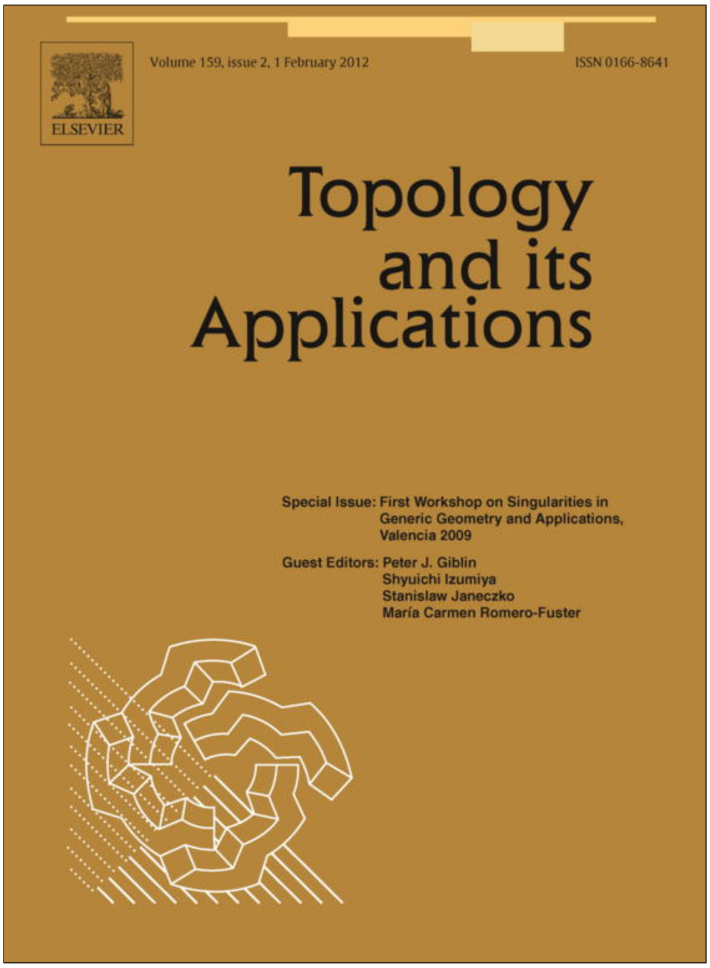
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Geometry of curves and surfaces through the contact map

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ABSTRACT

We introduce a new approach to the study of affine equidistants and centre symmetry sets via a family of maps obtained by reflexion in the midpoints of chords of a submanifold of affine space. We apply this to surfaces in $\mathbb{R}^3$, previously studied by Giblin and Zakalyukin, and then apply the same ideas to surfaces in $\mathbb{R}^4$, elucidating some of the connexions between their geometry and the family of reflexion maps. We also point out some connexions with symplectic topology.

1. Introduction

A number of recent works have studied curves and surfaces in affine space from the point of view of a bifurcation of 'central symmetry'. Some of these concentrate on purely local, or multi-local results—depending on small neighbourhoods of two or more points [7–9,14]—while others involve more global, topological properties such as the total number of singular points [4,6]. A closed manifold M, such as a planar ellipse or an ellipsoid in 3-space, which is globally symmetric about a point O—which has, in other words, global central symmetry—is invariant under reflexion through O. Given $p \in M$, we can take a neighbourhood U of p in M: its reflexion in O coincides exactly with a neighbourhood of the reflected point $p' \in M$. When M no longer has a global centre O we can still consider a chord pp' joining points p, p' of M and reflect the neighbourhood U in the midpoint of this chord: instead of the reflected neighbourhood exactly coinciding with a neighbourhood of p' there will be some measurable *contact* between these two. It turns out that this simple construction has very close connexions with previous work on bifurcations of central symmetry, helps to explain some of the results of that work in a simple geometrical way, and opens up a number of interesting avenues for further investigation. It is the purpose of this article to introduce the 'contact' approach, and to give some preliminary results and applications.

Previous work in this area has concentrated on two basic affine constructions, both of which involve the consideration of chords pp' of M for which the tangent lines or planes at the endpoints p and p' are parallel. We shall relax this condition below, but to see how it arises at all see Fig. 1, left.

The organization is as follows: In Section 2 we motivate the new approach by considering a case studied in some detail in [8], namely that of surfaces in affine space $\mathbb{R}^3$. From Section 3 we apply this approach to a new application, that of a surface M in affine space $\mathbb{R}^4$: the contact map in Section 3, the midpoint map which associates to any pair of points of M

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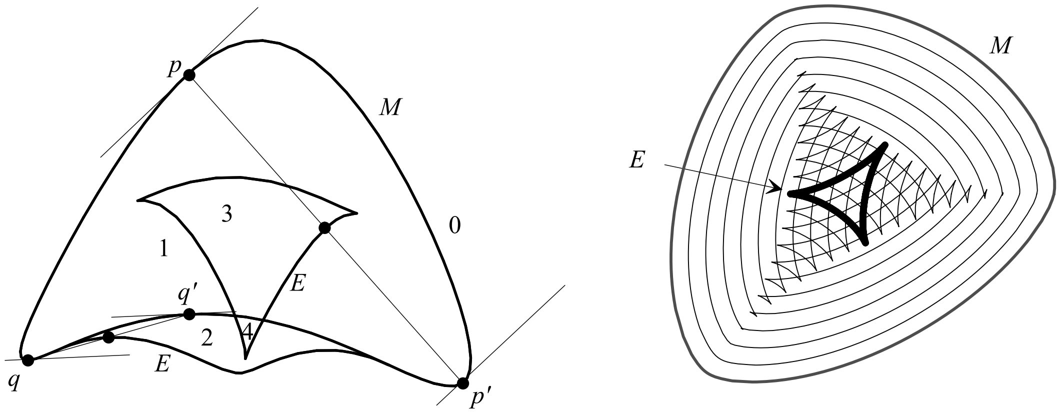

Fig. 1. Left: E is the 'half-way equidistant', having two components, for the two-lobed curve M: it is the locus of midpoints of *parallel tangent chords* of M such as pp' and qq'. Points outside M and E are the midpoints of *zero* chords of M, and crossing M this number changes by 1 while crossing E it changes by 2. (Compare [5].) The union $M \cup E$ is the bifurcation set of the count of midpoints. Right: for this (convex) M many equidistants are drawn: each one is the locus of points at a fixed ratio of distance along a parallel tangent chord, the half-way equidistant E being drawn heavily. The cusps on the equidistants trace out the *centre symmetry set* (*CSS*), also known as the *Minkowski set*. The CSS is also the envelope of parallel tangent chords.

their midpoint in $\mathbb{R}^4$ in Section 4 and finally in Section 5 we study 'vanishing chords', asking how the geometry of M at p is related to the contact map for arbitrarily small chords close to p.

2. The contact map for surfaces in $\mathbb{R}^3$

Consider a surface M given locally by $z = f(x, y)$ in $\mathbb{R}^3$, and take M in Monge form, that is f and its first derivatives with respect to x and y vanish at $(0, 0)$.

For given points $p = (s, t, f(s, t))$ and $p' = (u, v, f(u, v))$ on M we reflect a neighbourhood of p on the surface M in the midpoint of the chord joining p and p', and then write down the contact between this reflected surface, which passes through p', and M itself at p'. The midpoint of the chord is $(\frac{1}{2}(s+u), \frac{1}{2}(t+v), \frac{1}{2}(f(s,t)+f(u,v)))$, and reflecting the point $(s+X, t+Y, f(s+X, t+Y))$, close to p, in this midpoint gives

$$(x, y, z) = \big(u - X, v - Y, f(s,t) + f(u,v) - f(X+s, Y+t)\big). \tag{1}$$

For fixed s, t, u, v this parametrizes the reflected surface close to p'. To find the contact with M at p' we write down an equation for M, namely $f(x, y) - z = 0$ and substitute for x, y, z from (1). This gives the *contact function*

$$F_{(s,t,u,v)} : \big(\mathbb{R}^2, \mathbf{0}\big) \to (\mathbb{R}, 0), \quad F_{(s,t,u,v)}(X, Y) = f(X+s, Y+t) + f(u-X, v-Y) - f(s,t) - f(u,v). \tag{2}$$

Because we are measuring contact it is strictly the contact class of F at $X = Y = 0$ which we need, but for functions this is the same as the right-equivalence class, in particular there are two normal forms for A_{2k-1}, namely $X^2 \pm Y^{2k}$. Since (using subscripts without brackets for partial derivatives)

$$F_X(0,0) = f_x(s,t) - f_x(u,v), \qquad F_Y(0,0) = f_y(s,t) - f_y(u,v), \tag{3}$$

it is clear that F is singular at $(0, 0)$ if and only if the tangent planes to M at p and p' are parallel.

We shall also be interested in s, t, u, v all tending to 0, and we shall ask which singularities of the contact function can persist as the chords shrink to zero, referring to this as the *singularities of vanishing chords*: note that the chords need to be *nontrivial*, that is they need to have distinct endpoints, so either $s \neq u$ or $t \neq v$. Thus we are considering chords whose midpoints lie very close to M; compare Fig. 1.

Note that we can also put $s = t = u = v = 0$ in the family, giving

$$F_0(X, Y) = f(X, Y) + f(-X, -Y). \tag{4}$$

This function measures the contact at the origin between M and the reflexion of M through the origin. It has only terms of even degree in X and Y. Clearly, F_0 is singular at $(0, 0)$, that is both $F_{0X}(0, 0)$ and $F_{0Y}(0, 0)$ are zero there. Also the hessian of F_0 is zero at the origin, that is F_0 has type A_3 at least, if and only if $f_{xx}f_{yy} = f_{xy}^2$ there, that is to say if and only if the origin is parabolic on M. There are indeed two cases, according as F_0 has type A_3^+ or A_3^-. In the first case the intersection of the two surfaces is a single point while in the latter case it is two curves with ordinary tangency. See Fig. 2, left, for the A_3^- case.

Assuming then that the origin is parabolic, we can take f in the form

$$f(x, y) = f_{20}x^2 + f_{30}x^3 + f_{21}x^2y + f_{12}xy^2 + f_{03}y^3 + f_{40}x^4 + \cdots + f_{04}y^4 + \cdots \quad (f_{20} \neq 0), \tag{5}$$

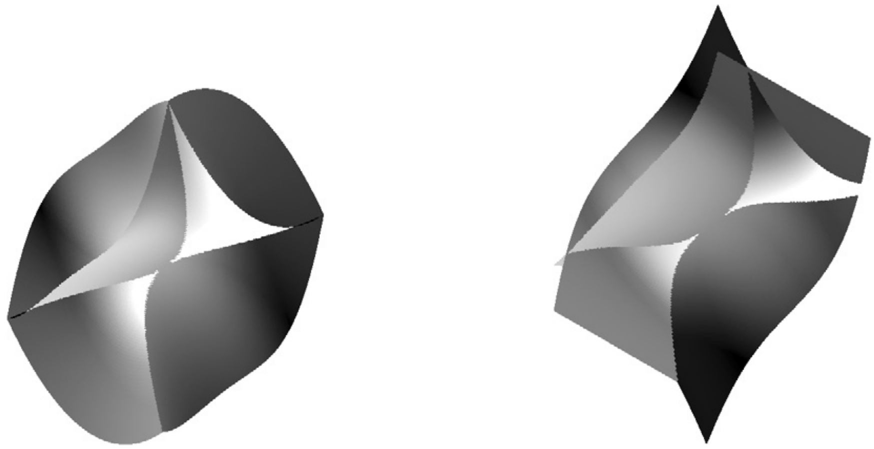

Fig. 2. Left: a surface and its reflexion in the origin having A_3^- contact. Right: A_5^- contact.

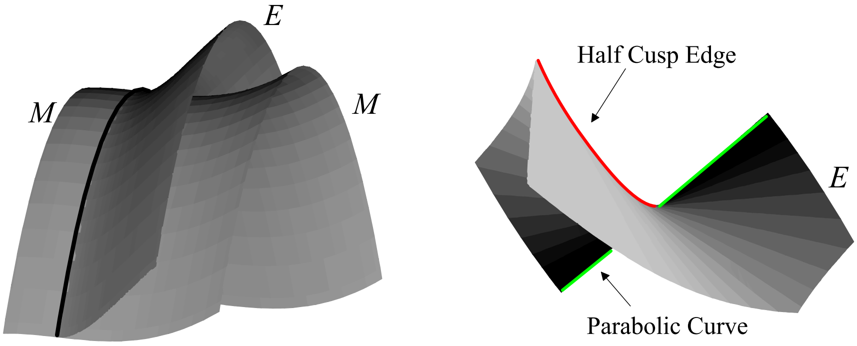

Fig. 3. Left: the half-way equidistant E at ordinary parabolic points is a smooth surface with boundary along the parabolic curve. Right: at an A_2^* point the half-way equidistant acquires a 'half-cuspidal edge'. This model does not show the surface and the parabolic set has become a straight line.

and then F_0 omits all the odd degree terms (and doubles the even degree terms), so that F_0 has type A_5 at least if and only if $f_{04} = 0$. Provided the origin is not a cusp of Gauss (that is, provided $f_{03} \neq 0$), the condition $f_{04} = 0$ is the condition for what is called in [9] an 'A_2^* point'. (Note that the A_2 here refers to a singularity of the *height function* at the origin, not to the function F_0! Parabolic points are 'A_2-points' in this sense and cusps of Gauss are 'A_3-points'.)

Proposition 2.1. *If the origin is a non-parabolic point of M, then F_0 is a Morse function. A parabolic point, not a cusp of Gauss, is of type A_2^* only if F_0 has type A_5.* □

Remarks 2.2. 1. The last condition in the proposition is not quite 'if and only if'. This is because the precise condition for an A_2^* point, as in [9], requires certain open conditions on the *odd degree* terms of the expansion of f which of course cannot affect the type of F_0 at the origin. However the open conditions on the *even degree* terms of f which are required by an A_2^* point guarantee that the contact is of type A_5 exactly. Also, while the contact can have type A_5^+ or A_5^-, no such distinction appears in [9].

2. The special A_2^* parabolic points have a significant impact on the half-way equidistant—the locus of midpoints of parallel tangent chords—as detailed in [16,9]. See Fig. 3 for an illustration of this distinction. However Proposition 2.1 appears to be the first simple interpretation of the special nature of these points, which are not detected by the usual flat geometry of a surface in $\mathbb{R}^3$.

2.1. *Vanishing chords for surfaces in $\mathbb{R}^3$*

Now let us turn to the family of functions F in (2). As above, (3), F is singular at $(X, Y) = (0, 0)$ if and only if the tangent planes to M at the points with parameters (s, t) and (u, v) are parallel. In particular, if $s = u$ and $t = v$ then these partial derivatives are always zero. Eliminating these 'trivial chords' is a familiar problem of 'avoiding the diagonal'. To help in this we shall change coordinates in F by

$$s + u = 2a, \qquad s - u = 2b, \qquad t + v = 2c, \qquad t - v = 2d,$$
$$\text{so that} \quad s = a + b, \qquad u = a - b, \qquad t = c + d, \qquad v = c - d; \tag{6}$$

the diagonal $s=u$, $t=v$ then becomes $b=d=0$. We continue to use F for the resulting family,

$$F_{(a,b,c,d)}(X,Y)=f(X+a+b,Y+c+d)+f(a-b-X,c-d-Y)-f(a+b,c+d)-f(a-b,c-d).$$

The objective in what follows is to determine which singularities of $F_{(a,b,c,d)}$ at $X=Y=0$ occur for pairs of *distinct* points (that is, b,d not both zero) arbitrarily close to $(a,b,c,d)=\mathbf{0}$. So we are interested in the contact singularities corresponding to families of *vanishing chords*, with distinct endpoints ('nontrivial chords'), close to the given point, the origin, on M. The answer, as with Proposition 2.1 for the function F_0, will depend on the geometry of M at the origin.

Consider first the map

$$G:\mathbb{R}^4,\mathbf{0}\to\mathbb{R}^2,\mathbf{0}\quad\text{given by}\quad(a,b,c,d)\mapsto\left(\frac{\partial F}{\partial X}(0,0),\frac{\partial F}{\partial Y}(0,0)\right).$$

The jacobian matrix of this map at $(a,b,c,d)=\mathbf{0}$ is

$$\begin{pmatrix}0&4f_{20}&0&2f_{11}\\0&2f_{11}&0&4f_{02}\end{pmatrix}.$$

It follows that, if the origin is not parabolic, then this matrix has rank 2 and furthermore b,d can be expressed uniquely as functions of a and c close to the origin. But we know that $G(a,0,c,0)\equiv 0$ so these unique functions are in fact $b=0$, $d=0$. Thus:

Proposition 2.3. *If the origin is a non-parabolic point of M, there is a neighbourhood of* $(a,b,c,d)=\mathbf{0}$ *in which* $F_{(a,b,c,d)}$ *is singular at* $(X,Y)=(0,0)$ *if and only if the chord is trivial*: $b=d=0$. □

Thus at a non-parabolic point there are no families of nontrivial vanishing chords giving singular contact maps at all. In a sense this is 'obvious' since it says that sufficiently close to a non-parabolic point there are no pairs of *distinct* points with parallel tangent planes. But note that excluding the diagonal $b=d=0$ was a nontrivial step.

Suppose now that the origin is a parabolic point of M, and write f as in (5). The condition for $F_X=\partial F/\partial X$ at $(X,Y)=(0,0)$ to be zero has an expansion about the origin in a,b,c,d-space of the form $0=4f_{20}b+$ higher order terms, so that this condition can be solved uniquely for b as a smooth function $b=\overline{B}(a,c,d)$. But we know that $b=d=0$ identically satisfies this condition, so $\overline{B}(a,c,d)=d\overline{B}_1(a,c,d)$ for a smooth function $\overline{B}_1$. Substituting this solution for b into the condition for $F_Y=\partial F/\partial Y$ at $(X,Y)=(0,0)$ the result must be divisible by d since we know that $d=0$ (which implies $b=0$ now) identically satisfies this condition. We find in fact that the condition, divided by d, takes the form $0=4f_{12}a+12f_{03}c+$higher order terms.

Assume now that the origin is not a cusp of Gauss on M. Then $f_{03}\neq 0$ and we can solve the last condition for c as a smooth function of a,d, and hence also b as such a function. Let us write

$$b=dB_1(a,d)=d\left(\frac{(f_{12}^2-3f_{03}f_{21})a}{3f_{20}f_{03}}+\text{h.o.t.}\right),\qquad c=C(a,d)=-\frac{f_{12}}{3f_{03}}a+\text{h.o.t.},$$

as the unique solutions obtained in this way. Thus the solutions of $\partial F/\partial X=\partial F/\partial Y=0$ at $(X,Y)=(0,0)$ take the form

$$\text{(i)}\quad(a,0,c,0),\quad\text{or}\quad\text{(ii)}\quad\bigl(a,dB_1(a,d),C(a,d),d\bigr),\quad\text{intersecting in points}\quad\text{(iii)}\quad\bigl(a,0,C(a,0),0\bigr).\tag{7}$$

The set of points (ii) when $d\neq 0$ consists of the 'genuine' pairs of *distinct* points—that is, nontrivial chords—where tangent planes are parallel; as $d\to 0$ these tend to pairs of repeated points, which are parametrized by (iii). Thus (iii) represents the parabolic curve on M; more precisely $(a,C(a,0))$ is a parametrization of the parabolic curve on M.

Proposition 2.4. *At a parabolic point O of M, not a cusp of Gauss, the nontrivial vanishing chords giving singular contact maps are those with* a,b,c,d *as in* (7)(ii). □

We now go on to consider the condition for F to have a non-Morse singularity at $X=Y=0$, for points (a,b,c,d) of type (7)(ii). Let

$$\widetilde{H}(a,b,c,d)=\bigl(F_{XX}F_{YY}-F_{XY}^2\bigr)(0,0),\quad\text{and}\quad H(a,d)=\widetilde{H}\bigl(a,dB_1(a,d),C(a,d),d\bigr).$$

Now $H(a,0)$ evaluates H at $(a,0,C(a,0),0)$ and from the definition of F, evaluating at $X=Y=0$,

$$F_{XX}=f_{xx}(a+b,c+d)+f_{xx}(a-b,c-d),\qquad F_{XY}=f_{xy}(a+b,c+d)+f_{xy}(a-b,c-d),$$
$$F_{YY}=f_{yy}(a+b,c+d)+f_{yy}(a-b,c-d).$$

Consequently, putting $b=d=0$, we find that $H(a,0)=4(f_{xx}f_{yy}-f_{xy}^2)$, evaluated at $(a,C(a,0))$, and this is zero since, as remarked above, $(a,C(a,0))$ is the parabolic curve on M. Thus $H(a,0)$ is identically 0 and this shows H is divisible by d.

We can go on to show that H is divisible by d^2. The definitions of B_1 and C are that, at $X = Y = 0$,

$$F_X\big(a, dB_1(a,d), C(a,d), d\big) \equiv 0, \qquad F_Y\big(a, dB_1(a,d), C(a,d), d\big) \equiv 0,$$

and these equations can be differentiated with respect to d (or indeed a) to obtain other identities. Careful inspection of $\partial B_1/\partial d$ and $\partial C/\partial d$ shows that these are identically zero when evaluated at $(a, C(a,0))$. It follows by a direct calculation that $\partial H/\partial d(a,0)$ is identically zero, which now implies that $H(a,d)$ is divisible by d^2: $H(a,d) = d^2 H_1(a,d)$ for a smooth function H_1. Furthermore, calculation shows that

$$H(a,d) = d^2(64 f_{04} f_{20} + \text{h.o.t.}).$$

This now allows us to avoid the diagonal, since that is represented here by $d = 0$. We conclude that, when the origin is a parabolic point and not a cusp of Gauss on M, and when $f_{04} \neq 0$, which means that the origin is not an A_2^* point in the notation of [9], there are *no* nontrivial vanishing chords for which the singularity of F is non-Morse. Hence we have another geometrical interpretation of A_2^* points:

Proposition 2.5. *Suppose that O is a parabolic point, not a cusp of Gauss, on M. Then the existence of nontrivial chords arbitrarily close to O (vanishing chords) for which the contact singularity is degenerate (non-Morse) implies that O is an A_2^* point of M.* □

Note that vanishing chords therefore distinguish points of the parabolic curve which are not distinguished by the flat geometry of the surface M.

Remark 2.6. We shall not pursue the above any further than A_2^* points. It is clearly of interest to compare higher singularities of the contact function for vanishing chords with the classification of equidistants undertaken in [9]. This will be done elsewhere.

3. The contact map F for surfaces in $\mathbb{R}^4$

We now turn to a new situation, that of a surface in affine space $\mathbb{R}^4$. Much of the above analysis can also be carried out in this situation and we shall summarize the results here. Our references for the flat geometry of surfaces in $\mathbb{R}^4$ are [1,12].

Because of the additional complications as compared with $\mathbb{R}^3$ we start with the simplest situation of two surface patches M, N and points $a_0 = (-1,0,0,0) \in M$, $b_0 = (1,0,0,0) \in N$, with midpoint of the chord a_0b_0 at the origin. As in Section 2 we consider the contact between the reflexion of M in this midpoint and N at b_0. Let the surfaces be in the following form:

$$M\colon \quad \big\{\gamma(s,t) = \big(p(s,t) - 1, q(s,t), s, t\big)\big\}, \qquad N\colon \quad \big\{\delta(u,v) = \big(f(u,v) + 1, g(u,v), u, v\big)\big\}, \tag{8}$$

where $s, t; u, v$ are small, and zero at a_0, b_0 respectively.

The reflexion of M in the origin consists of points $(-p(s,t)+1, -q(s,t), -s, -t)$ and to measure the contact of this with N at b_0 we consider the composite (see [11])

$$\begin{array}{ccccc}
\mathbb{R}^2 & \longrightarrow & \mathbb{R}^4 & \longrightarrow & \mathbb{R}^2 \\
(s,t) & \longmapsto & (-p+1, -q, -s, -t) & & \\
 & & (w,x,y,z) & \longmapsto & \big(f(y,z) + 1 - w, g(y,z) - x\big).
\end{array}$$

This composite is the contact map for the chord a_0b_0:

$$F : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{where } F(s,t) = \big(p(s,t) + f(-s,-t), q(s,t) + g(-s,-t)\big). \tag{9}$$

It should be noted that, since we are measuring contact between two surfaces in $\mathbb{R}^4$, it is the contact class of this map which is relevant. In particular the corank 1 singularities will be with normal forms $(x,y) \mapsto (x, y^2)$ (fold), (x, y^3) (cusp), (x, y^4) (swallowtail). We shall not follow up these cases here except to note two propositions (Propositions 3.1 and 3.2), the first of which is a straightforward calculation.

Proposition 3.1. *The contact map is singular at a_0 if and only if the tangent planes at a_0 and b_0 are 'weakly parallel', that is have a nonzero tangent vector in common.* □

3.1. The generating function and its caustic, the CSS

The generating function, as in [7], is the following. Consider two surfaces M, N as in (8). Then the generating function is

$$\mathcal{F}(s,t,u,v,P,\lambda,Q) = \lambda\big\langle\gamma(s,t) - Q, P\big\rangle + (1-\lambda)\big\langle\delta(u,v) - Q, P\big\rangle,$$

where P is a direction in $\mathbb{R}^4$, $Q \in \mathbb{R}^4$ and $\langle\ \rangle$ can be interpreted as scalar product or as evaluation of a covector on a vector. For a given λ, the equidistant is the set of points $Q = (Q_1, Q_2, Q_3, Q_4)$ of the form $Q = \lambda a + (1-\lambda)b$ where a, b are points of M, N respectively at which the tangent planes are weakly parallel. See Fig. 1 for an illustration in $\mathbb{R}^2$. As before, λ and $1-\lambda$ give the same equidistant, and $\lambda = \frac{1}{2}$ is then special. The family of equidistants is precisely the set of points

$$\left\{(\lambda, Q)\colon\ \mathcal{F} = \frac{\partial \mathcal{F}}{\partial s} = \frac{\partial \mathcal{F}}{\partial t} = \frac{\partial \mathcal{F}}{\partial u} = \frac{\partial \mathcal{F}}{\partial v} = \frac{\partial \mathcal{F}}{\partial P} = 0\right\},$$

where the last equation represents three equations. The common normal to M and N when $s = t = u = v = 0$ is $(-q_t(0,0), p_t(0,0), 0, 0) = (-q_{01}, p_{01}, 0, 0)$, in the notation of Section 3, so we take $P = (-q_{01}/p_{01} + P_1, 1, P_3, P_4)$ as a local parametrization of P for s, t, u, v small, assuming, as before that $p_{01} \neq 0$. The 'base value' of Q_1 is given by $\mathcal{F} = 0$ and is $(1-2\lambda, 0, 0, 0)$. The base values of Q_2, Q_3, Q_4 are 0 since the point Q always lies on the chord.

To find the envelope point(s) (Centre Symmetry Set, or CSS point(s)) on the chord a_0b_0 we need to impose the additional condition that the matrix of second partial derivatives of $\mathcal{F}$ is singular. Write p_{ij} for the coefficient of s^it^j in the Taylor expansion of p at $(0,0)$, and similarly for q, f, g. The determinant of this matrix, evaluated at $s = t = u = v = P_1 = P_3 = P_4 = 0$, $Q_1 = 1-2\lambda$, $Q_2 = Q_3 = Q_4 = 0$ comes to

$$2(1-\lambda)^3\lambda^3 p_{01}\big(\lambda(f_{20}q_{01} - g_{20}p_{01} - p_{20}q_{01} + q_{20}p_{01}) - q_{20}p_{01} + p_{20}q_{01}\big).$$

Thus apart from $\lambda = 0, 1$, which always give CSS points (they are the points of M, N) there is a unique, always real (but possibly infinite), value of λ on the envelope of chords. The value of λ is

$$\lambda = \frac{q_{20}p_{01} - p_{20}q_{01}}{f_{20}q_{01} - g_{20}p_{01} - p_{20}q_{01} + q_{20}p_{01}}, \quad \text{i.e. } 1 - \frac{1}{\lambda} = \frac{f_{20}q_{01} - g_{20}p_{01}}{p_{20}q_{01} - q_{20}p_{01}}. \tag{10}$$

The condition for the CSS point to be at the midpoint of the chord is that $\lambda = \frac{1}{2}$ and this comes to

$$g_{20}p_{01} - q_{01}p_{20} - q_{01}f_{20} + q_{20}p_{01} = 0,$$

which, by calculation, is precisely the same as the cusp condition on the contact map. Hence the CSS is closely related to contact:

Proposition 3.2. *The contact map has the type of a cusp* (*or worse*) *if and only if the* (*unique*) *CSS point on the chord is at the midpoint of the chord.*

Remark 3.3. The generating family $\mathcal{F}$ defines the fibre homogeneous Lagrangian variety $L_{\mathcal{F}}$ in $T^*\mathbb{R}^4$, which projects onto critical values of the λ-point map m_λ and is described by equations:

$$\frac{\partial \mathcal{F}}{\partial s} = 0\colon\quad \lambda\left\langle\frac{\partial\gamma}{\partial s}, P\right\rangle = 0; \qquad \frac{\partial \mathcal{F}}{\partial t} = 0\colon\quad \lambda\left\langle\frac{\partial\gamma}{\partial t}, P\right\rangle = 0;$$
$$\frac{\partial \mathcal{F}}{\partial u} = 0\colon\quad (1-\lambda)\left\langle\frac{\partial\delta}{\partial u}, P\right\rangle = 0; \qquad \frac{\partial \mathcal{F}}{\partial v} = 0\colon\quad (1-\lambda)\left\langle\frac{\partial\delta}{\partial v}, P\right\rangle = 0;$$
$$\frac{\partial \mathcal{F}}{\partial P} = 0\colon\quad Q = \lambda(\gamma - \delta) + \delta,$$

and the cotangent bundle lifting, $(\overline{Q}, Q) \in T^*\mathbb{R}^4$,

$$\overline{Q} = \frac{\partial \mathcal{F}}{\partial Q} = -P.$$

The first four equations define the critical set C_m of λ-point map, i.e. C_m, given by

$$\det\left|\frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t}, \frac{\partial\delta}{\partial u}, \frac{\partial\delta}{\partial v}\right| = 0 \tag{11}$$

and P belongs to the co-normal bundle of $\Sigma_m = m_\lambda(C_m)$ and is homogeneous. After projectivization $T^*\mathbb{R}^4 \to \mathbb{P}T^*\mathbb{R}^4$ we get a Legendrian variety which is smooth on the maximal strata of Σ_m and has singularities along $\operatorname{Sing}\Sigma_m$.

Singularities of the Legendrian projection of $\mathbb{P}L_{\mathcal{F}}$ are defined by second derivatives of $\mathcal{F}$ and finally the equation

$$\det\begin{pmatrix} B & A^t \\ A & 0 \end{pmatrix} = (\det A)^2 = 0,$$

and

$$A = \left(\lambda\frac{\partial\gamma}{\partial s}, \lambda\frac{\partial\gamma}{\partial t}, (1-\lambda)\frac{\partial\delta}{\partial u}, (1-\lambda)\frac{\partial\delta}{\partial v}\right)$$

which on the basis of (11) says that there are no other singularities of Legendre projection besides Σ_m.

4. The midpoint map

There is a construction which is closely related to the contact map described in Section 3. To start with the planar case, that of a simple closed smooth curve M in the affine plane $\mathbb{R}^2$ parametrized by γ, say, consider the map m which associates to a pair (s,t) of parameter values the midpoint of the chord, $\frac{1}{2}(\gamma(s)+\gamma(t))$. This is a map $\mathbb{R}^2 \to \mathbb{R}^2$, and its singularities are closely connected with the half-way equidistant, the centre symmetry set (envelope of parallel tangent chords) and the contact map. Let us consider just the local map m at a point (s,t), $s \neq t$. Then it is easy to check the following.

Proposition 4.1.

(i) *The midpoint map m is singular at (s,t) if and only if the tangents to the curve M are parallel at $\gamma(s)$ and $\gamma(t)$, so that the set of critical values is the half-way equidistant*;
(ii) *m has a cusp singularity* (*or worse*) *if and only if in addition to* (i) *the* (*unique*) *CSS point on this chord is at the midpoint*;
(iii) *m has a swallowtail singularity* (*or worse*) *if and only if in addition to* (ii) *the CSS is singular at the midpoint*;
(iv) *m has a lips/beaks singularity* (*or worse*) *if and only if in addition to* (i) *both $\gamma(s)$ and $\gamma(t)$ are inflexion points of M.*

In the above situation, unlike that of Section 3, swallowtail and lips/beaks arise from distinct phenomena (iii) and (iv). Note that these will occur generically only in a 1-parameter family of curves.

For a surface M in $\mathbb{R}^4$, the midpoint map is a map $m: M\times M \to \mathbb{R}^4$, thus locally a map $m:\mathbb{R}^4 \to \mathbb{R}^4$. It is closely related to the contact map; for example writing down the Jacobian matrix of m it is clear that the corank of m and the corank of the contact map, as defined locally in (9), are equal and in particular must be 0, 1 or 2. Furthermore it is not difficult to check the following:

Proposition 4.2.

(i) *For a surface in $\mathbb{R}^4$ the critical set Σ_m of m is the set of pairs of points of M at which the tangent planes are weakly parallel*; *the critical values form the half-way equidistant.*
(ii) *The condition for m to have a singularity of type $\Sigma^{1,1}$* (*or worse*)*, that is for m restricted to Σ_m to be singular, is the same as the condition that the contact map have a cusp singularity*; *see Proposition* 3.2.
(iii) *When m has exactly type $\Sigma^{1,1}$ the image of the critical set of m, that is the half-way equidistant, is a 3-dimensional cuspidal edge. Its singular set is smooth, and, using the setup as in* (8)*, its slices by hyperplanes given by the third coordinate in $\mathbb{R}^4$ equal to a small constant are cuspidal edges in $\mathbb{R}^3$.* $\square$

Remark 4.3. The affine midpoint map is relevant for the construction of the semiclassical Wigner function on symplectic space which completely describes a quantum state in the classical limit. In this case we deal with a Lagrangian surface L in $\mathbb{R}^4$ endowed with the canonical symplectic structure in Darboux form $\omega = dy_1\wedge dx_1 + dy_2 \wedge dx_2$, $\omega|_L = 0$ (see [2, p. 130]).

Now the two pieces M, N of generic L are generated by two independent generating functions G_M, and G_N (resp.),

$$M:\quad \gamma(s,t) = \left(\frac{\partial G_M}{\partial s} - 1, \frac{\partial G_M}{\partial t}, s, t\right), \quad \gamma^*\omega = 0,$$

$$N:\quad \delta(u,v) = \left(\frac{\partial G_N}{\partial u} + 1, \frac{\partial G_N}{\partial v}, u, v\right), \quad \delta^*\omega = 0.$$

It was shown in [4] that the midpoint map m_L lifts to a lagrangian map into cotangent bundle $T^*\mathbb{R}^4$ and its singularities are singularities of Lagrangian projection, i.e. the following diagram commutes

$$\begin{tikzcd}
 & (T^*\mathbb{R}^4, \Omega) \arrow[d, "\pi"'] \\
\mathbb{R}^2 \times \mathbb{R}^2 \arrow[ur, "\mathcal{M}_L"] \arrow[r, "m_L"] & (\mathbb{R}^4, \omega)
\end{tikzcd}$$

where Ω is a Liouville symplectic structure of $T^*\mathbb{R}^4$, $\mathcal{M}_L^*\Omega = 0$ and a generating Morse family of $\mathcal{M}_L$ reads as follows

$$F(x,y,\alpha) = G_M(x-\alpha) - G_N(x+\alpha) - \alpha y,$$

where $\alpha \in \mathbb{R}^2$ is a Morse parameter.

5. Vanishing chords for surfaces in $\mathbb{R}^4$

As in Section 2.1 we consider a neighbourhood of a point, say the origin O, on a surface N in $\mathbb{R}^4$. For chords whose distinct endpoints both tend to O we ask which singularities can occur for the contact map, defined relative to the chords as in Section 3. For a surface in $\mathbb{R}^3$ we could identify specific geometrical restrictions on N which allow such vanishing chords with associated singularities of different types. The types in question for a surface in $\mathbb{R}^4$ are fold, cusp and swallowtail. We cover here the first two of these, and leave the third one for discussion elsewhere.

We begin with some standard facts about surfaces in $\mathbb{R}^4$; for more details see [1,3,12]. Let us start with a surface N in 'Monge form'

$$N: \quad \delta(x,y) = \big(f(x,y), g(x,y), x, y\big)$$

where f and g have no constant or linear terms. Specifically,

$$f(x,y) = f_{20}x^2 + f_{11}xy + f_{02}y^2 + \cdots, \qquad g(x,y) = g_{20}x^2 + g_{11}xy + g_{02}y^2 + \cdots, \tag{12}$$

the coefficient of $x^i y^j$ being in general f_{ij} or g_{ij}. Thus unit tangent vectors to N at the origin are of the form $(0,0,\cos\theta,\sin\theta)$. We measure contact between a line ℓ_θ through the origin in this direction and N by projecting N orthogonally along the direction to the orthogonal 3-space in $\mathbb{R}^4$. This gives a map germ $\mathbb{R}^2,0 \to \mathbb{R}^3,0$ which is $\mathcal{A}$-equivalent to $h_\theta : (x,y) \mapsto (f(x,y), g(x,y), x\sin\theta - y\cos\theta)$. (To see this, assume $\sin\theta \neq 0$ and parametrize the orthogonal 3-space by its first three coordinates. Then the point where the line through $(f,g,x,y) \in N$ in the direction $(0,0,\cos\theta,\sin\theta)$ meets this 3-space has coordinates $(f, g, x\sin^2\theta - y\cos\theta\sin\theta, y\cos^2\theta - x\cos\theta\sin\theta)$. Taking the first three coordinates and dividing by $\sin\theta \neq 0$ gives the stated form. If $\sin\theta = 0$ then parametrize by the first, second and fourth coordinates.)

The map h_θ will generically be stable ($\mathcal{A}$-equivalent to a crosscap). The condition for it to be unstable—that is for the line to have higher contact with N, which says that the line ℓ_θ is in an *asymptotic direction* at the origin—is obtained as in [13, p. 341]. Namely, change variables from (x,y) to $(u,y) = (x\sin\theta - y\cos\theta, y)$ (assuming again that $\sin\theta \neq 0$) so that h_θ takes the form say

$$\big(a_{20}u^2 + a_{11}uy + a_{02}y^2 + \cdots, b_{20}u^2 + b_{11}uy + b_{02}y^2 + \cdots, u\big),$$

where the terms in u^2 can be ignored for $\mathcal{A}$-equivalence. The 'asymptotic direction' condition is then $a_{11}b_{02} = a_{02}b_{11}$. In our case this comes to

$$(f_{20}g_{11} - f_{11}g_{20})\cos^2\theta + 2(f_{20}g_{02} - f_{02}g_{20})\cos\theta\sin\theta + (f_{11}g_{02} - f_{02}g_{11})\sin^2\theta = 0. \tag{13}$$

If (13) has two real solutions then the point of N (here taken to be the origin) is called *hyperbolic*; if there are no real solutions it is *elliptic* and if there is a repeated solution it is *parabolic*. In the hyperbolic and parabolic cases the real solution(s) are the asymptotic direction(s) at the point of N.

Now we proceed to examine vanishing chords. We choose two points $\delta(s,t)$ and $\delta(u,v)$ and reflect the surface N in the midpoint of the chord joining these two points. We write down the contact function between the two surfaces at the point $\delta(s,t)$ say (we could equally well do it at the other point, $\delta(u,v)$). For fixed s,t,u,v the contact function is, for (x,y) close to (s,t),

$$\begin{aligned}(x,y) \mapsto \big(&f(u,v) + f(s,t) - f(x,y) - f(u+s-x, v+t-y),\\ &g(u,v) + g(s,t) - g(x,y) - g(u+s-x, v+t-y)\big).\end{aligned}$$

We want to evaluate this at $x = s$, $y = t$ so substitute $X = x - s$, $Y = y - t$ and the function becomes

$$\begin{aligned}F(X,Y) = \big(&f(u,v) + f(s,t) - f(X+s, Y+t) - f(u-X, v-Y),\\ &g(u,v) + g(s,t) - g(X+s, Y+t) - g(u-X, v-Y)\big).\end{aligned}$$

We are interested in the singularity of this at $X = Y = 0$. In particular we want to know whether, for s,t,u,v all tending to 0, F can have, say, a fold singularity, or a cusp, or some higher singularity. Thus we need to write down conditions for F to have one of these singularities, as a function of s,t,u,v and of course the Monge coefficients in the functions f and g.

Remark 5.1. Note that F is a 4-parameter family of functions $\mathbb{R}^2 \to \mathbb{R}^2$, parametrized by s,t,u,v. As in Section 2 we can consider the function F_0 with $s = t = u = v = 0$:

$$F_0(X,Y) = \big(-f(X,Y) - f(-X,-Y), -g(X,Y) - g(-X,-Y)\big),$$

which has corank 2, starting with quadratic terms in both components. Also only even degree terms in the power series expansions of f, g will appear. This map F_0 can be interpreted as the contact between N and its reflexion through the origin.

In fact the contact type of this function is indeed related to the flat geometry of the surface N at O, giving different types for a hyperbolic point, a parabolic point, and a special parabolic point very similar to the A_2^* points which arose in Section 2. But here we shall concentrate on the vanishing chords and their contact singularities.

We shall, as in Section 2, use the substitution in (6) to rewrite F as a function, still called F, of a, b, c, d, X, Y. Thus

$$F_{(a,b,c,d)}(X,Y) = \big(f(a+b, c+d) + f(a-b, c-d) - f(X+a+b, Y+c+d) - f(a-b-X, c-d-Y),\\ g(a+b, c+d) + g(a-b, c-d) - g(X+a+b, Y+c+d) - g(a-b-X, c-d-Y)\big).$$

5.1. *Fold of F for s, t, u, v, or $a, b, c, d \to 0$*

The condition for the contact map F above to be singular—at least a fold—at $X = Y = 0$ is that its Jacobian determinant J is singular there. This gives a condition say $G(a,b,c,d) = 0$. The lowest degree terms of G come to

$$\begin{aligned} G_2(a,b,c,d) &= 8\big((f_{20}g_{11} - f_{11}g_{20})b^2 + 2(f_{20}g_{02} - f_{02}g_{20})bd + (f_{11}g_{02} - f_{02}g_{11})d^2\big) \\ &= \alpha b^2 + 2\beta bd + \gamma d^2, \quad \text{say.} \end{aligned} \tag{14}$$

Note that these terms exactly mirror (13)[3]. Since (b,d) represents twice the difference of the parameter values of the two points this strongly suggests that, in the limit, the chord joining them must be in an asymptotic direction and hence cannot exist at an elliptic point. We make this precise below.

The key observation is that, writing $F = (F_1, F_2)$ in terms of a, b, c, d,

$$\left.\frac{\partial F_1}{\partial X}\right|_{X=Y=0} = -f_X(a+b, c+d) + f_X(a-b, c-d),$$

which is identically zero when $b = d = 0$, so that this function can be written as $bB + dD$ where B, D are functions of a, b, c, d. The same applies to the other three partial derivatives $F_{1Y} = \partial F_1/\partial Y$, $F_{2X} = \partial F_2/\partial X$, $F_{2Y} = \partial F_2/\partial Y$ and therefore G can be expressed as

$$G(a,b,c,d) = b^2\big(\alpha + \alpha_1(a,b,c,d)\big) + 2bd\big(\beta + \beta_1(a,b,c,d)\big) + d^2\big(\gamma + \gamma_1(a,b,c,d)\big), \tag{15}$$

for smooth functions $\alpha_1, \beta_1, \gamma_1$ vanishing at $a = b = c = d = 0$, that is without constant terms.

The condition for $G_2 = 0$ (14) to have real solutions other than $b = d = 0$ is that its discriminant $\beta^2 - \alpha\gamma$ is $\geqslant 0$, namely, as above, that the origin is a *hyperbolic* or *parabolic* point of N. Suppose then that the origin is an *elliptic* point of N. Then the only real solutions to $G_2 = 0$ are $b = d = 0$. Choose a neighbourhood U of the origin in a, b, c, d-space small enough that, for all $(a,b,c,d) \in U$,

$$\big(\beta + \beta_1(a,b,c,d)\big)^2 < 4\big(\alpha + \alpha_1(a,b,c,d)\big)\big(\gamma + \gamma_1(a,b,c,d)\big).$$

Then every solution to $G(a,b,c,d) = 0$ for $(a,b,c,d) \in U$ must have $b = d = 0$. Hence

Proposition 5.2. *Suppose that the origin O is an elliptic point of N. Then there is a neighbourhood of O such that no nontrivial chords both of whose (distinct) endpoints lie in that neighbourhood give a singular contact function. In other words: there are no nontrivial vanishing chords giving a singular contact function. Or again: there are no nontrivial vanishing chords at whose ends the tangent planes to N are weakly parallel.* □

Suppose now that the origin O is a hyperbolic of N, which amounts to $\beta^2 > \alpha\gamma$ in the notation above. There will now be two real solutions to $G_2 = 0$ (14) in the b, d plane, and it is to be expected that these extend to solutions (a,b,c,d) through $a = b = c = d = 0$. In this case, the Morse lemma with parameters establishes that there are two families of solutions, with say d a function of a, b, c, and that these solutions form a set in (a,b,c,d)-space which is locally diffeomorphic to $\{(a,b,c,d)\colon\ d = \pm b\}$.

5.2. *Cusp of F*

We now ask the question: given a hyperbolic or parabolic point O of N, what is the condition that there exist vanishing nontrivial chords for which the contact function F has a cusp singularity? The calculations are similar to those above, and they are sketched below. Again the crucial observation is that the additional criterion for a cusp singularity, beyond that for a fold, has the form $b^2(\cdots) + bd(\cdots) + d^2(\cdots) = 0$ for suitable functions of a, b, c, d in the brackets.

There is no single condition for a cusp of a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ at the origin; rather there are two alternative forms which hold in different circumstances. The precise condition (compare for example [15, Chapter 3] or [10, 5.2]) is that the critical set of F passes through the origin, is smooth there, and is tangent, with 2-point contact, to the kernel line of the map at the origin. We shall concentrate here on the conditions that the critical set is smooth and tangent to the kernel line: that is the ‘cusp or worse’ condition.

[3] We thank the referee for pointing out this interesting fact.

Using J for the Jacobian determinant of $F_{(a,b,c,d)}(X,Y)$ the additional cusp condition, apart from open conditions, is (writing partial derivatives as suffices)

$$J_X F_{1Y} - J_Y F_{1X} = 0 \quad \text{at } X = Y = 0 \quad \text{if } F_{1X}, F_{1Y} \text{ are not both zero there}$$

and similarly with F_2 replacing F_1 if F_{2X}, F_{2Y} are not both zero at $X = Y = 0$. At least one of these conditions will hold since the cusp has corank 1. Now $J = F_{1X}F_{2Y} - F_{2X}F_{1Y}$ and, as above, at $X = Y = 0$, $F_{1X}(a, 0, c, 0)$ and the other three similar derivatives are identically zero. We deduce that each of the possible cusp conditions, say $H(a,b,c,d)$, has the form $b^2(\cdots) + bd(\cdots) + d^2(\cdots)$. In fact calculation shows that we can write either of the cusp conditions in the form

$$H(a,b,c,d) = b^2\big(A + A_1(a,b,c,d)\big) + bd\big(B + B_1(a,b,c,d)\big) + d^2\big(C + C_1(a,b,c,d)\big).$$

Here A, B, C are real numbers, giving the 'lowest terms' of H, so that A_1, B_1, C_1 are smooth functions without constant terms.

Remark 5.3. There is an especially delicate case here, namely when the quadratic parts of f and g are both squares. The reason this can affect the calculations is that if, say, the quadratic part of f is a square, then there are points arbitrarily close to the origin in (a,b,c,d)-space, and away from the diagonal $b = d = 0$, at which the first cusp condition is invalid since $F_{1X} = F_{1Y} = 0$ at $X = Y = 0$. If both f and g have quadratic parts which are squares then both conditions can be invalid, but on the other hand at such points the corank of F will be 2 so that F cannot have a cusp singularity. It seems likely that the condition that the quadratic parts of f, g are not both squares can be removed from the result below, but we will err on the side of caution.

Consider the quadratic terms of the fold and an appropriate one of the cusp conditions,

$$Ab^2 + Bbd + Cd^2 = 0 \quad \text{and} \quad \alpha b^2 + \beta bd + \gamma d^2 = 0 \quad \big(\text{see } (14)\big).$$

Calculation shows that, if these share a common root other than $b = d = 0$, then the origin O is parabolic on the surface N. We can now apply an argument similar to that in Section 5.1 to show that, assuming the origin is *not* parabolic on N and a, b, c, d are sufficiently small, there will be no common solutions to $G = 0$ and $H = 0$ other than $b = d = 0$. Hence:

Proposition 5.4. *Suppose that the origin O is a hyperbolic point of N, and that the quadratic parts of f, g are not both squares. Then there is a neighbourhood of O such that no nontrivial chords, both of whose (distinct) endpoints lie in that neighbourhood, give a contact function with a cusp singularity or 'worse'. In other words: there are no nontrivial vanishing chords giving a contact function with a cusp or 'worse'.* □

The situation when O is a parabolic point is more complex, and depends on subtle properties of the parabolic points analogous to the A_2 versus A_2^* cases of Section 2.1. This and an investigation of swallowtail points of the contact map, will appear elsewhere.

References

[1] J.W. Bruce, F. Tari, Families of surfaces in $\mathbb{R}^4$, Proc. Edinb. Math. Soc. 45 (2002) 181–203.
[2] A.M. Ozorio de Almeida, J.H. Hannay, Geometry of two dimensional tori in phase space: projections, sections and the Wigner function, Ann. Physics 138 (1982) 115–154.
[3] R.A. Garcia, D.K.H. Mochida, M.C.R. Fuster, M.A.S. Ruas, Inflection points and topology of surfaces in 4-space, Trans. Amer. Math. Soc. 352 (2000) 3029–3043.
[4] W. Domitrz, P. de M. Rios, Singularities of equidistants and global centre symmetry sets of Lagrangian submanifolds, preprint, Technical University of Warsaw, 2010, arXiv:1007.1939.
[5] P.J. Giblin, Affinely invariant symmetry sets, in: Caustics '06, in: Banach Center Publ., vol. 82, 2008, pp. 71–84.
[6] P.J. Giblin, P.A. Holtom, The centre symmetry set, in: S. Janeczko, V.M. Zakalyukin (Eds.), Geometry and Topology of Caustics, in: Banach Center Publ., vol. 50, Polish Acad. Sci., Warsaw, 1999, pp. 91–105.
[7] P.J. Giblin, V.M. Zakalyukin, Singularities of centre symmetry sets, Proc. Lond. Math. Soc. 90 (2005) 132–166.
[8] P.J. Giblin, V.M. Zakalyukin, Recognition of centre symmetry set singularities, Geom. Dedicata 130 (2007) 43–58.
[9] P.J. Giblin, J.P. Warder, V.M. Zakalyukin, Bifurcations of affine equidistants, Proc. Steklov Inst. Math. 267 (2009) 57–75.
[10] Yung-Chen Lu, Singularity Theory and an Introduction to Catastrophe Theory, Springer-Verlag, 1976.
[11] J.A. Montaldi, On generic composites of maps, Bull. Lond. Math. Soc. 23 (1991) 81–85.
[12] D.K.H. Mochida, M.C.R. Fuster, M.A.S. Ruas, The geometry of surfaces in 4-space from a contact viewpoint, Geom. Dedicata 54 (1995) 323–332.
[13] D.M.Q. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, Proc. Lond. Math. Soc. 50 (1985) 333–369.
[14] G. Reeve, V. Zakalyukin, Singularities of the Minkowski set and affine equidistants for a curve and a surface, Topology Appl. 159 (2) (2012) 555–561.
[15] Farid Tari, Some applications of singularity theory to the geometry of surfaces, PhD thesis, University of Liverpool, 1990.
[16] J. Paul Warder, PhD thesis, University of Liverpool, 2009; available at: http://www.liv.ac.uk/~pjgiblin.