_3M_y + 2a_{13}sM_y + 2a_{23}M_x^2M_y + 4a_{33}M_y^3 + 2M_y\alpha_{,3}(s, M_x^2, M_y^2).$$

Setting $H_x = H_y = 0$ and eliminating s and M_y from (2.13), we obtain the following relation between temperature and spontaneous magnetization,

$$t = \frac{\delta}{a_{12}} M_x^2 + o(M_x^4).$$

Note that $M_y = 0$ if $H_y = 0$ (note that $a_3 \neq 0$ in the third equation of (2.13)) and $M_x \neq 0$ only for $t < 0$ (as $\delta/a_{12} < 0$).

2.2. Magnetization in the easy axis direction. Now we consider $H_x \neq 0$ and $H_y = 0$. We have from (2.13):

$$t = 2a_{11}s + a_{12}M_x^2 + \alpha_{,1}(s, M_x^2, 0), \qquad H_x = 2a_{12}M_xs + 4a_{22}M_x^3 + 2M_x\alpha_{,2}(s, M_x^2, 0).$$

The projection of this surface of equilibrium states onto coordinates (H_x, t, M_x) is presented in Fig. 6. The hysteresis region is bounded by a cusp curve and at $H_x = 0$, $t = 0$ we observe a continuous phase transition [8], [18], [19], [27], [30].

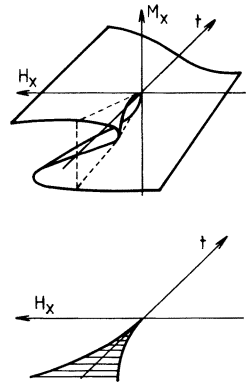

FIG. 6. *Equilibrium states for* $H_y = 0$.

2.3. The line of points of continuous phase transition. This line coincides with the line of the cusp points (I_2 on Fig. 5(b)) and is given by the following equations:

$$t = 2a_{11}s + a_{12}M_y^2 + \alpha_{,1}(s, 0, M_y^2),$$
$$0 = a_{12}s + a_{23}M_y^2 + \alpha_{,2}(s, 0, M_y^2),$$
$$H_y = 2a_3M_y + 2a_{13}sM_y + 4a_{33}M_y^3 + 2M_y\alpha_{,3}(s, 0, M_y^2).$$

After some simple calculations we obtain:

$$t = \frac{\Delta}{(2a_3)^2 a_{12}} H_y^2 + o(H_y^4).$$

Hence, in the case when $\Delta < 0$, $a_{12} > 0$ we obtain the line of the shape qualitatively compatible with experiment [30], [31].

We would like to add (without proof) the following.

Remark 2.1. The structurally stable local form of mole internal energy reproducing the experimentally observed behavior near points P and P' in Fig. 3 is (D_4) (see Remark 1.1).

3. Classification of equivariant Lagrangian singularities. In § 1 we have introduced equivalent mole internal energies. But for an effective classification we need the equivalence itself: a type of operator which for a given internal energy would create another one equivalent to it. This would enable the process of subsequent simplifications leading to normal forms. In this section these goals are achieved by a simple geometrization which establishes:

(i) The correspondence between mole internal energy or equations of state (1.2)-(1.3) and the elements of an (open) subset of the space of proper Lagrangian submanifolds of the phase space.

(ii) A correspondence between weak equivalences of mole internal energy and elements of the group of equivariant Lagrangian equivalences of the phase space.

3.1. Basic definitions. All objects in this Section are assumed to be C^∞. By $C^\infty(n)$ we denote the ring of smooth functions $\mathbb{R}^n \to \mathbb{R}$, by $\mathscr{E}(n)$ the ring of germs of functions from $C^\infty(n)$ at $0 \in \mathbb{R}^n$ and by $\mathfrak{m}^k(n)$ the ideal in $\mathscr{E}(n)$ consisting of all germs vanishing at 0 together with all their partial derivatives up to the order $k-1$. To simplify the notation we denote functions from $C^\infty(n)$ and their germs at 0 by the same symbols (although for germs at $x \neq 0$ we use the notation (F, x)). The ideal in $\mathscr{E}(n)$ generated

by $F_1, F_2, \cdots, F_m \in \mathscr{E}(n)$ is denoted by $\langle F_1, F_2, \cdots, F_m \rangle$. Partial derivatives $\partial F/\partial x_i$, $\partial^2 F/\partial x_i\, \partial x_j$, etc., of function F are denoted by $F_{,i}$, $F_{,ij}$, etc.; their values at 0 by f_i, f_{ij}, etc.

Let $(x_1, x_2, x_3, y_1, y_2, y_3)$ be a fixed coordinate system in $\mathbb{R}^6$. Let $\omega^2 := \sum dx_i \wedge dy_i$ be a symplectic form. With respect to these coordinates, $\pi(x, y) := y$ is the second projection. The group $G = Z_2 \oplus Z_2$ operates on $\mathbb{R}^6$ as follows $(\varepsilon, \delta): (x, y) \to (x_1, \varepsilon x_2, \delta x_3, y_1, \varepsilon y_2, \delta y_3)$.

Any 3-dimensional C^∞ submanifolds $L \subset \mathbb{R}^6$ which is Lagrangian and G-invariant (i.e., the pull-back of ω^2 vanishes on L and $g(L) = L$ for all $g \in G$) is called *LG-manifold.* Examples of *LG*-manifolds are submanifolds of $\mathbb{R}^6$ of the form

$$L_u = \left\{ (x, y): y = \frac{\partial u}{\partial x} \right\}, \tag{3.1}$$

with $u \in C^\infty_G(3)$, i.e., the submanifold given by equations of state (1.2)-(1.4). The function $u(x)$ is called the *generating function of L.*

Since we are interested in local properties of *LG*-manifolds near G-invariant points we consider germs (L, m), where L is a *LG*-manifold and m is a G-invariant point of the form $(x_1, 0, 0, y_1, 0, 0)$.

By *LG-equivalence* we mean a symplectic diffeomorphism $\Phi: \mathbb{R}^6 \to \mathbb{R}^6$ preserving fibration π and commuting with the action of G on $\mathbb{R}^6$, i.e., a mapping $\Phi: \mathbb{R}^6 \to \mathbb{R}^6$ such that $\Phi * \omega^2 = \omega^2$, $g \circ \Phi = \Phi \circ g$ for any $g \in G$ and $\pi \circ \Phi(\pi^{-1}\{y\})$ is a "singleton". Note that the group of *LG*-equivalences acts transitively on the set of G-invariant points of $\mathbb{R}^6$.

Two *LG*-manifolds (or germs of *LG*-manifolds) are called *LG-equivalent* if there exists an *LG*-equivalence which carries one onto the other.

LEMMA 3.1. *Any germ (L, m) of an LG-manifold L at a G-invariant point m is LG-equivalent to a germ $(L_1, 0)$ of LG-manifold L_1 with a generating function $u \in C^\infty_G(3)$.*

Proof. Applying, if necessary, the translation $(x, y) \to (x - x_0, y - y_0)$ and a transformation $(x, y) \to (x - \lambda y, y)$ for suitable $\lambda \in \mathbb{R}$ [2] we reduce the proof to the case that L has a generating germ $u(x)$. As L is G-symmetric the equations

$$u_{,2}(x_1, x_2, x_3) = -u_{,2}(x_1, -x_2, x_3), \qquad u_{,3}(x_1, x_2, x_3) = -u_{,3}(x_1, x_2, -x_3)$$

hold. Now by integration we get $u(x_1, x_2, x_3) = u(x_1, \varepsilon x_2, \delta x_3)$ for $\varepsilon, \delta = \mp 1$. Q.E.D.

3.2. *LG*-equivalences and equivalences of germs. Following Zakalyukin [33] any Lagrangian equivalence Φ could be identified with a pair (ϕ, γ), where $\phi: \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth diffeomorphism and $\gamma \in C^\infty(3)$, in the following way: if $\Phi: (x, y) \to (X, Y)$, then $Y = (\phi y)$, $X = (\phi'^*)^{-1}(x + \partial\gamma/\partial y)$. In the equivariant case $\Phi \simeq (\phi, \gamma)$ is an *LG*-equivalence if and only if ϕ is an equivariant diffeomorphism and $\gamma \in C^\infty_G(3)$. Sufficiency can be checked directly. For necessity note that $\phi = \pi \circ \Phi$, and since $\pi: \mathbb{R}^6 \to \mathbb{R}^3$ is an equivariant transformation, ϕ must be also. Now consider the composition $\Psi \circ \Phi$ where Ψ is an *LG*-equivalence determined by $(\phi^{-1}, 0)$. Obviously $\Psi \circ \Phi: (x, y) \to (x + \partial\gamma/\partial y, y)$ is an *LG*-equivalence and $\pi_x \circ \Psi \circ \Phi \circ i_y$, where $\pi_x: (x, y) \to x$ and $i_y: y \to (0, y)$, is an equivariant transformation $\mathbb{R}^3 \to \mathbb{R}^3$ of the form $y \to \partial\gamma/\partial y$. Thus similarly as in the proof of Lemma 3.1 we establish that $\gamma \in C^\infty_G(3)$.

For typical Lagrangian manifold L_u of the forms (3.1) the covering $\pi \mid L_u: L_u \to \mathbb{R}^3$ is finite. Thus there exists a family of functions (Gibbs potentials) $\{\Gamma_\alpha\}$, where each Γ_α is a smooth function on an open subset $V_\alpha \subset \mathbb{R}^3$, such that

$$L = \text{closure} \bigcup_\alpha \left\{ \left(\frac{-\partial \Gamma_\alpha}{\partial y}, y \right) \right\}.$$

The functions $\{\Gamma_\alpha\}$ are routinely obtained in thermodynamics by Legendre transformations of internal energy u, i.e., by elimination of x-variables from the density of the Gibbs potential Γ_u given by (1.7), by means of equations (1.2)-(1.4). Using the above explicit characterizations of LG-equivalence Φ by the pair (ϕ, γ) it is easy to find that $\tilde{L} := \Phi(L_u)$ has a family of Gibbs potentials $\{\tilde{\Gamma}_\alpha\}$ of the form [23] $\tilde{\Gamma}_\alpha = (\Gamma_\alpha - \gamma) \circ \phi^{-1}$ each defined on $\tilde{V}_\alpha = \phi(V_\alpha)$.

The condition $|d^2u| = 0$ at x in terms of the geometry means that projection $\pi | L_u : L_u \to \mathbb{R}^3$ has a singular point at $(x, \partial u/\partial x)$. Since $\phi \circ \pi | L_u = \pi \circ \Phi | L_u$, for LG-equivalence $\Phi = (\phi, \gamma)$ the diffeomorphism Φ maps singular points of $\pi | L_u$ onto singular points of $\pi | \Phi(L_u)$. Thus we obtain the next proposition.

PROPOSITION 3.1. *Two mole internal energies from $C^\infty_G(3)$ are weakly equivalent if and only if corresponding Lagrangian submanifolds of the form* (3.1) *are LG-equivalent.*

Remark. The above considerations prove Proposition 3.1 only in the case when the covering $\pi | L_u : L_u \to \mathbb{R}^3$ is finite, but the proof can be generalized.

3.3. An infinitesimal condition for germ of $\mathscr{E}(3)$. Taking advantage of Lemma 3.1 we may restrict our attention to the germs at $0 \in \mathbb{R}^6$ of LG-manifolds which possess generating functions of the form $u(x) = F \circ z(x)$ with $F \in C^\infty(3)$.

DEFINITION 3.1. Let x, $\tilde{x}$ be two points of $\mathbb{R}^3$ and $\zeta = z(x)$, $\tilde{\zeta} = z(\tilde{x})$. Two germs (F, ζ) and $(\tilde{F}, \tilde{\zeta})$ of functions from $C^\infty(3)$ are called *equivalent* (*LG-equivalent*) if the germs $(F \circ z, x)$ and $(\tilde{F} \circ z, \tilde{x})$, are equivalent (weakly equivalent). The germ (F, ζ) is called *stable* (*LG-stable*) if $(F \circ z, \zeta)$ is stable (weakly stable).

DEFINITION 3.2. The germ $F \in \mathscr{E}(3)$ is called *infinitesimally LG-stable* if for every germ $\alpha \in \mathscr{E}(3)$ there exists decomposition

$$\alpha(x) = F_{,1}(x)h_1(x) + c_0 + c_1x_1 + \sum_{i=2}^{3} (x_i F^2_{,i}(x)h_i(x) + x_i F_{,i}(x)c_i) \tag{3.2}$$

with $h_i \in \mathscr{E}(3)$ and $c_i \in \mathbb{R}$.

Remarks. (i) Condition (2) is an equivariant form of the infinitesimal stability condition for Lagrangian mappings [2]. More precisely: let $L \subset \mathbb{R}^6$ be a Lagrangian manifold with generating function $u \in \mathscr{E}(3)$. First, Arnold [2] defines: a germ $(L, 0)$ or, equivalently, the Lagrangian map $x \to \partial u/\partial x$ as being infinitesimally stable if for every $\beta \in \mathscr{E}(3)$ there exists a Hamiltonian $H(x, y)$ on $\mathbb{R}^6$ of the form

$$H(x, y) = \sum_1^3 a_i(y)x_i + b(y),$$

where a_i, $b \in \mathscr{E}(3)$, such that

$$\beta(x) = H\left(x, \frac{\partial u}{\partial x}\right). \tag{$*$}$$

Next by application of the Malgrange-Mather theorem [4], [9], [20], we obtain the following condition: for every $\alpha \in \mathscr{E}(3)$ there exists a decomposition

$$\alpha(x) = \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x_i} h_i(x) + c_i x_i\right) + c_0$$

where $h_i \in \mathscr{E}(3)$, $c_i \in \mathbb{R}$.

If we just add the requirement that u, H and β in $(*)$ are G-invariant and then apply the Equivariant Division Theorem [22] we obtain (3.2) as an equivalent form of $(*)$.

PROPOSITION 3.2. *A germ $F \in \mathscr{E}(3)$ is infinitesimally LG-stable if and only if one of the following conditions are satisfied*:

(A_0) $f_1 \neq 0,$

(A_1) $f_1 = 0, \quad f_2 f_3 f_{11} \neq 0,$

(A_2) $f_1 = f_{11} = 0 \quad \textit{and} \quad f_2 f_3 f_{111} \neq 0,$

$(A_{3'})$ $f_1 = f_2 = 0 \quad \textit{and} \quad f_3 f_{11} f_{12}(f_{12}^2 - f_{11} f_{22}) \neq 0,$

$(A_{3''})$ $f_1 = f_3 = 0 \quad \textit{and} \quad f_2 f_{11} f_{13}(f_{13}^2 - f_{11} f_{33}) \neq 0.$

Proof. Necessity: It is straightforward to check that one of these conditions must be fulfilled if (3.2) is satisfied mod $\mathfrak{m}^3$. Sufficiency: For $\alpha \in \mathscr{E}(3)$ we show how to define germs h_i and constants c_i satisfying (3.2).

CASE (A_0). It is enough to take $h_1 = \alpha / F_{,1}$, $h_2 \equiv h_3 \equiv 0$, $c_0 = \alpha(0)$, $c_i = 0$ for $i = 1, 2, 3$. So, from now on, let us assume $f_1 = 0$.

CASE (A_1). Let (A_1) be satisfied, α_i, $U_i \in \mathscr{E}(3)$ be such that $\alpha(x) = \alpha(0) + \sum_1^3 x_i \alpha_i(x)$, $F_{,1}(x) = \sum_1^3 x_i U_i(x)$. Then $U_1(0) = f_{11} \neq 0$, $F_{,2}(0) = f_2 \neq 0$, $F_{,3}(0) = f_3 \neq 0$. Hence $c_0 = \alpha(0)$, $c_1 = c_2 = c_3 = 0$, $h_1 = \alpha_1 / U_1$, $h_2 = (\alpha_2 - U_2 h_1)/F_{,2}^2$, $h_3 = (\alpha_3 - U_3 h_1)/F_{,3}^2$ satisfy (3.1).

CASE (A_2). Let (A_2) be satisfied and U_i, $\alpha_i \in \mathscr{E}(3)$ be such that $F_{,1}(x) = x_1^2 U_1(x) + x_2 U_2(x) + x_3 U_3(x)$, $\alpha(x) = \alpha(0) + x_1 \alpha_{,1}(0) + x_1^2 \alpha_1(x) + x_2 \alpha_2(x) + x_3 \alpha_3(x)$. As $U_1(0) = f_{111} \neq 0$, $F_{,2}(0) = f_2 \neq 0$, $F_{,3}(0) = f_3 \neq 0$ it suffices to put $c_0 = \alpha(0)$, $c_1 = \alpha_{,1}(0)$, $h_1 = \alpha_1 / U_1$, $h_2 = (\alpha_2 - U_2 h_1)/F_{,2}^2$, $h_3 = (\alpha_3 - U_3 h_1)/F_{,3}^2$.

CASE $(A_{3'})$ (for $(A_{3''})$ the procedure is similar). Assume that conditions $(A_{3'})$ are satisfied. The germ (at $0 \in \mathbb{R}^3$) of

$$\beta(x) = \alpha(x) - c_0 - c_1 x_1 - c_2 F_{,2}(x) x_2 + F_{,1}(x)(g_0 + g_1 x_1 + g_2 x_2)$$

belongs to the ideal $\langle x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_3 \rangle$ provided that $c_0 = \alpha(0)$ and constants c_1, c_2, g_0, g_1, g_2 satisfy the following system of linear equations (solvable iff $f_{11} f_{12}(f_{12}^2 - f_{11} f_{22}) \neq 0$):

$$\begin{aligned}
\alpha_{,1}(0) &= f_{11} g_0 + c_1, \\
\alpha_{,2}(0) &= f_{12} g_0, \\
\alpha_{,11}(0) &= f_{111} g_0 + 2 f_{11} g_{11}, \\
\alpha_{,12}(0) &= f_{112} g_0 + f_{12} g_1 + f_{11} g_2 + f_{12} c_2, \\
\tfrac{1}{2} \alpha_{,22}(0) &= f_{112} g_0 + f_{12} g_2 + f_{22} c_2.
\end{aligned}$$

Now consider germs U_{ij}, β_i such that

$$F_{,i}(x) = x_1 U_{i1}(x_1) + x_2 U_{i2}(x_1, x_2) + x_3 U_{i3}(x_1, x_2, x_3),$$

for $i = 1, 2$ and

$$\beta(x) = x_1^3 \beta_1(x) + x_1^2 x_2 \beta_2(x) + x_1 x_2^3 \beta_3(x) + x_2^3 \beta_4(x) + x_3 \beta_5(x).$$

Let germs k_1, k_2, k_3, $h_2 \in \mathscr{E}(3)$ be the solutions of the following system of linear equations (with germ coefficients):

$$\begin{aligned}
\beta_1 &= U_{11} k_1, \\
\beta_2 &= U_{12} k_1 + \quad + U_{11} k_3 + U_{21}^2 h_2, \\
\beta_3 &= \quad U_{11} k_2 + U_{12} k_3 + 2 U_{21} U_{22} h_2, \\
\beta_4 &= \quad U_{12} k_2 \quad + U_{22}^2 h_2.
\end{aligned}$$

The above equations are solvable since the system determinant at 0 is equal to $f_{11}(f_{12}^2 - f_{11} f_{22})^2 \neq 0$. It can be easily checked that the germ $\gamma(x) := \beta(x) - F_{,1}(x_1)(x_1^2 k_1(x) + x_2^2 k_2(x) + x_1 x_2 k_3(x)) - x_2 F_{,2}^2(x) h_2(x)$ belongs to the ideal $\langle x_3 \rangle$

in $\mathscr{E}(3)$, i.e., it is of the form $\gamma(x) = x_3\gamma'(x)$, where $\gamma' \in \mathscr{E}(3)$. Finally we observe that c_0, c_1, c_3, $h_2(x)$ defined as above, $c_3 := 0$ and

$$h_1(x) := g_0 + g_1x_1 + g_2x_2 + x_1^2f_1(x) + x_2^2f_2(x) + x_1x_2f_3(x),$$

$$h_3(x) := \gamma'(x)/F_{,3}^2(x),$$

satisfy (3.2). Q.E.D.

3.4. Germs of typical generating functions $u = F \circ z$. Consider a manifold $J^2(\mathbb{R}^3, \mathbb{R})$ of all 2-jets of functions from $C^\infty(3)$ [9] with a coordinate system $(x_i, f^*, f_i^*, f_{ij}^*)$ corresponding to $(x_i, F(x), F_{,i}(x), F_{,ij}(x))$. Let M_1 be a submanifold of $J^2(\mathbb{R}^3, \mathbb{R})$ defined by $x_2 = x_3 = 0$ and with f^*, f_i^*, f_{ij}^* satisfying the second part of condition (A_1) from Proposition 3.2 (f_1^* need not be zero!). Submanifolds M_2, $M_{3'}$ and $M_{3''}$ are defined analogously with conditions (A_2), $(A_{3'})$ and $(A_{3''})$, respectively, replacing (A_1). The codimensions of M_1, M_2, $M_{3'}$, $M_{3''}$ are 2, 3, 3 and 3, respectively. The subset of those 2-jets at $x = (x_1, 0, 0)$, $x_1 \in \mathbb{R}$, which does not belong to $\cup M_i$ is a finite union of submanifolds of codimension four. Thus from Thom's transversality theorem [4], [9], the following proposition results.

PROPOSITION 3.3. *For typical function $F \in C^\infty(3)$ all germs $(j^2F)(x_1, 0, 0)$ belong to $\cup M_i$.*

Denote $\mathscr{E}_i$, for $i = 1, 2, 3', 3''$, the subset of all germs $F \in \mathscr{E}(3)$ satisfying condition (A_i) of Proposition 3.2 together with $F(0) = F_{,1}(0) = 0$ (for such a germ F the point $0 \in \mathbb{R}^6$ belongs to the manifold $L_{F\circ z}$). Using an appropriate translation in $\mathbb{R}^6$ we easily obtain the following.

PROPOSITION 3.4. *Let $F \in C^\infty(3)$, $x_0 = (x_1, 0, 0) \in \mathbb{R}^3$ and $i \in \{1, 2, 3', 3''\}$. If $(j^2F)(x_0) \in M_i$, then the germ (F, x_0) is LG-equivalent to a germ from $\mathscr{E}_i$.*

3.5. Inf-homotopic germs. Our present aim is to find classes of LG-equivalent germs in $\mathscr{E}_i$. For this purpose we introduce the next definition.

DEFINITION 3.3. Let J be an open interval in $\mathbb{R}$. A smooth function on $\mathbb{R}^3 \times J$, $F(x, t) = F(x, t) = F_t(x)$, is called *inf-homotopy* if all germs $(F_t, 0)$ belong to the same class $\mathscr{E}_i$ (note that $F(0, t) = (\partial F/\partial t)(0, t) = 0$ for any inf-homotopy $F(x, t)$). The germs F_a, F_b, $a, b \in J$, are called *inf-homotopic.*

PROPOSITION 3.5. *Any germ from $\mathscr{E}_i$ is inf-homotopic to one of the following forms*:

$$(\mathscr{E}_1) \qquad F = \pm x_1^2 \pm x_2 \pm x_3,$$

$$(\mathscr{E}_2) \qquad F = \pm x_1^3 \pm x_2 \pm x_3,$$

$$(\mathscr{E}_{3'}) \qquad F = \pm x_2^2 \pm (x_1 \pm x_2)^2 \pm x_3,$$

$$(\mathscr{E}_{3''}) \qquad F = \pm x_3^2 \pm (x_1 \pm x_3)^2 \pm x_2.$$

Proof. We will prove only the case $\mathscr{E}_{3'}$. Conditions $\operatorname{sgn} f_{11} = \pm 1$, $\operatorname{sgn} f_2 = \pm 1$, $\operatorname{sgn} f_3 = \pm 1$ and $\operatorname{sgn}(f_{12}^2 - f_{11}f_{22}) = \pm 1$, divided the 4-dimensional space of coefficients $(f_{11}, f_{12}, f_3, f_{22}) := (F_{,11}, F_{,12}, F_{,3}, F_{,22})|_0$ into sixteen open convex regions. So, if germs F', $F'' \in \mathscr{E}_{3'}$ correspond to the same region, the following function

$$F(x, t) := tF'(x) + (1 - t)F''(x)$$

is an inf-homotopy between them. It is straightforward to check that to each of these sixteen regions belongs one of the forms given by $(\mathscr{E}_{3'})$ which completes the proof. Q.E.D.

PROPOSITION 3.6. *Let* $F(x, t)$, $(x, t) \in R^3 \times J$, *be an inf-homotopy,* $S(x, t) := F(z(x), t)$ *and* $t_0 \in J$ *be a fixed point. Then there exists an open neighborhood* $U \times I$ *of* $(0, t_0)$ *and a smooth function* $a_i(x, t)$, $b(x, t)$ *on* $R^3 \times R$, *with compact supports, such that*

$$a_1(x, t) = \frac{\partial b}{\partial x_1}(0, t) \quad \text{for } t \in I, \tag{3.3}$$

and

$$-\frac{\partial S}{\partial t}(x, t) = H\left(x, \frac{\partial S}{\partial x_1}(x, t), t\right) \quad \text{for } (x, t) \in U \times I, \tag{3.4}$$

where

$$H(x, y, t) = a_1(z(y), t)x_1 + \sum_2^3 a_i(z(y), t)x_i y_i + b(z(y), t) \tag{3.5}$$

for $(x, y, t) \in R^3 \times R^3 \times R$.

Proof. Assume $t_0 = 0$. From the proof of Proposition 3.2 it follows that every germ $\alpha \in \mathscr{E}(4)$ has a decomposition

$$\begin{aligned} \alpha(x, t) &= F_{,1}(x, t)h_1(x, t) + c_1(t)x_1 + c_0(t) \\ &\quad + \sum_{i=2}^{3} [x_i F_{,i}^2(x, t)h_i(x, t) + x_i F_{,i}(x, t)c_i(t)] \end{aligned} \tag{3.6}$$

with $c_i \in \mathscr{E}(1)$, $h_i \in \mathscr{E}(4)$. Substituting $c_i(t) = c_i(0) + tc_i^*(t)$ $(i = 0, 1, 2, 3)$ and

$$h(x, t) := c_0^*(t) + c_1^*(t)x_1 + \sum_{i=2}^{3} x_i F_{,i}(x, t)c_i^*(t),$$

we obtain

$$\begin{aligned} \alpha(x, t) &= F_{,1}(x, t)h_1(x, t) + c_1(0)x_1 + c_0(0) \\ &\quad + \sum_{i=2}^{3} x_i F_{,i}^2(x, t)h_i(x, t) + x_i F_{,i}(x, t)c_i(0) + th(x, t). \end{aligned}$$

From the Malgrange–Mather Preparation Theorem ([4], [9], [20]) applied to the germ $g: (R^4, 0) \to (R^4, 0)$, where

$$g(x, t) = (F_{,1}(x, t), 4x_2 F_{,2}^2(x, t), 4x_3 F_{,3}^2(x, t), t)$$

for $(x, t) \in R^3 \times R$, the existence of the following decomposition results:

$$-\frac{\partial F}{\partial t}(x, t) = a_1 \circ g(x, t)x_1 + \sum_{i=2}^{3} 2a_i \circ g(x, t)x_i F_{,i}(x, t) + b \circ g(x, t), \tag{3.7}$$

with a_i, $b \in \mathscr{E}(4)$ (we may take for these germs representatives with compact supports). Now, if we consider equation (3.7) at $(z(x), t)$ and observe that $g(z(x), t) = (z(\partial S/\partial x)(x, t), t)$, we easily get (3.4).

It remains to show (3.3). In the case $\mathscr{E}_2$ we have $F_{,1}(0, t) = F_{,11}(0, t) = 0$ and $F_{,111}(0, t) \neq 0$. Taking derivatives $\partial/\partial x_1$ and $\partial^2/\partial x_1^2$ of (7) at $(0, t)$ we obtain $0 = a_1(0, t)$ and $0 = b_{,1}(0, t)F_{,111}(0, t)$, thus (3.3) results.

In the case $\mathscr{E}_3$, we have $F_{,1}(0, t) = F_{,2}(0, t) = 0$, and $F_{,12}(0, t) \neq 0$. Taking the derivative $\partial/\partial x_2$ of (3.7) at $(0, t)$ we have $0 = b_{,1}(0, t) \cdot F_{,12}(0, t)$, so $b_{,1}(0, t) = 0$. Now differentiation of (3.7) with respect to x_1 at $(0, t)$ yields $0 = a_1(0, t)$.

For $\mathscr{E}_1$ we have $F_{,1}(0, t) = 0 \neq F_{,11}(0, t)$, so taking $\partial/\partial x_1$ of (3.7) at $(0, t)$ we get

$$0 = a_1(0, t) + b_{,1}(0, t)F_{,11}(0, t).$$

Hence, if $a_1(0, t) = 0$, then $b_{,1}(0, t) = 0$. Thus it is enough to show that the decomposition (3.7) with $a_1(0, t) = 0$ is always possible. As the jacobian $\partial g/\partial(x, t) \neq 0$ at $(x, t) = (0, 0)$, there exists $X_1 \in \mathscr{E}(4)$ such that $x_1 = X_1 \circ g(x, t)$. If we set $a_1^*(z, t) := a_1(z, t) - a_1(0, t)$ and $b^*(z, t) := b(z, t) + a_1(0, t)X_1(z, t)$, we may substitute a_1^* and b^* into (3.7) for a_1 and b, respectively. As $a_1^*(0, t) = 0$, this completes the proof. Q.E.D.

3.6. *LG*-equivalences of inf-homotopic germs. Let $F(x, t)$, $S(x, t)$, $H(x, y, t) = H_t(x, y)$, etc., be as in Proposition 3.6. We assume $t_0 = 0$ and $I = (-\varepsilon, \varepsilon)$ for simplicity. Let us consider a time-dependent Hamiltonian vector field on $\mathbb{R}^6$

$$X_{H_t} = \sum_{i=1}^{3} \frac{\partial H}{\partial y_i}(x, y, t) \frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}(x, y, t) \frac{\partial}{\partial y_i}$$

as well as the vector field $\tilde{X}_{H_t} = \partial/\partial t + X_H$ on $\mathbb{R}^6 \times \mathbb{R}$. The vector field X_{H_t} has the global flow g_t, $t \in \mathbb{R}$ (i.e., there exists the smooth mapping (x, y, t) $(x, y, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \to g_t(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that $(d/dt)g_t(x, y) = X_{H_t}(g_t(x, y))$ and $g_0(x, y) = (x, y)$ for $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$). This results from (i) compactness of supports of a_i and b, (ii) the independence of "y"-component of X_{H_t} from x (so $y(t)$ could be found independently of x), (iii) linearity of "x"-component of X_{H_t} with respect to x.

LEMMA 3.2. *g_t is a LG-equivalence for every $t \in \mathbb{R}$.*

Proof. Since $\omega_H^2 = \omega^2 + dH \wedge dt$ is an invariant form of $\tilde{X}$ [1], g_t is a symplectomorphism for every t.

Take $\sigma \in G$. As $H_t(x, y) = H_t \circ \sigma(x, y)$, $\sigma \circ X_{H_t} = X_{H_t} \circ \sigma$ and

$$\frac{d}{dt}(\sigma \circ g_t - g_t \circ \sigma) = \sigma \circ X_{H_t} - X_{H_t} \circ \sigma = 0,$$

for every $t \in \mathbb{R}$. Hence $\sigma \circ g_t = g_t \circ \sigma$ holds for every $t \in \mathbb{R}$ since $g_0 = id_{\mathbb{R}^6}$.

Finally g_t preserves the fibration π because the "y"-component of X_{H_t} is independent of x. Q.E.D.

Let us define a mapping $\Phi: \mathbb{R}^3 (-\varepsilon, \varepsilon) \to \mathbb{R}^6$ as $\Phi(x, t) = \Phi_t(x) := (x, (\partial u/\partial x)(x, t))$ and let a LG-manifold $\Phi(\mathbb{R}^3 \times \{t\}) = \{(x, (\partial u/\partial x)(x, t))\}$ be denoted by L_t.

LEMMA 3.3. *g_t is a LG-equivalence of the germs $(L_0, 0)$ and $(L_t, 0)$ for $|t| < \varepsilon$.*

Proof. First we show that $g_t(L_0) = L_t$. It could be checked by simple calculation that the vector

$$A(x, t) := \frac{d}{dt}\Phi_t(x) - X_{H_t}(\Phi_t(x)) = \sum_{i=1}^{3} \frac{\partial H_t}{\partial y_i}(\Phi_t(x)) \left(\frac{\partial}{\partial x_i} + \frac{\partial^2 u_t}{\partial x_i \partial x_j}(x) \frac{\partial}{\partial y_j} \right)$$

is tangent to L_t at the point $\Phi_t(x)$ for every $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$. Let $B(x, t)$ be a smooth vector field on $\mathbb{R}^3 \times \mathbb{R}$ and let ε', $0 < \varepsilon' < \varepsilon$, be a number such that

$$\Phi_*(B(x, t)) = A(x, t) \text{ for every } (x, t) \in \mathbb{R}^3 \times (-\varepsilon', \varepsilon')$$

where Φ_* denotes the tangent map to $\Phi: \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^6$. Denote by h_s the flow of $-B(x, t) + \partial/\partial t$ on $\mathbb{R}^3 \times \mathbb{R}$ (assumed to be defined globally, for simplicity). Then

$$h_s(\mathbb{R}^3 \times \{t\}) = \mathbb{R}^3 \times \{t + s\} \quad \text{for } s, t \in \mathbb{R}.$$

Let us define $k_t: \mathbb{R}^3 \to \mathbb{R}^3$, $t \in \mathbb{R}$, by the formula

$$k_t(x) := \Phi(h_t(x, 0)).$$

It is easily checked that $(d/dt)k_t(x) = X_{H_t}(k_t(x))$ and $k_0(\mathbb{R}^3) = L_0$. Hence, we obtain by the uniqueness theorem for first order differential equations that $k_t = g_t$, and $g_t(L_0) = k_t(\mathbb{R}^3) = L_t$ for $|t| < \varepsilon'$.

To complete the proof it suffices to notice that $g_t(0,0)=(0,0)\in\mathbb{R}^3\times\mathbb{R}^3$ since $X_{H_t}(0,0)=0$ (by (3.3) and (3.5), and $g_0(0,0)=(0,0)$). Q.E.D.

Since *LG*-equivalences preserve the boundary of the stability region (see § 3.2), the above *LG*-equivalences g_t (Lemma 3.3) preserve (by continuity) the regions characterized by different spectra (= differences between the numbers of positive and negative eigenvalues) of the Hessian $d_x^2F(z(\cdot),t)=[\partial^2F(z,t)/\partial x_i\,\partial x_j]$. Thus from the above lemmas we obtain the following.

PROPOSITION 3.7. *Any two inf-homotopic germs in* $\mathscr{E}(3)$ *are equivalent.*

3.7. Normal form of *LG*-stable germs. It is easily verified that for any $F\in C^\infty(3)$ the mapping $j^2F\colon\mathbb{R}^3\to J^2(\mathbb{R}^3,\mathbb{R})$ is transversal to M_i, $i=1,2,3',3''$. Hence if $j^2F(x)\in M_i$, for every function $\tilde F$ sufficiently close to $F\in C^\infty(3)$ there exists a point $\tilde x\in\mathbb{R}^3$ close to x such that $j^2\tilde F(\tilde x)\in M_i$. Thus (F,x) and $(\tilde F,\tilde x)$ are equivalent to two mutually inf-homotopic germs in $\mathscr{E}_i$, and thus they are equivalent. Finally we obtain the following.

PROPOSITION 3.8. *Any germ* (F,x), *where* $F\in C^\infty(3)$, $x=(x_1,0,0)$, *and* $j^2F(x)\in M_i$ *is stable.*

Theorem 1.1 results from Propositions 3.3, 3.4 and 3.8. Theorems 1.2 and 1.3 can be proved immediately if we add the following proposition.

PROPOSITION 3.9. *Any stable germ* (F,x_0) *where* $F\in C^\infty(3)$ *and* $x_0=(x_{01},0,0)$, *is equivalent to the germ at* $0\in\mathbb{R}^3$ *of one of the following normal forms*:

$$(A_1)\qquad F=\pm(x_1^2+x_2+x_3),$$

$$(A_2)\qquad F=\pm x_1^3\pm(x_2+x_3),$$

$$(A_3)\qquad F=\pm x_2^2\pm((x_2+x_1)^2+x_3).$$

Proof. By Propositions 3.3, 3.4, 3.5, 3.7, 3.8 it is sufficient to show equivalences which carry forms of Proposition 3.5 onto the above normal forms. This could be easily achieved by equivalences of the form $(x,y)\to(\alpha_ix_i+\beta_iy_i,y_i)$ for suitable α_i, $\beta_i\in\{-1,0,1\}$. Q.E.D.

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REFERENCES

[1] R. ABRAHAM AND J. MARSDEN, *Foundations of Mechanics*, 2nd ed., W. A. Benjamin, Reading, MA, 1978.

[2] V. I. ARNOLD, *Normal forms of functions near degenerate critical points, the Weyl groups of* A_k, D_k, E_k *and Lagrangian singularities*, Functional. Anal. i Prilozn., 6 (1972), pp. 30–25. Functional Anal. Appl., 6 (1972), pp. 254–272.

[3] V. I. ARNOLD, S. M. GUSEIN-ZADE AND A. N. VARCHENKO, *Singularities of Differentiable Maps*, Vol. 1, Birkhauser, Boston, 1985. (English edition.)

[4] T. BRÖCKER AND L. LANDER, *Differentiable Germs and Catastrophes*, London Mathematical Society Lecture Notes 17, Cambridge University Press, Cambridge, 1975.

[5] H. B. CALLEN, *Thermodynamics*, John Wiley, New York, 1960.

[6] J. J. DUISTERMAAT, *Oscillatory integrals, Lagrange immersions and unfoldings of singularities*, Comm. Pure Appl. Math., 27 (1974), pp. 207–281.

[7] D. H. FOWLER, *The Riemann–Hugoniot catastrophe and van der Walls equation*, in Towards a Theoretical Biology, vol. 4, Waddington, C. H., ed., Edinburgh University Press, 1972.

[8] R. GILMORE, *Catastrophe Theory for Scientists and Engineers*, John Wiley, New York–Chichester–Brisbane–Toronto, 1981.

[9] M. GOLUBITSKY AND V. GUILLEMIN, *Stable Mappings and their Singularities*, Graduate Texts in Mathematics 14, Springer-Verlag, New York–Heidelberg, 1973.

[10] J. GUCKENHEIMER, *Catastrophe and partial differential equations*, Ann. Inst. Fourier (Grenoble), 23 (1973), pp. 31-59.
[11] S. JANECZKO, *Geometrical approach to phase transitions and singularities of Lagrangian submanifolds*, Demonstratio Math., 16 (1983), pp. 487-502.
[12] ———, *On G-Versal Lagrangian Submanifolds*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys., 31 (1983), pp. 183-190.
[13] S. JANECZKO AND A. KOWALCZYK, *Stability criteria and classification of singularities for equivariant Lagrangian submanifolds*, preprint, Max-Plank-Institut für Mathematik, Bonn, MPI/86-43 (1986), pp. 1-58.
[14] J. KIJOWSKI AND W. TULCZYJEW, *A Symplectic Framework for Field Theories*, Lecture notes in Physics 107, Springer-Verlag, Berlin-New York, 1981.
[15] KLAMUT, *Fizyka i chemia ciala stalego*, PWN, Warszawa, 1977. (In Polish).
[16] A. KOWALCZYK, *Lagrangian singularities with Z_2 symmetry*, to appear.
[17] L. D. LANDAU AND E. M. LIFSHITZ, *Electrodynamics of Continuous Media*, Pergamon Press, Oxford-London-New York-Paris, 1960.
[18] L. D. LANDAU, *On theory of phase transitions*, J. Exp. Theor. Phys., 7 (1937), pp. 19-32.
[19] D. A. LAVIS AND G. M. BELL, *Thermodynamic phase changes and catastrophe theory*, Bull. Inst. Math. Appl., 13 (1977), pp. 34-42.
[20] B. MALGRANGE, *Ideals of Differentiable Functions*, Oxford University Press, London, 1967.
[21] D. O'SHEA, *An exposition of catastrophe theory and its applications to phase transitions*, Queen's Papers in Pure and Appl. Math., 47, Queen's University, Kingston, Ontario, Canada.
[22] V. POÉNARU, *Singularitiés C^∞ en présence de symétrie*, Lecture Notes in Mathematics 510, Springer, Berlin-New York, 1976.
[23] T. POSTON AND I. STEWART, *Catastrophe Theory and its Applications*, Pitman, London-San Fransisco-Melbourne, 1978.
[24] G. SCHWARZ, *Smooth functions invariant under the action of a compact Lie group*, Topology, 14 pp. 63-68.
[25] P. SLODOWY, *Einige Bemerkungen zur Entfaltung Symmetrischer Funktionen*, Math. Z., 15B (1978), pp. 157-170.
[26] H. E. STANLEY, *Introduction to Phase Transitions and Critical Phenomenon*, Oxford University Press, London, 1971.
[27] L. TISZA, *Generalised Thermodynamics*, MIT Press, Cambridge, MA, 1977.
[28] R. THOM, *Structural Stability and Morphogenesis*, Benjamin-Addison-Wesley, New York, 1975.
[29] H. THOMAS, *Phase transitions in magnetic systems*, Proc. "Enrico Fermi" School, Course LIX, Amsterdam 1976, pp. 659-687.
[30] ———, *Phase transitions in uniaxial ferromagnet*, Phys. Rev., 187 (1969), pp. 630-637.
[31] W. M. TULCZYJEW, *The Legendre transformation*, Ann. Inst. h. Poincare Sect. A(N.S.), 27 (1977), pp. 101-114.
[32] G. WASSERMAN, *Classification of singularities with compact abelian symmetry*, preprint 1977.
[33] V. M. ZAKALYUKIN, *Lagrangian and Legendrian singularities*, Functional. Anal. i Prilozen., 10 (1976), pp. 26-36. Functional Anal. Appl., 10 (1976), pp. 23-31.
[34] E. C. ZEEMAN, *Catastrophe Theory, Selected Papers* 1972-1977, Addison-Wesley Publishing, Reading, MA, 1977.