Letters in Mathematical Physics **16** (1988) 301–311.

On Bifurcation Diagrams of Stochastic Dynamical Systems

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(Received: 7 March 1988)

Abstract. The topological type function for stationary probability density of stable stochastic dynamical systems is introduced. The corresponding bifurcation diagrams in the case of one dichotomic noise are derived. Examples encountered in physics and chemistry are given.

1. Introduction

Many systems of physical interest can be described by the following Langevin equation [2, 5, 15, 8],

$$\dot{x} = f(x) + g(x)u(t), \tag{1}$$

where $u(t)$ is the external noise and $f(x)$, $g(x)$ are deterministic vector fields. The main interest of recent bifurcation theory [3, 7, 13] has been to determine bifurcation diagrams and the corresponding structural changes in these types of systems with additional parameters. For the generic, finitely-determined models of singularity theory [4, 16, 13], these catastrophe sets are well known and their relevance for the understanding of various physical, chemical and biological systems has been exhaustively proved in a number of recent results [13, 3, 7, 16].

It appears, however, that in a realistic laser system [5], or various open or semi-open chemical or thermodynamical systems, random internal fluctuations play an important role [8, 9]. It is necessary to include in the model the stochastic component of the noise. Stochasticity of some control parameters of the ordinary dynamical models tends to completely change bifurcation diagrams. Their use is an important physical branch of the theory of stochastic processes (see, e.g. [15, 1, 12, 11]). Abstracting from the concrete physical models, we see that the structure of these stochastic bifurcation sets is interesting, even for the theory itself. In this Letter, we try to investigate the bifurcation sets for a class of one-dimensional stochastic dynamical systems, coming from standard singularity theory, with Markovian dichotomic noise. We do not pretend to give a complete description of such systems, but only to check some interesting examples and the universal mathematical problems suggested by them. In fact, the Letter only deals

★ The author was a visitor at the Department of Mathematics, Monash University, Australia, during part of the period when this paper was written.

with dichotomic fluctuations imposed on the systems under consideration, which is a serious restriction.

2. Dynamical Systems in the Stochastic Control Space

Let $u(t)$ be a stochastic process [6, 12]. Precisely, it is defined by a system of probability densities,

$$P(u(t)),\ P(u(t_1), u(t_2)),\ \ldots,\ P(u(t_1), \ldots, u(t_n)),\ \ldots, \tag{2}$$

for the ordered sequences of the time points, $t_1 < t_2 < \ldots < t_n < \ldots$. By the standard formula for the conditional probabilities of random variables, say ξ, η, i.e. $P(\eta, \xi) = P(\eta)P(\xi \mid \eta)$, we can write

$$\begin{aligned}
&P(u(t_1), u(t_2)) = P(u(t_1))P(u(t_2) \mid u(t_1)),\\
&P(u(t_1), u(t_2), u(t_3))\\
&\quad = P(u(t_1), u(t_2))P(u(t_3) \mid u(t_1), u(t_2))\\
&\quad = P(u(t_1))P(u(t_2) \mid u(t_1))P(u(t_3) \mid u(t_1), u(t_2)),\\
&\ldots\\
&P(u(t_1), \ldots, u(t_n))\\
&\quad = P(u(t_1))P(u(t_2) \mid u(t_1))P(u(t_3) \mid u(t_1), u(t_2))\\
&\qquad \ldots P(u(t_n) \mid u(t_1), \ldots, u(t_{n-1})).
\end{aligned} \tag{3}$$

The Markovian stochastic process is characterized by the following simplifying condition (for its interpretation see, e.g., [15, 6]),

$$P(u(t_k) \mid u(t_1), \ldots, u(t_{k-1})) = P(u(t_k) \mid u(t_{k-1})),$$

for all possible ordered points $t_1, \ldots, t_k$. We see that for the Markovian processes, the system (3) is determined by $P(u(t))$ and conditional probabilities $P(u(t) \mid u(t'))$, $t > t'$ (so-called transitional probabilities). For the purpose of this Letter, we consider a dichotomic Markov process (also called the telegraphic noise [1, 10]) $u(t)$. This type of process takes two possible values, a and $-a$ ($a > 0$ is called an amplitude of the process) with the transition rates, from a to $-a$ and vice-versa, equal to $\gamma/2$. Let $P_\pm(t) \overset{\text{def}}{=}$ Probability $\{u(t) = \pm a\}$, then the temporal evolution of $P_\pm(t)$, which characterizes the process completely, is given by the following equation (Master Equation [15]),

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P_+(t) \\ P_-(t) \end{pmatrix} = -\frac{\gamma}{2}\begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix}\begin{pmatrix} P_+(t) \\ P_-(t) \end{pmatrix}. \tag{4}$$

If $P_\pm(0) = \frac{1}{2}$, then $u(t)$ is a stationary process with the correlation function $\langle u(t)u(t') \rangle = a^2 \exp(-\gamma |t - t'|)$ and the mean value equal to zero, $\langle u(t) \rangle = 0$ (for detailed information, we refer to [15, 1, 11]).

Let us consider in general a parametrized stochastic dynamical system with one internal variable x [15],

$$\dot{x} = -\text{grad}_x V(x, \bar{u}, u_1(t), \ldots, u_n(t)) = F(x, \bar{u}, u_1(t), \ldots, u_n(t)), \tag{5}$$

where V is a smooth function; $V: \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}$, $u_1(t), \ldots, u_n(t)$ are independent stochastic processes and $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_k) \in \mathbb{R}^k$ are deterministic control variables (likewise the λ-parameter in the Verhulst model of population dynamics [11], p. 1235) of the system. We allow (5) to be a general nonlinear system. Because of its simplicity (this is a one-dimensional system) and extensive use in applications [1, 8, 10], we restrict our considerations to the Markovian dichotomic noises $u_1(t), \ldots, u_n(t)$. In this concrete case (usually an approximation of the real situation [1]), we have that the general stochastic system (5) is equal to one with stochastic variables entering linearly, i.e.

$$\dot{x} = f(x, \bar{u}) + \sum_{i=1}^{N} g_i(x, \bar{u}) w_i(t), \tag{6}$$

where $w_i(t)$ are, not necessary independent, stochastic variables also representing dichotomic processes as multiplicatively composed initial ones. In fact, for one dichotomic noise we have immediately the form (6), i.e.

$$F(x, \bar{u}, u(t))$$
$$= \tfrac{1}{2}(F(x, \bar{u}, a) + F(x, \bar{u}, -a)) + u(t)\, \frac{1}{2a}\, (F(x, \bar{u}, a) - F(x, \bar{u}, -a)).$$

By induction with respect to the number of telegraphs in F, we easily find the general form of (6).

DEFINITION. Let $u_1(t), \ldots, u_n(t)$ be dichotomic Markov noises (also called the telegraphic noises, or simply, telegraphs, cf. [15]). Let the potential function V depend on $\bar{u} \in \mathbb{R}^n$ and $\{u_i(t)\}_{i=1}^n$ as well, in the following way:

$$V(x, \bar{u}, u_1(t), \ldots, u_n(t)) = \bar{V}(x, \bar{u}_1 + u_1(t), \ldots, \bar{u}_n + u_n(t)).$$

Then (5) is called the stochastic dynamical system with telegraphically fluctuating control parameters.

For the deterministic dynamical system of type (5), its stationary surface (i.e. the set of zeroes of the corresponding field F) gives the basic information about the slow dynamics appearing in various dynamical models of physical systems [13, 16]. The general nonlinear system of type (5) can be very complicated with regard to its stationary properties. However, the theory of singularities [3, 4], by removing some very small class of 'pathological' (unstable) systems, provides complete information about the local stationary properties of typical nonlinear systems.

Let us recall some necessary facts from the standard singularity theory. Let $\dot{x} = -\text{grad}_x V(x, u)$ be a dynamical system with smooth potential $V: \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$. We say that this system is stable iff at each point $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}^k$, the following holds:

the quotient ring $\mathfrak{M}_{n+k}/J(V)$ is generated by $\{(\partial/\partial u_i)V(x,u)\}_{i=1}^{k}$ as an $\mathfrak{E}_k$-module (cf. [4]). Here by $\mathfrak{M}_{n+k} \subset \mathfrak{E}_{n+k}(\mathfrak{E}_k)$ we denote the maximal ideal of the ring $\mathfrak{E}_{n+k}$ of germs at (x_0, u_0) (u_0, respectively) of smooth functions $\mathbb{R}^{n+k}$, $(x_0, u_0) \to \mathbb{R}$ and we denote by $J(V)$ the corresponding ideal generated by $\{(\partial/\partial x_j)V(x,u)\}_{j=1}^{n}$. It appears that for structurally stable systems (by the Malgrange preparation theorem [4]), in a neighbourhood of each point (x_0, u_0), the system is equivalent (in its stationary dynamics) to the one given in the following form:

$$\dot{x} = -\mathrm{grad}_x V_\mu(x,u) = -\mathrm{grad}_x\left(g(x) + \sum_{i=1}^{\mu} u_i h_i(x)\right), \quad \mu \leqslant k, \tag{7}$$

where we assume that $(x_0, u_0) = (0, 0)$ and $h_i(x)$ are smooth functions generating the following space

$\mathfrak{M}_n/J(g)$ [4, 13].

The stable potentials for one-dimensional dynamical systems are completely classified (see, e.g., [13]) in all dimensions of control space, i.e.

$$A_{\mu+1}, \dot{x} = -\mathrm{grad}_x\left(x^{\mu+2} + \sum_{i=1}^{\mu} u_i x^{\mu-i+1}\right), \quad \mu \leqslant k. \tag{8}$$

Comparing the form of the above potentials to the stochastic one (6) with telegraphically fluctuating control parameters, we are encouraged also to restrict ourselves to consideration of dynamical systems with stable potentials (7). It suggests that we also consider additional stochastic noises entering additively in the deterministic parameters (u_i).

Thus, in what follows, we assume our stochastic dynamical system to be reconstructed from a stable (7) deterministic one by switching on the respective telegraphs (i.e. dichotomic noises) associated to each control parameter, the values of which are treated as the fluctuating centers of external forces. Thus we have

$$\dot{x} = -\mathrm{grad}_x\left(V_\mu(x, \bar{u}) + \sum_{i=1}^{\mu} u_i(t) h_i(x)\right),$$

where the functions $h_i(x)$ are determined by the stable deterministic potential V at the neighbourhood of each point $(x, \bar{u})$.

We now recall some notions indispensable for characterization of the stationary stochastic solutions of Equation (5) [15]. Let us consider, for simplicity, the Langevin equation (1). For some stochastic solution $x(t)$ of this equation, we can write the corresponding Liouville equation [6, 12],

$$\frac{\partial}{\partial t} p(x,t) = -\frac{\partial}{\partial x} f(x)p(x,t) - u(t)\frac{\partial}{\partial x} g(x)p(x,t), \tag{9}$$

fulfilled by the probability density $p(x, t) = \delta(x - x(t))$. Let $u(t)$ be a dichotomic Markov noise. Then the main statistical information about $x(t)$ is provided by an average $\langle p(x, t)\rangle$ over all realizations of $u(t)$ according to Equation (4). This average splits into two parts, denoted by $P_+(x, t)$ and $P_-(x, t)$, respectively (2^n-parts for n independent dichotomic noises). Combining Equations (9) and (4), we obtain the evolution equations for $P_+(x, t)$, $P_-(x, t)$ (Master Equations [15]),

$$\frac{\partial}{\partial t} P_+(x, t) = -\frac{\partial}{\partial x} f(x)P_+(x, t) - a\frac{\partial}{\partial x} g(x)P_+(x, t) - $$

$$ - \frac{\gamma}{2} P_+(x, t) + \frac{\gamma}{2} P_-(x, t),$$

$$\frac{\partial}{\partial t} P_-(x, t) = -\frac{\partial}{\partial x} f(x)P_-(x, t) + a\frac{\partial}{\partial x} g(x)P_-(x, t) + $$

$$ + \frac{\gamma}{2} P_+(x, t) - \frac{\gamma}{2} P_-(x, t), \tag{10}$$

where $P_\pm(x, t)$ denotes the so-called joint probability density for x at time t and $u(t) = \pm a$ ($a > 0$). By straightforward generalization of Equations (10) to the system endowed with n dichotomic noises [15, 9], we obtain the corresponding master equations for 2^n joint probability densities.

PROPOSITION 2.1. *For the generic stochastic system* (5) *with $u_i(t)$ being independent dichotomic noises ($i = 1, \ldots, n$), with amplitudes $a_i > 0$, i.e.*

$$\langle u_i(t)u_i(t')\rangle = a_i^2 \exp(-\gamma_i |t - t'|), \quad u_i^2(t) = a_i^2,$$

the corresponding system of equations for the joint stationary probability densities, considered in the complexified domain, reduces to a system of equations with regular singularities [3, 5].

Proof. A stochastic solution of (5), in the case of dichotomic noises $u_i(t)$, is characterized by its joint probability density $P_I(x, \bar{u}, t)$, $I = (i_1, \ldots, i_n)$, $i_j = \pm 1$. We recall the evolution equations for the dichotomic noises $u_i(t)$;

$$\frac{d}{dt} P_+^{(i)}(t) = \gamma_i \frac{1}{2}(P_-^{(i)} - P_+^{(i)})(t), \qquad \frac{d}{dt} P_-^{(i)}(t) = \gamma_i \frac{1}{2}(P_+^{(i)} - P_-^{(i)})(t),$$

where $i = 1, \ldots, n$ (cf. (4)). Thus, the evolution of $P_I(x, \bar{u}, t)$ is governed by the following master equation [15, 9],

$$\frac{\partial}{\partial t} P_I(x, \bar{u}, t) = -\frac{\partial}{\partial x} F(x, \bar{u}, A_I)P_I(x, \bar{u}, t) - \frac{1}{2}\sum_{j=1}^{n} \gamma_j(P_I - P_{I_j})(x, \bar{u}, t), \tag{11}$$

where $I_j = (i_1, \ldots, -i_j, \ldots, i_n)$ and A_I is a value of $u(t) = (u_1(t), \ldots, u_n(t))$ corresponding to multi-indice I, i.e. $A_I = (i_1 a_1, \ldots, i_n a_n)$. By (11), we obtain an equation for the stationary joint probability density $\overline{P}_I(x, \overline{u})$, namely,

$$\frac{\partial}{\partial x} F(x, \overline{u}, A_I)\overline{P}_I(x, \overline{u}) + \frac{1}{2} \sum_{j=1}^{n} \gamma_j(\overline{P}_I - \overline{P}_{I_j})(x, \overline{u}) = 0 . \tag{12}$$

Let us introduce an order into the set of indices I (say, $(+, +\ldots), (-, +\ldots), (+, -\ldots)$ etc.), then we can write

$$\tilde{P}(x, \overline{u}) \overset{\text{def}}{=} B(x, \overline{u})\overline{P}(x, \overline{u}) ,$$

as a column vector. Thus, for $\tilde{P}$, on the basis of (12), we have the following equation

$$\frac{\partial}{\partial x} \tilde{P}(x, \overline{u}) = B(x, \overline{u})\tilde{P}(x, \overline{u}) , \tag{13}$$

where $B(x, \overline{u}) = TH^{-1}(x, \overline{u})$, H is a matrix function with $F(x, \overline{u}, A_I)$ entries on the diagonal,

$$(H^{-1}(x, \overline{u}))_{JK} = \delta_{JK}/F(x, \overline{u}, A_J) \quad \text{and} \quad T_{IJ} = \frac{1}{2} \sum_{j=1}^{n} \gamma_j \delta_{JI_j} - \frac{1}{2}\left(\sum_{i=1}^{n} \gamma_i\right) \delta_{IJ} .$$

Taking into account the form of $B(x, \overline{u})$, on the basis of the Thom transversality lemma [4, 14], we see that for generic $\overline{u} \in \mathbb{R}^k$, (13) has only regular singularities.

On the basis of Proposition 2.1, one can use the theory of singularities of ordinary differential equations [3] and analyze the behaviour of a physically admitted solution $\tilde{P}$ in the neighbourhood of singular points (for the generic stochastic system without parameters, see [5]).

In the case of one dichotomic noise, we can take the new functions

$$P_{\mathrm{st}}(x, \overline{u}) \overset{\text{def}}{=} P_+(x, \overline{u}) + P_-(x, \overline{u}) ,$$
$$P_{\mathrm{st}}(x, \overline{u}) \overset{\text{def}}{=} P_+(x, \overline{u}) + P_-(x, \overline{u}) \tag{14}$$

and transform master equations (10) into the new form

$$\frac{\partial}{\partial x} f(x, \overline{u})P_{\mathrm{st}}(x, \overline{u}) + a \frac{\partial}{\partial x} g(x, \overline{u})Q_{\mathrm{st}}(x, \overline{u}) = 0 ,$$

$$\frac{\partial}{\partial x} f(x, \overline{u})Q_{\mathrm{st}}(x, \overline{u}) + a \frac{\partial}{\partial x} g(x, \overline{u})P_{\mathrm{st}}(x, \overline{u}) + \gamma Q_{\mathrm{st}}(x, \overline{u}) = 0 , \tag{15}$$

where

$$f(x, \overline{u}) = F(x, \overline{u}, a) + F(x, \overline{u}, -a) , \qquad g(x, \overline{u}) = \frac{1}{2} (F(x, \overline{u}, a) - F(x, \overline{u}, -a)) .$$

$P_{\mathrm{st}}(x, \overline{u})$ is expressed by the stationary solutions of (11) and can be interpreted as the stationary probability density of x parametrically depending on $\bar{u}$ (for the notation and physical meaning of $P_{st}(x, \bar{u})$ see [10], p. 378). We easily see that the master equation (15) for the stationary probability density $P_{st}(x, \bar{u})$, has the following general solution (see, e.g., [10, 11]),

$$P_{st}(x, \bar{u}) = N\left(\frac{1}{F_+(x, \bar{u})} - \frac{1}{F_-(x, \bar{u})}\right)\exp\left(-\frac{\gamma}{2}\int^x \left(\frac{1}{F_+(x', \bar{u})} + \frac{1}{F_-(x', \bar{u})}\right)\mathrm{d}x'\right), \tag{16}$$

where $F_\pm(x, \bar{u}) = F(x, \bar{u}, \pm a)$ and the stochastic dynamics is governed by two potentials $V_\pm(x, \bar{u}) = V(x, u, \pm a)$. Thus, one can completely characterize the topological structure of the support of the stationary probability density [5, 10]. The method of this analysis is based on the observation [5] that to make physical sense of P_+ and P_-, it is required that they are not negative functions. We easily see that this is fulfilled if and only if F_+ and F_- have the opposite signs.

Let x_0 be a finite critical point of one of the potentials $V_\pm$, i.e. minimum of order $2i$, or inflection point of order $2i + 1$ (which we denote by $(2i + 1)_\pm$ if this is a nondecreasing '+' or nonincreasing '−' function in the neighbourhood of x_0, including $i = 0$). Verifying the Lebesgue integrability of (16) in the critical point x_0 after straightforward calculations, we obtain the following proposition.

PROPOSITION 2.2. (A) *In the case of one dichotomic Markov noise, the following configurations of potentials V_+, V_-, with at most one having a critical point x_0, form (at x_0) the support boundary point (s.b.p. for short):*

1. *x_0 is a right s.b.p. if V_+ and V_- (or in the inverse order) are of order $2i$ or $(2i + 1)_-$ and $(1)_+$ respectively (we denote these two configurations by R^{2i+1}, R^{2i}).*
2. *x_0 is a left s.b.p. if V_+ and V_- (or in the inverse order) are of order $2i$ or $(2i + 1)_+$ and $(1)_-$ respectively (we denote these configurations by L^{2i+1}, L^{2i}).*

(B) *With the assumptions of part (A), and assuming that the potentials V_+, V_- are in the general position [14], we have that the only support boundary points which can appear have the type R^2, or L^2, with the corresponding index of divergence*

$$P_{st} \sim |x - x_0|\left(\frac{\gamma}{2F'_\pm(x_0, \bar{u})} - 1\right).$$

3. Stochastic Bifurcation Sets for Stable Dynamical Systems

Let us consider the following stochastic system

$$\dot{x} = F(x, \bar{u}, u(t)) = f(x) + \sum_{i=1}^{\mu} (\bar{u}_i + v_i u(t)) g_i(x), \tag{17}$$

where F is a universal unfolding of f [13] with the μ-dimensional control space, and $v \in S^{\mu-1}$ is a direction of the dichotomic fluctuation $u(t)$.

REMARK 3.1. The bifuration set of stationary points of (17), in the deterministic case, corresponds to the discriminant set (catastrophe set – generalized swallowtail [3]) of F. Investigation of the catastrophe sets for controlled dynamical systems is one of the aims of the bifurcation theory or catastrophe theory [14]. Applying the stochastic noise to control parameters, the nature of the dynamical system is drastically changed but the bifurcation set (diagram) for the stationary probability is still very important in the stochastic analysis of the system [11]. In this section, we investigate the bifurcations of (17) with dichotomic Markov noise and connect them to the standard catastrophe set of the corresponding deterministic system.

Now we can use formulae (16) and completely analyze the bifurcations of the topological structure of the domain of P_{st} as well as its divergence exponents on the boundaries. Let us consider $F_{\pm}(x, \bar{u}) = F(x, \bar{u}, \pm a)$, with fixed parameters $v \in S^{\mu-1}$, $a \in \mathbb{R}_+$. We define

$$\Sigma^i_{\pm} = \{\bar{u} \in \bar{U};\ \text{number of zeroes of } F_{\pm}(\cdot, \bar{u}),\ \text{i.e., } \#(\Phi^{\pm}_{\bar{u}} = \\ = \{x : F_{\pm}(x, \bar{u}) = 0\})\ \text{is equal } i\}, \tag{18}$$

where by $\bar{U}$ we denote the space of control parameters.

Obviously we have the canonical stratifications:

$$\bar{U} = \bigcup_{i=0}^{\mu+1} \Sigma^i_{+} = \bigcup_{j=0}^{\mu+1} \Sigma^j_{-}\,.$$

To each $x^{\pm}_k \in \Phi^{\pm}_{\bar{u}}$ we can associate $+1$ (-1) if it is a local minimum or inflection point of the potential $V_{\pm}$ (if it is a local maximum of $V_{\pm}$ respectively). We denote this function by $\operatorname{sgn} x^{\pm}_k$. Now we are ready to define the following integer-valued function

$$\chi\colon \bar{U} \to N \cup \{0\}\,,$$

$$\chi(\bar{u}) = \min\left\{\frac{1}{2}\sum_{k=1}^{i}(1 + \operatorname{sgn} x^{+}_k),\ \frac{1}{2}\sum_{k=1}^{j}(1 + \operatorname{sgn} x^{-}_k)\right\}, \tag{19}$$

where $\bar{u} \in \Sigma^i_{+} \cap \Sigma^j_{-}$.

By straightforward calculations for the A_k-singularities of (17) [3], we have the following proposition

PROPOSITION 3.2. (A) *For a generic stochastic system* (17) *and sufficiently small* $a > 0$ *the function* χ *defines the topological type of support of* P_{st}. *The value of* χ *measures the number of connected components of the support.* χ *is equal to zero if and only if* P_{st} *is not defined at all. Discontinuities of* χ *define the points of the bifurcation diagram for the corresponding system.*

(B) *The function* χ *is also a differential invariant of the stochastic dynamical system, i.e. it does not depend on isomorphic changes of unfoldings* [4].

Now we describe the bifurcation diagrams for the concrete perturbations of the fold and cusp [13] catastrophes. Let us consider the following system (the fold catastrophe)

$$\dot{x} = x^2 - (\bar{u} + u(t)) = -\operatorname{grad}_x V\,.$$

Geometrical analysis of the pairs of potentials V_+, V_- gives immediately the supports for integrable (thus, physically acceptable) stationary probability densities. The topological type function for this system is following

$$\chi(\bar{u}) = \begin{cases} 0, & \bar{u} < a\,, \\ 1, & \bar{u} \geqslant a\,. \end{cases}$$

Thus, $\bar{u} = a$ is a bifurcation point – the shifted one corresponding to the deterministic fold catastrophe.

Let us now consider the system (17) on the plane, $\mu = 2$, with two orthogonal directions of fluctuations $v_1 = (0, 1)$ and $v_2 = (1, 0)$, i.e.

$$\dot{x} = -x^3 - \bar{u}_1 x - (\bar{u}_2 + u(t)) \quad \text{and} \quad \dot{x} = -x^3 - (\bar{u}_1 + u(t))x - \bar{u}_2\,,$$

respectively, (i.e. modelled on the cusp catastrophe). The corresponding stratifications of $\bar{U}$ (cf. [13]) are as follows

$$\begin{array}{ll} v_1: & v_2: \\ \quad \Sigma^1_\pm : \{\Delta^1_\pm > 0\} & \quad \Sigma^1_\pm : \{\Delta^2_\pm > 0\} \\ \quad \Sigma^2_\pm : \{\Delta^1_\pm = 0\} & \quad \Sigma^2_\pm : \{\Delta^2_\pm = 0\} \\ \quad \Sigma^3_\pm : \{\Delta^1_\pm < 0\} & \quad \Sigma^3_\pm : \{\Delta^2_\pm < 0\}\,, \end{array}$$

where

$$\Delta^1_\pm = \frac{1}{4}\,(\bar{u}_2 \pm a)^2 + \frac{1}{27}\,\bar{u}_1^3\,, \qquad \Delta^2_\pm = \frac{1}{4}\,\bar{u}_2 + \frac{1}{27}\,(\bar{u}_1 \pm a)^3\,.$$

The topological type functions have the following form

$$\chi_{1,2}(\bar{u}) = \begin{cases} 1, & \bar{u} \in \Sigma^1_+ \cup \Sigma^1_-\,, \\ 2, & \bar{u} \in \bar{\Sigma}^3_+ \cap \bar{\Sigma}^3_-\,. \end{cases}$$

They are illustrated in Figures 1(a), (b). The configurations of potentials corresponding to the points α, β, γ of control space are illustrated in Figure 2, (α), (β), (γ), respectively. The corresponding bifurcation set B for the first perturbation v_1 is described parametrically by the following formulae

$$(\bar{u}_1, \bar{u}_2) = (-3s^2, \mp 2s^3 \pm a)\,, \quad \text{for } s \geqslant (a/2)^{1/3}\,.$$

REMARK 3.3. In the case of unstable systems, which are always induced from stable ones by appropriate morphisms, say $F \circ \Phi(x, \bar{u})$ [4], the function χ gives only an upper bound for the topological type of support. As an example of such a system, we can take the Verhulst chemical reaction equation [11],

$$\dot{x} = \bar{F}(x, \bar{u}, u(t)) = -x^2 + x(\bar{u} + u(t))\,,$$

where the deterministic $\bar{F}$ is induced from the stable fold singularity $F(x, \bar{u}) = -x^2 + \bar{u}$

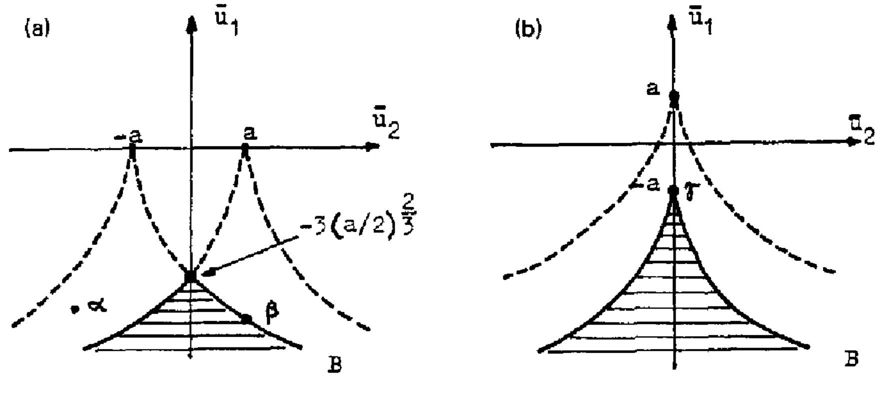

Fig. 1.

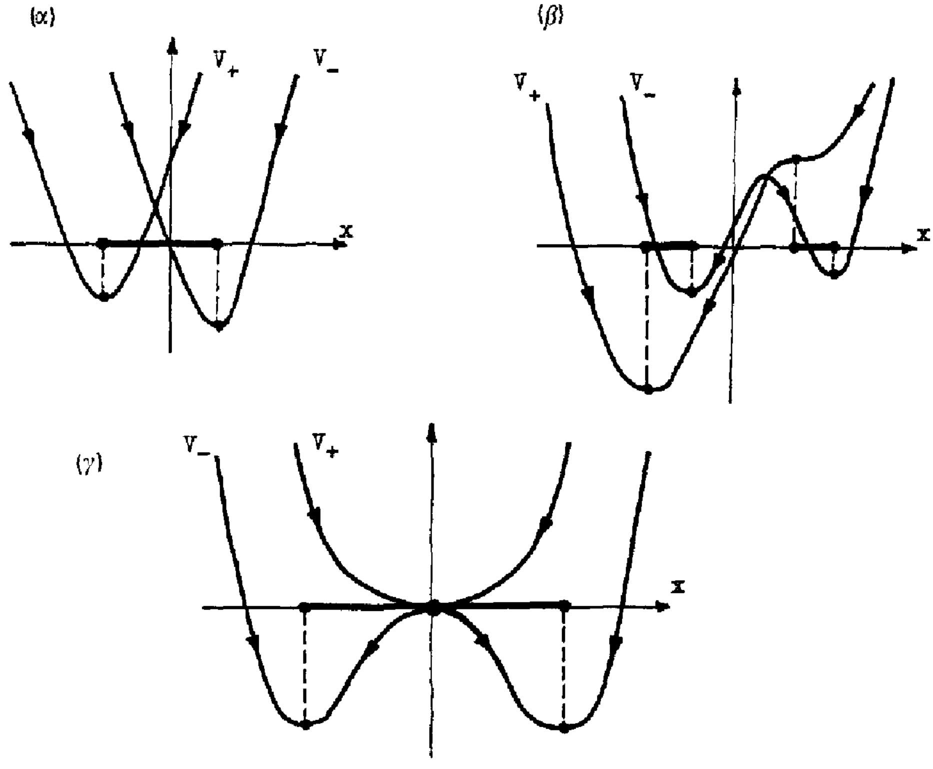

Fig. 2.

by the morphism

$$\Phi(x, \bar{u}) = (x - \tfrac{1}{2}\bar{u}, \tfrac{1}{4}\bar{u}^2),$$

i.e. $\bar{F} = F \circ \Phi$. We easily check that $\chi(\bar{u}) \equiv 1$ but the existence of an integrable P_{st} only occurs for $\bar{u} \geqslant 0$.

Acknowledgements

The author is very grateful to J. W. Bruce and M. Roberts for many useful discussions and would also like to thank the referee for helpful comments.

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