ANNALES SOCIETATIS MATHEMATICAE POLONAE
Series I: COMMENTATIONES MATHEMATICAE XXVI (1986)
ROCZNIKI POLSKIEGO TOWARZYSTWA MATEMATYCZNEGO
Seria I: PRACE MATEMATYCZNE XXVI (1986)

STANISŁAW JANECZKO (Warszawa)

On Arnold's singularities of type B_k, C_k, F_4

Abstract. It is shown that using Tougeron's version of the Implicit Function Theorem (cf. [4]) the classification theorem of Arnold [1] can be completely proved.

1. Definitions. Let E_n denote the ring of germs at $0 \in \boldsymbol{R}^n$ of C^∞-functions, $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$ and $\mathfrak{m}_n$ the ideal consisting of all f with $f(0) = 0$. $\mathfrak{m}_n$ is the unique maximal ideal in E_n and $\mathfrak{m}_n = \langle x_1, \ldots, x_n \rangle$ the ideal generated by $x_1, \ldots, x_n$ (see [3]). Let L be a smooth submanifold of $\boldsymbol{R}^n$, $\dim L = n-1$, and $0 \in L$.

Two germs $f, g \in \mathfrak{m}_n$ are called *G_L-right equivalent* if there exists a C^∞-diffeomorphism $\Phi\colon (\boldsymbol{R}^n; 0) \to (\boldsymbol{R}^n, 0)$ such that $f = g \circ \Phi$ and $\Phi(L) = L$. Notation: $f \overset{G_L}{\sim} g$.

We choose a local chart in $(\boldsymbol{R}^n, 0)$, $\{x_1, \ldots, x_n\}$ such that the submanifold L has a form $L = \{(x_1, \ldots, x_n) \in \boldsymbol{R}^n;\ x_1 = 0\}$. For $f \in \mathfrak{m}_n^2$ (such f is called a *singularity*), we define:

The G_L-right codimension of f, $\operatorname{codim}_L(f) = \dim_{\boldsymbol{R}} \mathfrak{m}_n/I(f) - 1$, where the ideal $I(f)$ is defined as follows

$$I(f) = \left\langle x_1 \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle,$$

i.e., $I(f)$ is the ideal of E_n generated by $x_1 \dfrac{\partial f}{\partial x_1}$ and remaining partial derivatives of f (cf. for example [4]). The G_L-right codimension of f is equal to the codimension of the orbit of f in $\mathfrak{m}_n^2$ under the right-action of the group of all diffeomorphisms of $(\boldsymbol{R}^n, 0)$ preserving L. Examples: Direct computations gives us, $\operatorname{codim}_L(\pm x_1^k + x_2^2) = k-2$, $\operatorname{codim}_L(x_1 x_2 \pm x_2^k) = k-2$, $\operatorname{codim}_L(\pm x_1^2 + x_2^3) = 2$.

The quotient space $J^k = E_n/\mathfrak{m}_n^{k+1}$ is called the *jet space*. Let j^k be the canonical projection $j^k\colon E_n \to J^k$. By $j^k f$ we denote the k-jet of the germ f. A germ f is called *G_L-right* determined by its k-jet or G_L-k determined if $j^k f = j^k g \Rightarrow f \overset{G_L}{\sim} g$ for all $g \in E_n$. Let $\det_L(f)$ be the smallest integer such that f is G_L-right determined by its k-jet. If no such integer exists we write $\det_L(f) = \infty$.

Let $F\colon M^k \times G_L^k \to M^k$, be a right C^∞-action of the group of k-jets of diffeomorphisms belonging to G_L, acting on a C^∞-manifold $M^k = \mathfrak{m}_n/\mathfrak{m}_n^{k+1} \subset J^k$. $F\colon (u, s) \to F(u, s) := j^k(f \circ g)$, where f and g are two representants of $u \in M^k$ and $s \in G_L^k$ respectively. For each open subset U of M^k we denote by $\eta(U)$ the number of G_L^k-orbits of this action which meets U (the intersection of the orbit and the set U is not empty). Let $f \in \mathfrak{m}_n^2$ and $\eta(f, k) := \inf_{U \ni j^k f} \eta(U)$. If there exists $K, N \in \mathbf{N}$ such that for all $k \geqslant K$, $\eta(f, k) \leqslant N$, then f is called *simple*.

2. Right G_L-equivalence, simple singularities B_k, C_k, F_4. First of all we have an important restriction.

PROPOSITION 2.1. *If $f \in \mathfrak{m}_n^3$, where $n \geqslant 3$, then f is not simple.*

Proof. As we know $\eta(f, 3) = \infty$ for the group G of the germs of diffeomorphisms preserving zero (see e.g. [2]). The function $\eta(f, 3)$ for the group G_L is greater or equal than the function $\eta(f, 3)$ for the group G. Thus $f \in \mathfrak{m}_n^3$, $n \geqslant 3$, is not simple.

For further considerations we assume that $f \in \mathfrak{m}_n^2$. Let us consider the restriction of f to $L = \{x_1 = 0\}$, i.e., $\tilde{f}\colon (x_2, \ldots, x_n) \to f(0, x_2, \ldots, x_n)$. Let ϱ denote the rank of the Hessian matrix $(\partial^2 \tilde{f}/\partial x_i \partial x_j(0))$, $2 \leqslant i, j \leqslant n$. As we know, one can diagonalize the Hessian to put $\tilde{f}$ into the form, that $j^2 \tilde{f} = \sum_{i=n-\varrho+1}^{n} \varepsilon_i x_i^2$, where ε_i is 1 or -1.

PROPOSITION 2.2. *If $f \in \mathfrak{m}_n^2$ and ϱ is as above, then there exists $\Phi \in G_L$ such that:*

$$f \circ \Phi = \zeta + \sum_{i=n-\varrho+1}^{n} \varepsilon_i x_i^2,$$

where ζ has a representative which depends only on $x_1, x_2, \ldots, x_{n-\varrho}$, $1 \leqslant \varrho \leqslant n-1$.

Proof. We can write $f = g + \sum_{i=n-\varrho+1}^{n} \varepsilon_i x_i^2$ in some coordinate system, where $g \in \mathfrak{m}_n^2$ and $j^2 g$ is a quadratic form in the variables $x_1, x_2, \ldots, x_{n-\varrho}$. We treat $f_y(x) = f(y, x)$, $x = (x_{n-\varrho+1}, \ldots, x_n)$, $y = (x_1, \ldots, x_{n-\varrho})$ as a function of x parametrized by y. x_0 is a critical point of f_{y_0} if and only if $df_{y_0}(x_0) = d_x f(y_0, x_0) = 0$, i.e., $d_x\big(g(y_0, x_0) + \sum_{i=n-\varrho+1}^{n} \varepsilon_i x_i^2\big) = 0$. By the Implicit Function Theorem we can solve the last equation for x in terms of y, say $x = \varphi(y)$, near 0, $\varphi(0) = 0$. Hence, there exists a neighbourhood U of 0 in $R^{n-\varrho}$ such that the critical point $(y, \varphi(y))$ of f_y is non-degenerated for $y \in U$.

Let $G_L \ni \Phi_1: (y, x) \to (Y, X) = (y, x - \varphi(y))$. Then, there exists a neighbourhood V of 0 in R^n such that if $(Y, X) \in V$, then the only critical point of $f_y(\cdot) := f \circ \Phi_1^{-1}(y, \cdot)$ is at $x = 0$. Thus by Taylor's Theorem we have $f(y, x) = f(y, 0) + B(y, x) \cdot x^2$, where $B(y, x)$ is a bilinear form depending smoothly on (y, x). By the generalized Morse Lemma we have a local diffeomorphism Φ_2, $G_L \ni \Phi_2: (y, x) \to (Y, X) = (y, Q(y, x))$, such that $f(y, x) = f(Y, 0) + B(0, 0) \cdot X^2$. Thus taking ζ to be the germ of $Y \to f(Y, 0)$ we have the required result. ∎

COROLLARY 2.3. *The germ ζ in Proposition 2.2 has the form*:

$$\zeta(x_1, \ldots, x_{n-\varrho}) = x_1 P(x_1, \ldots, x_{n-\varrho}) + \varrho(x_2, \ldots, x_{n-\varrho}),$$

where $P \in \mathfrak{m}_{n-\varrho}$, $\varrho \in \mathfrak{m}^3_{n-\varrho-1}$.

DEFINITION 2.4. A germ $f \in \mathfrak{m}_n^2$ is said to be *stable G_L-equivalent to the germ $\zeta \in \mathfrak{m}^2_{n-\varrho}$* if there exists $\Phi \in G_L$ such that

$$f \circ \Phi = \zeta + \sum_{i=n-\varrho+1}^{n} \varepsilon_i x_i^2, \quad \varepsilon_i = \pm 1.$$

THEOREM 2.5 ([1]). *The complete list of simple singularities up to the stable G_L-equivalence is as follows*:

$$B_k^{\pm}: f = \pm x_1^k + x_2^2 \quad (k \geqslant 2),$$

$$C_k^{\pm}: f = x_1 x_2 \pm x_2^k,$$

$$F_4^{\pm}: f = \pm x_1^2 + x_2^3.$$

Before we proceed to the proof of this theorem we will prove a few results:

LEMMA 2.6. *Let Q be the quadratic form, $Q(y) = \sum_{i=n-\varrho+1}^{n} \varepsilon_i y_i^2$ and $x = (x_1, \ldots, x_{n-\varrho})$, $y = (y_{n-\varrho+1}, \ldots, y_n)$. The germ $f(x) + Q(y) \in \mathfrak{m}_n^2$ is simple if and only if $f \in \mathfrak{m}^2_{n-\varrho}$ is simple.*

Proof. (A) Let $k \geqslant 3$. At the first step we show the following equivalence

$$f(x) + Q(y) \in (g(x) + Q(y)) G_L^k \Leftrightarrow f \in g G^k_{\pi(L)},$$

where π is the projection, $\pi: R^n \to R^{n-\varrho}$, $g \in \mathfrak{m}^2_{n-\varrho}$.

We prove "$\Rightarrow$" only, because "$\Leftarrow$" is trivial.

By the assumption, there exists a diffeomorphism $\Phi \in G_L$ such that

(i) $$j^k(f+Q) = j^k((g+Q) \circ \Phi),$$

$\Phi: (x, y) \to (X(x, y), Y(x, y)) = (A_1 x + A_2 y (\bmod\ \mathfrak{m}_n^2), B_1 x + B_2 y (\bmod\ \mathfrak{m}_n^2))$, where A_1, A_2, B_1, B_2 are the matrices.

On the basis of (i) and Corollary 2.3

$$\big(x_1 P_1(x)+Q(y)\big)(\mathrm{mod}\ \mathfrak{m}_n^3) = \big(x_1 P_2\big(X(x, y)\big)+Q\big(Y(x, y)\big)\big)(\mathrm{mod}\ \mathfrak{m}_n^3).$$

From this relation it follows that

$$B_2 = I \text{ (identity)}, \quad B_1 = (b_{ij}), \quad b_{ij} = 0 \quad \text{if } j \neq 1.$$

$\Phi \in G_L$ implies: $A_1 = (a_{ij}^1)$, $a_{1j}^1 = 0$, $2 \leqslant j \leqslant n-\varrho$; $A_2 = (a_{ij}^2)$, $a_{ij}^2 = 0$, $1 \leqslant j \leqslant \varrho$. Now we solve the following equation:

$$Q\big(Y(\cdot, \varphi(\cdot))\big)-Q\big(\varphi(\cdot)\big) = 0, \qquad \varphi: \mathbf{R}^{n-\varrho} \to \mathbf{R}^{\varrho}.$$

After the substitution $Y(x, y) = y+\beta(x, y)$, $\beta(x, y) = B_1 x(\mathrm{mod}\ \mathfrak{m}_n^2)$ we get

$$\sum_{i=n-\varrho+1}^{n} \varepsilon_i \beta_i(x, y)\big(2y_i+\beta_i(x, y)\big) = 0, \qquad \beta = (\beta_{n-\varrho+1}, \ldots, \beta_n).$$

Thus, in the end, we obtain the equation $2\varphi(\cdot)+\beta\big(\cdot, \varphi(\cdot)\big) = 0$, which satisfies the assumptions of the Implicit Function Theorem. In some neighbourhood of $0 \in \mathbf{R}^{n-\varrho}$, $Q\big(Y(x, \varphi(x))\big) = Q\big(\varphi(x)\big)$.

Let $h(x) := g\big(X(x, \varphi(x))\big)$, so we have: $j^k f = j^k h$. It is easily seen that $\mathbf{R}^{n-\varrho} \ni x \to X\big(x, \varphi(x)\big) \in \mathbf{R}^{n-\varrho}$ belongs to $G_{\pi(L)}$.

(B) As we know (see e.g. [4]) the tangent space to the G_L^k-orbit of $\sigma = j^k \eta \in J^k$ has a form

$$T_\sigma(\sigma G_L^k) = j^k J(\eta),$$

where

$$J(\eta) = \left\langle x_1 \frac{\partial \eta}{\partial x_1} \right\rangle + \mathfrak{m}_n \left\langle \frac{\partial \eta}{\partial x_2}, \ldots, \frac{\partial \eta}{\partial x_n} \right\rangle.$$

We denote $W = \{\eta \in \mathfrak{m}_n^2 \colon \eta(x, 0) = 0\}$. It is easy to see that $W \subset J(f+Q)$. In this way the space $V \subset \mathfrak{m}_n^2/\mathfrak{m}_n^{k+1}$ generated by all monomials of variables $x = (x_1, \ldots, x_{n-\varrho})$ is complementary to $j^k J(f+Q)$ in $\mathfrak{m}_n^2/\mathfrak{m}_n^{k+1}$. Therefore $V' := j^k(f+Q)+V$ is transversal to the orbit $(f+Q)G_L^k$.

Let $\mathcal{O}$ be an open neighbourhood of $(f+Q)$ in $\mathfrak{m}_n^2$ such that, for every η in $\mathcal{O}$, $(D_y^2 \eta)|_0 \neq 0$. So, for every $\eta \in \mathcal{O}$ we have $J(\eta) \supset W$, i.e., every orbit in J^k intersecting $j^k \mathcal{O}$ is transversal to V'. Now we are ready to construct the following mapping $H\colon V' \to \mathfrak{m}_{n-\varrho}^2/\mathfrak{m}_{n-\varrho}^{k+1}$, $H\big(j^k(f+Q)+\sigma\big) := j^k f+\sigma$, H is an isomorphism, $DH = \mathrm{id}$. If $f+Q$ is simple, then f is simple because $\eta(f+Q, k) \underset{k}{<} \infty$ implies $\eta(f, k) \underset{k}{<} \infty$ by isomorphism H. The converse is true by the same argument and transversality of all orbits in $j^k \mathcal{O}$ to V', q.e.d.

PROPOSITION 2.7. *Let $f \in \mathfrak{m}_n^2$ and $L = \{x_1 = 0\}$; then: if f is simple, then there exists $\Phi \in G_L$ such that $f \circ \Phi = g(x_1, x_2)+Q$ and $g \in \mathfrak{m}_2^2 \setminus \mathfrak{m}_2^3$, Q is non-degenerated quadratic form in the remaining variables $x_3, \ldots, x_n$.*

Proof. At three steps.

Let corank $f|_{x_1=0} = 2$; then on the basis of Proposition 2.2 there exists $\Phi' \in G_L$ that $f \circ \Phi' = g'(x_1, x_2, x_3) + Q$, and $g'(x_1, x_2, x_3) = x_1 P(x_1, x_2, x_3) + \varrho(x_2, x_3)$, $\varrho \in \mathfrak{m}_2^3$, $P \in \mathfrak{m}_3$. Proposition 2.1 excludes the case $P \in \mathfrak{m}_3^2$. Hence we have the two forms to which g' can be reduced: either

(i) $$x_1 x_2 + \varrho_1(x_2, x_3), \quad \varrho_1 \in \mathfrak{m}_2^3,$$

or

(ii) $$x_1^2 + \varrho_2(x_1, x_2, x_3), \quad \varrho_2 \in \mathfrak{m}_3^3.$$

Step A. Let us consider case (i).

One can notice that (i) reduces to the form

(iii) $$f_1 = x_1 x_2 + \alpha x_2^3 + \beta x_2^2 x_3 + \gamma x_3^3 \pmod{\mathfrak{m}_3^4}.$$

So, we confine our considerations to such germs and show that at every open neighbourhood of (iii) there exists the germ

(iv) $$f_2 = x_1 x_2 + \alpha' x_2^3 + \beta' x_2^2 x_3 + \gamma' x_3^3 \pmod{\mathfrak{m}_3^4},$$

which is not equivalent to (iii):

Let us take

$$p\colon (x_1, x_2, x_3) \to \big(x_1(1 + a_1 x_1 + a_2 x_2 + a_3 x_3) \pmod{\mathfrak{m}_3^3},$$
$$x_2 + b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2 + b_4 x_1 x_2 + b_5 x_1 x_3 + b_6 x_2 x_3 \pmod{\mathfrak{m}_3^3},$$
$$c_1 x_1 + c_2 x_2 + c_3 x_3 \pmod{\mathfrak{m}_3^2}\big).$$

It is easy to verify that the condition $f_1 \circ p = f_2$ gives the following necessary conditions on the coefficients of the diffeomorphism p, $\gamma c_2 = 0$, $\alpha + \beta c_2 = \alpha'$, $\beta c_3 = \beta'$, $\gamma c_3^3 = \gamma'$, where $c_3 \neq 0$. By the first condition we have two possibilities:

(a) $\gamma = 0$ then the neighbouring germ: $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' \neq 0$ is not equivalent to f_1.

(b) $c_2 = 0$, then it suffices to take $\gamma' = \gamma$, $\beta' = \beta$ and α neighbouring to α'. In this case the germ f_2 is not equivalent to f_1.

Step B. For case (ii) we consider the following subspace of $\mathfrak{m}_3^2$: $V = \{bx_1^2 + \sum_{i+j+k=3} x_1^i x_2^j x_3^k a_{ijk}\} \subset J^3$. It is easy to see that the orbits of G_L^3 intersect V along the submanifolds of codimension $\geqslant 1$ and these submanifolds can be parametrized by d, a_i, b_i, c_i, $i = 1, 2, 3$, provided with the diffeomorphisms preserving V:

$$G_L \ni \Phi_{(d,\bar{a},\bar{b},\bar{c})}\colon (x_1, x_2, x_3) \to \big(x_1(d + a_1 x_1 + a_2 x_2 + a_3 x_3) \pmod{\mathfrak{m}_3^3},$$
$$b_1 x_1 + b_2 x_2 + b_3 x_3 \pmod{\mathfrak{m}_3^3},\ c_1 x_1 + c_2 x_2 + c_3 x_3 \pmod{\mathfrak{m}_3^3}\big).$$

Thus their dimensions are not greater than 10. On the other hand, $\dim V = 11$, so these facts prove non-simplicity of (ii).

Step C. In the previous steps we proved that: if $f \in \mathfrak{m}_n^2$ is simple, then f is equivalent to $g(x_1, x_2)+Q$.

Let $g \in \mathfrak{m}_2^3$, we consider the right action: $\mathfrak{m}_2^3/\mathfrak{m}_2^4 \times G_L \to \mathfrak{m}_2^3/\mathfrak{m}_2^4$. It is easy to see that every open subset of $\mathfrak{m}_2^3/\mathfrak{m}_2^4$ intersects infinitely many orbits, namely: The above defined action reduces to the action $\mathfrak{m}_2^3/\mathfrak{m}_2^4 \times G_L^1 \to \mathfrak{m}_2^3/\mathfrak{m}_2^4$, because for every $g \in \mathfrak{m}_2^3$ and $\Phi \in G_L$ $j^3(g \circ \Phi) = j^3 g \cdot j^1 \Phi$. As we know $\dim G_L^1 = 3$, and $\dim \mathfrak{m}_2^3/\mathfrak{m}_2^4 = 4$. Hence, every orbit of G_L^3 in $\mathfrak{m}_2^3/\mathfrak{m}_2^4$ has the codimension $\geqslant 1$, q.e.d.

On the basis of Proposition 2.7 we are interested in germs depending only in two variables.

COROLLARY 2.8. *The set of non-simple germs in* $\mathfrak{m}_2^2$ *has the codimension* 3.

Proof of Theorem 2.5.

LEMMA 2.9. *If* $f \in \mathfrak{m}_2^2$, $\operatorname{corank} f|_{x_1=0} = 0$ *and* f *is simple, then there exists* $\Phi \in G_L$ *such that*: $f \circ \Phi = \pm x_1^k \pm x_2^2$, $k \geqslant 2$.

These singularities are denoted by B_k (according to Arnold [1]).

Proof. At first we shall show that the k-jets $\pm x_1^k \pm x_2^2$ are sufficient. As is well known (e.g. [4]) the sufficient condition for the k-determinacy as follows

$$(*) \qquad \mathfrak{m}^{k+1} \subset \mathfrak{m}J(f) \Rightarrow f \text{ is } k\text{-determined}.$$

In our case $J(\pm x_1^k \pm x_2^2) = \langle x_1^k \rangle E + \mathfrak{m}\langle x_2 \rangle$. For proving $(*)$ we must verify that for every monomial $x_1^i x_2^j$, $i+j = k+1$ there is a decomposition: $x_1^i x_2^j = x_1^k h_1 + x_2 h_2$, $h_1 \in \mathfrak{m}$, $h_2 \in \mathfrak{m}^2$. It is true because for $i < k$, $i = k$, $i = k+1$ we can take respectively $h_1 = 0$, $h_2 = x_1^i x_2^{j-1}$; $h_1 = x_2$, $h_2 = 0$; $h_1 = x_1$, $h_2 = 0$.

If $\operatorname{corank} f|_{x_1=0} = 0$ we can reduce f to the form

$$f = x_1 P_1(x_1, x_2) \pm x_2^2, \qquad P_1(x_1, x_2) = a_1 x_1 + a_2 x_2 \ (\mathrm{mod}\ \mathfrak{m}^2).$$

Let us consider the following cases:

1° $a_1 \neq 0$, $a_1 \mp \frac{1}{4} a_2^2 \neq 0$, then

$$a_1 x_1^2 + a_2 x_1 x_2 \pm x_2^2 \ (\mathrm{mod}\ \mathfrak{m}^3) = \pm(x_2 \pm \tfrac{1}{2} a_2 x_1)^2 + (a_1 \mp \tfrac{1}{4} a_2^2) x_1^2 \ (\mathrm{mod}\ \mathfrak{m}^3)$$

is equivalent to $\pm x_2^2 \pm x_1^2 \ (\mathrm{mod}\ \mathfrak{m}^3) \in B_2$.

2° $a_1 = 0$, $a_2 \neq 0$, then

$$a_2 x_1 x_2 \pm x_2^2 \ (\mathrm{mod}\ \mathfrak{m}^3) = \pm(x_2 \pm \tfrac{1}{2} a_2 x_1)^2 \mp \tfrac{1}{4} a_2^2 x_1^2 \ (\mathrm{mod}\ \mathfrak{m}^3) \in B_2.$$

3° $a_1 \neq 0$, $a_1 \mp \frac{1}{4} a_2^2 = 0$ and $a_1 = 0$, $a_2 = 0$, then

$$x_1 P_1(x_1, x_2) \pm x_2^2 \sim x_1 P_2(x_1, x_2) \pm x_2^2, \qquad P_2 \in \mathfrak{m}^2 \qquad (\sim, \text{ equivalency}).$$

Let $P_2(x_1, x_2) = b_1 x_1^2 + x_2 Q_1(x_1, x_2) \ (\mathrm{mod}\ \mathfrak{m}^3)$ and

$$b_1 \neq 0;$$

then

$$x_1 P_2(x_1, x_2) \pm x_2^2 = b_1 x_1^3 \pm (x_2 \pm \tfrac{1}{2} x_1 Q_1)^2 \pmod{\mathfrak{m}^4} \in B_3;$$

$$b_1 = 0;$$

then

$$x_1 P_2(x_1, x_2) \pm x_2^2 \sim x_1 P_3(x_1, x_2) \pm x_2^2,$$

$$\text{where } P_3 \in \mathfrak{m}^3,\ P_3(x_1, x_2) = b_2 x_1^3 + x_2 Q_2(x_1, x_2) \pmod{\mathfrak{m}^4}.$$

If $b_2 \neq 0$, then $x_1 P_3 \pm x_2^2 \in B_4$.

If $b_2 = 0$, $b_3 = 0, \ldots, b_{k-1} \neq 0$, then $x_1 P_k(x_1, x_2) \pm x_2^2 \in B_{k+1}$.

Thus we obtain the singularities of type B_k.

COROLLARY 2.10. *The codimension of the singularity B_k in $\mathfrak{m}^2$ is equal to $k-2$.*

If $\operatorname{corank} f|_{x_1} = 1$, then, by Proposition 2.7, $f \in \mathfrak{m}_2^2 \setminus \mathfrak{m}_2^3$ and we can reduce f to the form:

$$f = x_1 P(x_1, x_2) \pm x_2^k, \quad k \geqslant 3, \quad \text{where } P \in \mathfrak{m}_2^1 \setminus \mathfrak{m}_2^2.$$

There are two cases for $P(x_1, x_2) = a_1 x_1 + a_2 x_2 + (\operatorname{mod} \mathfrak{m}_2^2)$,

(a) $$a_1 = 0.$$

It is easy to see that the k-jet $x_1 x_2 \pm x_2^k$ is sufficient. Thus we have the singularities of type C_k.

(b) $$a_2 = 0.$$

In this case f is equivalent to such germ that the k-jet of f has the form

(*) $$\pm x_1^2 + x_1 \sum_{i=2}^{k-2} c_i x_2^i \pm x_2^k.$$

It is easy to verify that this jet is k-determined. By the calculation method of Arnold [2] we see that germ (*) is not simple for $k \geqslant 4$, namely the equation

(**) $$\Big(\pm 2x_1^2 + x_1 \sum_{i=2}^{k-2} c_i x_2^i\Big) h_1 + \Big(x_1 \sum_{i=1}^{k-3} (i+1) x_2^i c_{i+1} \pm k x_2^{k-1}\Big) h_2 = x_1 x_2^2 \pmod{\mathfrak{m}^{k+1}}$$

has no solutions for $h_1, h_2 \in \mathfrak{m}$ if we assume the existence of the solutions h_1 and $h_2(x_1, x_2) = A_1 x_1 + A_2 x_2 \pmod{\mathfrak{m}^2}$, then by simple calculations on (**) we get the contradiction:

$$2c_2 A_2 = 1 \quad \text{and} \quad \pm k A_2 = 0.$$

Hence, in case (b) the only simple singularity is F_4: $\pm x_1^2 + x_2^3$, q.e.d.

References

[1] V. I. Arnold, *Critical points of functions on manifolds with boundaries, simple Lie groups B_k, C_k, F_4 and singularities of involutes*, Uspekhi Mat. Nauk 33 (1978), 91–105.

[2] —, *Normal forms of functions with simple critical points, the Weyl groups A_k, D_k, E_k and Lagrange manifolds*, Functional Anal. Appl. 6 (1972), 3–25.

[3] M. Golubitsky, V. Guillemin, *Stable mappings and their singularities*, Springer Verlag, Berlin 1973.

[4] S. Janeczko, *On algebraic criteria for k-determinacy of germs of smooth functions on manifold with boundary*, Demonstratio Matematica 4 (1982) (in preparation).

[5] D. Siersma, *Singularities of functions on boundaries, corners etc.*, Quart. J. Math., Oxford (2) 32 (1981), 119–127.

INSTYTUT MATEMATYCZNY
POLITECHNIKI WARSZAWSKIEJ
WARSAW, POLAND