COVARIANT SYMPLECTIC GEOMETRY OF BINARY FORMS AND SINGULARITIES OF SYSTEMS OF RAYS

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There is a method for classifying the polynomial covariants of binary forms by using umbral calculus. We generalize this method to obtain a general framework to describe the tensor invariants and the corresponding invariant subspaces of binary forms. As a consequence we obtain the unique invariant symplectic structure and Riemannian structure on the spaces of binary forms of odd and, respectively, even degree. In addition, the unique invariant contact structure on this space is found. Using the theory of common covariants of binary forms, the hierarchy of apolar coisotropic manifolds and generalized open swallowtails is derived. We identify the invariant Lagrangian varieties of the symplectic space of binary forms with the generic systems of rays on the generic obstacle surface.

1. Introduction

Let (T^*R^n, ω_{R^n}) be the phase space for the free particle (cf. [1]) with the Hamiltonian function $H(q,p) = 1/2(||p||^2 - 1)$. Let the smooth hypersurface $\widetilde{Z} \subset R^n$, representing an obstacle in the Euclidean space, be described by the smooth function $\widetilde{F} : R^n \to R$. The corresponding two hypersurfaces of $(T^*R^n, \omega_{R^n}) : Y = \{(q,p); H(q,p) = 0\}$ and $Z = \{(q,p); F(q,p) = \widetilde{F}(q) = 0\}$, define the sphere bundle $W = Y \cap Z$ over $\widetilde{Z}$. Let us denote by M the symplectic space of all integral lines of the characteristic distribution of Y. Let $\pi : Y \to M$ be its canonical characteristic projection. The critical points of $\pi|_W$ are given by $\Omega = \{(q,p) \in W; \{H,F\} = 0\}$. Ω forms the set of all coversors in R^n tangent to $\widetilde{Z}$ and $\Xi = \pi(\Omega) \subset M$ is the hypersurface of all lines tangent to $\widetilde{Z}$.

In the generic case of $\pi|_W$ being the Whitney's map of type A_1, A_2 or A_3, there exist only three normal forms for W, which in the appropriate Darboux coordinates (which do not preserve special symplectic structure) are described as follows (see [3], Theorem 3):

$$W_1 = \{(q,p) \in T^*R^n; \quad p_0^2 + p_1 = 0, q_0 = 0\},$$
$$W_2 = \{(q,p) \in T^*R^n; \quad p_0^3 + p_1p_0 + q_1 = 0, q_0 = 0\},$$
$$W_3 = \{(q,p) \in T^*R^n; \quad p_0^4 + p_1p_0^2 + q_2p_0 + p_2 = 0, q_0 = 0\}.$$

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In the variational obstacle problem (cf. [6, 14]) the natural geodesic flow γ on $\widetilde{Z}$ is defined by the mutual position of the source of radiation and the obstacle itself. The flow γ corresponds to the Hamiltonian flow of H on the hypersurface Ξ of M. The generic geodesic flows on the obstacle are classified by the corresponding Hamiltonian flows for the hypersurfaces $\Xi_i = \pi(W_i) \subset M$, $i = 1, 2, 3$. The most singular case $i = 3$, as well as its generalization, was precisely described by V.I. Arnold in [3] and [6]. He wrote down the corresponding geodesic trajectories on Ξ_3 and showed that the typical Lagrangian variety in $\Xi_3 \subset R^4$, corresponding to the generic system of gliding rays in the bioasymptotic point of the obstacle, is symplectomorphic to that one given by the following local model

$$\{(q_1, q_2, p_1, p_2) \in M;\ \tfrac{1}{5}\lambda^5 + \tfrac{1}{3}p_1\lambda^3 + \tfrac{1}{2}q_2\lambda^2 + p_2\lambda + \tfrac{1}{2}q_1 \text{ has a root of multiplicity } \geq 3\},$$

(cf. Remark 5.9).

It appeared (see [5, 6, 14]) that the singularities of wavefronts obtained in these models are also classified by the spaces of the singular orbits of reflection groups as in the standard A, D, E case (see [6, 13]). The analogous system of rays on the plane gliding along the curve having an inflection point is determined by the reflection group of symmetries of the icosahedron (see [5], p. 28).

The aim of this paper is to investigate the combinatorial aspects of the singularities considered above and to give the precise description of the analytical structure of these singularities in the general setting of symplectic geometry and invariant theory of binary forms. We show that in the theory of obstacle singularities, the natural symplectic structures are provided by symplectic reduction process (cf. [18]) from the unique $Sl_2(K)$-invariant symplectic structure which appears as a unique tensor invariant of degree two on the space of binary forms of odd degree (cf. [10]). In order to prove that fact, in Section 2 we formulate an appropriate umbral approach to the general investigations of the spaces of invariants of binary forms (cf. [12, 15]) and prove the useful polynomial identities appearing in the stabilized hierarchy of polynomial spaces (introduced in [6]) with the fixed root comultiplicity. In Section 3 we construct an umbral approach in order to classify explicitly tensor invariants of binary forms of degree n (cf. [16]). Following [12] we prove the fundamental theorems concerning the umbral, bracket representation of tensor invariants. In Section 4 we show the existence and uniqueness of the $Sl_2(K)$-invariant symplectic structure on the space of binary forms of odd degree and we write down its explicit normal form. Momentum mapping for the symplectic action of $Sl_2(K)$ in the symplectic space of binary forms is derived and the classical theorems of the theory of polynomial invariants of binary forms (Cayley) are reformulated using the canonical Poisson brackets. In an analogous way the $Sl_2(R)$-invariant contact structure of the projective space RP^n of all zero-dimensional submanifolds of degree n in the projective line is indicated. Section 5 is devoted to the generalization of the notion of Arnold's [6] open swallowtail in all dimensions. We extend the notion of Hamiltonian system generated by translation of one of the two variables of binary forms, to the general sequence of coisotropic submanifolds defined by the so-called apolar subspaces.

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2. Polynomial invariants of binary forms

Let M^{n+1} be the space of binary forms of degree n. A binary form $f(x,y)$ of degree n in the variables x and y is a homogeneous polynomial of degree n in x and y (cf. [16]). By

$$\theta_n = \sum_{k=0}^{n} \binom{n}{k} a_k x^k y^{n-k} \tag{2.1}$$

we denote the general binary form. It is convenient to view it as a mapping from K^{n+1} identified with M^{n+1}, to the space of homogeneous polynomials of degree n in x and y. For a given binary form with parameters a_k we also write $f(x,y) = \theta_n(f) = \sum\limits_{k=0}^{n} \binom{n}{k} a_k x^k y^{n-k}$.

One can view θ_n as an element of the space of polynomials $K[a_0, \ldots, a_n, x, y]$. Let us consider the standard action ν of $Gl_2(K)$ on M^{n+1} (cf. [12]) and extend this action canonically to $M^{n+1} \times K^2$. Let us denote the extension by $\widetilde{\nu}$, then θ_n is an invariant with respect to this action, i.e. for all $h \in Gl_2(K)$ we have $\widetilde{\nu}_h^* \theta_n = \theta_n$.

A nonconstant polynomial $I \in K[a_0, \ldots, a_n, x, y]$ is said to be a covariant of index g of binary forms of degree n if for all $h \in Gl_2(K)$ we have

$$\widetilde{\nu}_h^* I = (\det h)^g I. \tag{2.2}$$

A polynomial function I defined only on M^{n+1} and invariant with respect to ν, according to the formula (2.2), is said to be an invariant of binary forms. We assume that the coefficients of f belong to a field K of characteristic zero and the action ν of $Gl_2(K)$ is induced by the following transformations of variables x and y:

$$\begin{aligned} x &= c_{11}\overline{x} + c_{12}\overline{y}, \\ y &= c_{21}\overline{x} + c_{22}\overline{y}. \end{aligned} \tag{2.3}$$

A very effective method for indicating the polynomial covariants of binary forms comes from the so-called combinatorial "umbral calculus" (see [12, 15]), We recall now some of the basic properties of the umbral calculus applied in the invariant theory of binary forms. Using the umbral methods we can reduce computations with binary forms to the special case of binary forms of type $f(x,y) = (\alpha_1 x + \alpha_2 y)^n$ and obtain the uniform theory of covariants. The aim of this paper is to extend this theory with respect to the invariant (covariant) differential forms defined on the appropriate spaces of binary forms.

Let $p = \{\alpha, \beta, \ldots, \omega, u\}$ be an alphabet consisting of the supply of Greek letters followed by a single Roman letter u. To each Greek letter, say α, and the Roman letter u, we associate two variables α_1, α_2 and u_1, u_2 respectively. The ring of polynomials in these variables is a vector space called the standard umbral space $\mathcal{U}$. With every space of binary forms of degree n we associate a linear operator, say U_n, defined from the umbral space $\mathcal{U}$ to the space of polynomials $K[a_0, \ldots, a_n, x, y]$ in the following way.

We define the action of U_n on the corresponding monomials of $\mathcal{U} = K[\alpha_1, \alpha_2, \ldots$ $\ldots, u_1, u_2]$:

$$\langle U_n | \alpha_1^k \alpha_2^{n-k} \rangle = a_k, \qquad \langle U_n | \alpha_1^j \alpha_2^k \rangle = 0 \quad \text{if } j + k \neq n \tag{2.4}$$

for any Greek umbral letter α,

$$\langle U_n | u_1^k \rangle = (-y)^k, \qquad \langle U_n | u_2^k \rangle = x^k$$

and the multiplicative rule:

$$\langle U_n | \alpha_1^i \alpha_2^j \beta_1^k \beta_2^l \ldots u_1^p u_2^q \rangle = \langle U_n | \alpha_1^i \alpha_2^j \rangle \langle U_n | \beta_1^k \beta_2^l \rangle \ldots \langle U_n | u_1^p \rangle \langle U_n | u_2^q \rangle. \tag{2.5}$$

These rules uniquely define, by linearity, the umbral operator U_n on the umbral space $\mathcal{U}$.

Every polynomial $I(a_0, \ldots, a_n, x, y)$ can be written as $\langle U_n | Q(\alpha_1, \alpha_2, \ldots, u_1, u_2) \rangle$ for some polynomial $Q \in \mathcal{U}$. The polynomial Q is called an umbral representation of the polynomial I and I is called the umbral evaluation of Q. It is easy to see that, for the monomial $I = a_0^{d_0} a_1^{d_1} \ldots a_n^d n x^\delta y^\rho$ we have

$$I = \langle U_n | \underbrace{\alpha_1^0 \alpha_2^n \ldots \gamma_1^0 \gamma_2^n}_{d_0 \text{ times}} \underbrace{\delta_1^1 \delta_2^{n-1} \ldots \varepsilon_1^1 \varepsilon_2^{n-1}}_{d_1 \text{ times}} \ldots (-u_1^\delta) u_2^\rho \rangle.$$

Usually the umbral representation of a polynomial I is far from being unique.

The $\tilde{\nu}$-action of $Gl_2(K)$ is implied by the corresponding action of $Gl_2(K)$ on the umbral space $\mathcal{U}$. We derive this action in the following way. Let (c_{ij}) be defined as in (2.3). Then the corresponding change of umbral variables, say between the Greek letters $\alpha = (\alpha_1, \alpha_2)$ and $\overline{\alpha} = (\overline{\alpha}_1, \overline{\alpha}_2)$, is defined as follows:

$$\alpha_1 = [\overline{\alpha} c], \qquad \alpha_2 = [\overline{\alpha} d], \tag{2.6}$$

where $c = (-c_{21}, c_{11})$, $d = (-c_{22}, c_{12})$ and $[vw]$ is the determinant of a two by two matrix formed by two pairs $v = (v_1, v_2)$, $w = (w_1, w_2)$.

Using this notation we can easily express the umbral representation of any polynomial $I(\overline{a}_0, \ldots, \overline{a}_n, \overline{x}, \overline{y})$ in terms of the umbral representation of $I(a_0, \ldots, a_n, x, y)$, namely (see [12], p. 33), let $I \in K[a_0, \ldots, a_n, x, y]$, $I = \langle U_n | P(\alpha_1, \alpha_2, \ldots, u_1, u_2) \rangle$. Then we have

$$I(\overline{a}_0, \ldots, \overline{a}_n, \overline{x}, \overline{y}) = \left\langle U_n | P([\alpha c], [\alpha d], \ldots, \frac{[uc]}{[cd]}, \frac{[ud]}{[cd]}) \right\rangle. \tag{2.7}$$

By (2.7) we easily obtain the following explicit expression for the representation ν (cf. [12], p. 34),

$$\begin{aligned} \overline{a}_k &= \langle U_n | [\alpha c]^k [\alpha d]^{n-k} \rangle \\ &= \sum_{m=0}^{n} \left(\sum_{i=m-n+k}^{\min(m,k)} \binom{k}{i} \binom{n-k}{m-i} c_{11}^i c_{12}^{m-i} c_{21}^{k-i} c_{22}^{n-k-m+i} \right) a_m. \end{aligned} \tag{2.8}$$

We see that the bracket monomials, say $[\alpha\beta]$, $[\alpha u]$, etc. are covariants of index 1, i.e., $[\alpha\beta] = [cd][\overline{\alpha}\overline{\beta}]$, $[\alpha u] = [cd][\overline{\alpha}\overline{u}]$. Thus we can consider, in the umbral space $\mathcal{U}$, the subspace of bracket polynomials defined as the linear combinations of bracket monomials, i.e. the nonconstant polynomials in $\mathcal{U}$ which can be written as a product of brackets, say $[\alpha\beta][\alpha\delta]\ldots[wu]$.

The fundamental theorem of invariant theory of binary forms in the umbral approach can be formulated as follows.

THEOREM 2.1. (cf. [12]). (A) *The umbral evaluation $\langle U_n|P\rangle$ of a bracket polynomial P, for which in every bracket monomial M the number of brackets in M containing only Greek letters is constant and equal to $g \in N$, is a covariant of index g.*

(B) *Let I be a covariant of index g of binary forms of degree n. Then there exists a bracket polynomial P of index g (i.e. with the same number of brackets containing only Greek letters in each monomials involved in P) such that $I = \langle U_n|P\rangle$.*

One can find the proof of this theorem as well as the exhaustive account of its applications in [12].

EXAMPLE 2.2. Using the bracket representation we can easily calculate, by the appropriate algorithm, the corresponding basic invariants for binary quadric, binary cubic and binary quartic forms, namely

1. Binary quadric:

$$D = \langle U_2|[\alpha\beta]^2\rangle \qquad (= a_0a_2 - a_1^2 \text{ — discriminant}).$$

2. Binary cubic:

$$\Delta = \langle U_3|[\alpha\beta]^2[\alpha\gamma][\beta\delta][\gamma\delta]^2\rangle.$$

3. Binary quartic:

$$I = \langle U_4|[\alpha\beta]^4\rangle,$$
$$J = \langle U_4|[\alpha\beta]^2[\alpha\gamma]^2[\beta\gamma]^2\rangle.$$

In what follows we need to express evaluations of covariants on the subspaces of binary forms with multiple linear factors. Thus we introduce the so-called apolar covariant (see [12], p. 60). Let M^{n+1}, M^{m+1}, $n \geq m$, be two spaces of binary forms of degree n and m, respectively. Umbrally one can write $\theta_n = \langle U_n|[\alpha u]^n\rangle$, $\theta_m = \langle U_m|[\beta u]^m\rangle$. We define the umbral operator $U_{n,m}$ acting on the space of polynomials $K[\alpha,\beta]$, namely: $\langle U_{n,m}|\alpha_1^i\alpha_2^j\rangle = a_i$ if $i + j = n$ and zero otherwise, and $\langle U_{n,m}|\beta_1^k\beta_2^l\rangle = b_k$ if $k + l = m$ and zero otherwise. Thus we define the apolar covariant

$$\langle\theta_n|\theta_m\rangle = \langle U_{n,m}|[\alpha\beta]^m[\alpha u]^{n-m}\rangle, \qquad ([12], \text{p. } 60)$$

as a bilinear mapping from $M^{n+1} \times M^{m+1}$ to M^{n-m+1}, which is jointly covariant on $M^{n+1} \times M^{m+1}$. If $f \in M^{n+1}$, $g \in M^{m+1}$ are two binary forms we also write the corresponding evaluation

$$\langle\theta_n|\theta_m\rangle(f,g) = \langle f|g\rangle. \tag{2.9}$$

If $\langle f|g\rangle$ is zero form then f and g are said to be *apolar forms*. Let us define $\partial_x^k f = \partial^k f/\partial x^k$ and, by abuse of notation, we will view $\langle f|\partial_x^k f\rangle = \langle\theta_n|\theta_{n-k}\rangle(f,\partial_x^k f)$ as a covariant of binary forms of degree n.

By the straightforward calculations using the explicit formula for (2.9) we obtain another expression of polynomial identities mentioned in [9] (Theorem 2).

PROPOSITION 2.3. (a) *Let n be even. Then the apolar covariant $\langle f|f\rangle$ is an invariant of binary forms of degree n. It can be expressed by*

$$\langle f|f\rangle = \frac{1}{n!}(-1)^n \sum_{k=0}^{n}(-1)^k \partial_x^k f \partial_x^{n-k} f = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k} a_k a_{n-k}. \tag{2.10}$$

(b) *Let n be odd, say $n = 2j+1$. Then the second apolar covariant $\langle f|\partial_x f\rangle$ can be expressed by*

$$\langle f|\partial_x f\rangle = \frac{1}{n!}\sum_{k=0}^{j}(-1)^k(n-2k)\partial_x^k f \partial_x^{n-k} f$$
$$= \sum_{k=0}^{n-1}(-1)^k n\binom{n-1}{n-k-1} a_{n-k}(x a_{k+1} + y a_k). \tag{2.11}$$

COROLLARY 2.4. (a) *Let n be even and let f have a linear factor of degree greater than $n/2$. Then $\langle f|f\rangle = 0$.*

(b) *Let $n = 2j+1$ and f have a linear factor of degree greater than $j+1$. Then $\langle f|\partial_x f\rangle = 0$ for all (x, y).*

Proof: (a) $\langle f|f\rangle$ is invariant so we can choose the variables x, y, in which x^m, $m > n/2$, is a factor of f so $a_i = 0$ for $i = 0, \ldots, m$. Now by (2.10) we obtain $\langle f|f\rangle = 0$ (this is Hilbert zero-form [19]).

(b) We see that on one side $\langle f|\partial_x f\rangle$ has a multiple linear factor of degree at least two (because $k \leq j$ in differential factors $\partial_x^k f$ of the formula (2.11). However, from the right-hand side of (2.11) $\langle f|\partial_x f\rangle$ is a linear form, which gives a contradiction unless $\langle f|\partial_x f\rangle$ is zero. ∎

Remark 2.5: Instead of binary forms we can consider polynomials $f(x, 1)$. Let us denote $\sum_k(n)$ the set of all polynomials of degree n having at least one root of comultiplicity ($= n$ — multiplicity) less than or equal to k. We see that differentiation of polynomials preserves the comultiplicity of roots and defines the mapping $\phi_k(n) : \sum_k(n+1) \to \sum_k(n)$. Using Proposition 2.3 and Corollary 2.4 it can be proved (see e.g. [9], p. 14) that if $k \leq \frac{n-1}{2}$, then ϕ_k is a diffeomorphism of affine algebraic varieties. In this paper we use the umbral methods to prove fundamental results concerning the symplectic geometry of polynomials, some of them already mentioned in [4, 6], and to show the effective directions for further generalizations.

3. Umbral derivation of tensor invariants of binary forms

Let $\widetilde{\mathcal{U}}$ be the elementary umbral space with one umbral letter α. $\mathcal{U}_n(\alpha)$ denotes the subspace of $\widetilde{\mathcal{U}}$ of all homogeneous polynomials of degree n. By $D_n(\alpha)$ we denote the vector space of all differential symbols $a_1(\alpha)d\alpha_1 + a_2(\alpha)d\alpha_2$ with coefficients a_1, a_2 belonging to $\mathcal{U}_{n-1}(\alpha)$. The corresponding action, say μ, of $Gl_2(\boldsymbol{K})$ on $D_n(\alpha)$ is canonically induced from the standard $Gl_2(\boldsymbol{K})$-action on $\boldsymbol{K}^2$, namely

$$Gl_2(\boldsymbol{K}) \times \boldsymbol{K}^2 \ni \left(\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right) \to \begin{pmatrix} [\alpha c] \\ [\alpha d] \end{pmatrix} = \begin{pmatrix} \alpha_1 c_{11} + \alpha_2 c_{21} \\ \alpha_1 c_{12} + \alpha_2 c_{22} \end{pmatrix} \in \boldsymbol{K}^2, \quad (3.1)$$

(cf. (2.6)).

Let $E_n(\alpha) \subset D_n(\alpha)$ be the subspace of exact differential formal 1-forms. $E_n(\alpha)$ is generated by the following linearly independent elements: $\{d(\alpha_1^k \alpha_2^{n-k})\}_{k=0}^n$. Let $K_n(\alpha)$ denote the subspace of $D_n(\alpha)$ generated by $\{\alpha_1^r \alpha_2^{n-r-2}[\alpha d\alpha]\}_{r=0}^{n-2}$, where $[\alpha d\alpha]$ is an ordinary bracket with $d\alpha = (d\alpha_1, d\alpha_2)$.

PROPOSITION 3.1. *There exists a uniquely defined $Gl_2(\boldsymbol{K})$-equivariant projection $\boldsymbol{P}$: $D_n(\alpha) \to E_n(\alpha)$, $\operatorname{Ker} \boldsymbol{P} = K_n(\alpha)$.*

Proof: $E_n(\alpha)$, $K_n(\alpha)$ are $Gl_2(\boldsymbol{K})$-invariant subspaces of $D_n(\alpha)$, $\dim E_n(\alpha) = n + 1$, $\dim K_n(\alpha) = n - 1$, $\dim D_n(\alpha) = 2n$. Let us choose the following basis in $D_n(\alpha)$:

$$e_{p,i} = \alpha_1^p \alpha_2^{n-p-1} d\alpha_i, \qquad p = 0, \ldots, n-1, \qquad i = 1, 2.$$

Then the corresponding generators for $E_n(\alpha)$ and $K_n(\alpha)$ can be expressed as

$$d(\alpha_1^k \alpha_2^{n-k}) = k e_{k-1,1} + (n-k) e_{k,2}; \qquad k = 0, \ldots, n$$

and

$$\alpha_1^r \alpha_2^{n-r-2}[\alpha d\alpha] = e_{r+1,2} - e_{r,1}; \qquad r = 0, \ldots, n-2$$

respectively.

The associated $2(n-1) \times 2(n-1)$ matrix

$$\left(\begin{array}{ccc|ccc} 1 & & 0 & n-1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 1 & 0 & & 1 \\ \hline -1 & & 0 & 1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & -1 & 0 & & 1 \end{array} \right)$$

has the same rank as the following matrix:

$$\left(\begin{array}{ccc|ccc} 1 & & 0 & n-1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & n-1 & 0 & & 1 \\ \hline & & & n & & \\ & & & & \ddots & \\ & 0 & & & & \frac{n}{n-1} \end{array} \right)$$

Thus $D_n(\alpha) = E_n(\alpha) \oplus K_n(\alpha)$. The action μ restricted to $E_n(\alpha)$ and $K_n(\alpha)$ is irreducible, which implies the uniqueness of the projection $\boldsymbol{P}$. ∎

Now we can define the umbral operator U_α^* into the space of differential 1-forms over the space of binary forms M^{n+1}. Let U be the standard umbral operator (see Section 2) defined on $\widetilde{\mathcal{U}} = \boldsymbol{K}[\alpha_1, \alpha_2]$. Let U' be the restriction of U to the space $\mathcal{U}_n(\alpha)$. Now we introduce an operator $\overline{U}$ defined only on $E_n(\alpha) \subset D_n(\alpha)$ and satisfying the following commutation relation:

$$d \circ U' = \overline{U} \circ d. \tag{3.2}$$

Thus for the elements of the basis of $E_n(\alpha)$ we have

$$\langle \overline{U} | d(\alpha_1^k \alpha_2^{n-k}) \rangle = da_k, \qquad k = 0, \dots, n. \tag{3.3}$$

DEFINITION 3.2. The linear operator

$$U_\alpha^* := \overline{U} \circ \boldsymbol{P} \colon D_n(\alpha) \to V^{n+1}, \tag{3.4}$$

defined from the umbral space $D_n(\alpha)$ to the space V^{n+1} of differential 1-forms on M^{n+1}, is called the *elementary umbral operator* for the space of binary forms of degree n.

PROPOSITION 3.3. *U_α^* is a $Gl_2(\boldsymbol{K})$-equivariant linear operator.*

Proof: On the basis of Proposition 3.1, $\boldsymbol{P}$ is $Gl_2(\boldsymbol{K})$-equivariant so we have to prove that $\overline{U}$ is also $Gl_2(\boldsymbol{K})$-equivariant. In fact, using the bracket notation (3.1) we have (cf. [12]) that for any polynomial $I \in \boldsymbol{K}[a_0, \dots, a_n]$ and its umbral representation $P \in \boldsymbol{K}[\alpha_1, \alpha_2, \dots, \beta_1, \beta_2]$ the change of variables (c, d) is expressed (by the formula (2.7)) as follows:

$$\begin{aligned} I(\bar{a}_0, \dots, \bar{a}_n) &= \langle U(\bar{f}) | P(\alpha_1, \alpha_2, \dots, \beta_1, \beta_2) \rangle \\ &= \langle U(f) | P([\alpha c], [\alpha d], \dots, [\beta c], [\beta d]) \rangle. \end{aligned}$$

Thus for the operator $\overline{U}$ we have (see (3.2)),

$$\begin{aligned} \langle \overline{U}(f) | d([\alpha c]^k [\alpha d]^{n-k}) \rangle &= d\langle U'(f) | [\alpha c]^k [\alpha d]^{n-k} \rangle \\ &= d\langle U'(\bar{f}) | \alpha_1^k \alpha_2^{n-k} \rangle = d\bar{a}_k, \qquad k = 0, \dots, n, \end{aligned}$$

where by $\bar{f}(\bar{x}, \bar{y}) = \sum_{k=0}^{n} \binom{n}{k} \bar{a}_k \bar{x}^k \bar{y}^{n-k}$ we denote the transformed binary forms in the new variables, and $U(f)$, $U(\bar{f})$ denote the corresponding umbral operator U written using these two types of variables. This completes the proof. ∎

By the extension of U_α^* to the tensor product of p factors, say $W_{n,p} = D_n(\alpha) \otimes \dots \otimes D_n(\beta)$, we obtain the partial umbral operator for representing the corresponding tensor invariants of degree p:

$$\begin{gathered} U_{(\alpha, \dots, \beta)}^* : D_n(\alpha) \otimes \dots \otimes D_n(\beta) \to \otimes^p V^{n+1}, \\ \langle U_{(\alpha, \dots, \beta)}^* | w_1(\alpha) \otimes \dots \otimes w_p(\beta) \rangle = \langle U_\alpha^* | w_1(\alpha) \rangle \otimes \dots \otimes \langle U_\beta^* | w_p(\beta) \rangle. \end{gathered} \tag{3.5}$$

The formula computing the effect of a change of variables in the standard umbral representations of polynomial invariants (cf. [12, 16]) as well as Proposition 3.3, suggest a subspace of the umbral space $W_{n,p}$ whose umbral evaluations are obviously invariants. Let us define the bracket monomials, say for two umbral letters:

$$\begin{aligned} [\alpha\beta] &= \alpha_1 \beta_2 - \alpha_2 \beta_1 \\ [\alpha d\beta] &= \alpha_1 d\beta_2 - \alpha_2 d\beta_1 \\ [d\alpha \otimes d\beta] &= d\alpha_1 \otimes d\beta_2 - d\alpha_2 \otimes d\beta_1. \end{aligned} \tag{3.6}$$

DEFINITION 3.4. A bracket monomial $q \in W_{n,p}$ is a nonconstant polynomial in $W_{n,p}$ which can be written as an appropriate product of brackets (3.6). A bracket polynomial is a linear combination of bracket monomials. The linear subspace of $W_{n,p}$ formed by the bracket polynomials is denoted by $B_{n,p}$.

The total umbral space is defined as $W_n = \bigoplus_{p \in \mathbf{N}} W_{n,p}$ and the corresponding umbral operator U^* is defined as the respective direct sum of the partial umbral operators. Let $g \in \mathbf{N}$. A nonconstant polynomial $Q = \bigoplus_{p \in \mathbf{N}} \overset{p}{\otimes} V^{n+1}$ is said to be an invariant of index g if for all binary forms $f(x,y)$ of degree n and for all linear changes of variables (c,d) the following identity holds:

$$\overline{Q} = [cd]^g Q.$$

The aim of this section is to provide the methods for determining, as explicitely as possible, all the tensor invariants of binary forms.

The index of a bracket monomial $q \in W_n$ is the number of brackets in q. The bracket polynomials in W_n which are linear combinations of bracket monomials all of the same index g, are called *bracket polynomials* of index g.

THEOREM 3.5. *Let $U^*_{(\alpha,\ldots,\beta)}$ be the umbral operator into the tensor space $\overset{p}{\otimes} V^{n+1}$. Let $\phi \in B_{n,p}$ be the bracket polynomial of index g. Then the umbral evaluation of ϕ, $\langle U^*_{(\alpha,\ldots,\beta)} | \phi \rangle$, is an invariant of index g.*

Proof: Let q be a bracket monomial of index g. Let us change the umbral variables, i.e.

$$[\overline{\alpha}\overline{\beta}] = \det \begin{pmatrix} [\alpha c] & [\beta c] \\ [\alpha d] & [\beta d] \end{pmatrix} = [cd][\alpha\beta]$$

$$[\alpha d\beta] = \det \begin{pmatrix} [\alpha c] & [d\beta c] \\ [\alpha d] & [d\beta d] \end{pmatrix} = [cd][\alpha d\beta]$$

$$[d\alpha \otimes d\beta] = \det \begin{pmatrix} [d\alpha c] & [d\beta c] \\ [d\alpha d] & [d\beta d] \end{pmatrix} = [cd][d\alpha \otimes d\beta].$$

Thus, for any binary form $f(x,y)$ and for any change of variables (c,d), we have on the basis of Proposition 3.3,

$$\begin{aligned} \langle U^*_{(\alpha,\ldots,\beta)}(\bar{f}) | q \rangle &= \langle U^*_{(\alpha,\ldots,\beta)}(f) | [cd]^g q \rangle \\ &= [cd]^g \langle U^*_{(\alpha,\ldots,\beta)}(f) | q \rangle, \end{aligned}$$

which implies that $\langle U^*_{(\alpha,\ldots,\beta)} | \phi \rangle$ is a tensor invariant of index g and degree p. ∎

The converse of this theorem is also true.

THEOREM 3.6. *Let Q be a tensor invariant of index g and degree p for binary forms of degree n. Then there exists a bracket polynomial $\phi \in B_{n,p}$ of index g such that*

$$Q = \langle U^*_{(\alpha,\ldots,\beta)} | \phi \rangle.$$

Proof: In order to prove this theorem we must first study the combinatorics of bracket polynomials (cf. [12]). Let us order the umbral alphabet in such a way that $\alpha < \beta < \ldots <$

$d\alpha < d\beta < \ldots$. Let q be a bracket monomial. Thus q is a product, say $[\alpha\beta][\alpha\gamma]\ldots[\delta d\delta]\otimes[d\varepsilon\otimes d\rho]$ of s brackets. Using the notation introduced in [12] we rewrite q as a tableau of height s:

$$\begin{bmatrix} \alpha\ \beta \\ \alpha\ \gamma \\ \vdots \\ \delta\ d\delta \\ d\varepsilon\ d\rho \end{bmatrix} = [\alpha\beta][\alpha\gamma]\ldots[\delta d\delta]\otimes[d\varepsilon\otimes d\rho].$$

We will call such a tableau ordered if the letters in each row are increasing from left to right, and the letters in each column are nondecreasing from top down.

We easily see, using the *syzygy* relations

$$\begin{aligned} [\alpha\delta][\beta\gamma] &= [\alpha\gamma][\beta\delta] - [\alpha\beta][\gamma\delta], \\ [\gamma\delta][d\alpha\otimes d\beta] &= [d\alpha\gamma]\otimes[d\beta\delta] - [d\alpha\delta]\otimes[d\beta\gamma], \\ [\gamma\delta][d\alpha\beta] &= [d\alpha\gamma][\beta\delta] - [d\alpha\delta][\beta\gamma] \end{aligned}$$

and antisymmetry

$$\begin{aligned} [\alpha\beta] &= -[\beta\alpha], \\ [d\alpha\otimes d\beta] &= -[d\beta\otimes d\alpha] = d\alpha_1\otimes d\beta_2 - d\alpha_2\otimes d\beta_1 \end{aligned}$$

that the ordered bracket monomials form a basis for the vector space of all bracket polynomials. In fact, treating the differentials combinatorially as ordered symbols and assuming the existence of the nontrivial linear dependence relation between ordered bracket monomials with smallest number of distinct symbols and smallest height of bracket monomials, we come easily to a contradiction setting two highest symbols to be equal (cf. [12], p. 37).

Let Q be a tensor invariant of index g and degree p, let $p \in W_{n,p}$ be its umbral representation. As Q is an invariant, we have for any change of variables (c, d)

$$\overline{Q} = [cd]^g Q = \langle U^*_{(\alpha,\ldots,\beta)} | P([\alpha c],[\alpha d],\ldots,[\beta c],[\beta d])\rangle. \tag{i}$$

This identity is true as a polynomial identity in the variables c_1, c_2, d_1, d_2. Using this fact we can prove that $P([\alpha c],[\alpha d],\ldots,[\beta c],[\beta d]) = [cd]^g R(\alpha_1,\alpha_2,\ldots,\beta_1,\beta_2)$, where $R \in B_{n,p}$. We can easily see that the polynomial P may be so chosen that the letter c, as well as the letter d, occurs exactly g times in each of the ordered monomials q_k contained in the expansion of $P([\alpha c],[\alpha d],\ldots)$ as a linear combination of ordered bracket monomials. Here the new alphabet $\{c, d, \alpha, \beta, \ldots, d\alpha, d\beta, \ldots, d\varepsilon\}$ is ordered as follows: $c < d < \alpha < \beta \ldots < d\varepsilon$. In fact, replacing c_1 and c_2 by rc_1 and rc_2, we obtain the polynomial identity

$$r^g[cd]^g Q = \langle U^*_{(\alpha,\ldots,\beta)} | \sum_k b_k r^{c(k)} q_k\rangle,$$

where $c(k)$ is the number of occurrences of c in the bracket monomial q_k. Equating coefficients of r^g we see that the bracket monomials with $c(k) \neq g$ can be omitted in P. Let q_k be a bracket monomial in this improved expansion of P as a linear combination of ordered bracket monomials. Let $s(k)$ be the number of brackets $[cd]$ occurring in q_k.

Let s be the minimum of these integers $s(k)$. We see that $s \leq g$. If $s = g$ we can simply cancel $[cd]^g$ from both sides of (i). Let us suppose $s < g$, writing $q_k = [cd]^s q'_k$ we can cancel $[cd]^s$ from both sides of (i) to obtain

$$[cd]^{q-s}Q = \langle U^*_{(\alpha,\ldots,\beta)} | \sum_k b_k q'_k \rangle. \tag{ii}$$

Treating (ii) as a polynomial identity in the variables c_1, c_2, d_1, d_2, we can therefore set $c_1 = d_1$, $c_2 = d_2$. This yields the identity:

$$\langle U^*_{(\alpha,\ldots,\beta)} | \sum_k b_k \widehat{q}_k \rangle = 0, \tag{iii}$$

where $\widehat{q}_k$ is obtained from q'_k by setting $c = d$.

As we know the ordered bracket monomials q'_k as well as $\widehat{q}_k$ are linearly independent. Because $\sum_k b_k \widehat{q}_k \in \operatorname{Ker} U^*_{(\alpha,\ldots,\beta)}$ and $\widehat{q}_k$ are linearly independent, each monomial $\widehat{q}_k$ can be written as a tableau

$$\begin{bmatrix} c\ * \\ \vdots \\ c\ * \\ \vdots \\ *\ * \\ \vdots \\ \varepsilon\ d\varepsilon \\ \vdots \\ *\ * \end{bmatrix}, \tag{iv}$$

where c occurs in the first $2(g{-}s)$ rows and an asterisk stands for the rest of letters. By inspection of (iv) we can deduce the corresponding elements q'_k. They can be written in the form

$$\begin{bmatrix} c\ * \\ \vdots \\ c\ * \\ d\ * \\ \vdots \\ d\ * \\ *\ * \\ \vdots \\ \varepsilon\ d\varepsilon \\ \vdots \\ *\ * \end{bmatrix},$$

where c occurs as the first letter in the first $g{-}s$ rows and d occurs as the first letter in the next g–s rows with the additional bracket $[\varepsilon d\varepsilon]$ standing also in (iv). Thus we have that

$$\sum_k b_k q'_k \in \operatorname{Ker} U^*_{(\alpha,\dots,\beta)}$$

and we can cancel in the original polynomial P all terms giving the smallest number s of brackets $[cd]$ occurring in q_k. Repeating this procedure we obtain the result of Theorem 3.6. ∎

COROLLARY 3.7. *On the space of binary forms of odd degree the odd degree tensor invariants do not exist.*

Proof: On the basis of Theorem 3.6, for the tensor invariant Q of degree p and index g we can write

$$\overline{Q} = \langle U^*_{(\alpha,\dots,\beta)} | R([\alpha c], [\alpha d], \dots)\rangle = [cd]^g Q,$$

where R is the corresponding bracket polynomial. On the other hand, taking the new parameters $c \to tc$, $d \to td$ we obtain the following equality (polynomial in t):

$$t^{np}\langle U^*_{(\alpha,\dots,\beta)} | R([\alpha c], [\alpha d], \dots)\rangle = t^{2g}[cd]Q.$$

Thus we see that for odd numbers n only for even number p the integer g can exists. ∎

4. Invariant symplectic structure on the space of binary forms

Let us give now the complete classification of the tensor invariants of degree two, i.e. we assume $p = 2$.

THEOREM 4.1. *There exists only one (up to constant multiples) tensor invariant of degree two on the space of binary forms.*

Proof: Let n be the degree of binary forms. $B_{n,2}$ is generated by the following basis of ordered bracket monomials: $v_1 = [\alpha\beta]^{n-1}[d\alpha \otimes d\beta]$, $v_2 = [\alpha\beta]^{n-2}[\alpha d\alpha] \otimes [\beta d\beta]$. We see that the third admissible bracket monomial $w = [\alpha\beta]^{n-2}[\beta d\alpha] \otimes [\alpha d\beta]$, by the appropriate *syzygy*, can be written as follows:

$$w = [\alpha\beta]^{n-2}([\alpha d\alpha] \otimes [\beta d\beta] - [\alpha\beta][d\alpha \otimes d\beta]) = v_2 - v_1.$$

We see also that $v_2 \in \operatorname{Ker} U^*_{(\alpha,\beta)}$. Thus

$$\dim(J_{n,2} = \operatorname{Im} U^*_{(\alpha,\beta)}|_{B_{n,2}}) = 1,$$

which completes the proof. ∎

Now we are asking for the normal forms of the corresponding tensor invariants.

PROPOSITION 4.2. *All tensor invariants of degree two on the space of binary forms of degree n are proportional to the following basic invariant*:

$$Q = \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^{j+1}((-1)^n da_{j+1} \otimes da_{n-j-1} + da_{n-j-1} \otimes da_{j+1}), \qquad (4.1)$$

i.e. $J_{n,2} = \{Q\}$.

Proof: On the basis of Theorem 4.1, as a generator of $J_{n,2}$ we can take the following invariant:

$$Q = \langle U^*_{(\alpha,\beta)}|n^2[\alpha\beta]^{n-1}[d\alpha \otimes d\beta]\rangle = \langle U^*_{(\alpha,\beta)}|n^2 \sum_{j=0}^{n-1} \binom{n-1}{j}(-1)^{n-j-1}$$
$$(e_{j,1} \otimes \widetilde{e}_{n-j-1,2} - e_{j,2} \otimes \widetilde{e}_{n-j-1,1})\rangle, \qquad \text{(i)}$$

where we denoted

$$e_{p,i} = \alpha_1^p \alpha_2^{n-p-1} d\alpha_i, \qquad \widetilde{e}_{p,i} = \beta_1^p \beta_2^{n-p-1} d\beta_i, \qquad p = 0, \ldots, n-1, \qquad i = 1, 2.$$

Recalling (3.5) we obtain the action of $U^*_{(\alpha,\beta)}$ on the basic elements:

$$\begin{aligned}
\langle U^*_{(\alpha,\beta)}|e_{k,1} \otimes \widetilde{e}_{p,1}\rangle &= (1/n^2) da_{k+1} \otimes da_{p+1}, \\
\langle U^*_{(\alpha,\beta)}|e_{k,1} \otimes \widetilde{e}_{p,2}\rangle &= (1/n^2) da_{k+1} \otimes da_{p}, \\
\langle U^*_{(\alpha,\beta)}|e_{k,2} \otimes \widetilde{e}_{p,1}\rangle &= (1/n^2) da_{k} \otimes da_{p+1}, \\
\langle U^*_{(\alpha,\beta)}|e_{k,2} \otimes \widetilde{e}_{p,2}\rangle &= (1/n^2) da_{k} \otimes da_{p}, \qquad 0 \le p, k \le n-1.
\end{aligned}$$

Applying these formulae to (i) we obtain the desired invariant Q of the proposition. ∎

From the formula (4.1) we see that Q is a symmetric tensor on the space of binary forms of even degree and antisymmetric if n is odd. Thus we have

COROLLARY 4.3. *On the space of binary forms of odd degree there exists only one (up to constant multiples) $Sl_2(\boldsymbol{K})$-invariant symplectic structure.*

Proof: Taking $n = 2k + 3$ in (4.1), after straightforward calculations we obtain

$$Q = \sum_{r=0}^{k+1} \binom{n}{r} (-1)^{r+1} da_r \wedge da_{n-r}, \tag{4.2}$$

which is a closed, nondegenerate, $Sl_2(\boldsymbol{K})$-invariant two-form. ∎

Remark 4.4: On the basis of (4.2) we can choose the symplectic form on M^{n+1} as follows:

$$\omega = n!(-1)^{k-1} \sum_{r=0}^{k+1} \binom{n}{r} (-1)^{r+1} da_r \wedge da_{n-r}. \tag{4.3}$$

Choosing the new coordinates

$$q_r = \frac{n!}{r!} a_{n-r}, \qquad p_r = (-1)^{k-r} \frac{n!}{(n-r)!} a_r, \qquad r = 0, \ldots, k+1, \tag{4.4}$$

on M^{n+1}, we can write (4.3) in the following Darboux form: (cf. [1, 18])

$$\omega = \sum_{j=0}^{k+1} dp_j \wedge dq_j,$$

where the elements of M^{n+1} can be written as follows:

$$\begin{aligned} M^{n+1} \ni f(x,y) &= q_0 \frac{x^{2k+3}}{(2k+3)!} + \ldots + q_{k+1} \frac{x^{k+2}y^{k+1}}{(k+2)!} - p_{k+1} \frac{x^{k+1}y^{k+2}}{(k+1)!} \\ &+ \ldots + (-1)^{k+2} p_0 y^{2k+3}. \end{aligned} \tag{4.5}$$

This is exactly the $\mathrm{Sl}_2(\boldsymbol{K})$-invariant symplectic structure mentioned only in [9] in the context of the generalized Newton equation as well as in the obstacle problem [5].

$\mathrm{Sl}_2(\boldsymbol{K})$ acts symplectically on (M^{n+1}, ω). Thus we have

PROPOSITION 4.5. *The momentum mapping corresponding to the standard* $\mathrm{Sl}_2(\boldsymbol{K})$*-action on* (M^{n+1}, ω) *is the* Ad^**-equivariant quadratic momentum mapping (cf.* [2]*). In the coordinates of* (4.5) *it can be written as follows*:

$$J : M^{n+1} \to \mathrm{sl}_2(\boldsymbol{K})^*; \qquad J(\overline{p}) = (H_+, H_-, H_d)(\overline{p}),$$

where

$$\begin{aligned} H_+(\overline{p}) &= \sum_{r=1}^{k+1} p_r q_{r-1} + \frac{1}{2} q_{k+1}^2, \\ H_-(\overline{p}) &= \sum_{r=0}^{k} (2k+3-r)(r+1) p_r q_{r+1} - \frac{1}{2}(k+1)^2 p_{k+1}^2, \\ H_d(\overline{p}) &= \sum_{r=0}^{k+1} (2r-2k-3) q_r p_r, \end{aligned} \tag{4.6}$$

and $\{H_+, H_-\} = H_d$, $n = 2k+3$.

Proof: Taking the standard decomposition of $\mathrm{Sl}_2(\boldsymbol{K})$ onto the three one-parameter subgroups (cf. [19])

$$A_+ : \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}; \quad A_- : \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}; \quad D : \begin{bmatrix} d^{-1} & 0 \\ 0 & d \end{bmatrix}, \quad \alpha, \beta, d \in \boldsymbol{K}, \tag{4.7}$$

we obtain the three corresponding Hamiltonian vector fields, say $\overline{A}_+$, $\overline{A}_-$, $\overline{D}$, with the corresponding Hamiltonians H_+, H_-, H_d. Thus, after straightforward calculations, the momentum mapping follows immediately (cf. [1, 2]).

Remark 4.6: Taking into account the relation $\{H_+, H_-\} = H_d$ we can reformulate the fundamental theorem (Cayley, Sylvester [16]) of the theory of invariants of binary forms of odd degree, i.e. we have: A polynomial $\phi(q,p)$ on the space of binary forms M^{n+1} is $\mathrm{Sl}_2(\boldsymbol{K})$-invariant if and only if the following identities are fulfilled: $\{H_+, \phi\} = 0$, $\{H_d, \phi\} = 0$. Thus we easily see that the algebra of polynomial $\mathrm{Sl}_2(\boldsymbol{K})$ invariants of binary forms is endowed with the canonical Poisson structure (cf. [18]).

In the obvious way, by the multiplicative rule, we can extend the umbral operator U^* (see Section 3) to have an umbral representation of tensor invariants with polynomial coefficients. In fact we define

$$\widetilde{W}_n = \bigoplus_{p,s\in \boldsymbol{N}} W_{n,p} \otimes_{\boldsymbol{K}} \boldsymbol{K}_s[\gamma,\dots,\delta],$$

where "s" is a number of umbral letters $\gamma,\dots,\delta$. Thus the corresponding extension of U^*, say $\widetilde{U}^*$, on the homogeneous elements of $\widetilde{W}_n$ is defined as follows:

$$\langle \widetilde{U}^* | \phi \otimes \gamma_1^i\gamma_2^j \dots \delta_1^k\delta_2^l\rangle = \langle U^*|\phi\rangle\langle U|\gamma_1^i\gamma_2^j\rangle \dots \langle U|\delta_1^k\delta_2^l\rangle$$

(cf. (2.5)).

After straightforward reformulations we immediately obtain the fundamental classification theorems, analogous to Theorem 3.5 and Theorem 3.6 of Section 3. As a simple example, we apply these theorems to classify all tensor invariants of degree one with coefficients being the linear polynomials in variables $a_0,\dots,a_n$.

PROPOSITION 4.7. *All tensor invariants of degree one with linear polynomial coefficients on the space of binary forms of degree* n *are proportional to the following basic invariant*:

$$I = \sum_{j=0}^{n} (-1)^{n-j-1}\binom{n}{j} a_{n-j}da_j. \tag{4.8}$$

Proof: Let us take the relations between elementary umbral monomials (cf. Proposition 3.1):

$$\begin{aligned} e_{k,1} &= \frac{1}{n}(w_{k+1} - (n-k-1)g_k),\\ e_{k,2} &= \frac{1}{n}(w_k + kg_{k-1}), \qquad k = 0,\dots,n-1, \end{aligned}$$

where

$$e_{k,i} = \alpha_1^k\alpha_2^{n-k-1}d\alpha_i, \qquad w_k = d(\alpha_1^k\alpha_2^{n-k}), \qquad g_k \in \operatorname{Ker} U_\alpha^*.$$

It is easy to see that the space of irredundant bracket polynomials in $D_n(\alpha)\otimes U_n(\beta)$ is spanned by the bracket monomial $n[\alpha\beta]^{n-1}[\beta d\alpha]$. Thus we have

$$\begin{aligned} \langle \widetilde{U}^*_{(\alpha;\beta)} | n[\alpha\beta]^{n-1}[\beta d\alpha]\rangle &= n\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n-1}{j}\langle U_\alpha^*|e_{j,2}\rangle\langle U|\beta_1^{n-j}\beta_2^j\rangle\\ &\quad - n\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n-1}{j}\langle U_\alpha^*|e_{j,1}\rangle\langle U|\beta_1^{n-j-1}\beta_2^{j+1}\rangle = I. \qquad \blacksquare \end{aligned}$$

COROLLARY 4.8. *Let* $n = 2k+3$. *Then the corresponding* $\mathrm{Sl}_2(\boldsymbol{K})$ *invariant one-form on the space of binary forms of degree* n, *in Darboux coordinates has the following form*:

$$\theta = \sum_{j=0}^{k+1}(p_j dq_j - dp_j q_j). \tag{4.9}$$

Remark 4.9: We know (cf. [4], p. 306) that the projective space $\boldsymbol{R}P^n$ of all zero-dimensional submanifolds of degree n in the projective line is endowed with the natural $\mathrm{Sl}_2(\boldsymbol{R})$-invariant contact structure (cf. [1, 6]). Indeed, we see that the appropriate $\mathrm{Sl}_2(\boldsymbol{R})$-invariant field of hyperplanes in $\boldsymbol{R}P^n$ is defined by the $\mathrm{Sl}_2(\boldsymbol{R})$-invariant 1-form $V \lrcorner\, \omega$, where $V = \sum_{i=0}^{n} a_i \frac{\partial}{\partial a_i}$ is $\mathrm{Sl}_2(\boldsymbol{R})$-invariant and ω is given in Corollary 4.3. In the affine part of $\boldsymbol{R}P^n$ formed by the zero-dimensional submanifolds of $\boldsymbol{R}P^1$, which do not contain the point in infinity $y = 0$, this contact structure is given by (see Corollary 4.8)

$$\theta|_{q_0=1} = \sum_{j=1}^{k+1}(p_j dq_j - q_j dp_j) - dp_0.$$

This is the canonical contact structure on the space of polynomials

$$\left\{\frac{x^{2k+3}}{(2k+2)!} + q_1\frac{x^{2k+2}}{(2k+2)!} + \ldots + q_{k+1}\frac{x^{k+2}}{(k+2)!} - p_{k+1}\frac{x^{k+1}}{(k+1)!} + \ldots + (-1)^{k+2}p_0\right\}.$$

Thus all the results concerning the symplectic geometry of polynomial spaces have a direct reformulation in terms of the above introduced contact geometry (cf. [4]). The more precise analysis of this case will be discussed in a forthcoming paper.

5. The hierarchy of apolar coisotropic manifolds and generalized open swallowtails

Let (M, ω) be a symplectic manifold. The new symplectic structures associated to (M, ω) are provided by the so-called coisotropic submanifolds in M (cf. [1, 18]). We recall that a submanifold $C \subseteq M$ is coisotropic if at each $x \in C$ we have $(T_xC)^{\S} = \{v \in T_xM; \langle v \wedge u, \omega\rangle = 0$, for every $u \in T_xC\} \subseteq T_xC$.

The distribution $\Gamma = \coprod_{x\in C}(T_xC)^{\S}$ is the characteristic distribution of $\omega|_C$. Let B be the space of characteristics of it and $\rho : C \to B$ be its canonical projection. It is known (cf. [18]) that if B admits a differentiable structure and ρ is a submersion, then there is a unique symplectic structure β on B such that $\rho^*\beta = \omega|_C$. The symplectic manifold (B, β) associated in this way with the triplet (M, ω, C) is called the reduced symplectic manifold (cf. [1]).

Let (M^{n+1}, ω) be the symplectic space of binary forms (see Section 4). The canonical subspaces in M^{n+1}, say $C^{(l)}$, $0 \le l \le \frac{n-1}{2}$ of all binary forms apolar to its l-derivatives with respect to x are called the canonical apolar subspaces.

PROPOSITION 5.1. *The canonical apolar subspaces* $C^{(l)}$, $(0 \le l \le \frac{n-1}{2})$ *form coisotropic varieties of* (M^{n+1}, ω).

Proof: We see that $C^{(l)}$ is described by the following system of $l + 1$ equations:

$$P_0^{(l)} = \binom{n-l}{0}a_na_0 - \binom{n-l}{1}a_{n-1}a_1 + \ldots \pm \binom{n-l}{n-l}a_la_{n-l} = 0,$$

$$P_1^{(l)} = \binom{n-l}{0}a_na_1 - \binom{n-l}{1}a_{n-1}a_2 + \ldots \pm \binom{n-l}{n-l}a_la_{n-l+1} = 0,$$

$$P_l^{(l)} = \binom{n-l}{0} a_n a_l - \binom{n-l}{1} a_{n-1} a_{l+1} + \ldots \pm \binom{n-l}{n-l} a_l a_n = 0.$$

After straightforward calculations we find that

$$\{P_i^{(l)}, P_j^{(l)}\}|_{C^{(l)}} = 0,$$

where $\{\phi, \psi\}$ is the appropriate Poisson bracket (cf. [1]);

$$\{\phi, \psi\} = \frac{1}{n!} \sum_{i=0}^{k+1} (-1)^{k-i} \binom{n}{i}^{-1} \left(\frac{\partial \phi}{\partial a_i} \frac{\partial \psi}{\partial a_{n-i}} - \frac{\partial \phi}{\partial a_{n-i}} \frac{\partial \psi}{\partial a_i} \right), \qquad n = 2k + 3.$$

Thus the system of equations $P_i^{(l)}(a_0, \ldots, a_n) = 0$, $i = 0, \ldots, l$ ($\langle f | f_x^{(l)} \rangle = 0$) form the coisotropic subvariety of (M^{n+1}, ω). ∎

Now we pay more attention to the particular case $l = 1$.

COROLLARY 5.2. *Let $l = 1$, then the second apolar coisotropic variety $C^{(1)}$ can be expressed as follows*:

$$C^{(1)} = \{f \in M^{n+1}; \langle f | f_x' \rangle = n(P_0^{(1)} y + P_1^{(1)} x) = (-1)^{k+1} \frac{1}{n!} (y H_d + 2 H_+ x) \underset{x,y}{\equiv} 0\},$$

where

$$\{P_1^{(1)}, P_0^{(1)}\} = \frac{1}{nn!} (-1)^{k+1} P_1^{(1)} \quad \textit{and} \quad \{H_+, H_d\} = H_+.$$

Proof: Immediate, on the basis of (4.6), Proposition 5.1 and simple but tedious calculations (see Proposition 2.3, see also Example 2 in [5] p. 45). ∎

To the space of binary forms of degree n one can easily associate the corresponding spaces of polynomials of one variable putting $y = 1$ in (2.1). In order to have the polynomial symplectic spaces adapted to the investigations of singularities in the variational obstacle problem (see [3, 5, 6, 14]) we associate to every symplectic space (M^{n+1}, ω) the canonically reduced symplectic space Q^{n-1} of polynomials of degree $n - 1$ where the leading term has constant coefficient $\frac{1}{(n-1)!}$. $Q^{n-1} = C_0/\sim$, where "$\sim$" is given by the coisotropic submanifold $C_0 := \{f \in M^{n+1}; n! a_n = 1\}$. Q^{n-1} is identified canonically with the space of derivatives $\frac{d}{dx}(f(x, 1))$, $f \in M^{n+1}$ belonging to C_0, namely

$$\begin{aligned} Q^{n-1} \ni \phi(x) = \frac{x^{2k+2}}{(2k+2)!} + q_1 \frac{x^{2k+1}}{(2k+1)!} \\ + \ldots + q_{k+1} \frac{x^{k+1}}{(k+1)!} - p_{k+1} \frac{x^k}{k!} + \ldots + (-1)^{k+1} p_1 \end{aligned} \tag{5.1}$$

endowed with the reduced symplectic structure

$$\omega' = \sum_{j=1}^{k+1} dp_j \wedge dq_j.$$

PROPOSITION 5.3. *The apolar subspaces $C^{(l)}$ ($l = 1, \ldots, \frac{n-1}{2}$) of (M^{n+1}, ω) induce the corresponding coisotropic subspaces of (Q^{n-1}, ω'), say $\widetilde{C}^{(l)}$ ($l = 1, \ldots, \frac{n-1}{2}$), described by*

$$\widetilde{C}^{(l)} = \{\phi \in Q^{n-1}; \widetilde{P}_s^{(l)}(q,p) = 0, s = 1, \ldots, l\}, \qquad l = 1, \ldots, k+1,$$

where

$$\widetilde{P}_s^{(l)} = \frac{(-1)^k}{n!} \sum_{i=1}^{k-s+1} \binom{n-l}{i} \binom{n}{i}^{-1} q_i p_i + \frac{1}{n!^2} \sum_{i=k-s+2}^{k+1} (-1)^i \binom{n-l}{i} i!$$
$$(n-s-i)! q_i q_{n-s-i} + \frac{(-1)^k}{n!^2} \sum_{i=k+2}^{n-l} \binom{n-l}{i} i!(n-s-i)! p_{n-i} q_{n-s-i} + a_s.$$

Proof: Let us observe that $(P_0^{(l)})^{-1}(0) \cap C_0$ is transversal to the characteristic distribution Γ of C_0. Thus $P_0^{(l)}$ does not give any constraint on the reduced space Q^{n-1}. It is easy to see also that the functions $P_s^{(l)}|_{C_0}$ ($l \geq s \geq 1$) are constant along the integral manifolds of distribution Γ. Thus we obtain the new coisotropic constraints defined by these functions on Q^{n-1}, which after straightforward calculations are expressed in the form (5.1). ∎

Now we investigate the properties of the symplectic space induced by the coisotropic submanifold $\widetilde{C}^{(1)}$ in (Q^{n-1}, ω'), $n = 2k+3$.

PROPOSITION 5.4. *The reduced symplectic space corresponding to the triplet $(Q^{n-1}, \omega', \widetilde{C}^{(1)})$ is identified with the following space of polynomials*

$$Z = \left\{ \frac{x^{2k+1}}{(2k+1)!} + q_1 \frac{x^{2k-1}}{(2k-1)!} + \ldots + q_k \frac{x^k}{k!} - p_k \frac{x^{k-1}}{(k-1)!} + \ldots + (-1)^k p_1 \right\} \tag{5.2}$$

endowed with the reduced symplectic form $\overline{\omega} = \sum_{i=1}^{k} dp_i \wedge dq_i$.

Proof: The function $\widetilde{P}_1^{(1)}$ as well as the Hamiltonian H_+ (see Corollary 5.2) corresponding to the one-parameter subgroup A_+ (cf. (4.7)) generates translations along variable x. Thus the space of characteristics of the coisotropic submanifold $\widetilde{C}^{(1)}$ can be immediately identified with the derivatives of polynomials:

$$\frac{x^{2k+2}}{(2k+2)!} + \overline{q}_1 \frac{x^{2k+1}}{(2k+1)!} + \ldots + \overline{q}_{k+1} \frac{x^{k+1}}{(k+1)!} - \overline{p}_{k+1} \frac{x^k}{k!} + \ldots + (-1)^{k+1} \overline{p}_1 \tag{5.3}$$

with an additional condition that the sum of all roots is equal to zero ($\overline{q}_1 = 0$) (cf. [6, 11]). This completes the proof. ∎

As a polynomial parametrization of characterictics of $\widetilde{C}^{(1)}$ by the parameter t, described in Proposition 5.4, we can write the following identification (cf. [10, 1], and (5.3) above),

$$\frac{(x-t)^{2k+1}}{(2k+1)!} + \overline{q}_1 \frac{(x-t)^{2k}}{(2k)!} + \ldots + \overline{q}_{k+1} \frac{(x-t)^k}{k!} - \overline{p}_{k+1} \frac{(x-t)^{k-1}}{(k-1)!} + \ldots + (-1)^k \overline{p}_2$$

$$= \frac{x^{2k+1}}{(2k+1)!} + q_1\frac{x^{2k-1}}{(2k-1)!} + \ldots + q_k\frac{x^k}{k!} - p_k\frac{x^{k-1}}{(k-1)!} + \ldots + (-1)^k p_1.$$

Thus we immediately have

COROLLARY 5.5. *Let $m \geq \left[\frac{n}{2}\right]$. Then the set of polynomials of Z having a root of multiplicity m, say $L^{(n)}_{m-1}$, form an isotropic (see [18]) variety in $(Z, \overline{\omega})$. The maximal isotropic variety, i.e. $m = \left[\frac{n}{2}\right]$, is a Lagrangian variety (cf. [11]) symplectomorphic, in the case of $n = 7$, to the system of rays on the smooth obstacle, with the highest generic singularity, so-called open swallowtail singularity (cf. [5, 6] and Fig. 1, below)*

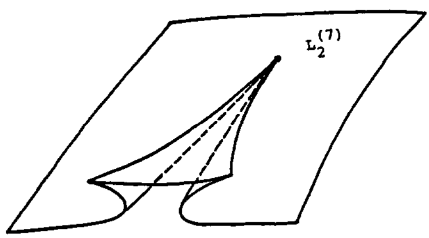

Fig. 1

Remark 5.6: Let us notice that the open swallowtail singularities in $(Z, \overline{\omega})$ are connected with the structure of the space of the Hilbert's zero-forms (cf. [17, 19]), and are quite exceptional. We can easily see that the variety V of polynomials in $(Z, \overline{\omega})$ with maximal possible number of double roots is not Lagrangian (cf. [5], p. 37).

One can easily check this for $k = 2$. In fact, we have

$$\frac{1}{5!}x^5 + q_1\frac{1}{3!}x^3 + q_2\frac{1}{2}x^2 - p_2x + p_1 = \frac{1}{5!}(x-\alpha)^2(x-\beta)^2(x-\gamma)$$

and the corresponding immersion of the smooth strata of V is the following:

$$q_1 = \tfrac{1}{20}(2w - 3z^2), \qquad q_2 = \tfrac{1}{30}(wz + z^3),$$
$$p_1 = \tfrac{1}{60}w^2 z, \qquad p_2 = \tfrac{1}{120}(4wz^2 - w^2),$$

where $z = \alpha + \beta$, $w = \alpha\beta$, $2\alpha + 2\beta + \gamma = 0$.

By straightforward calculations we obtain

$$\overline{\omega}|_V = dp_1 \wedge dq_1 + dp_2 \wedge dq_2|_V = \left(\tfrac{1}{450}w^2 + \tfrac{23}{1800}wz^2 - \tfrac{1}{300}z^4\right)dz \wedge dw \neq 0.$$

Following the theory of generating families for the germs of Lagrangian varieties presented in [11] one can describe the analytical structure of open swallowtails, i.e. $L^{(n)}_k$, using the polynomial functions. Let us recall that the function $F : Q \times \boldsymbol{R}^s \to \boldsymbol{R}$ is a generating family (with s-parameters) for the germ of Lagrangian variety $L \subseteq (T^*Q, \omega_Q)$ if L can be locally written in the following way (cf. [18]):

$$L = \{(q,p) \in T^*Q; \exists_{\lambda \in \boldsymbol{R}^s},\ \frac{\partial F}{\partial q}(q,\lambda) = p,\ \frac{\partial F}{\partial \lambda}(q,\lambda) = 0\}. \tag{5.4}$$

We see that $(Z,\bar{\omega})$ has a canonical cotangent bundle structure, $(Z,\bar{\omega}) \simeq (T^*Q,\omega_Q)$. Thus we are able to calculate the global generating families for the general open swallowtails $L_k^{(n)}$.

PROPOSITION 5.7. *An open k-dimensional swallowtail $L_k^{(n)} \subseteq (Z,\bar{\omega})$ is represented, in the form* (5.4), *by the following one-parameter generating family $G_k : Q \times \boldsymbol{R} \to \boldsymbol{R}$;*

$$\begin{aligned}
G_k(q,\lambda) = {} & \sum_{i=-1}^{k-2}\sum_{s=2}^{k-i-1}\sum_{u=2}^{s}\sum_{r=2}^{k-i} D^{(k)}_{k-i,s} A_{s-u} A_{k-i-r}\left(q_{u-1} + (-1)^u \frac{u-1}{u!}\lambda^u\right) \\
& \left(q_{r-1} + (-1)^r \frac{r-1}{r!}\lambda^r\right)\lambda^{n-u-r} + \frac{1}{2}\sum_{i=0}^{k-2}\sum_{u=2}^{k-i}\sum_{r=2}^{k-i} D^{(k)}_{k-i,k-i} A_{k-i-u} A_{k-i-r} \\
& \left(q_{u-1} + (-1)^u \frac{u-1}{u!}\lambda^u\right)\left(q_{r-1} + (-1)^r \frac{r-1}{r!}\lambda^r\right)\lambda^{n-u-r} + \sum_{i=0}^{k-2}\sum_{r=2}^{k-i} E^{(k)}_{k-i} A_{k-i-r} \\
& \left(q_{r-1} + (-1)^r \frac{r-1}{r!}\lambda^r\right)\lambda^{n-r} + \frac{1}{2} D^{(k)}_{k+1,k+1}\sum_{i=2}^{k+1}\sum_{r=2}^{k+1} A_{k+1-i}A_{k+1-r} \\
& \left(q_{i-1} + (-1)^i \frac{i-1}{i!}\lambda^i\right)\left(q_{r-1} + (-1)^r \frac{r-1}{r!}\lambda^r\right)\lambda^{n-i-r} + E^{(k)}_{k+1}\sum_{i=2}^{k+1} A_{k+1-i} \\
& \left(q_{i-1} + (-1)^i \frac{i-1}{i!}\lambda^i\right)\lambda^{n-i} - \frac{E_2^{(k)}}{2k+3}\lambda^{2k+3},
\end{aligned}$$

where

$$D^{(k)}_{r,s} = (-1)^{k-r}\sum_{j=s}^{k+1}\frac{(-1)^{j-s}}{(j-s)!(n-j-r)!},$$

$$E^{(k)}_r = (-1)^{k-r}\left(\frac{1}{(n-r)!} - \sum_{j=2}^{k+1}\frac{(-1)^j(j-1)}{j!(n-j-r)!}\right), \qquad 1 \le r,s \le k+1$$

and the numbers A_k are given by the following recurrential formulae:

$$A_0 = 1, \qquad A_k = \sum_{i=1}^{k}\frac{1}{i!}(-1)^{i+1}A_{k-i}.$$

Proof: On the basis of Proposition 4.2 in [11] and the formulae for the characteristic curves of $\widetilde{C}^{(1)}$. After straightforward calculations we obtain the corresponding generating one-parameter families for the open swallowtails in all dimensions. ∎

EXAMPLE 5.8. Let be $k = 1,2$, then the corresponding generating families for the

cusp singularity (one-dimensional open swallowtail) and the standard (two-dimensional [3]) open swallowtail singularity of Lagrangian varieties can be written directly, by Proposition 5.7, in the following way:
cusp:

$$G_1(q,\lambda) = -\tfrac{1}{40}\lambda^5 - \tfrac{1}{6}\lambda^3 q - \tfrac{1}{2}\lambda q^2, \tag{5.5}$$

open swallowtail:

$$G_2(q_1,q_2,\lambda) = -\tfrac{1}{576}\lambda^7 - \tfrac{1}{30}\lambda^5 q_1 - \tfrac{1}{24}\lambda^4 q_2 - \tfrac{1}{6}\lambda^3 q_1^2 - \tfrac{1}{2}\lambda^2 q_1 q_2 - \tfrac{1}{2}\lambda q_2^2. \tag{5.6}$$

Remark 5.9: (singularities in the obstacle geometry [6]): Let Q be a hypersurface in $\boldsymbol{R}^3$. $T^*\boldsymbol{R}^3$ is the phase space of free particle. We take the hypersurface Y; $Y = \{(x,p) \in T^*\boldsymbol{R}^3;\ H(x,p) = \frac{1}{2}(|p|^2 - 1) = 0\}$. Let M denote the symplectic manifold of integral lines of the characteristic distribution of Y. $\pi : Y \to M$ is the canonical projection along the integral lines. M can be identified with symplectic manifold of oriented lines in $\boldsymbol{R}^3$, $M \cong T^*S^2$ (cf. [6]). Let γ be a geodesic flow on Q (determined by the point source of light in the space, [14]). Let $\widetilde{L} \subseteq Y$ be the submanifold formed by versors tangent to geodesics of γ along the surface Q.

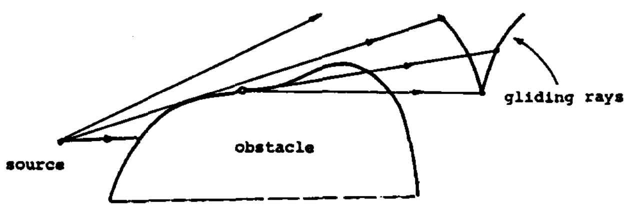

Fig. 2

PROPOSITION (cf. [3, 14]). (A) $L = \pi(\widetilde{L})$ *is a Lagrangian subvariety of* $(M,\widetilde{\omega})$. L *is singular in the asymptotic points of* γ (*i.e. the corresponding line of* L *is also an asymptotic direction on* Q) *in a hyperbolic region of* Q. *Typically the asymptotic points of* γ *form a curve, say* $\ell \subseteq Q$.

(B) *Let* $P_0 \in \ell$ *be such that the corresponding geodesic of* γ *going through* p_0 *is tangent to* ℓ *in* p_0. *Then the corresponding germ of Lagrangian variety* $(\pi(\widetilde{L}), w_0)$ *is the open swallowtail singularity* (*cf.* [3]) *symplectomorphic to* $L_2^{(7)}$ *described in Corollary* 5.5 (*the corresponding variety of rays gliding along the obstacle on the plane with the inflection point, is illustrated in Fig.* 2). ∎

Using the Huyghens principle (cf. [7, 8]) one can express the asymptotic intensity of radiation in the presence of an obstacle by the appropriate rapidly oscillating integrals with singular stationary varieties represented by the corresponding phase functions (optical distances), say

$$\int_{\boldsymbol{R}^n} e^{i\tau\phi(x,\lambda)} a(x,\lambda,\tau) d\lambda, \qquad \tau \to \infty. (5.7)$$

For the open swallowtail singularities the phase functions (families) are indicated, by Proposition 5.7, in the following way:

Let us take the product symplectic manifold

$$\Xi = (T^*\boldsymbol{R}^3 \times M, \tilde{\omega}\Theta\omega_{\boldsymbol{R}^3}),$$

(see [11]).

We know that graph $\pi \subseteq \Xi$ is a Lagrangian submanifold of Ξ. Then there exists its local Morse family (cf. [18]), say $K : \boldsymbol{R}^3 \times X \times \boldsymbol{R}^\mu \to \boldsymbol{R} : (x, q, \mu) \to K(x, q, \mu)$, where T^*X is an appropriate local cotangent bundle structure on M (see [1]). Let $G_k(q, \lambda)$ be the generating family for $L_k^{(n)}$ given in Proposition 5.7. Then the corresponding phase family in (5.7) is a generating family for the pullback (cf. [11])

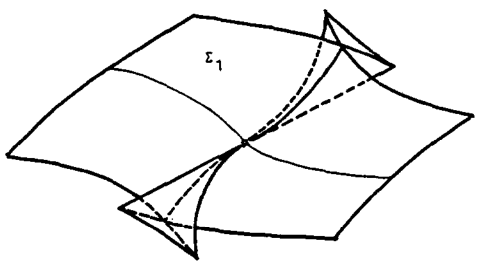

Fig. 3

$$(\text{graph}\,\pi)^t(L_k^{(n)}).$$

Thus the corresponding optical distance (time), say $\psi_k(x)$, is described by the following equations:

$$\psi_k(x) = \text{Stat}_{q,\mu,\lambda}(G_k(q,\lambda) - K(x,q,\mu)). \tag{5.8}$$

EXAMPLE 5.10. Now we exactly calculate the planar case $k = 1$. In this case the local Morse family for graph π is the following:

$$K(x_1, x_2, q) = x_2 q - x_1\sqrt{1-q^2}, \qquad q \neq 1.$$

Thus taking the generating family (5.5) for $L_1^{(7)}$ we obtain the corresponding family of optical distance functions (cf. (5.7))

$$\phi(x, \lambda_1, \lambda_2) = -\frac{1}{40}\lambda_2^5 - \frac{1}{6}\lambda_2^3\lambda_1 - \frac{1}{2}\lambda_2\lambda_1^2 - x_2\lambda_1 - x_1\sqrt{1-\lambda_1^2}$$

and the graph of phase function $\psi_1(x)$, $\Sigma_1 = \{\psi_1(x) - t = 0\}$ (see Fig. 3). By the straightforward calculations, using this family, we obtain the corresponding family of wave fronts parametrized by the optical time t;

$$x_1 = \left(\frac{1}{10}\mu^5 - t\right)\sqrt{1 - \frac{1}{4}\mu^4},$$
$$x_2 = \frac{1}{3}\mu^3 - \frac{1}{2}\mu^2\left(\frac{1}{10}\mu^5 - t\right) \quad \text{(see Fig. 4)},$$

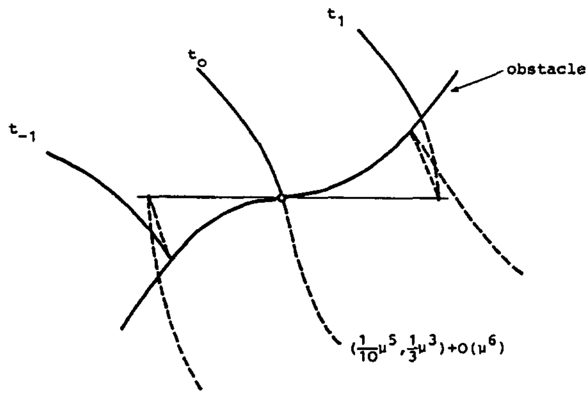

Fig. 4

which are exactly the level-sets of the phase function $\psi_1(x)$ in the planar obstacle problem (see Fig. 2) with inflection point [5].

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