# CHARACTERIZATION OF DIFFEOMORPHISMS THAT ARE SYMPLECTOMORPHISMS

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Abstract. We consider diffeomorphism invariance of symplectic data on submanifolds of symplectic manifolds. We prove that for the canonical restrictions for toruses or general submanifolds of compact symplectic manifold they still exist as the symplectic submanifolds or isotropic toruses. Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be symplectic manifolds or compact symplectic manifolds of dimension $2n > 2$. Let us fix a number $s$ with $0 < s < n$ and assume that a diffeomorphism $\Phi : X \to Y$ maps all $2s$-dimensional symplectic submanifolds of $X$ to symplectic submanifolds of $Y$ or it maps all isotropic $k$-dimensional toruses of $X$ to isotropic toruses of $Y$ $(1 < k \leq n)$. We prove that in these both cases (symplectic and isotropic ones) $\Phi$ is a conformal symplectomorphism, i.e., there is a constant $c \neq 0$ such that $\Phi^* \omega_Y = c\omega_X$.

## 1. Introduction and main results.

Let $(X, \omega_0)$ be the standard symplectic vector space over $\mathbb{R}$ of dimension $2n$, i.e., $X \cong \mathbb{R}^{2n}$ and $\omega_0 = \sum_i dx_i \wedge dy_i$ is the standard non-degenerate skew-symmetric form on $X$. The group of automorphisms of $(X, \omega_0)$ is called the symplectic group and is denoted by $\mathbf{Sp}(X)$.

Our first result shows that any element of $\mathbf{Sp}(X)$ can be finitely decomposed into elements of the family of elementary automorphisms.

**Theorem 1.** *The symplectic group* $\mathbf{Sp}(X)$ *is generated by the family of elementary symplectic automorphisms:*

$$\{L_i(c_i), L_{ij}(c_{ij}), R_i(d_i), R_{ij}(d_{ij}) : 0 < i < j \leq n, \quad c_i, c_{ij}, d_i, d_{ij} \in \mathbb{R}\}$$

*defined by*

1) $L_i(c_i)(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_n, y_1, ..., y_{i-1}, y_i + c_i x_i, y_{i+1}, ..., y_n)$,

2) $L_{ij}(c_{ij})(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_n, y_1, ..., y_{i-1}, y_i + c_{ij} x_j, y_{i+1}, ..., y_{j-1},$
$y_j + c_{ij} x_i, y_{j+1}, ..., y_n)$,

3) $R_i(d_i)(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_{i-1}, x_i + d_i y_i, x_{i+1}, ..., x_n, y_1, ..., y_n)$,

4) $R_{ij}(d_{ij})(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_{i-1}, x_i + d_{ij} y_j, x_{i+1}, ..., x_{j-1}, x_j + d_{ij} y_i,$
$x_{j+1}, ..., x_n, y_1, ..., y_n)$.

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Let $H : X \times \mathbb{R} \ni (z,t) \mapsto \mathbb{R}$ be a smooth function and consider system of differential equations (Hamiltonian system)

$$\frac{d}{dt}\phi(t,z) = J_0(\nabla_z H)(\phi(t,x),t),\ \phi(0,z) = z,$$

where $z = (x_1, \dots, x_n, y_1, \dots, y_n)$ and $J_0$ is the $2n \times 2n$ matrix of $\omega_0$. Then for the smooth solution $\phi(t,z)$ we have that $\Phi(z) = \phi(1,z)$ is a diffeomorphism preserving $\omega_0$, i.e. symplectomorphism, which is called *hamiltonian symplectomorphism* with Hamiltonian $H$ (cf. [10]). The next basic result we get is

**Theorem 2.** *For any linear symplectomorphism $L : (\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^{2n}, \omega_0)$ there exists a polynomial Hamiltonian*

$$H_L(z,t) = \sum_{i,j=1}^{2n} a_{i,j}(t) z_i z_j,$$

*where $a_{i,j}(t) \in \mathbb{R}[t]$ are polynomials of one variable $t$. Moreover $H_L$ can be computed effectively.*

The purpose of this article is twofold: First we provide the new results in still basic linear symplectic geometry which were already formulated. Second we characterize general symplectic manifolds and their structure groups through family of isotropic or symplectic submanifolds and their basic invariance. This leads to a complete geometric characterization of symplectomorphisms and to a reinterpretation of symplectomorphisms as diffeomorphisms acting purely on isotropic or symplectic submanifolds (cf. [2],[8]).

Recall that a submanifold $Z \subset X$ is isotropic if $\omega_X|_{TZ} = 0$. We will call $Z$ a *symplectic submanifold* of $X$ if it is closed and the pair $(Z, \omega_X|_{TZ})$ is a symplectic manifold. Existence of isotropic or symplectic submanifolds with the fixed prescribed data in a compact symplectic manifold is a very fundamental geometric property of symplectic structures. The main result in this direction we prove specifies the canonical restrictions for toruses or general submanifolds of compact symplectic manifold which may exist still as the symplectic submanifolds or isotropic toruses.

**Theorem 3.** *Let $(X,\omega)$ be a symplectic manifold of dimension $2n$ (compact symplectic manifold of dimension $2n$). Let $a_1, ..., a_m$ be a family of points of $X$. Take $0 < k \leq n$ ($0 < s \leq n$). For every $i = 1, ..., m$ choose a linear $k$-dimensional isotropic subspace ($2s$-dimensional symplectic subspace) $H_i \subset T_{a_i}X$. Then there is a closed isotropic $k-$dimensional torus (closed symplectic $2s$-dimensional submanifold) $Y \subset X$ such that*

1) $a_i \in Y$,

2) $T_{a_i}Y = H_i$.

Linear symplectomorphisms of $(X, \omega_0)$ are characterized in [5] as linear automorphisms of $X$ preserving some minimal, complete data defined by $\omega_0$ on systems of linear subspaces. In this way the linear symplectic group $\mathbf{Sp}(X)$ may be characterized geometrically together with its natural conformal and anti-symplectic extensions. It is the natural task to put the linear considerations of symplectic invariants into a more general context (cf. [4], [7]). Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be symplectic manifolds of dimension $2n$ (all manifolds in this paper are assumed to be connected). We say that a diffeomorphism $F : X \to Y$ is a *conformal symplectomorphism* (cf. [9]) if there is a non-zero constant $c \in \mathbb{R}$ such that $F^*\omega_Y = c\omega_X$.

**Theorem 4.** *Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be symplectic manifolds of dimension $2n > 2$ (compact symplectic manifolds of dimension $2n > 2$). Fix a number $1 < k \leq n$ $(0 < s < n)$. Assume that $\Phi : X \to Y$ is a diffeomorphism which transforms all $k-$dimensional isotropic toruses of $X$ ($2s$-dimensional symplectic, closed submanifolds of $X$) onto isotropic toruses of $Y$ (symplectic, closed submanifolds of $Y$). Then $\Phi$ is a conformal symplectomorphism.*

In other words, for any fixed $k$ (or $s$) as above, the conformal symplectic structure on $X$ is uniquely determined by the family of all $k-$dimensional isotropic toruses ($2s$-dimensional, closed symplectic submanifolds) of $X$.

## 2. Generators of the group $Sp(2n)$

Here we recall some basic facts about the linear symplectic group. Let $(X, \omega)$ be a symplectic vector space. There exists a basis of $X$, called a symplectic basis, $u_1, \dots, u_n, v_1, \dots, v_n$, such that

$$\omega(u_i, u_j) = \omega(v_i, v_j) = 0, \quad \omega(u_i, v_j) = \delta_{ij}.$$

Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be symplectic vector spaces. We say that a linear isomorphism $F : X \to Y$ is a *symplectomorphism* (or is *symplectic* on $X$) if $F^*\omega_Y = \omega_X$, i.e., $\omega_X(x, y) = \omega_Y(F(x), F(y))$ for every $x, y \in X$. The group of automorphisms of $(X, \omega)$ is called the symplectic group and is denoted by $\mathbf{Sp}(X, \omega)$. Via a symplectic basis, $X$ can be identified with the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ and $\mathbf{Sp}(X, \omega)$ can be identified with the group of $2n \times 2n$ real matrices $A$ which satisfy $A^T J_0 A = J_0$, where $J_0$ is the $2n \times 2n$ matrix of $\omega_0$ (in the standard basis), i.e.,

$$J_0 = \begin{bmatrix} 0 & \dots & 0 & -1 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 \\ 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}.$$

We can define the following "elementary" symplectomorphisms:

1) $L_i(c_i)(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_n, y_1, ..., y_{i-1}, y_i + c_i x_i, y_{i+1}, ..., y_n)$,

2) $L_{ij}(c_{ij})(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_n, y_1, ..., y_{i-1}, y_i + c_{ij} x_j, y_{i+1}, ..., y_{j-1}, y_j + c_{ij} x_i, y_{j+1}, ..., y_n)$,

3) $R_i(d_i)(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_{i-1}, x_i + d_i y_i, x_{i+1}, ..., x_n, y_1, ..., y_n)$,

4) $R_{ij}(d_{ij})(x_1, ..., x_n, y_1, ..., y_n) = (x_1, ..., x_{i-1}, x_i + d_{ij} y_j, x_{i+1}, ..., x_{j-1}, x_j + d_{ij} y_i, x_{j+1}, ..., x_n, y_1, ..., y_n)$,

where $c_i, c_{ij}, d_i, d_{ij}$ are real numbers and $1 \leq i < j \leq n$.

We have the following basic result:

**Theorem 2.1.** *Let $X = (\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space. Then the group $\mathbf{Sp}(X)$ is generated by the following family of elementary symplectomorphisms:*

$$\{L_i(c_i), L_{ij}(c_{ij}), R_i(d_i), R_{ij}(d_{ij}) :\ 0 < i < j \leq n \text{ and } c_i, c_{ij}, d_i, d_{ij} \in \mathbb{R}\},$$

*i.e. if $g \in \mathbf{Sp}(X)$ then $g = \prod_{i=1}^m e_i$, where $e_i$ is one of the elementary symplectomorphisms and $m \in \mathbb{N}$.*

*Proof.* We reason by induction. For $n = 1$ we have $\mathbf{Sp}(\mathbb{R}^2) = \mathbf{SL}(2)$ and the result is well known from linear algebra. Assume $n > 1$.

Let $S : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear symplectomorphism. Denote coordinates by $x_1, y_1, ...., x_n, y_n$ (where $\omega_0 = \sum_i dx_i \wedge dy_i$ ). We have

$$S(x_1, y_1, ..., x_n, y_n) = (\sum_i a_{1,i}x_i + \sum_j b_{1,j}y_j, \dots, \sum_i a_{2n,i}x_i + \sum_j b_{2n,j}y_j).$$

Observe how the rows of the matrix of $S$ are transformed under composition $S \circ L$ with an elementary symplectomorphism $L$ (for simplicity we consider only the first row and we take the coordinates $x_1, ..., x_n, y_1, ..., y_n$). After composition

with $L_i(c)$ we have:

1) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1i} + cb_{1i}, ..., a_{1n}, b_{11}, ..., b_{1n})$,

with $L_{ij}(c)$ we have:

2) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1i} + cb_{1j}, ..., a_{1j} + cb_{1i}, ..., a_{1n}, b_{11}, ..., b_{1n})$,

with $R_i(c)$ we have:

3) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1n}, b_{11}, ..., b_{1i} + ca_{1i}, ..., b_{1n})$,

with $R_{ij}(c)$ we have:

4) $(a_{11}, ..., a_{1n}, b_{11}, ..., b_{1n}) \to (a_{11}, ..., a_{1n}, b_{11}, ..., b_{1i} + ca_{1j}, ..., b_{1j} + ca_{1i}, ..., b_{1n})$.

Transformations 1) - 4) will be called *elementary operations*. Now we show that using only elementary operations we can transform the first row of $S$ to $(1, 0, ..., 0)$ and the second to $(0, ..., 0, 1, 0, ..., 0)$ (here the unit corresponds to $b_{1n}$).

First note that rows $\mathbf{r_1}, ..., \mathbf{r_{2n}}$ of the matrix $S$ form a symplectic basis. Now, consider the first row. Of course it has a non-zero element, say $b_{1s}$. Using $L_s(c)$ we can assume that also $a_{1s} \neq 0$. Now using $L_{is}(c)$ and $R_{js}(d)$ for sufficiently general $c$ and $d$ we can assume that all elements of the first row are non-zero. Again applying $R_i(c)$ for $i > 1$ we can now transform the first row to $(a_{11}, ..., a_{1n}, 1, 0, ..., 0)$. Using $L_{1j}(c)$ we can transform this row to $(1, 0, ..., 0, 1, 0, ..., 0)$ and finally using $R_1(-1)$ we obtain $(1, 0, ...., 0)$. Now consider the row $\mathbf{r_{n+1}}$ (after these transformations): $\mathbf{r_{n+1}} = (a_{n+11}, ..., a_{n+1n}, b_{n+11}, ..., b_{n+1n})$. We can apply our method to the subrow $(a_{n+12}, ..., a_{n+1n}, b_{n+12}, ..., b_{n+1n})$ (if it is non-zero) and obtain finally the row $(a_{n+11}, 1, 0, ...., 0, b_{n+11}, 0, ..., 0)$

(or $(a_{n+11}, 0, ...., 0, b_{n+11}, 0, ..., 0))$. Since the value of $\omega_0$ on these two rows is 1 we conclude that $b_{n+11} = 1$. Now (in the first case) we can use $L_{12}(-1)$ to obtain a row of the form $(a_{n+_1 1}, 0, ..., 0, 1, 0, ..., 0)$. Finally applying $L_1(-a_{12})$ we get $(0, ..., 0, 1, 0, ..., 0)$.

Thus under all these compositions the matrix of $S$ has the form

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & \dots & b_{21} & \dots & b_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & b_{31} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \dots & 0 \\ a_{n+11} & a_{n+12} & a_{n+13} & \dots & b_{n+11} & \dots & b_{n+1n} \\ a_{n+21} & a_{n+22} & b_{n+22} & \dots & b_{n+21} & \dots & b_{4n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n1} & a_{2n2} & a_{2n3} & \dots & b_{2n1} & \dots & b_{2nn} \end{bmatrix}.$$

For $j \neq 1, n+1$ we have $\omega_0(\mathbf{r}_1, \mathbf{r}_j) = 0$ and $\omega_0(\mathbf{r}_{n+1}, \mathbf{r}_j) = 0$. We can easily conclude that for all such $j$ elements $a_{j1}$ and $b_{j1}$ in the matrix of $S$ are 0. This implies that the matrix

$$\begin{bmatrix} a_{22} & a_{23} & \dots & b_{21} & \dots & b_{2n} \\ a_{32} & a_{33} & \dots & b_{31} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n2} & a_{n3} & \dots & b_{n1} & \dots & b_{nn} \\ a_{n+22} & a_{n+23} & \dots & b_{n+21} & \dots & b_{n+2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2n2} & a_{2n3} & \dots & b_{2n1} & \dots & b_{2nn} \end{bmatrix}$$

is a symplectic matrix we can apply the induction hypothesis. □

We conclude this section by recalling (and extending) some result from [5].

**Definition 2.2.** Let $\mathcal{A}_{l,2r} \subset G(l, 2n)$ denote the set of all $l$-dimensional linear subspaces of $X$ on which the form $\omega$ has rank $\leq 2r$.

Of course $\mathcal{A}_{l,2r} \subset \mathcal{A}_{l,2r+2}$ if $2r + 2 \leq l$. We have the following important (see [5], Theorem 6.2):

**Theorem 2.3.** *Let $(X, \omega)$ be a symplectic vector space of dimension $2n$ and let $F : X \to X$ be a linear automorphism. Let $0 < 2r < 2n$. Assume $F$ transforms $\mathcal{A}_{2r,2r-2}$ into $\mathcal{A}_{2r,2r-2}$. Then there is a non-zero constant $c$ such that $F^*\omega = c\omega$.*

From Theorem 2.3 we can deduce the following interesting facts:

**Proposition 2.4.** *Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be symplectic vector spaces of dimension $2n$ and let $F : X \to Y$ be a linear isomorphism. Fix a number $s :\ 0 < s < n$ and assume that $F$ transforms all $2s$-dimensional symplectic subspaces of $X$ onto symplectic subspaces of $Y$. Then there is a non-zero constant $c$ such that $F^*\omega_Y = c\omega_X$.*

*Proof.* Via a symplectic basis we can assume that $(X,\omega_X)\cong(\mathbb{R}^{2n},\omega_0)\cong(Y,\omega_Y)$. By assumption the mapping $F^*$ induced by $F$ transforms the set $A=\mathcal{A}_{2s,2s}\setminus\mathcal{A}_{2s,2s-2}$ into the same set $A$. Of course $F^*:A\to A$ is an injection. Since $A$ is a smooth algebraic variety and $F^*$ is regular, the Borel Theorem (see [1]) implies that $F^*$ is a bijection. This means that $F$ transforms $\mathcal{A}_{2s,2s-2}$ into the same set, and we conclude the proof by applying Theorem 2.3. □

**Proposition 2.5.** *Let $(X,\omega_X)$ and $(Y,\omega_Y)$ be symplectic vector spaces of dimension $2n$ and let $F:X\to Y$ be a linear isomorphism. Fix a number $k:\ 1<k\le n$ and assume that $F$ transforms all $k$-dimensional isotropic subspaces of $X$ onto isotropic subspaces of $Y$. Then there is a non-zero constant $c$ such that $F^*\omega_Y=c\omega_X$.*

*Proof.* For $k=2$ it follows immediately from Theorem 2.3. Assume that $k>2$. Take a plane $H$ belonging to $\mathcal{A}_{2,0}$. Since $H$ is isotropic then we can extend $H$ to $k-$dimensional isotropic subspace $L$. By assumption $L$ is transformed onto isotropic subspace $F(L)$. Observe that $F(H)$ is contained in $F(L)$ then $F(H)$ is also isotropic. In particular $F(H)\in\mathcal{A}_{2,0}$. Then on the basis of Theorem 2.3 we have the thesis. □

We end this section by:

**Proposition 2.6.** *Let $X$ be a vector space of dimension $2n$ and let $\omega_1,\ \omega_2$ be two symplectic forms on $X$. If $\mathbf{Sp}(X,\omega_1)\subset\mathbf{Sp}(X,\omega_2)$, then there exists a non-zero constant $c$ such that $\omega_2=c\omega_1$.*

*Proof.* If $n=1$, then theorem is obvious. Assume that $n>1$. Let $\mathcal{A}_1$ $(\mathcal{A}_2)$ be a set of all $\omega_1$ $(\omega_2)$ symplectic 2 dimensional subspaces of $X$. These sets are open and dense in the Grassmannian $G(2,2n)$. Hence $\mathcal{A}_1\cap\mathcal{A}_2\neq\emptyset$. Take $H\in\mathcal{A}_1\cap\mathcal{A}_2$. We have $\mathcal{A}_1=\mathbf{Sp}(X,\omega_1)H\subset\mathbf{Sp}(X,\omega_2)H=\mathcal{A}_2$. Now apply Proposition 2.4 to $X=(X,\omega_1),\ Y=(X,\omega_2)$ and $F=\textit{identity}$. □

## 3. Hamiltonian symplectomorphisms

Let $X=(\mathbb{R}^{2n},\omega_0)$ be the standard symplectic vector space. In $X$ we consider the norm $\|(a_1,\dots,a_{2n})\|=\max_{i=1}^{2n}|a_i|$. Take a smooth function $H:X\times\mathbb{R}\ni(z,t)\to\mathbb{R}$ and consider a system of differential equations (cf. [3])

$$\frac{d}{dt}\phi(t,z)=J_0(\nabla_z H)(\phi(t,z),t),\ \phi(0,z)=z.$$

Assume that this system has a solution $\phi(t,z)$ for every $z$ and every $t$ (this is satisfied, e.g., if supports of all functions $H_t,\ t\in\mathbb{R}$ are contained in a compact set). Then we can define the diffeomorphism

$$\Phi(z)=\phi(1,z) \tag{3.1}$$

It is not difficult to check that $\Phi$ is a symplectomorphism.

**Definition 3.1.** Let $\Phi : X \to X$ be a symplectomorphism. We say that $\Phi$ is a *hamiltonian symplectomorphism* if it is given by the formula (3.1) for some smooth function $H$. We also say that $H$ is a Hamiltonian of $\Phi$.

**Lemma 3.2.** *All elementary linear symplectomorphisms are hamiltonian symplectomorphisms.*

*Proof.* Indeed, we have:

1) $L_i(c)$ is given by the Hamiltonian $H(x, y) = (c/2)x_i^2$,

2) $L_{ij}(c)$ is given by the Hamiltonian $H(x, y) = cx_i x_j$,

3) $R_i(c)$ is given by the Hamiltonian $H(x, y) = -(c/2)y_i^2$,

4) $R_{ij}(c)$ is given by the Hamiltonian $H(x, y) = -cy_i y_j$. □

Now we show how to compute a Hamiltonian of a linear symplectomorphism:

**Theorem 3.3.** *Let $L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear symplectomorphism. Then $L$ has a polynomial Hamiltonian*

$$H_L(z, t) = \sum_{i,j=1}^{2n} a_{i,j}(t) z_i z_j, \tag{3.2}$$

*where $a_{i,j}(t) \in \mathbb{R}[t]$ are polynomials of one variable $t$. Moreover, we can compute $H_L$ effectively.*

*Proof.* Let $L = L_m \circ \cdots \circ L_1$ where $L_i$ are elementary symplectomorphisms. We proceed by induction with respect to $m$. If $m = 1$ then we can use Lemma 3.2. In this case the flow $L_1(t)$ depends linearly on $t$.

Now consider $L' = L_{m-1} \circ \cdots \circ L_1$. By the induction hypothesis $L'(t) = L_{m-1}(t) \circ \cdots \circ L_1(t)$ is given by the Hamiltonian $H'$ of the form 3.2. Let $H''$ be the Hamiltonian of $L_m$ (as in Lemma 3.2). Now the flow $L(t) = L_m(t) \circ L'(t)$ is given by the Hamiltonian

$$H(z, t) = H''(z) + H'(L_m(t)^{-1}(z), t).$$

Of course it has also the form 3.2. Since we can decompose $L$ into the product $L = L_m \circ \cdots \circ L_1$ effectively (see the proof of Theorem 2.1), we can also compute $H$ in effective way. □

**Proposition 3.4.** *Let $L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a hamiltonian symplectomorphism given by the flow $z \to \phi(t, z)$; $t \in \mathbb{R}$. Assume that $\phi(t, 0) = 0$ for $t \in [0, 1]$. For every $\eta > 0$ there is an $\epsilon > 0$ and a hamiltonian symplectomorphism $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that*

*1) $\Phi(z) = L(z)$ for all $z$ with $\|z\| \leq \epsilon$,*

*2) $\Phi(z) = z$ for all $z$ with $\|z\| \geq \eta$.*

*Proof.* We know that $L(z) = \phi(1, z)$, where $\phi(t, z)$ is the solution of some differential equation

$$\frac{d}{dt}\phi(t, z) = J_0(\nabla_z H)(\phi(t, z), t);\ \phi(0, z) = z.$$

Since $\phi(t,0)=0$ for every $t\in[0,1]$, we can find $\epsilon>0$ so small, that all trajectories $\{\phi(t,z),\ 0\le t\le 1\}$, which start from the ball $B(0,\epsilon)$ are contained in the ball $B(0,\eta/2)$. Let $\sigma:\mathbb{R}^{2n}\to\mathbb{R}$ be a smooth function such that

$$\sigma(z)=\begin{cases}1 & \text{if } \|z\|\le\eta/2,\\ 0 & \text{if } \|z\|\ge\eta.\end{cases}$$

Take $S=\sigma H$. The hamiltonian symplectomorphism $\Phi$ given by the differential equation

$$\frac{d}{dt}\phi(t,z)=J_0(\nabla_z S)(\phi(t,z),t),\ \phi(0,z)=z,$$

is well defined on the whole of $\mathbb{R}^{2n}$ and

$$\Phi(z)=\begin{cases}L(z) & \text{if } ||z||\le\epsilon,\\ z & \text{if } \|z\|\ge\eta.\end{cases}$$

□

Now Theorem 3.3 easily yields the following:

**Corollary 3.5.** *Let $L:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ be a linear symplectomorphism. For every $\eta>0$ there is an $\epsilon>0$ and a hamiltonian symplectomorphism $\Phi:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ such that*

*1) $\Phi(z)=L(z)$ for all $z$ with $\|z\|\le\epsilon$,*

*2) $\Phi(z)=z$ for all $z$ with $\|z\|\ge\eta$.*

## 4. Characterization of symplectomorphisms.

Before we formulate our next result we need the following (well-known):

**Lemma 4.1.** *Let $X=(\mathbb{R}^{2n},\omega_0)$ be the standard symplectic vector space. Fix $\eta>0$ and let $a,b\in B(0,\eta)$. Then there exists a symplectomorphism $\Phi:X\to X$ such that*

$$\Phi(a)=b \textit{ and } \ \Phi(z)=z \ \textit{ for } \ \|z\|\ge 2\eta.$$

*Proof.* Let $c=(c_1,\dots,c_{2n})=b-a$. Define a sequence of points as follows:

1) $a_0=a$,

2) $a_i=a_{i-1}+(0,\dots,0,c_i,0,\dots,0)$.

Of course $a_i\in B(0,\eta)$ and $a_{2n}=b$. Now consider the translation

$$T_i:\mathbb{R}^{2n}\ni(x,y)\mapsto(x,y)+(0,\dots,0,c_i,0,\dots,0)\in\mathbb{R}^{2n}.$$

We have $T_i(a_{i-1})=a_i$ for $i=1,...,2n$.

The translation $T_i$ is a hamiltonian symplectomorphism given by the Hamiltonian

$$H_i(x,y)=\begin{cases}-c_iy_i & \text{if } i\le n,\\ c_ix_{i-n} & \text{if } i>n.\end{cases}$$

Let $V_i$ be the symplectic vector field which is determined by the Hamiltonian $H_i$. Since the ball $B(0,r)$ is a convex set, all trajectories $\phi(t),\ 0\le t\le 1$, of the symplectic vector fields $V_i$, which begin at $a_i$ lie in the ball $B(0,\eta)$. Let $\sigma:\mathbb{R}^{2n}\to\mathbb{R}$ be a smooth function such that

$$\sigma(z)=\begin{cases}1 & \text{if } \|z\|\le\eta,\\ 0 & \text{if } \|z\|\ge 2\eta.\end{cases}$$

Now let $F_i:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ be the hamiltonian symplectomorphism given by the Hamiltonian $G_i=\sigma H_i$. Then

$$G_i(a_{i-1})=a_i \text{ and } G_i(z)=z \text{ if } \|z\|\ge 2\eta.$$

Now it is enough to take $\Phi=G_{2n}\circ G_{2n-1}\circ\cdots\circ G_1$. □

We apply Proposition 3.4 to the general case:

**Theorem 4.2.** *Let $(X,\omega)$ be a symplectic manifold. Let $a_1,...,a_m$ and $b_1,...,b_m$ be two families of points of $X$. For every $i=1,...,n$ choose a linear symplectomorphism $L_i:T_{a_i}X\to T_{b_i}X$. Then there is a symplectomorphism $\Phi:X\to X$ such that*

*1) $\Phi(a_i)=b_i$,*

*2) $d_{a_i}\Phi=L_i$*

*for every $i=1,...,m$.*

*Proof.* By the Darboux Theorem every point $z\in X$ has an open neighborhood $V_z$ which is symplectically isomorphic to the ball $B(0,r_z)$ in the standard vector space $(\mathbb{R}^{2n},\omega_0)$. Denote by $U_z\subset V_z$ the open set which corresponds to the ball $B(0,r_z/3)$.

Since $\dim X\ge 2$ the manifold $X\setminus\{a_2,...,a_m\}$ is also connected. Hence there exists a smooth path $\gamma:I\to X$ such that $\gamma(0)=a_1,\ \gamma(1)=b_1$ and $\{a_2,...,a_m\}\cap\gamma(I)=\emptyset$. Additionally we can assume that the sets $V_z$ which cover $\gamma(I)$ are also disjoint from $\{a_2,...,a_m\}$.

Let $\epsilon$ be a Lebesgue number for the function $\gamma:I\to X$ with respect to the cover $\{U_z\}_{z\in X}$ and choose an integer $N$ with $1/N<\epsilon$. If $I_k:=[k/N,(k+1)/N]$, then $\gamma(I_k)$ is contained in some $\{U_z\}$; denote it by $U_k$, the set $V_z$ by $V_k$, and $r_z$ by $r_k$. Let $A_k:=\gamma(k/N)$, in particular $A_0=a_1, A_N=b_1$.

Since $V_k\cong B(0,r_k)$ and $A_k,A_{k+1}\in B(0,r_k/3)$ we can apply Lemma 4.1 to obtain a symplectomorphism $\Phi:B(0,r_k)\to B(0,r_k)$ such that

$$\Phi(A_k)=A_{k+1} \text{ and } \ \Phi(z)=z \quad \text{for } \|z\|\ge(2/3)r_k.$$

We can extend $\Phi$ to the whole of $X$ (we glue it with the identity); denote this extension by $\Phi_k$. Put

$$\Psi=\Phi_N\circ\Phi_{N-1}\circ\cdots\circ\Phi_0.$$

Then $\Psi(a_1)=b_1$ and $\Psi(a_i)=a_i$ for $i>1$. Repeating this process, we finally arrive at a symplectomorphism $\Sigma:X\to X$ such that $\Sigma(a_i)=b_i$ for $i=1,\dots,m$. In a similar way using Proposition 3.5 we can construct a symplectomorphism $\Pi:X\to X$ such that

1) $\Pi(b_i)=b_i$,

2) $d_{b_i}\Pi=L_i\circ(d_{a_i}\Sigma)^{-1}$.

Now it is enough to take $\Phi = \Pi \circ \Sigma$. □

**Remark 4.3.** The point "1)" of in the thesis of Theorem 4.2 is well known, however the new ingredient is given by the point "2)" of this theorem.

Since for compact symplectic manifold $(X, \omega)$ of dimension $2n$ it is well known (cf. [6]) that for a fixed number $0 < s \leq n$ there exists a closed $2s-$dimensional symplectic submanifold $Z \subset X$, we can use Theorem 4.2 to obtain:

**Corollary 4.4.** *Let $(X, \omega)$ be a compact symplectic manifold of dimension $2n$. Let $a_1, ..., a_m$ be a family of points of $X$. Take $0 < s \leq n$. For every $i = 1, ..., m$ choose a linear $2s$-dimensional symplectic subspace $H_i \subset T_{a_i}X$. Then there is a closed symplectic $2s$-dimensional submanifold $Y \subset X$ such that*

*1) $a_i \in Y$,*

*2) $T_{a_i}Y = H_i$.*

In a similar way we get:

**Corollary 4.5.** *Let $(X, \omega)$ be a symplectic manifold of dimension $2n$. Let $a_1, ..., a_m$ be a family of points of $X$. Take $0 < k \leq n$. For every $i = 1, ..., m$ choose a linear $k$-dimensional isotropic subspace $H_i \subset T_{a_i}X$. Then there is a closed isotropic $k$-dimensional torus $Y \subset X$ such that*

*1) $a_i \in Y$,*

*2) $T_{a_i}Y = H_i$.*

## 5. Diffeomorphisms that are symplectomorphisms

Finally we show that a symplectomorphism can be described as a diffeomorphism which preserves symplectic or isotropic submanifolds of given fixed dimension.

**Theorem 5.1.** *Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be compact symplectic manifolds of dimension $2n > 2$. Fix a number $0 < s < n$. Assume that $\Phi : X \to Y$ is a diffeomorphism which transforms all $2s$-dimensional symplectic submanifolds of $X$ onto symplectic submanifolds of $Y$. Then $\Phi$ is a conformal symplectomorphism, i.e., there exists a non-zero number $c \in \mathbb{R}$ such that*

$$\Phi^* \omega_Y = c\omega_X.$$

*Proof.* Fix $z \in X$ and let $H \subset T_zX$ be a $2s$-dimensional symplectic subspace of $T_zX$. By Proposition 4.4 (applied for $m = 1$, $a_1 = z$ and $H_1 = H$) there exists a $2s$-dimensional symplectic submanifold $M$ of $X$ such that $z \in M$ and $T_zM = H$.

Let $\Phi(M) = M'$, $z' = \Phi(z)$. By assumption the submanifold $M' \subset Y$ is symplectic. This means that the space $d_z\Phi(H) = T_{z'}M'$ is symplectic. Hence the mapping $d_z\Phi$ transforms all linear

$2s$-dimensional symplectic subspaces of $T_zX$ onto subspaces of the same type. By Proposition 2.4 this implies that $d_z\Phi$ is a conformal symplectomorphism. i.e.,

$$(d_z\Phi)^*\omega_Y = \lambda(z)\omega_X,$$

where $\lambda(z) \neq 0$. This means that there is a smooth function $\lambda : X \to \mathbb{R}^*$ $(= \mathbb{R} \setminus \{0\})$ such that

$$\Phi^*\omega_Y = \lambda\omega_X.$$

But since the form $\omega_X$ is closed, so is $\Phi^*\omega_Y$. Since $n > 1$ this implies that the derivative $d\lambda$ vanishes, i.e., the function $\lambda$ is constant. □

**Theorem 5.2.** *Let $(X, \omega_X)$ and $(Y, \omega_Y)$ be symplectic manifolds of dimension $2n > 2$. Fix a number $1 < k \leq n$. Assume that $\Phi : X \to Y$ is a diffeomorphism which transforms all $k$-dimensional isotropic tori of $X$ onto isotropic tori of $Y$. Then $\Phi$ is a conformal symplectomorphism, i.e., there exists a non-zero constant $c \in \mathbb{R}$ such that*

$$\Phi^*\omega_Y = c\omega_X.$$

*Proof.* Fix $z \in X$ and let $H \subset T_zX$ be a $k$-dimensional isotropic subspace of $T_zX$. By Theorem 4.5 (applied for $m = 1$, $a_1 = z$ and $H_1 = H$) there exists a $k$-dimensional isotropic torus $M$ of $X$ such that $z \in M$ and $T_zM = H$.

Let $\Phi(M) = M'$, $z' = \Phi(z)$. By assumption the torus $M' \subset Y$ is isotropic. This means that the space $d_z\Phi(H) = T_{z'}M'$ is isotropic. Hence the mapping $d_z\Phi$ transforms all linear $k$-dimensional isotropic subspaces of $T_zX$ onto subspaces of the same type. By Proposition 2.5 this implies that $d_z\Phi$ is a conformal symplectomorphism. The rest of the proof is the same as in the case of Theorem 5.1 above. □

**Remark 5.3.** Let us note that in particular if $\Phi$ maps Lagrangian tori onto tori of the same type then $\Phi$ is a conformal symplectomorphism.

**Corollary 5.4.** *Let $X$ be a compact manifold of dimension $2n > 2$. Let $\omega_1$ and $\omega_2$ be two symplectic forms on $X$. Fix a number $1 < k < n$. Assume that the family of all $2k$-dimensional $\omega_1$-symplectic submanifolds of $X$ is contained in the family of all $2k$-dimensional $\omega_2$-symplectic submanifolds of $X$. Then there exists a non-zero number $c \in \mathbb{R}$ such that*

$$\omega_1 = c\omega_2.$$

*Proof.* It is enough to apply Theorem 5.1 to $X = (X, \omega_1)$, $Y = (X, \omega_2)$ and $\Phi = identity$. □

**Corollary 5.5.** *Let $(X, \omega)$ be a compact symplectic manifold of dimension $2n > 2$. Fix a number $1 < k < n$. Assume that $\Phi : X \to X$ is a diffeomorphism which transforms all $2k$-dimensional symplectic submanifolds of $X$ onto submanifolds of the same type. Then $\Phi$ is a symplectomorphism or antisymplectomorphism, i.e., $\Phi^*\omega = \pm\omega$. If $\Phi$ preserves an orientation and $n$ is odd, then $\Phi$ is a symplectomorphism. Moreover, if $n$ is even, then $\Phi$ has to preserve the orientation.*

*Proof.* Indeed, we have $\Phi^*\omega = c\omega$. And we can write

$$vol(X) = \int_X \omega^n = \pm \int_X \Phi^*\omega^n = \pm c^n \int_X \omega^n \tag{5.1}$$

hence $c = \pm 1$. Moreover, if $\Phi$ preserves an orientation and $n$ is odd, then we get that $c = 1$. If $n$ is even then $(-\omega)^n = \omega^n$ and $\Phi$ has to preserve the orientation. □

**Remark 5.6.** The same corollaries like Corollary 5.4 and Corollary 5.5 are true for compact symplectic manifold $X$ in the case of isotropic tori. Also the similar concept of geometric characterization of symplectomorphisms was already used for diffeomorphisms preserving capacity, which imply symplectic and antisymplectic diffeomorphisms (cf. [4],[6]).

**Example 5.7.** We show that in the general case $\Phi$ need not to be a symplectomorphism. Let $Y = (S^2, \omega)$ (where $\omega$ is a standard volume form on the sphere) and let $(X_n, \omega_n) = \prod_{i=1}^n Y$ be a standard symplectic product. Further let $\sigma : S^2 \ni (x, y, z) \to (x, y, -z) \in S^2$ be a mirror symmetry. Of course $\sigma^*\omega = -\omega$. More general if $\Sigma = \prod_{i=1}^n \sigma : X_n \to X_n$, then $\Sigma^*\omega_n = -\omega_n$. Hence it is possible that $\Phi$ from Corollary 5.5 is an antisymplectomorphism.

However, in any case either $\Phi$ or $\Phi \circ \Phi$ is a symplectomorphism.

Now let $(X, \omega)$ be a symplectic manifold and let us denote by $\mathbf{Symp}(X, \omega)$ the group of symplectomorphisms of $X$. At the end of this note we show that this group also determine a conformal symplectic structure on $X$:

**Theorem 5.8.** *Let $X$ be a smooth manifold of dimension $2n > 2$ and let $\omega_1, \omega_2$ be two symplectic forms on $X$. If $\mathbf{Symp}(X, \omega_1) \subset \mathbf{Symp}(X, \omega_2)$, then there exists a non-zero constant $c$ such that $\omega_2 = c\omega_1$.*

*Proof.* Take $z \in X$ and consider symplectic vector spaces $V_1 = (T_zX, \omega_1)$ and $V_2 = (T_zX, \omega_2)$. By Theorem 4.2 we have that for every linear symplectomorphism $S$ of $V_1$, there is a symplectomorphism $\Phi_S \in \mathbf{Symp}(X, \omega_1)$, such that

a) $\Phi_S(z) = z$,

b) $d_z\Phi_S = S$.

Since $\mathbf{Symp}(X, \omega_1) \subset \mathbf{Symp}(X, \omega_2)$ we easily obtain that $\mathbf{Sp}(V_1) \subset \mathbf{Sp}(V_2)$. Consequently by Proposition 2.6 there exist a non-zero number $\lambda(z)$ such that $\omega_2(z) = \lambda(z)\omega_1(z)$. Now we finish the proof as in the proof of Theorem 5.1. □

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