Equivalence of isotropic submanifolds and symmetry

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It is shown that any two generating families of the same isotropic submanifold are equivalent. Also the singularity theory of symmetric isotropic submanifolds is developed and the basic classification theorems on prenormal forms, in particular for $\mathbf{Z}_2$ action, are proved. © *1997 American Institute of Physics.* [S0022-2488(97)03209-X]

I. INTRODUCTION

The genesis of this paper lies in theoretical questions in geometrical diffraction theory where a central role is played by the systems of rays passing through a boundary of an obstacle (aperture) (cf. Refs. 1,2). It is explained in Refs. 1,3,4 why the proper isotropic submanifolds of cotangent boundles (phase spaces) do occur in geometrical diffraction and why the symmetry group of these objects appear as a natural feature of existing optical systems (cf. Refs. 5,6).

Let $F:R^2\times R^2\times X\rightarrow R$, $F(x,y,a,b,q_1,q_2,q_2)$ be the optical distance function from the wave front $\{(x,y,z):z=\phi(x,y),\ \phi(0)=0,\ \phi'(0)=0\}$ in the presence of an aperture parametrized by $\{(a,b)\in R^2: f(a,b)\geqslant 0\}$ to the configurational point $(q_1,q_2,q_3)\in X$. If the incident ray goes from the point $(x,y,\phi(x,y))$ to the point of an edge $\{f(a,b)=0\}$ of the aperture, then the diffracted rays form a cone in X (cf. Refs. 2,6). The natural subsystems of diffracted rays form those rays that are straight continuations of the incident rays. The system of incident rays passing through an edge of the aperture form an isotropic two-dimensional submanifold of T^*X. This submanifold is described by the following equations:

$$\left.\frac{\partial F}{\partial a}\right|_{\{f=0\}}=0,\quad \left.\frac{\partial F}{\partial b}\right|_{\{f=0\}}=0,\quad \left.\frac{\partial F}{\partial x}\right|_{\{f=0\}}=0,\quad \left.\frac{\partial F}{\partial y}\right|_{\{f=0\}}=0,$$

and

$$p_i=\frac{\partial F}{\partial q_i}(x,y,a,b,q)\bigg|_{\{f=0\}},\quad i=1,2,3,\quad (q,p)\in T^{\star}X.$$

It appears that the typical singularities of the proper (sub-Lagrangian) isotropic submanifolds are classified mainly by the singularities of functions on varieties.[7–9] The general approach to the classification problem of isotropic submanifolds and the begining of the list of simple normal forms in small dimensions was given by Ref. 1. The complete list of simple normal forms was announced in Ref. 10. In the present paper we generalize this approach to include the symmetric, with respect to the compact Lie group action, isotropic submanifolds.

The technical content of the paper is rather close to that of Refs. 1 and 4, where the primary singularities of isotropic projections and symmetric Lagrangian projections were listed. In Sec. II the basic notion of the isotropic submanifold was introduced and the generating families for such submanifolds were constructed. The basic theorem on equivalence of generating families representing the same isotropic submanifold was shown. The infinitesimal stability condition and the prenormal form theorem for symplectic equivariant equivalence of symmetric isotropic submanifolds was derived in Secs. III and IV. In Sec. V the $\mathbf{Z}_2$-symmetry case is explicitly calculated and the generic singularities of isotropic projections in small dimensions are classified.

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II. CLASSIFICATION OF ISOTROPIC SUBMANIFOLDS

Let I be a stratifable subset of a symplectic manifold (M,ω). We call this subset isotropic if for each stratum of I, say I_i,

$$\omega|_{I_i}=0.$$

If I is smooth and $\dim I=\frac{1}{2}\dim M$, then I is called Lagrangian (see Ref. 11 for the theory of Lagrangian singularities). In this paper we consider proper isotropic subsets, i.e., $\dim I<\frac{1}{2}\dim M$, and study their local structure, so we assume $M\equiv T^{\star}X$ for some smooth manifold X. In what follows we also take $X\equiv R^n(\mathbf{C}^n)$.

In case of smooth $I\subset T^{\star}X$ the corresponding I-Morse families were introduced in Ref. 1. The smooth function germ $H:(X\times R^L\times R^K,0)\to R$ is called an I-Morse family if the smooth map,

$$X\times R^K\ni(q,\lambda)\to\left(\frac{\partial H}{\partial\beta_i}(q,0,\lambda),\frac{\partial H}{\partial\lambda_j}(q,0,\lambda)\right),\quad 1\leqslant i\leqslant L,\quad 1\leqslant j\leqslant K, \tag{1}$$

is nonsingular on the stationary set,

$$\Sigma_H^I=\left\{(q,\lambda):\frac{\partial H}{\partial\beta_i}(q,0,\lambda)=0,\quad\frac{\partial H}{\partial\lambda_j}(q,0,\lambda)=0,\quad 1\leqslant i\leqslant L,\quad 1\leqslant j\leqslant K\right\}.$$

Then the set

$$I^{n-L}=\left\{(q,p)\in T^{\star}X:\exists_{\lambda\in R^K},\quad p_j=\frac{\partial H}{\partial q_j}(q,0,\lambda),\quad\frac{\partial H}{\partial\beta_i}(q,0,\lambda)=0,\quad\frac{\partial H}{\partial\lambda_I}(q,0,\lambda)=0\right\}, \tag{2}$$

for $1\leqslant i\leqslant L$, $1\leqslant j\leqslant n$, $1\leqslant l\leqslant K$, is a smooth immersed isotropic submanifold of $T^{\star}X$.

If $H:(X\times R^L\times R^K,0)\to R$ is an I-Morse family generating the germ $(I^{n-L},0)$ then

$$\widetilde{H}(q,\beta,\lambda)=H(q,\beta,\lambda)+\varphi(q,\beta,\lambda), \tag{3}$$

where

$$\varphi\in\mathscr{H}=\langle\beta_1,\ldots,\beta_L\rangle^2\mathscr{E}_{(q,\beta,\lambda)}+\left\langle\frac{\partial H}{\partial\beta_1}(q,\beta,\lambda),\ldots,\frac{\partial H}{\partial\beta_L}(q,\beta,\lambda)\right\rangle^2\mathscr{E}_{(q,\beta,\lambda)}$$

is also a generating family for the same germ $(I^{n-L},0)$. Here $\mathscr{E}_{(q,\beta,\lambda)}$ denotes the ring of smooth function germs and $\mathbf{m}_{(q,\beta,\lambda)}$ denotes its unique maximal ideal. Moreover if $\Phi:(X\times R^L\times R^K,0)\to(R^L\times R^K,0)$ is a smooth family of diffeomorphism germs $\Phi(q,\cdot,\cdot)$ preserving the plane $\{(\beta,\lambda):\beta_i=0,\ i=1,\ldots,L\}$, then

$$\widetilde{H}(q,\Phi(q,\beta,\lambda)) \tag{4}$$

is also a generating family for the initial isotropic germ $(I^{n-L},0)$.

Let $(I^{n-L},0)$ be a germ of an isotropic submanifold and by $H:(X\times R^L\times R^K,0)\to R$ we denote its generating I-Morse family. By the reduction procedure (3) and elimination of λ parameters (cf. Ref. 11), we get the following minimal form for H:

$$H(q,\beta,\lambda)=f(q,\lambda)+\sum_{i=1}^{L}\beta_i g_i(q,\lambda),$$

where $f,g_i \in \mathbf{m}^2_{(q,\lambda)}$, and the number of λ parameters is minimal, i.e., $(\partial^2 f/\partial\lambda_i\,\partial\lambda_j)(0)=0$, $(\partial g_l/\partial\lambda_s)(0)=0$, $1\leqslant l\leqslant L$, $1\leqslant s\leqslant K$, $1\leqslant i$, $j\leqslant K$.

The family H is an I-Morse family, so by regularity of (1) we have $L+K\leqslant n$ and $\det(\partial^2 f/\partial\lambda_i\,\partial q_j)(0)=0$ for $1\leqslant i\leqslant K$ and $j\in J$, $J\subset\{1,...,n\}$, where J has K elements. Because $p_i=(\partial f/\partial q_i)(q,\lambda)$ on I^{n-L}, so we can take the new variables on $(X\times R^L\times R^K,0)$,

$$\Theta:(X\times R^L\times R^K,0)\ni(q,\beta,\lambda)\to\left(q,\beta,\frac{\partial f}{\partial q_J}(q,\lambda)\right),$$

which preserve $(I^{n-L},0)$ in the sense of (4), $[\partial f/\partial q_J=(\partial f/\partial q_{i_1},\ldots,\partial f/\partial q_{i_K}),i_1,\ldots,i_K\in J]$. Thus we came to the I-Morse family $\widetilde{H}=H\circ\Theta^{-1}$. By P_J we denote the coordinate space spanned by $\{p_{i_1},\ldots,p_{i_K}\}$.

Proposition II.1: Let H_1, H_2: $(X\times R^L\times R^K,0)\to(R,0)$ *be two minimal I-Morse families for the germ* $(I^{n-L},0)\subset T^\star X$. *Then there is a family of diffeomorphisms* $\Phi(q,\cdot,\cdot)$: $(R^L\times R^K,0)\to(R^L\times R^K,0)$ preserving the plane $\{\beta_i=0\}$, such that

$$H_1(q,\beta,\lambda)+\varphi(q,\beta,\lambda)=H_2(q,\Phi(q,\beta,\lambda)),$$

for some $\varphi\in\mathscr{H}$.

Proof: By diffeomorphism Θ we get $\widetilde{H}_1$, $\widetilde{H}_2:(X\times R^L\times P_J,0)\to(R,0)$,

$$\widetilde{H}_1(q,\beta,p_J)=H_1\circ\Theta_1^{-1}(q,\beta,p_J),\widetilde{H}_2(q,\beta,p_J)=H_2\circ\Theta_2^{-1}(q,\beta,p_J).$$

The isotropic germ $(I^{n-L},0)$ is given by the equations

$$\tilde{g}^l_k(q,p_J)=\frac{\partial\widetilde{H}_l}{\partial\beta_k}(q,\beta,p_J)\bigg|_{\{\beta=0\}}=0,\quad p_i=\frac{\partial\widetilde{H}_l}{\partial q_i}(q,0,p_J),\quad \frac{\partial\widetilde{H}_l}{\partial p_j}(q,0,p_J)=0,$$

for $l=1$ and as well for $l=2$. So the projection of I^{n-L} onto $X\times p_J$, represented by $\Sigma^I_{\widetilde{H}_1}$, $\Sigma^I_{\widetilde{H}_2}$, give the same germ. Thus we can write the differential

$$d(\widetilde{H}_1-\widetilde{H}_2)|_{\{\partial\tilde{f}_1/\partial p_j=0,\tilde{g}^1_k=0,\beta_i=0\}}=0,$$

and we can deduce immediately that, modulo some element from $\mathscr{H}|_{\{\beta=0\}}$, we have

$$\tilde{f}_1-\tilde{f}_2\in\mathbf{m}_{(q,p_J)}\left\langle\frac{\partial\widetilde{H}_1}{\partial p_J}\right\rangle^2.$$

Thus by the Tougeron's implicit function theorem (cf. Ref. 12, p. 206) we get a diffeomorphism,

$$\Xi(q,p_J)=(q,\widetilde{\Xi}(q,p_J)),$$

such that $\tilde{f}_1\circ\Xi=\tilde{f}_2$ and $\langle g^1_k\circ\Xi\rangle=\langle g^2_k\rangle$. So there exists a diffeomorphism of (β), $\kappa(\beta,q,p_J),\kappa(0,q,p_J)\equiv0$, such that

$$\Phi:(q,\beta,p_J)\to(q,\kappa(\beta,q,p_J),\widetilde{\Xi}(q,p_J))$$

form the necessary equivalence of $\widetilde{H}_1$ and $\widetilde{H}_2$. Q.E.D.

It is obvious that instead of minimal I-Morse families we can consider the pairs of functions/mappings f and $g=(g_1,\ldots,g_L)$. The corresponding I equivalence of such pairs is induced by the above defined equivalence of I-Morse families. The beginning of the classification procedure of simple normal forms for isotropic submanifolds was done in Ref. 1. The complete list of simple singularities was announced in Ref. 10. Now there is a natural way to generalize the notion of an I-Morse family and pass to the generation of no necessary smooth isotropic varieties.

Let $H:(X\times R^N,0)\to R$ be a smooth function germ. We consider an analytic (algebraic) subset V of $(X\times R^N,0)$, $V=F^{-1}(0)$, and an analytic map $F:(X\times R^N,0)\to(R^k,0)$.

Definition II.2: The germ of an isotropic variety $I\subset(T^\star X,\omega_X)$ *defined by the pair of functions* (H,F),

$$I=\left\{\overline{p}\in T^\star X;\exists_{(q,\lambda)\in V},p_i=\frac{\partial H}{\partial q_i}(q,\lambda),\quad \frac{\partial H}{\partial \lambda_j}(q,\lambda)=0,\quad i=1,...,n,\quad j=1,...,N\right\},\tag{5}$$

is called the diffractional isotropic variety. $\overline{p}=(p,q)\in T^\star X$.

Remark II.3:

(1) *If* $V=R^N$ *then I is a usual representation of Lagrangian varieties by generating families (cf. Refs. 11,13,14). In this case there is no extra condition, which makes I the proper isotropic.*

(2) *If V is a smooth submanifold of* R^N *then this case was exploited (see Ref. 1) to generate all germs of smooth isotropic submanifolds of* $(T^\star X,\omega_X)$. *This is a generalization of the standard notion of the generating family for Lagrangian submanifolds.*

(3) *The isotropic varieties that we are studying here first appeared in geometrical diffraction on apertures (see Refs. 6,2) and always are connected to some distance function properties and the geometrical structure of the boundary of an aperture. Diffractional isotropic varieties that we investigate in this paper, unless otherwise stated, are proper isotropic (not Lagrangian).*

(4) *To see that (5) is an isotropic variety we consider the stratum of the critical set*,

$$\Sigma_{H,V}=\left\{(q,\lambda);\frac{\partial H}{\partial \lambda}(q,\lambda)=0,(q,\lambda)\in V\right\},$$

and repeat the standard lines of singularity theory techniques usually applied in symplectic geometry (cf. Ref. 11).

Behind the definition introduced above there is the following construction. Let $\mathscr{T}=(T^\star X\times T^\star R^N,\pi_2^\star\omega_{R^N}-\pi_1^\star\omega_X)$ be the product symplectic structure. Let L $\subset\mathscr{T}$ be a Lagrangian submanifold of $\mathscr{T}$ transversal to the fibers of the fibering $T^\star(X\times R^N)\to X\times R^N$. Let S be a subset of $T^\star X$ then we define the image

$$L(S)=\{\overline{\mu}\in T^\star R^N:\exists_{\overline{p}\in S}(\overline{p,\mu})\in L\}.$$

In an analogous way we define the counterimage $L^t(\tilde{\Lambda})$ of the subset $\tilde{\Lambda}$ of $T^\star R^N$. The image and counterimage preserve the symplectic properties of S and $\tilde{\Lambda}$, respectively. Now instead of L we take the isotropic intersection $I_{(L,\Lambda)}=L\cap\Lambda$, where $\Lambda=\{(\overline{p,\mu}):F(q,\lambda)=0\}$, $F:X\times R^N\to R^k$. If L is generated by the function $(q,\lambda)\to H(q,\lambda)$ and R^N is a zero section of $T^\star R^N$, then the pair (L,Λ) [or the pair (H,F)] called a Λ pair defines a diffractional isotropic variety as the symplectic counterimage $I^t_{(L,\Lambda)}(R^N)$.

Example II.4: A_2*-type isotropic varieties. A natural class of isotropic varieties are those varieties that are critical sets of the Lagrangian projections. These are typically cone-like varieties described by the following generating families, H:* $(X\times R\times R^K,0)\to R$:

$$H(q,\beta,\lambda)=f(q,\lambda)+\beta\ \det\left(\frac{\partial^2 f}{\partial\lambda_i\ \partial\lambda_j}(q,\lambda)\right).$$

To be more concrete let us consider the D_5*-singular Lagrangian projection in* $T^\star R^4$. *Then its* A_2*-type isotropic variety is generated by*

$$H_{A_2}(q,\beta,\lambda_1,\lambda_2)=\lambda_1^2\lambda_2+\lambda_2^4+q_1\lambda_2^3+q_2\lambda_2^2-q_3\lambda_1-q_4\lambda_2+\beta(6\lambda_2^3+3q_1\lambda_2^2+q_2\lambda_2-2\lambda_1^2).$$

We can continue the above procedure and describe the corresponding sub-Lagrangian $A_k,D_k,...,$ varieties. The A_3-singular isotropic variety for D_5-Lagrangian projection is generated by the family

$$H_{A_3}(q,\beta_1,\beta_2,\lambda_1,\lambda_2)=\lambda_1^2\lambda_2+\lambda_2^4+q_1\lambda_2^3+q_2\lambda_2^2-q_3\lambda_1-q_4\lambda_2$$
$$+\beta_1(6\lambda_2^3+3q_1\lambda_2^2+q_2\lambda_2-2\lambda_1^2)+\beta_2(10\lambda_2^3+4q_1\lambda_2^2+q_2\lambda_2).$$

Example II.5: Another interesting example of an isotropic variety is a so-called C-Lagrangian manifold in the Maslov theory of the complex canonical operator.[15] *If $I^n\equiv R^n\subset(\mathbf{C}^{2n},\omega)$ is an isotropic submanifold of a canonical complex symplectic space then we have, for every $p\in I^n$,*

$$T_p^cI=T_pI\cap(iT_pI)=T_pI\cap(T_pI)^L,$$

where $(T_pI)^L$ is an ω-orthogonal subspace to T_pI. Any real (i.e., $T_p^cI=0$) isotropic (C-Lagrangian) submanifold $I^n\subset(\mathbf{C}^{2n},\omega)$ can be locally generated by the generating family F: $\mathbf{C}^n\times R^n\times\mathbf{C}^k\to\mathbf{C}$,

$$F(q,\beta,\lambda)=f(q,\lambda)+\sum_{i=1}^{n}\beta_ig_i(q,\lambda),$$

where f is holomorphic and g_i are real analytic; moreover, $\{g_i=0\}$ form a real hypersurface in the critical set $\Sigma=\{(\partial f/\partial\lambda)(q,\lambda)=0\}$.

III. SYMMETRIC ISOTROPIC SUBMANIFOLDS

Let G be a compact Lie group acting smoothly on M. This action extends naturally to a symplectic action of G on the cotangent bundle $T^\star M$, preserving the cotangent bundle structure. Because our considerations are local we may identify M with R^n and assume that $0\in R^n$ is a fixed point of the action of G. We also assume that the action of G on $(R^n,0)$ is linear and orthogonal. We shall denote R^n with this action of G by V. We identify $T^\star V$ with $V\oplus V^\star$, where $V^\star$ is the dual of V. If ν denotes a representation of G in V, then the natural symplectic action of G on $T^\star V$ is given by the symplectic lifting $\overline{\nu}=\nu\oplus\nu$, i.e.

$$T^\star V\ni(q,p)\to\overline{\nu_g}(q,p)=(\nu_g(q),\nu_g(p))\in T^\star V.$$

If $(I,0)$ is a G-invariant isotropic submanifold germ, then the image of the associated G-invariant isotropic projection $\pi_I=\pi_V|_I$: $(I,0)\to(V,0)$ is the germ of a G-invariant subvariety in $(V,0)$ called a *symmetric quasicaustic* of $(I,0)$. Also Ker $D\pi_I(0)=T_0I\cap T_0V^\star$ is a G-invariant subspace of $V^\star$ (we identify $T_0V^\star\equiv V^\star$). The existance of G-invariant I-Morse families for G-invariant isotropic submanifolds is given by the noninvariant existence result (cf. Ref. 1, Proposition 1.2) and the methods of constructing invariant Morse families for Lagrangian submanifolds used in Ref. 6. We can formulate this result in the following way.

Proposition III.1: Let $(I^G,0)$ be a G-invariant isotropic submanifold of $T^\star V$. There exists a smooth G-invariant function-germ of the I^G-Morse family,

$$F:(V\times R^L\times R^K,(0,0,0))\to R,$$

invariant with respect to a component-wise, linear action κ of G on $V\times R^L\times R^K$, $\kappa=\nu\oplus\mu\oplus\rho$, such that $(I^G,0)$ is defined by (2). Conversely, every such G-invariant function germ generates a G-invariant isotropic submanifold $(I,0)$.

In all further considerations we assume that an I-Morse family is defined with the presence of minimal number of parameters K, i.e.

$$\left(\frac{\partial^2 F}{\partial\lambda_i\ \partial\lambda_j}(0,0,0)\right)=0.$$

To each G-invariant isotropic submanifold $(I^G,0)$, generated by I^G-Morse family F, we associate the pair of G-invariant Lagrangian submanifolds (L^G,N^G), defined by the corresponding G-Morse families (cf. Ref. 1),

$$L^G:\quad \widetilde{F}(q,\lambda)=F(q,0,\lambda),$$

$$N^G:\quad \widetilde{F}(q,\mu)=F(q,\mu_1,\mu_2),$$

where we denote $\mu=(\mu_1,\mu_2)$. These two manifolds intersect along I^G and define two G-invariant subspaces of $V^\star$:

$$W^\star_{L^G}=\mathrm{Ker}\ D\pi_{L^G}(0),$$

$$W^\star_{N^G}=\mathrm{Ker}\ D\pi_{N^G}(0).$$

We see that the invariant subspace $W^\star=\mathrm{Ker}\ D\pi_I(0)$ is an intersection of both of these subspaces.

If V' is a representation space of G that has a G-invariant subspace isomorphic to V, then the invariant I-Morse family $F:V\times R^L\times R^K\rightarrow R$ also defines an invariant isotropic submanifold I' of $T^\star V'$.

Let $q_1,\ldots,q_n$ be coordinates on the subspace isomorphic to V and extend these to a system $q_1,\ldots,q_{n'}$ on V'. Then the equations for I' are obtained by supplementing the equations for I by $p_j=0$ for $j=n+1,...,n'$. We will say that the isotropic submanifold I' is a *trivial extension* of I.

We see that the functions $F(q,\bullet,\bullet)$ are only invariant under G_q, which is the isotropy subgroup at q of the action of G on V. If $V^{(G)}$ denotes the space of fixed points of the action of G, then the restriction $F|V^{(G)}\times R^L\times R^K$ is a family of G-invariant functions on $R^L\times R^K$. Any such family can be extended to a family on $V\times R^L\times R^K$. Also, a generic property of the restricted families can be regarded as a generic property of the full family. As an example, one can easily show that the generic invariant quasicaustics do not pass through isolated points of the action of G on V. In this case $V^{(G)}=\{0\}$ and the generic G-invariant I-Morse families F has a nondegenerate critical point at 0, i.e.

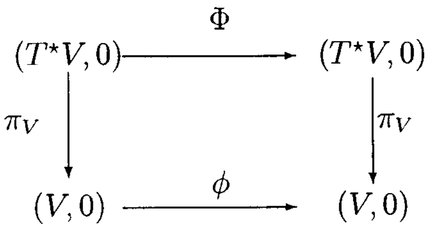
$$\det\begin{pmatrix}\frac{\partial^2 F}{\partial\beta\ \partial\beta} & \frac{\partial^2 F}{\partial\beta\ \partial\lambda}\\ \frac{\partial^2 F}{\partial\lambda\ \partial\beta} & \frac{\partial^2 F}{\partial\lambda\ \partial\lambda}\end{pmatrix}(0)\neq 0.$$

IV. SYMPLECTIC EQUIVARIANT EQUIVALENCE

In this section we introduce the equivalence relation that is used to classify G-invariant isotropic submanifolds. In the absence of the group action, the corresponding theory was presented in the preceding section (cf. Ref. 1).

Definition IV.1: Two G-invariant germs of isotropic submanifolds $(I_j^G,0)\subset(T^\star V,0)$, $(j=1,2)$ *are called equivalent if there exist germs of a G-equivariant symplectomorphism* Φ*:* $(T^\star V,0)\rightarrow(T^\star V,0)$ *and a G-equivariant diffeomorphism* ϕ*:* $(V,0)\rightarrow(V,0)$ *such that (i) the fol-*

lowing diagram commutes:



and

(ii) $\Phi(I_1^G)\subset I_2^G$.

Let $\mathscr{E}^G_{(q,\beta,\lambda)}$ (respectively, $\mathscr{E}^G_q$) denote the ring of germs of G-invariant functions on $V\times R^L \times R^K$ (respectively, on V). By $\mathbf{m}^G_{(q,\beta,\lambda)}$ we denote the maximal ideal of $\mathscr{E}^G_{(q,\beta,\lambda)}$. By $B^G_{(q,\beta,\lambda)}\subset\langle\beta_1,\ldots,\beta_L\rangle\mathscr{E}_{(q,\beta,\lambda)}$, we denote the ideal of invariant germs vanishing on the space $V \times\{0\}\times R^K$. By $B\mathscr{D}^G_{(\beta,\lambda)}$ we denote the space of germs of equivariant diffeomorphisms preserving the subspace

$$\Lambda=\{(\beta,\lambda):\beta_1=0,\ldots,\beta_L=0\},$$

in $R^L\times R^K$.

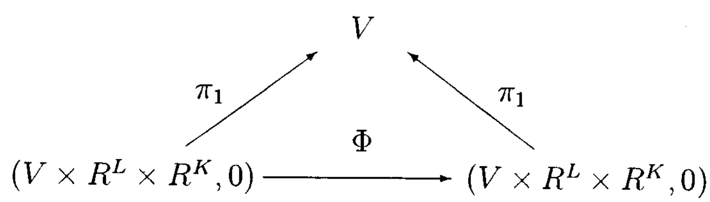
Definition IV.2: Two G-invariant I-Morse families,

$$F_{1,2}:(V\times R^L\times R^K,0)\rightarrow R,$$

are called β-equivalent (or simply equivalent) if there exist germs of an equivariant diffeomorphism,

$$\Phi:(V\times R^L\times R^K,0)\rightarrow(V\times R^L\times R^K,0),$$

$\Phi(q,\cdot,\cdot)\in B\mathscr{D}^G_{(\beta,\lambda)}$ *and a smooth function germ* $\alpha\in\overline{B^G_{(q,\beta,\lambda)}}$, *where* $\overline{B^G_{(q,\beta,\lambda)}}$ *denotes the space of invariant function germs belonging to* $\langle\beta_1,\ldots,\beta_L\rangle^2\mathscr{E}^G_{(q,\beta,\lambda)}$, *such that the following diagram commutes:*



and

$$F_1=F_2\circ\Phi+\alpha.$$

We say that F_1, F_2 are G equivalent if under the conditions introduced above,

$$\alpha\in\hat{B}^G_{(q,\beta,\lambda)}F_1,$$

where $\hat{B}^G_{(q,\beta,\lambda)}F_1$, denotes the space of G-invariant function germs belonging to $\langle\beta_1,\ldots,\beta_L\rangle^2 \mathscr{E}^G_{(q,\beta,\lambda)}+\langle\partial F_1/\partial\beta_1,\ldots,\partial F_1/\partial\beta_L\rangle^2\mathscr{E}^G_{(q,\beta,\lambda)}$.

Remark IV.3:

(A) *If μ is trivial then* $B^G_{(q,\beta,\lambda)}=\langle\beta_1,\ldots,\beta_L\rangle^2\mathscr{E}^G_{(q,\beta,\lambda)}$, *and*

$$\hat{B}^G_{(q,\beta,\lambda)}F_1=\langle\beta_1,\ldots,\beta_L\rangle^2\mathscr{E}^G_{(q,\beta,\lambda)}+\left\langle\frac{\partial F_1}{\partial\beta_1},\ldots,\frac{\partial F_1}{\partial\beta_L}\right\rangle^2\mathscr{E}^G_{(q,\beta,\lambda)}.$$

(B) *Let F:* $(V\times R^L\times R^K,0)\rightarrow R$ *be an I-Morse family for an isotropic submanifold* $(I,0)$, codim $I=n+L$. *Then* $(I,0)$ *is an intersection of* $L+1$ *Laqrangian submanifolds defined by the following Morse families:*

$$\overline{F_0}(q,\lambda)=F(q;\lambda_1,\lambda_2),\quad \lambda=(\lambda_1,\lambda_2)\in R^L\times R^K,$$

$$\overline{F_i}(q,\mu)=F(q;\mu_1,\ldots,\overset{i}{0},\ldots,\mu_{L-1},\ldots,\mu_{L+K-1}),\quad i=1,...,L.$$

The group of equivalences of I-Morse families defined in Ref. 1 keeps all these Lagrangian submanifolds identical. This group instead of $B\mathscr{D}_{(\beta,\lambda)}$ *contains the space of diffeomorphisms preserving the hyperplanes* $\{(\beta,\lambda):\ \beta_i=0\}$ *and the L-dimensional corner* $\mathscr{H}=\{\beta\in R^L:\beta_i\geqslant 0,$ $i=1,\ldots,L\}$ *in* $R^L\times R^K$. *It is a subgroup of the group of equivalences defined above (cf. Definition IV.2).*

One can easily check that the corresponding two isotropic submanifolds defined by G-equivalent I-Morse families F_1, F_2 are identical. Exactly as in the nonequivariant case we can prove the following result.

Proposition IV.4: Two G-invariant I-Morse families F_j: $(V\times R^L\times R^K,0)\rightarrow R$, $(j=1,2)$ *generate equivalent G-invariant isotropic submanifolds if and only if there is a G-equivariant diffeomorphism germ* ϕ: $(V,0)\rightarrow(V,0)$, *a G-invariant function germ g:* $(V,0)\rightarrow R$ *such that* $F_1\circ(\phi,id_{R^L\times R^K})+g\circ\pi_1$ *and* F_2 *are G equivalent.*

The group of β equivalences will be denoted by $I.\mathscr{R}^+_G$. This is an equivalence group we can operate with using the standard lines of infinitesimal stability theory (cf. Ref. 11). Now we describe the tangent space for this equivalence relation.

Let $\{\alpha_1,\ldots,\alpha_r\}$ denote a generating set for the $\mathscr{E}^G_{(q,\beta,\lambda)}$ module $\Xi^G_{\pi_2}$ consisting of germs of G-equivariant vector fields along the projection $\pi_2:V\times R^L\times R^K\rightarrow R^L\times R^K$ tangent to Λ. These are G-equivariant vector fields of the form

$$\sum_{i=1}^{L}a_i(q,\beta,\lambda)\,\frac{\partial}{\partial\beta_i}+\sum_{j=1}^{K}b_j(q,\beta,\lambda)\,\frac{\partial}{\partial\lambda_j},$$

with $a_i\in B^G_{(q,\beta,\lambda)}$. Let $\{\gamma_1,\ldots,\gamma_s\}$ denote a generating set for the $\mathscr{E}^G_q$ module Ξ^G_q of germs of G-equivariant vector fields on $(V,0)$. We will regard the direct sum,

$$\mathscr{L}I.\mathscr{R}^+_G=\Xi^G_{\pi_2}\oplus\Xi^G_q\oplus\overline{B^G_{(q,\beta,\lambda)}}\oplus\mathscr{E}^G_q,$$

as the Lie algebra of the group $I.\mathscr{R}^+_G$. The first two summands of $\mathscr{L}I.\mathscr{R}^+_G$ correspond to infinitesimal coordinate changes Φ, ϕ and the two next summands correspond to functions α and g as in Definition IV.2 and Proposition IV.4.

For any G-invariant I-Morse family we define the tangent space,

$$T^I_G(F)=\mathscr{L}I.\mathscr{R}^+_GF=\mathscr{E}^G_{(q,\beta,\lambda)}\{\alpha_1F,\ldots,\alpha_rF\}+\mathscr{E}^G_q\{\gamma_1F,\ldots,\gamma_sF,1\}+\overline{B^G_{(q,\beta,\lambda)}}.$$

The first term is the ideal in $\mathscr{E}^G_{(q,\beta,\lambda)}$ generated by $\{\alpha_1F,\ldots,\alpha_rF\}$, the second term is the $\mathscr{E}^G_q$ submodule of $\mathscr{E}^G_{(q,\beta,\lambda)}$, thought of as the $\mathscr{E}^G_q$ module, generated by $\{\gamma_1F,\ldots,\gamma_sF,1\}$ the third term is the ideal in $\mathscr{E}^G_{(q,\beta,\lambda)}$.

We define the infinitesimal stability for G-invariant I-Morse families in the following way.

Definition IV.5: A G-invariant I-Morse family function germ F: $(V\times R^L\times R^K,0)\rightarrow R$ *is infinitesimally* $I.\mathscr{R}_G^+$ *stable iff*

$$T_G^I(F)=\mathscr{E}^G_{(q,\beta,\lambda)}.$$

Let $f\in \mathbf{m}_{(\beta,\lambda)}$. We define the corresponding analog of the Jacobi ideal of f for the group of equivalences in $R^L\times R^K$ preserving Λ,

$$\delta_{L,K}(f)=\left\langle \beta_1\frac{\partial f}{\partial \beta_1},\dots,\beta_i\frac{\partial f}{\partial \beta_j},\dots,\beta_L\frac{\partial f}{\partial \beta_L},\frac{\partial f}{\partial \lambda_1},\dots,\frac{\partial f}{\partial \lambda_K}\right\rangle \mathscr{E}_{(\beta,\lambda)}.$$

We say that f has a finite codimension c if

$$\operatorname{cod}(f)=c=\dim_R \frac{\mathbf{m}_{(\beta,\lambda)}}{\delta_{L,K}(f)+\langle \beta_1,\dots,\beta_L\rangle^2\mathscr{E}_{(\beta,\lambda)}}<\infty.$$

If c is finite then c is the minimal dimension of a versal unfolding of f. If $g_1,\dots,g_c\in \mathbf{m}_{(\beta,\lambda)}$ are polynomial representations of a generating set of

$$\frac{\mathbf{m}_{(\beta,\lambda)}}{\delta_{L,K}(f)+\langle \beta_1,\dots,\beta_L\rangle^2\mathscr{E}_{(\beta,\lambda)}},$$

then the $I.\mathscr{R}$-minimal unfolding of f is written as follows:

$$H(x,\beta,\lambda)=f(\beta,\lambda)+\sum_{i=1}^{c} x_i g_i(\beta,\lambda).$$

We see that if $f\in \mathscr{E}^G_{(\beta,\lambda)}$, then $\delta_{L,K}(f)$ is invariant under the natural action of G on $\mathscr{E}_{(\beta,\lambda)}$. Then we have the following result.

Proposition IV.6: If F is an infinitesimally $I.\mathscr{R}_G^+$ *stable I-Morse family, then* $F|_{V^G\times R^L\times R^K}$ *is a* $\mathscr{R}_G$*-versal unfolding of* $F(0,\bullet,\bullet)$ *in* $\mathbf{m}^G_{(\beta,\lambda)}$.

Proof: $I.\mathscr{R}_G^+$-infinitesimal stability of F gives us the following surjective mapping:

$$\mathscr{E}_q^G\{\gamma_1F,\dots,\gamma_sF,1\}\rightarrow \frac{\mathscr{E}^G_{(q,\beta,\lambda)}}{\mathscr{E}^G_{(q,\beta,\lambda)}\{\alpha_1F,\dots,\alpha_rF\}+B^G_{(q,\beta,\lambda)}}.$$

This implies that the mapping,

$$\frac{\mathscr{E}_q^G\{\gamma_1F,\dots,\gamma_sF,1\}}{(\mathbf{m}_q\mathscr{E}_{(q,\beta,\lambda)})^G\cap\mathscr{E}_q^G\{\gamma_1F,\dots,\gamma_sF,1\}} \tag{6}$$

$$\rightarrow \frac{\mathscr{E}^G_{(q,\beta,\lambda)}}{\mathscr{E}^G_{(q,\beta,\lambda)}\{\alpha_1F,\dots,\alpha_rF\}+(\mathbf{m}_q\mathscr{E}_{(q,\beta,\lambda)})^G+B^G_{(q,\beta,\lambda)}} \tag{7}$$

is also surjective. If $\{q_1,\dots,q_n\}$ denote coordinates on V^G and $f(\bullet,\bullet)=F(0,\bullet,\bullet)$, then the above condition can be written in the form

$$R\left\{\left.\frac{\partial F}{\partial q_i}\right|_{q=0}\right\}_{i=1,\dots,a}=\frac{\mathscr{M}^G_{(\beta,\lambda)}}{\delta_{L,K}(f)^G+B^G_{(\beta,\lambda)}},$$

which is the infinitesimal criteria of $I.\mathscr{R}_G^+$ versality. Q.E.D.

For G-invariant infinitesimally $I\mathscr{R}_G^+$-stable I-Morse families we have the following useful criteria.

Proposition IV.7: Suppose G is a finite group and F: $V\times R^L\times R^K\rightarrow R$ *is* $I\mathscr{R}_G^+$*-infinitesimally stable G-invariant I-Morse family, then the following applies.*

(1) $F|_{V\times\{0\}\times R^K}$ *is G-invariant* $\mathscr{R}_G^+$*-infinitesimally stable Morse family (cf. Ref. 4).*
(2) *Let* $\tilde{f}(0)=F(0,0,\bullet)\in\mathscr{E}_\lambda^G$; $f(\bullet,\bullet)=F(0,\bullet,\bullet)$ *is finitely determined. Let*

$$U=\frac{\mathbf{m}_{(\beta,\lambda)}}{d_{L,K}(f)+\langle\beta_1,\dots,\beta_L\rangle^2\mathscr{E}_{(\beta,\lambda)}},\quad \dim U<\infty.$$

U is endowed with the induced action of G. Then there exists a G-invariant unfolding,

$$\mathscr{F}:U\times R^L\times R^K\rightarrow R,$$

such that for any representation of G with representation space V and any G-invariant unfolding F: $V\times R^L\times R^K\rightarrow R$ *of f there exists a G-invariant map* ϕ*: (V,0)→(U,0) such that F(q,β,λ) is* $I\mathscr{R}_G^+$ *equivalent to* $\mathscr{F}(\phi(q),\beta,\lambda)$.

F is $I\mathscr{R}_G^+$*-infinitesimally stable if and only if F is* $I\mathscr{R}_G^+$ *equivalent to the unfolding,*

$$\mathscr{F}(\phi(q),\beta,\lambda)=\overline{\mathscr{F}}(\phi(q),\lambda)+\sum_{i=1}^{L}\beta_i\psi_i(\phi(q),\lambda)),\quad \psi_i\in\mathscr{E}_{(u,\lambda)}^G,$$

where ϕ*: (V,0)→(U,0) is an infinitesimally* $\mathscr{R}_G$*-stable map,* $\overline{\mathscr{F}}:U\times R^K\rightarrow R$ *is a trivial extension of the G-invariant versal unfolding of* $\tilde{f}(\bullet)$ *in the space* $\mathscr{E}_\lambda$ *constructed in Ref. 16.*

V. Z_2 SYMMETRY

Let $G=\mathbf{Z}_2=\{1,g\}$ and $\mathbf{Z}_2$ acts on $V\cong R^n$ by

$$g(x_1,\dots,x_r,y_1,\dots,y_s)=(-x_1,\dots,-x_r,y_1,\dots,y_s),\quad n=r+s.$$

$\mathbf{Z}_2$ will also act nontrivially on R^L and R^K. Let $F:V\times R^L\times R^K\rightarrow R$ be a G-invariant I-Morse family. The number $K+L$ will be called corank of the I-Morse family F (we already assumed that F is minimal). In what follows we assume $K+L=2$ and at first we assume that μ is trivial, i.e., $g(\beta,\lambda)=(\beta,-\lambda)$.

Proposition V.1: The generic corank 2, $\mathbf{Z}_2$*-invariant I-Morse families on* $V\times R\times R$ *with the trivial* μ *action of* $\mathbf{Z}_2$ *are equivalent to families of the form*

$$F(x,y,\beta,\lambda)=\beta\lambda^{2k}+\lambda^{2t}+\sum_{i=1}^{t-1}y_i\lambda^{2i}+\sum_{a=1}^{\min\{k,t\}}y_{a+t-1}\beta\lambda^{2a-2}+\sum_{j=1}^{t-1}\phi_j(x,y)\lambda^{2j-1}+\sum_{b=1}^{\kappa_{tk}}\overline{\phi_b}(x,y)\beta\lambda^{2b-1},\tag{8}$$

where $t-1+\min\{t,k\}\leq s$, $\kappa_{tk}=\min\{t,k+1\}-1+\delta_{tk}$, $\delta_{tk}=1$ *if* $t=k$, $\delta_{tk}=0$ *if* $t\neq k$, *or*

$$F(x,y,\beta,\lambda)=\lambda^{2k}+\sum_{i=1}^{k-1}y_i\lambda^{2i}+\sum_{a=1}^{k}y_{a+k-1}\beta\lambda^{2a-2}+\sum_{j=1}^{k-1}\psi_j(x,y)\lambda^{2j-1}+\sum_{b=1}^{k-1}\overline{\psi_b}(x,y)\beta\lambda^{2b-1},\tag{9}$$

where $2k-1\leq s$ *and* ϕ_j, $\overline{\phi_b}$, ψ_j, $\overline{\psi_b}$ *are smooth functions,*

$$\phi_j(x,y)=\sum_{c=1}^{r} \widetilde{\kappa}_{jc}(x,y)x_c, \quad \overline{\phi_b}(x,y)=\sum_{c=1}^{r} \overline{\kappa_{bc}}(x,y)x_c,$$
$$\psi_j(x,y)=\sum_{c=1}^{r} \rho_{jc}(x,y)x_c, \quad \overline{\psi_b}(x,y)=\sum_{c=1}^{r} \overline{\rho_{bc}}(x,y)x_c, \tag{10}$$

and $\widetilde{\kappa}_{jc}$, $\overline{\kappa_{bc}}$, ρ_{jc}, $\overline{\rho_{bc}}$ *are* $\mathbf{Z}_2$*-invariant functions of* x *and* y.

Proof: We know that the restriction $F|_{V^{\mathbf{Z}_2}\times R\times R}$ of the generic $\mathbf{Z}_2$-invariant I-Morse family must be a germ of $I.\mathscr{R}_{\mathbf{Z}_2}$-versal unfolding of $f(\beta,\lambda)=F(0,\beta,\lambda)$. Here $I.\mathscr{R}_{\mathbf{Z}_2}$ denotes the group $B\mathscr{D}^{\mathbf{Z}_2}_{(\beta,\lambda)}$ of $\mathbf{Z}_2$-equivariant diffeomorphisms of $R\times R$. Thus, generic $\mathbf{Z}_2$-invariant I-Morse families of corank 2 will be represented by unfoldings of $K^{2k}_{2t}:f(\beta,\lambda)=\beta\lambda^{2k}+\lambda^{2t}$ and $F_{2p+1}:f(\beta,\lambda)=\lambda^{2p}$ (simple orbits of the action of the group $B\mathscr{D}^{\mathbf{Z}_2}_{(\beta,\lambda)}$ in the space of $\mathbf{Z}_2$-invariant functions on $R\times R$). By Proposition IV.7 and straightforward calculations they will be equivalent to the families (based on the versal unfoldings of K^{2k}_{2t} and F_{2p+1})

$$F(x,y,\beta,\lambda)=\beta\lambda^{2k}+\lambda^{2t}+\sum_{i=1}^{t-1} \delta_i(x,y)\lambda^{2i}+\sum_{a=1}^{\min\{t,k\}} \delta_{a+t-1}(x,y)\beta\lambda^{2a-2}$$
$$+\sum_{j=1}^{t-1} \phi_j(x,y)\lambda^{2j-1}+\sum_{b=1}^{\kappa_{tk}} \overline{\phi_b}(x,y)\beta\lambda^{2b-1}, \tag{11}$$

where $\kappa_{tk}=\min\{t,k+1\}-1+\delta_{tk}$, and

$$F(x,y,\beta,\lambda)=\lambda^{2p}+\sum_{i=1}^{p-1} \gamma_i(x,y)\lambda^{2i}+\sum_{a=1}^{p} \gamma_{p-1+a}(x,y)\beta\lambda^{2a-2}$$
$$+\sum_{j=1}^{p-1} \psi_j(x,y)\lambda^{2j-1}+\sum_{b=1}^{p-1} \overline{\psi_b}(x,y)\beta\lambda^{2b-1}, \tag{12}$$

where ϕ_j, $\overline{\phi_b}$, ψ_j, $\overline{\psi_b}$ are expanded in (10) and κ_{jc}, $\overline{\kappa_{bc}}$, ρ_{jc}, $\overline{\rho_{bc}}$, δ_i, γ_j are $\mathbf{Z}_2$-invariant functions of x and y.

Because the restriction of F to $V^{\mathbf{Z}_2}\times R\times R$ ($V^{\mathbf{Z}_2}=\{(0,...,0,y_1,\dots,y_s)\}$) is $\mathscr{R}_{\mathbf{Z}_2}$ versal, hence the mappings δ and γ are submersions. This implies that we can choose coordinates $(y_1,\dots,y_s)$ so that $\delta_i=y_i$ and $\gamma_j=y_j$, in both families (11), (12). Q.E.D.

Using the similar methods and arguments (as we used above), we can prove the completing result for the nontrivial μ action of $\mathbf{Z}_2$, $g(\beta,\lambda)=(-\beta,-\lambda)$.

Proposition V.2: The generic corank 2, $\mathbf{Z}_2$*-invariant I-Morse families on* $V\times R\times R$ *with the nontrivial* μ *action of* $\mathbf{Z}_2$ *are equivalent to families of the form*

$$F(x,y,\beta,\lambda)=\beta\lambda^{2k-1}+\lambda^{2t}+\sum_{i=1}^{t-1} y_i\lambda^{2i}+\sum_{a=1}^{\min\{k,t\}} \overline{\phi_a}(x,y)\beta\lambda^{2a-2}+\sum_{j=1}^{t-1} \phi_j(x,y)\lambda^{2j-1}$$
$$+\sum_{b=1}^{\min\{t,k\}-1} y_{t-1+b}\beta\lambda^{2b-1}, \tag{13}$$

where $t-2+min\{t,k\}\leqslant s$, *or*

$$F(x,y,\beta,\lambda)=\lambda^{2k}+\sum_{i=1}^{k-1} y_i\lambda^{2i}+\sum_{a=1}^{k} \overline{\psi_a}(x,y)\beta\lambda^{2a-2}+\sum_{j=1}^{k-1} \psi_j(x,y)\lambda^{2j-1}+\sum_{b=1}^{k-1} y_{b+k-1}\beta\lambda^{2b-1}, \tag{14}$$

where $2k-2\leqslant s$ *and* ϕ_j, $\overline{\phi_b}$, ψ_j, $\overline{\psi_b}$ *are smooth functions,*

$$\begin{aligned}\phi_j(x,y)&=\sum_{c=1}^{r} \widetilde{\kappa}_{jc}(x,y)x_c, \quad \overline{\phi_b}(x,y)=\sum_{c=1}^{r} \overline{\kappa_{bc}}(x,y)x_c,\\ \psi_j(x,y)&=\sum_{c=1}^{r} \rho_{jc}(x,y)x_c, \quad \overline{\psi_b}(x,y)=\sum_{c=1}^{r} \overline{\rho_{bc}}(x,y)x_c,\end{aligned} \tag{15}$$

and $\widetilde{\kappa}_{jc}$, $\overline{\kappa_{bc}}$, ρ_{jc}, $\overline{\rho_{bc}}$ *are* $\mathbf{Z}_2$*-invariant functions of x and y.*

The former Proposition V.1 gives us the prenormal form for generic $\mathbf{Z}_2$-invariant I-Morse families of corank 2. Now, under some additional conditions, we can derive the special infinitesimally stable normal forms.

Proposition V.3: If $r\geqslant s+1$ *then generic* $\mathbf{Z}_2$*-invariant I-Morse families of corank 2 with trivial* μ *are infinitesimally stable and equivalent to trivial extensions of the following families:*

$$\lambda^{2k}+\sum_{i=1}^{k-1} y_i\lambda^{2i}+\sum_{a=1}^{k} y_{k-1+a}\beta\lambda^{2a-2}+\sum_{j=1}^{k-1} x_j\lambda^{2j-1}+\sum_{b=1}^{k-1} x_{k-1+b}\beta\lambda^{2b-1},$$

$2k-1\leqslant s$, and

$$\beta\lambda^{2k}+\lambda^{2t}+\sum_{i=1}^{t-1} y_i\lambda^{2i}+\sum_{a=1}^{\min\{k,t\}} y_{t-1+a}\beta\lambda^{2a-2}+\sum_{j=1}^{t-1} x_j\lambda^{2j-1}+\sum_{b=1}^{\kappa_{tk}} x_{t+b}\beta\lambda^{2b-1},$$

$t-1+\min\{t,k\}\leqslant s.$

Proof: In the considered case $r\geqslant t-1+\kappa_{tk}$, $s\geqslant t-1+\min\{t,k\}$, the $\mathbf{Z}_2$-equivariant, infinitesimally $\mathscr{R}_{\mathbf{Z}_2}$-stable mappings,

$$\Psi(x,y)=(y_1,\ldots,y_{2k-1},\psi_1(x,y),\ldots,\psi_{k-1}(x,y),\overline{\psi_1}(x,y),\ldots,\overline{\psi_{k-1}}(x,y))\in R^{4k-3},$$

and

$$\begin{aligned}\Phi(x,y)=(&y_1,\ldots,y_{t-1+\min\{k,t\}},\phi_1(x,y),\ldots,\phi_{t-1}(x,y),\overline{\phi_1}(x,y),\ldots,\overline{\phi_{\kappa_{tk}}}(x,y))\\ &\in R^{2(t-1)+\min\{t,k\}+\kappa_{tk}},\end{aligned}$$

are submersions, and so may be reduced to the standard normal form and plugged into the families of Proposition V.1. Q.E.D.

In the similar way we get the normal forms in the case of nontrivial representation μ.

Proposition V.4: If $r\geqslant s+1$ *then the generic corank 2,* $\mathbf{Z}_2$*-invariant I-Morse families with the nontrivial* μ *are infinitesimally stable and equivalent to the trivial extension of the following families*:

$$\beta\lambda^{2k-1}+\lambda^{2t}+\sum_{i=1}^{t-1} y_i\lambda^{2i}+\sum_{a=1}^{\min\{k,t\}} x_{t-1+a}\beta\lambda^{2a-2}+\sum_{j=1}^{t-1} x_j\lambda^{2j-1}+\sum_{b=1}^{\min\{t,k\}-1} y_{t-1+b}\beta\lambda^{2b-1},$$

where $t-2+min\{t,k\}\leqslant s$, *or*

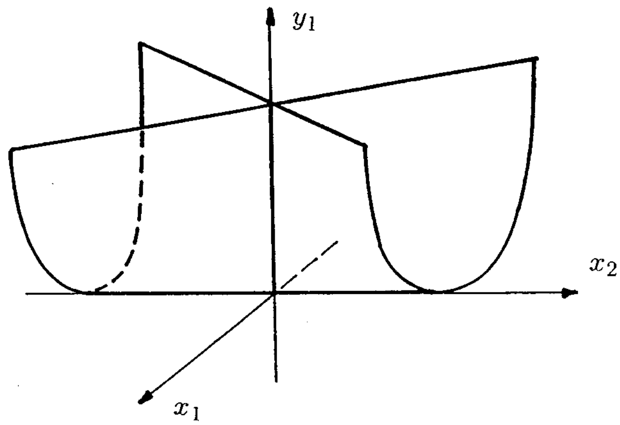


FIG. 1. Z_2-symmetric Whitney's cross-cap.

$$\lambda^{2k}+\sum_{i=1}^{k-1} y_i\lambda^{2i}+\sum_{a=1}^{k} x_a\beta\lambda^{2a-2}+\sum_{j=1}^{k-1} x_{j+k}\lambda^{2j-1}+\sum_{b=1}^{k-1} y_{b+k-1}\beta\lambda^{2b-1},$$

where $2k-2\leqslant s$.

Remark V.5: We know that (Ref. 4) the swallowtail (which is a Z_2*-symmetric set) cannot be realized as a* Z_2*-symmetric caustic. In contrast, the Whitney's cross-cap (which is a* Z_2*-symmetric set illustrated in Fig. 1) can be realized as a* Z_2*-symmetric quasicaustic. Its generating family may be reduced to the following form*:

$$\lambda^3+x_1\beta+x_2\beta\lambda-y_1\lambda,$$

with the action $(\beta,\lambda)\to(-\beta,\lambda)$.

As an interesting illustration (see Fig. 2) *in small dimensions, we present the* Z_2*-symmetric section* $\Sigma_F=Q_F\cap\{y_1=y_2=0\}$ *through the quasicaustic of the family,*

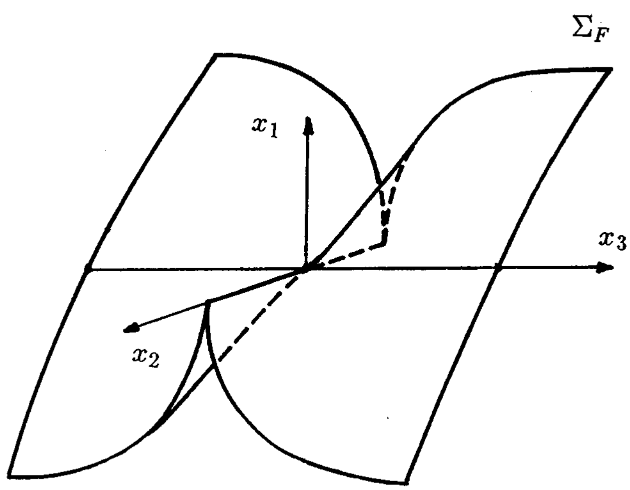


FIG. 2. Z_2-symmetric section $\Sigma_F=Q_F\cap\{y_1=y_2=0\}$ through the quasicaustic Q_F of (16).

$$\lambda^4+y_1\lambda^2+y_2\beta\lambda+x_1\beta+x_2\beta\lambda^2+x_3\lambda, \tag{16}$$

$$\Sigma_F=\{(x_1,x_2,x_3):x_1=-s\lambda^2,x_2=s,x_3=-4\lambda^3,\quad (s,\lambda)\in R^2\}.$$

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