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A note on singular lagrangian submanifolds

STANISŁAW JANECZKO

Institute of Mathematics,
Technical University of Warsaw
Pl. Jednósci Robotniczej 1, 00 661 Warsaw, Poland

Abstract. *Classification of singular lagrangian submanifolds which appear as images of a regular one under a symplectic relation, is considered from the point of view of standard singularity theory. The classification is carried out in small dimensions and restricted to special types of symplectic objects. Normal forms for singular pull-backs and pushforwards are given using an appropriate symplectic equivalence group. It is shown that the general classification problem reduces to the classification problem for appropriate mapping diagrams. An approach to the classical theories of phase transition is given based on the geometry of singular lagrangian images. The variational open swallowtails and regularly intersecting pairs of holonomic components are resolved using an appropriate reduction relation. Examples are given of singularities encountered in physics.*

1. INTRODUCTION

Let (T^*X, ω_X), (T^*Y, ω_Y) be cotangent bundles endowed with the canonical symplectic structure [1]. The product

$$\Omega = (T^*Y \times T^*X, pr_Y^* \omega_Y - pr_X^* \omega_X),$$

where $pr_Y : T^*Y \times T^*X \to T^*Y$, $pr_X : T^*Y \times T^*X \to T^*X$ are the natural projections, is a symplectic manifold. Let $f : X \to Y$ be a smooth mapping. We consider the canonical lift T^*f of the graph of f to a lagrangian submanifold of Ω[see [6], [23]]. In more general context [cf. [17], [14]], the lagrangian submanifold $T^*f \subset \subset \Omega$ is called a symplectic relation from (T^*Y, ω_Y) to (T^*X, ω_X). Let L be a

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lagrangian submanifold of T^*Y, we call the set

$$T^*f(L)=\{p_2\in T^*X;\text{there exists } p_1\in L:(p_1,p_2)\in T^*f\}$$

the image [pullback] of L by the relation T^*f. In particular if we define ${}^t(T^*f)=$ $=\{(p_2,p_1)\in T^*X\times T^*Y;\ (p_1,p_2)\in T^*f\}$ and take a lagrangian submanifold $N\subset T^*X$ then the inverse image [pushforward] of N by the relation T^*f is defined as ${}^t(T^*f)(N)$ [cf. [5], [10]].

It is well known that if the appropriate transversality [10] [clean intersection [22]] conditions are fulfilled then $T^*f(L)$ is an immersed lagrangian submanifold of (T^*X,ω_X). It turns out [see [19], [11], [4]] that the general situation, when $T^*f(L)$ is not even a submanifold, is important for applications of symplectic geometry to physics. Most of the motivating examples for this subject are introduced in [19], [4], [12]. A very interesting thermodynamical example of singular image [cf. [11]] was also suggested by W.M. Tulczyjew. In a different physical context singular lagrangian submanifolds have appeared in [4], where they form the sets of rays tangent to the geodesic flows on a hypersurface. This is connected to the theory of nested hypersurfaces in a symplectic manifold describing the geodesics on a Riemannian manifold with boundary as well as to the problem of the shortest bypassing of an obstacle represented by a smooth hypersurface [4]. It turned out that the generic singularities in these problems, so-called open swallowtails, can be conveniently obtained as images from the regular lagrangian submanifolds by an appropriate symplectic relation.

The next motivation, for investigations presented here, comes from thermodynamics of phase transitions [11]. Let us consider the simple one-component thermodynamical system [cf. [18]] and admit the calss of deformations onto two isolated subsystems of the same sample. The phase space for such deformations is the following

$$(T^*Y_1\times T^*Y_2,-S_1\mathrm{d}T_1-p_1\mathrm{d}V_1+\mu_1\mathrm{d}N_1-S_2\mathrm{d}T_2-p_2\mathrm{d}V_2+\mu_2\mathrm{d}N_2),$$

where T^*Y_1, $\{V_1,T_1,N_1,-p_1,-S_1,\mu_1\}$; T^*Y_2, $\{V_2,T_2,N_2,-p_2,-S_2,\mu_2\}$ are the phase spaces of the respective subsystems and V_i, T_i, N_i, p_i, S_i, μ_i are the standard thermodynamical coordinates. Let a lagrangian submanifold $L_1\times L_2\subset$ $\subset T^*Y_1\times T^*Y_2$ be a space of equilibrium states of a composite isolated system. After removing [chemical, thermal and mechanical] constraints the virtual states of the system are defined by the coisotropic submanifold $C\subseteq T^*Y_1\times T^*Y_2$ [cf. [5]],

$$C=\{T_1=T_2,p_1=p_2,\mu_1=\mu_2,N_1+N_2=N=\text{const.},N_1>0,N_2>0\}.$$

C provides the canonical characteristic submersion, say ρ, onto the phase space of the composite system $(T^*Y,-S\mathrm{d}T-p\mathrm{d}V)$ [cf. [18]], $\rho:C\to T^*Y$, $\rho(V_1,T_1,N_1,$

$p_1, S_1, \mu_1, V_2, T_2, N_2, p_2, S_2, \mu_2) = (V_1 + V_2, T_1, p_1, S_1 + S_2)$. Hence the space of equilibrium states of the composite system is an image $\rho(L_1 \times L_2)$, which for the Van der Waals gas forms a singular lagrangian submanifold in T^*Y [see Fig. 1] well known in thermodynamics of coexistence states [11]. It is obvious that the symplectic relation [reduction relation [5]] associated to ρ can be represented as an appropriate lifting [cf. §3]. This example suggestes that the very singular constitutive sets in thermodynamics can be derived as the symplectic images of constitutive sets for sufficiently deformed composite systems. The other examples of singular lagrangian images coming from microlocal analysis of holonomic systems [13], [16] and control of static mechanical systems we present in the next sections.

The aim of this paper is to set up a method of formalizing and generalizing the above observations and derive the first results for further applications. We now outline the organization of the paper. In section 2 we introduce some known but perhaps unfamiliar results of symplectic geometry, which we shall need later. Section 3 is devoted to the more precise symplectic treatment of the thermodynamic phase transitions. Then, in Sections 4, 5, 6, we formulate the problem of classification of images of lagrangian submanifolds by means of special classes of symplectic relations, namely these ones most frequently encountered in physics, i.e., pushforwards and pullbacks of smooth stable mappings. Here we obtain a classification of normal forms for pullbacks and pushforwards of regular lagrangian submanifolds in the case when dim X = dim $Y \leqslant 3$, the mapping f is stable and L has a fold singularity. The main point of the technique is that we reduce the classification of normal forms of pullbacks and pushforwards to the classification of normal forms of an appropriate mapping diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\psi"'] & Y \arrow[d, "\varphi"'] \arrow[dr, "g"] & \\ X \arrow[r, "\underline{f}"] & Y \arrow[r, "g'"'] & \mathbb{R}^k \end{tikzcd}$$

for pullbacks, and

$$\begin{tikzcd} Y \arrow[d, "\psi"'] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\varphi"'] & \mathbb{R}^k \times \mathbb{R} \arrow[d, "(\mathrm{id}, \mathrm{id} + \alpha \circ f)"] \\ Y & X \arrow[l, "f'"'] \arrow[r, "g'"] & \underline{\mathbb{R}^k \times \mathbb{R}} \end{tikzcd}$$

for pushforwards, where ψ, φ are diffeomorphisms, α is a smooth function and rankDg is maximal.

One of the purposes of this paper is to give an effective approach to the resolution problem for singular constitutive sets [see [19]]. The resolution or prepresentation as an image by means of smooth sumplectic objects means that all information about singularity structure is contained in the appropriate smooth parametric potential. So the classification presented here can be used to characterize the typical properties of resolving singularities. In §6 we show that the holonomic regular interactions [15] and open swallowtails [4] can be resolved in the canonical way.

2. DEFINITIONS AND PRELIMINARIES

Let (P, ω) be a symplectic manifold. A submanifold L of P such that $\omega|_L = 0$ and $\dim P = 2 \dim L$ is called a lagrangian submanifold of (P, ω). Let (P_1, ω_1) and (P_2, ω_2) be symplectic manifolds and let π_1 and π_2 denote the canonical projections of $P_2 \times P_1$ onto P_1 and P_2 respectively. The two-form $\omega_2 \ominus \omega_1 = \pi_2^* \omega_2 - \pi_1^* \omega_1$ is clearly a symplectic form on $P_2 \times P_1$. A symplectic relation from a symplectic manifold (P_2, ω_2) to (P_1, ω_1) is a lagrangian submanifold of the symplectic manifold $(P_2 \times P_1, \omega_2 \ominus \omega_1)$ [cf. [17], [6]]. For example the graph of a symplectic diffeomorphism Φ of (P_1, ω_1) onto (P_2, ω_2) is a symplectic relation [for more detailed description of the properties of symplectic relations and their applications see e.g. [6], [5], [12], [14], [17], [23]].

Let $R \subseteq (P_2 \times P_1, \omega_2 \ominus \omega_1)$ be a symplectic relation and let $L \subseteq (P_2, \omega_2)$ be a lagrangian submanifold of P_2, the set $R(L) = \{p_1 \in P_1;$ there exists $p_2 \in L$ such that $(p_2, p_1) \in R\}$ is called the image of L under the symplectic relation R [cf. [5]]. If the transversality conditions [or clean intersection conditions [22]] between R and $L \times P_1$ are fulfilled then this image is an immersed lagrangian submanifold of (P_1, ω_1) [see [10] p. 147]. If the transversality conditions are not fulfilled then the image of L can be singular, i.e. $R(L)$ in this case is a subset of P_1 but not a smooth submanifold. We limit all considerations to locally algebraic [semi-algebraic] subsets of P_1. For such subsets there exist [not unique] partition-stratifications [see [9]] into smooth submanifolds of P_1 [called the strata] which satisfy the local finiteness condition, i.e. every point in $R(L)$ has a neighbourhood in P_1 which meets only finitely many strata.

Let us assume that $R \cap L \cap P_1$ is an algebraic [or semi-algebraic] subset of $P_2 \times P_1$ then $R(L) = \pi_1(R \cap L \cap P_1)$ is also a semi-algebraic subset of P_1 [cf. [9]]. We show that all strata of this subset are isotropic submanifolds of (P_1, ω_1). A submanifold $K \subset P_1$ is called an isotropic submanifold of (P_1, ω_1) if $\omega_1|_K = 0$, cf. [23]. Let X be a stratum of $R \cap L \times P_1$ and let $x \in X$, then for all $\zeta, \eta \in T_x X$ we have $0 = \langle \zeta \wedge \eta, \pi_2^* \omega_2 - \pi_1^* \omega_1 \rangle = \langle \zeta \wedge \eta, \pi_2^* \omega_2 \rangle - \langle \zeta \wedge \eta, \pi_1^* \omega_1 \rangle$, where $\langle \zeta \wedge \eta, \pi_2^* \omega_2 \rangle = 0$ because of $\zeta, \eta \in T_x(L \times P_1)$, thus we obtain

$$0 = \langle \zeta \wedge \eta, \pi_1^* \omega_1 \rangle = \langle T\pi_1 \zeta \wedge T\pi_1 \eta, \omega_1 \rangle,$$

but this is the condition for isotropy of the image of the stratum X by the projection π_1.

We consider a typical example of a symplectic manifold namely the cotangent bundle [symplectic manifolds found in most applications are isomorphic to cotangen bundles [14], [11], [19]] $(T^*X, \mathrm{d}\vartheta_X)$ where X is a smooth manifold and ϑ_X is the Liouville form [cf. [1]]. The structure and properties of linear symplectic relations have been thoroughly investigated in [6]; however, it turns out that in physical applications nonlinear symplectic relations are important. In this paper we shall not consider the categorial aspect of symplectic formalism but rather we shall concentrate on a concrete class of relations [defined below] and develop a classification of images of lagrangian submanifolds under relations belonging to this class.

Let X, Y be smooth manifolds. Then $(T^*Y \times T^*X, \mathrm{d}\vartheta_Y \ominus \mathrm{d}\vartheta_X)$ is a symplectic manifold. If $f: X \to Y$ is a differentiable mapping then the set

$$T^*f = \{((y, \eta), (x, \zeta)) \in T^*Y \times T^*X; y = f(x), T_x^*f\eta = \zeta\} \tag{1}$$

is a symplectic relation from T^*Y to T^*X [cf. [23], [10]]. It is well known that the relation T^*f operates on certain lagrangian submanifolds of (T^*Y, ω_Y) to give lagrangian submanifolds of (T^*X, ω_X). In particular if $L \subset T^*Y$ has a generating function $S: Y \to \mathbb{R}$ then $T^*f(L) \subset T^*X$ has $S \circ f$ as its generating function [see [10]]. Let $F: Y \times \mathbb{R}^k \to \mathbb{R}$ be a Morse family [see [23], p. 25] generating $L \subset T^*Y$ locally, then

$$L = \left\{(y, \eta) \in T^*Y; \eta = \frac{\partial F}{\partial y}(y, \lambda), 0 = \frac{\partial F}{\partial \lambda}(y, \lambda)\right\}.$$

By using (1) and matrix notation we obtain for T^*f the formula

$$\zeta = {}^tDf(x)\eta. \tag{2}$$

Inserting $y = f(x)$ into equations of L and substituting these into (2) we obtain for $T^*f(L)$ the equations

$$\zeta = {}^tDf(x)\frac{\partial F}{\partial y}(f(x), \lambda) = \frac{\partial}{\partial x}F(f(x), \lambda)$$

$$0 = \frac{\partial F}{\partial \lambda}(f(x), \lambda).$$

Thus a generating family for the image of L under T^*f has the form

$$G(x, \lambda) = F(f(x), \lambda). \tag{3}$$

If f is a submersion then any lagrangian submanifold of T^*Y may be operated upon by T^*f providing a lagrangian submanifold of T^*X. If f is not a submersion then the respective transversality condition is not fulfilled for some class of lagrangian submanifolds and singularities of $T^*f(L)$ can appear. In this paper we propose an approach which would allow us to describe some generic classes of singularities of these images. We consider the space of pairs (L, T^*f) [rather than the space of images $T^*f(L)$]. In this space we introduce an equivalence relation and then reduce the problem of finding local models for singularities of images to the well known problem in the theory of singularities of composite mapping [see [2], [7]]. Similarly, we search the normal forms for the inverse images ${}^t(T^*f)(N)$ [called pushforwards].

3. COEXISTENCE OF PHASES IN THERMODYNAMICAL COMPOSITE SYSTEMS

Now we give the physically simple model [continuation of the one started in Introduction], which can form the partial motivation for investigation of singular lagrangian submanifolds, as well as the respective images. This section is an account of some research suggested by W.M. Tulczyjew.

Let us consider the one-component simple thermodynamical system of volume V [see [18]]. We look at the composite system as the volume element V of the container subdivided into spatially isolated cells. Let the phase space of the composite system be as follows: $(T^*X, \mathrm{d}\vartheta_X)$, with local coordinates, say $X \equiv \mathbb{R}^{3k}$, $T^*X \equiv \mathbb{R}^{6k}$, $T^*X : \{v_i, s_i, n_i, \pi_i, \tau_i, \tilde{\mu}_i\}$ and the symplectic form

$$\vartheta_X = \sum_{i=1}^{k} (\pi_i \mathrm{d}v_i + \tau_i \mathrm{d}s_i + \tilde{\mu}_i \mathrm{d}n_i),$$

where n_i are the mole numbers of respective cells, v_i, s_i are the molar volume and the molar entropy of the i-th cell. Thus according to classical thermodynamics π_i, τ_i, $\tilde{\mu}_i$ are defined as follows

$$\pi_i = -p_i n_i,\ \ \tau_i = T_i n_i,\ \ \tilde{\mu}_i = \mu_i - p_i v_i + T_i s_i,$$

where p_i, T_i, μ_i are the pressure, the temperature, and the chemical potential of the i-th cell. We use also the following notation $(p_i) = (p_1, \ldots, p_k)$, $(v_i) = (v_1, \ldots, v_k)$, etc.

Let the internal energy $(v_i, s_i, n_i) \to u(v_i, s_i, n_i)$ be the generating function for the space of equilibrium states of the composite system $u(v_i, s_i, n_i) = \sum_i n_i u(v_i, s_i)$. The configurational domain K [attainable states] of the system can be defined as follows

$$K = \left\{ (v_i, s_i, n_i) \in X;\ v_i > 0,\ s_i > 0,\ n_i \geqslant 0, \text{ for } i = 1, \ldots, k, \sum_i n_i = n > 0 \right\}.$$

Now we describe the set of equilibrium states of this composite system [where the constraints of deformation are only virtual]. Let $(T^*Y, \mathrm{d}\vartheta_Y)$ be the phase space of the system, i.e. the phase space of the starting system, before deformation. Local coordinates on $T^*Y : \{V, S, -p, T\}$. Let $W \subset T^*Y$ denotes the set of equilibrium states of the system actuals realized by the external forces [constitutive set]. Hence, following [19], we can adapt the variational definition of constitutive set introduced there and write for W: $(V, S, -p, T) \in W$ if there exists $(v_i, s_i, n_i) \in K$ such that

$$\begin{aligned} &1^\circ,\ f(v_i, s_i, n_i) = \left(\sum_i n_i v_i, \sum_i n_i s_i \right) = (V, S), \\ (4) \qquad & \\ &2^\circ,\ -p\delta \sum_i n_i v_i + T\delta \sum_i n_i s_i \leqslant \delta u(v_i, s_i, n_i) \end{aligned}$$

for all displacements $(\delta v_i, \delta s_i, \delta n_i)$ compatible with K, i.e. there exists an integral curve $\gamma : \mathbb{R} \to X$ of $v \in TX$, $v = (\delta v_i, \delta s_i, \delta n_i)$ such that $\gamma([0, \epsilon[) \in K$ for some $\epsilon > 0$.

It is easy to see the following fact,

REMARK 3.1. A constitutive set $W \subset T^*Y$ is an image, by means of T^*f, of the following resolving constitutive set $\widetilde{W} = \{ p \in T^*X; \pi_X(p) \in K, \langle v, p \rangle \leqslant \langle v, \mathrm{d}u \rangle$ for each $v \in TK$ such that $\tau_X(v) = \pi_X(p)\}$, where TK is defined as follows [cf. [19]]

$$TK = \{ v \in T^*X; \text{there exists an integral curve } \gamma : \mathbb{R} \to X \text{ of } v \text{ such that } \gamma([0, \epsilon[) \subset K \text{ for some } \epsilon > 0\}.$$

Now we have to take into consideration the following cases. ∎

K'), $n_i > 0$, for $i = 1, \ldots, k$ [k subsystems].

So $TK' = \{ (\delta v_i, \delta s_i, \delta n_i); \sum_i \delta n_i = 0\}$. Taking $\delta n_i \underset{i}{\equiv} 0$, from 2° we obtain

$$(5) \qquad p = -\frac{\partial u}{\partial v_i}(v_i, s_i), \qquad T = \frac{\partial u}{\partial s_i}(v_i, s_i),$$

but inserting into 2°, $\delta v_1 = 0$, $\delta s_m = 0$, for $1 \leqslant 1$, $m \leqslant k$ and $\delta n_i + \delta n_j = 0$, where $\delta n_r = 0$ if $r \neq i, j$, we obtain

$$-p(v_i - v_j) + T(s_i - s_j) - u(v_i, s_i) + u(v_j, s_j) = 0. \tag{6}$$

For the Van der Waals system at constant (p, T) there are possible at most three solutions to the equations (5). By the equation (6) only one pair of solutions can coexist. Thus the possible solutions to the equations (5), (6) can be formulated as follows,

(i) all subsystems are in the same phase $v_i = v_j, s_i = s_j$, $1 \leqslant i, j \leqslant k$,

(ii) each subsystem is in one of the two possible phases: only two different pairs, say (v_i^1, s_i^1), (v_i^2, s_i^2).

Let I_1 be a subset of $\mathbb{K} = \{1, \ldots, k\}$ corresponding to the solution (v^1, s^1) and $I_2 = \mathbb{K} - I_1$ the subset corresponding to the solution (v^2, s^2) respectively. Hence we can write down: $V = N_1 v^1 + N_2 v^2$, $S = N_1 s^1 + N_2 s^2$; $N_1 = \sum_{i \in I_1} n_i$, $N_2 = \sum_{i \in I_2} n_i$ and $-p = \dfrac{\partial u}{\partial v}(v^1, s^1)$, $T = \dfrac{\partial u}{\partial s}(v^1, s^1)$, which are well known equations for coexistence of phases [see [11]]. Moreover by (6) we obtain, as a consequence, the Maxwell principle of equal areas.

$K'')$, $n_1 = 0$ *for some* $1 \in I_1$.

Let us suppose that a cell, say 1-th one, became empty, i.e. $K'' = \{(v_i, s_i, n_i);\ v_i > 0, s_i > 0, n_1 = 0$ for some $1 \in I_1, n_i > 0$ for $i \in \mathbb{K} - \{1\}\}$. Hence at the points of K'', by (4) we obtain

$$-p = \frac{\partial u}{\partial v_i}(v_i, s_i), \quad T = \frac{\partial u}{\partial s_i}(v_i, s_i), \text{ for } i \in \mathbb{K} - \{1\},$$

$$V = v^1 \sum_{j \in I_1 - \{1\}} n_j + v^2 \sum_{i \in I_2} n_i, \; S = s^1 \sum_{j \in I_1 - \{1\}} n_j + s^2 \sum_{i \in I_2} n_i$$

and

$$TK|_{K''} = \left\{(\delta v_i, \delta s_i, \delta n_i);\ \sum_{i \in \mathbb{K}} \delta n_i = 0, \delta n_1 \geqslant 0\right\}. \qquad [1 \text{ is fixed}]. \tag{7}$$

Let $j \in \mathbb{K}$ and $j \neq 1$ then by (7)

$$\delta n_j = -\sum_{i \in \mathbb{K} - \{j\}} \delta n_i.$$

Substituting this formula in 2° of (4) we obtain the following inequality

$$\sum_{i \in \mathbf{K} - \{j\}} (-pv_i + Ts_i + pv_j - Ts_j)\, \delta n_i \leqslant$$

$$\leqslant \sum_{i \in \mathbf{K} - \{j\}} (u(v_i, s_i) - u(v_j, s_j))\, \delta n_i .$$

In fact, by independency of variations δn_i we have the following equations

$$-p(v_i - v_j) + T(s_i - s_j) = u(v_i, s_i) - u(v_j, s_j), \quad \text{for } i, j \neq 1, \tag{8}$$

and inequalities

$$-p(v_1 - v_j) + T(s_1 - s_j) \leqslant u(v_1, s_1) - u(v_j, s_j), \qquad 1 \leqslant j \leqslant k. \tag{9}$$

It is easy to check that (9) is trivially fulfilled. If the 1-th cell is the last one where the first phase exists then repeating the above calculations we obtain

$$p = -\frac{\partial u}{\partial v}(v_i, s_i), \quad T = \frac{\partial u}{\partial s}(v_i, s_i),$$

$$(v_i, s_i) \underset{i,j \neq 1}{=} (v_j, s_j) = (v^2, s^2),$$

$$V = v^2 N, S = s^2 N.$$

Thus the equations (8) and inequalities (9) are trivially fulfilled. As a consequence we obtain the following proposition.

PROPOSITION 3.2. *The space of equilibrium states for the Van der Waals system is a singular lagrangian submanifold represented as an image of a regular one describing an appropriately deformed composite system.* [see Fig. 1].

Further applications of singular images in the critical region and in chemical equilibria we leave to a forthcoming paper.

4. EQUIVALENCE FOR PULLBACKS AND PUSHFORWARDS OF LAGRANGIAN SUBMANIFOLD

In this section we give the more precise, geometric basis for the above introduced notions and formulate the problem.

Let us consider the set Θ of symplectic relations in Ω, which are defined by smooth mappings $f : X \to Y$, i.e. every relation of Θ has a form T^*f. In the present paper we are interested only in local properties of symplectic relations and images of lagrangian submanifolds. Hence X, Y will be open subsets of $\mathbb{R}^n$, and $\mathbb{R}^m$

$\psi, \alpha)(T^*f) = (\varphi_1, \alpha_1, \psi_1, \alpha_1 \circ \varphi \circ f \circ \psi^{-1})(\varphi, \alpha, \psi, \alpha \circ f)(T^*f) = (\varphi_1 \circ \varphi, \alpha_1 \circ \varphi + \alpha, \psi_1 \circ \psi, \alpha_1 \circ \varphi \circ f + \alpha \circ f) = (\varphi_1 \circ \varphi, \psi_1 \circ \psi, \alpha_1 \circ \varphi + \alpha)(T^*f)$. ■

In this way we are ready to identify the action of Γ in the space of images [called pullbacks or pushforwards] of lagrangian submanifolds $L \subset (T^*Y, \omega_Y)$ [and $N \subset (T^*X, \omega_X)$ respectively] with respect to the relation T^*f [as defined below]. For this purpose we associate with the pair (L, T^*f) the pullback of a lagrangian submanifold L with respect to T^*f, and to the pair (T^*f, N) the appropriate pushforward of N.

DEFINITION 4.4. Two pullbacks (L_1, T^*f_1), (L_2, T^*f_2) [pushforwards (T^*f_1, N_1), (T^*f_2, N_2)] are called equivalent if there exists $g = (\varphi, \psi, \alpha) \in \Gamma$ such that

$$\begin{aligned} &(L_2, T^*f_2) = (\Phi(L_1), g(T^*f_1)), \\ &[(T^*f_2, N_2) = (g(T^*f_1), \Psi(N_1)) \text{ resp.}], \end{aligned} \tag{13}$$

where $\Phi \in G_Y$ [$\Psi \in G_X$] is the symplectomorphism of T^*Y [T^*X] defined by (φ, α) [defined by $(\psi, \alpha \circ f_1)$ respectively]. ■

REMARK 4.5. Let us take $(\mathrm{id}_Y, 0, \mathrm{id}_X, \beta) \in G_Y \times G_X$ and $T^*f \in \Theta$. We see that $(\mathrm{id}_Y, 0, \mathrm{id}_X, \beta)\ (T^*f) = \{((y, \eta), (x, \zeta)) \in T^*Y \times T^*X; y = f(x), \zeta + d\beta(x) = {}^tDf(x)\eta\}$ and, if $\beta \neq 0$ then we have $(\mathrm{id}_Y, 0, \mathrm{id}_X, \beta)\,(T^*f) \notin \Theta$. Thus the action of $G_Y \times G_X$ moves out of Θ. Now we try to extend the set Θ to retain the action of $G_Y \times G_X$. Let $f : X \to Y$ be a smooth map. We denote graph $f \subset Y \times X$ by K. Let $\bar{A}$ be a differentiable function on K. The set [see [17], [20]]

$$\begin{aligned} &\{p \in T^*(Y \times X); \pi_{Y \times X}(p) \in K \text{ and } \langle u, p\rangle = \langle u, d\bar{A}\rangle \text{ for each } u \in \\ &\in TK \subset T(Y \times X) \text{ such that } \tau_{Y \times X}(u) = \pi_{Y \times X}(p)\} \end{aligned} \tag{14}$$

is a lagrangian submanifold of $(T^*(Y \times X), d(\vartheta_Y \ominus \vartheta_X))$.

Let us denote by Θ' the set of symplectic relations in $(T^*(Y \times X), d\vartheta_Y \ominus d\vartheta_X)$ defined by (14). Every relation $R \in \Theta'$ is defined by the pair $(f, \bar{A})$. It is easy to see that $\Theta \subset \Theta'$ and every element of Θ corresponds to a pair $(f, 0)$. The symplectic relations belonging to Θ' are generated by the Morse families of the form

$$G(y, x; \lambda) = \tilde{A}(y, x) + \sum_{i=1}^{m} \lambda_i(y_i - f_i(x)),\ m = \dim Y, \qquad [\text{locally}], \tag{15}$$

where $\tilde{A}$ is the local expression of arbitrary continuation of the function $\bar{A}$ to $Y \times X$. Obviously, we can take the pullback of $\tilde{A}$ to X by means of f, namely

$G_Y \times G_X$ is defined as a system of functions and diffeomorphisms: $(\varphi, \alpha, \psi, \beta)$ with an appropriate composition formula.

Let (φ, ψ, α) be a triplet of mappings, $\varphi : Y \to Y$, $\psi : X \to X$ are diffeomorphisms and $\alpha : Y \to \mathbb{R}$ is a smooth function. The set Γ of such triplets defines an equivalence group in Ω preserving the set Θ.

DEFINITION 4.1. Let $g \in \Gamma$, then the equivalence relation in Θ is defined by the following action:

$$(g, T^*f) \to g(T^*f) = (\varphi, \alpha, \psi, \alpha \circ f)(T^*f), \tag{11}$$

where $(\varphi, \alpha, \psi, \alpha \circ f)$ defines an element (Φ, Ψ) of $G_Y \times G_X$ as in (10).

PROPOSITION 4.2. *Let $T^*f \in \Theta$ and $g = (\varphi, \psi, \alpha) \in \Gamma$ then the result of the action of g on T^*f is the symplectic relation $T^*(\varphi \circ f \circ \psi)^{-1}$, hence it belongs to Θ.*

Proof. According to Definition 4.1 and formulae (10) we have

$$\begin{aligned}(\varphi, \alpha, \psi, \alpha \circ f)(T^*f) &= \{((\varphi \circ f)(\bar{x}), {}^tD\varphi(f(\bar{x}))^{-1}(\bar{\eta} + \\ &+ \mathrm{d}\alpha(f(\bar{x}))), \psi(\bar{x}), {}^tD\psi(\bar{x})^{-1}({}^tDf(\bar{x})\bar{\eta} + Df(\bar{x})\mathrm{d}\alpha(f(\bar{x})))); \\ &\bar{x} \in X, (f(\bar{x}), \bar{\eta}) \in T^*Y\} = \{((\varphi \circ f \circ \psi^{-1}(x), {}^tD\varphi(f \circ \\ &\circ \psi^{-1}(x))^{-1}(\bar{\eta} + \mathrm{d}\alpha(f \circ \psi^{-1}(x))), x, {}^tDf \circ \psi^{-1}(x)(\bar{\eta} + \\ &+ \mathrm{d}\alpha(f \circ \psi^{-1}(x)))); (f \circ \psi^{-1}(x), \bar{\eta}) \in T^*Y\}.\end{aligned} \tag{12}$$

Let η denote ${}^tD\varphi(f \circ \psi^{-1}(x))^{-1}(\bar{\eta} + \mathrm{d}\alpha(f \circ \psi^{-1}(x)))$, then

$$\bar{\eta} + \mathrm{d}\alpha(f \circ \psi^{-1}(x)) = {}^tD\varphi(f \circ \psi^{-1}(x))\,\eta.$$

Inserting this in the last term of (12) we obtain:

$$\begin{aligned}(\varphi, \alpha, \psi, \alpha \circ f)(T^*f) &= \{((y, \eta), (x, \zeta)) \in T^*Y \times T^*X; y = \varphi \circ f \circ \\ &\circ \psi^{-1}(x), \zeta = {}^tDf \circ \psi^{-1}(x){}^tD\varphi(f \circ \psi^{-1}(x))\eta\} = \{((y, \eta), (x, \zeta)) \in \\ &\in T^*Y \times T^*X; y = \varphi \circ f \circ \psi^{-1}(x), {}^tD\varphi \circ f \circ \psi^{-1}(x)\eta\}.\end{aligned}$$

This completes the proof of the proposition. ∎

COROLLARY 4.3. *The composition of symplectomorphisms corresponding to $(\varphi_1, \psi_1, \alpha_1)$, (φ, ψ, α) is the symplectomorphism corresponding to the triplet $(\varphi_1 \circ \circ \varphi, \psi_1 \circ \psi, \alpha_1 \circ \varphi + \alpha)$. This provides a formula for the group operation in Γ.*

Proof. On the basis of Definition 4.1 and (12) we can write $(\varphi_1, \psi_1, \alpha_1) \cdot (\varphi,$

$\psi, \alpha)(T^*f) = (\varphi_1, \alpha_1, \psi_1, \alpha_1 \circ \varphi \circ f \circ \psi^{-1})(\varphi, \alpha, \psi, \alpha \circ f)(T^*f) = (\varphi_1 \circ \varphi, \alpha_1 \circ \varphi + \alpha, \psi_1 \circ \psi, \alpha_1 \circ \varphi \circ f + \alpha \circ f) = (\varphi_1 \circ \varphi, \psi_1 \circ \psi, \alpha_1 \circ \varphi + \alpha)(T^*f)$. ■

In this way we are ready to identify the action of Γ in the space of images [called pullbacks or pushforwards] of lagrangian submanifolds $L \subset (T^*Y, \omega_Y)$ [and $N \subset (T^*X, \omega_X)$ respectively] with respect to the relation T^*f [as defined below]. For this purpose we associate with the pair (L, T^*f) the pullback of a lagrangian submanifold L with respect to T^*f, and to the pair (T^*f, N) the appropriate pushforward of N.

DEFINITION 4.4. Two pullbacks (L_1, T^*f_1), (L_2, T^*f_2) [pushforwards (T^*f_1, N_1), (T^*f_2, N_2)] are called equivalent if there exists $g = (\varphi, \psi, \alpha) \in \Gamma$ such that

$$(L_2, T^*f_2) = (\Phi(L_1), g(T^*f_1)),$$
$$[(T^*f_2, N_2) = (g(T^*f_1), \Psi(N_1)) \ \text{resp.}], \tag{13}$$

where $\Phi \in G_Y$ $[\Psi \in G_X]$ is the symplectomorphism of T^*Y $[T^*X]$ defined by (φ, α) [defined by $(\psi, \alpha \circ f_1)$ respectively]. ■

REMARK 4.5. Let us take $(\mathrm{id}_Y, 0, \mathrm{id}_X, \beta) \in G_Y \times G_X$ and $T^*f \in \Theta$. We see that $(\mathrm{id}_Y, 0, \mathrm{id}_X, \beta)\,(T^*f) = \{((y, \eta), (x, \zeta)) \in T^*Y \times T^*X; y = f(x), \zeta + \mathrm{d}\beta(x) = {}^tDf(x)\eta\}$ and, if $\beta \neq 0$ then we have $(\mathrm{id}_Y, 0, \mathrm{id}_X, \beta)\,(T^*f) \notin \Theta$. Thus the action of $G_Y \times G_X$ moves out of Θ. Now we try to extend the set Θ to retain the action of $G_Y \times G_X$. Let $f : X \to Y$ be a smooth map. We denote graph $f \subset Y \times X$ by K. Let $\bar{A}$ be a differentiable function on K. The set [see [17], [20]]

$$\{p \in T^*(Y \times X); \pi_{Y \times X}(p) \in K \text{ and } \langle u, p\rangle = \langle u, \mathrm{d}\bar{A}\rangle \text{ for each } u \in TK \subset T(Y \times X) \text{ such that } \tau_{Y \times X}(u) = \pi_{Y \times X}(p)\} \tag{14}$$

is a lagrangian submanifold of $(T^*(Y \times X), \mathrm{d}(\vartheta_Y \ominus \vartheta_X))$.

Let us denote by Θ' the set of symplectic relations in $(T^*(Y \times X), \mathrm{d}\vartheta_Y \ominus \mathrm{d}\vartheta_X)$ defined by (14). Every relation $R \in \Theta'$ is defined by the pair $(f, \bar{A})$. It is easy to see that $\Theta \subset \Theta'$ and every element of Θ corresponds to a pair $(f, 0)$. The symplectic relations belonging to Θ' are generated by the Morse families of the form

$$G(y, x; \lambda) = \tilde{A}(y, x) + \sum_{i=1}^{m} \lambda_i (y_i - f_i(x)),\ m = \dim Y, \qquad [\text{locally}], \tag{15}$$

where $\tilde{A}$ is the local expression of arbitrary continuation of the function $\bar{A}$ to $Y \times X$. Obviously, we can take the pullback of $\tilde{A}$ to X by means of f, namely

$A(x) = \widetilde{A}(f(x), x)$. Then, more simply our symplctic relations [at least locally] are reproduced by a smooth function on X, say $A : X \to \mathbb{R}$, and a smooth mapping $f : X \to Y$ defining the respective morse family $G(y, x; \lambda) = A(x) + \sum_{i=1}^{m} \lambda_i (y_i - f_i(x))$. For the pairs (f, A) determining the respective symplectic relations we have the following transformation law [according to the above introduced action of $G_Y \times G_X$]:

$$(f, A) \xrightarrow{(\underline{\varphi, \alpha}, \underline{\psi, \beta})} (\varphi \circ f \circ \psi^{-1}, A \circ \psi^{-1} + \alpha \circ f \circ \psi^{-1} - \beta \circ \psi^{-1}). \tag{16}$$

If the pair (L, R) is a pullback of the lagrangian submanifold $L \subset T^*Y$ with respect to the symplectic relation $R = (f, A) \subset (T^*Y \times T^*X, \mathrm{d}\vartheta_Y \ominus \mathrm{d}\vartheta_X)$ and if $F : Y \times \Lambda \to \mathbb{R}$ is a Morse family generating L then the generating family for the equivalent pullback $R'(L')$ [with respect to the group $G_Y \times G_X$] has the following form

$$H : X \times \Lambda \to \mathbb{R}, H = F \circ (\varphi^{-1}, \mathrm{id}'_\Lambda)(\varphi \circ f \circ \psi^{-1}, \mathrm{id}_\Lambda) + (\beta - A) \circ \psi^{-1}. \tag{17}$$

To obtain normal forms for pullbacks and pushforwards of lagrangian submanifolds we use the standard equivalence of Morse families [cf. [24]], namely the one defined by the diagram

$$\begin{tikzcd} Y \times \Lambda \arrow[rr, "\Xi"] \arrow[dr, "\pi_Y"'] & & Y \times \Lambda \arrow[dl, "\pi_Y"] \\ & Y & \end{tikzcd} \tag{18}$$

Two Morse families $F, F' : Y \times \Lambda \to \mathbb{R}$ are called equivalent if there exists a diffeomorphism Ξ [as in the diagram (18)] such that

$$F = F' \circ \Xi. \tag{19}$$

Applying the formula (17) to the family Θ we set $A = 0$ and $\beta = \alpha \circ f$. The resulting expression for the generating family of the equivalent pullback is

$$H = (F + \alpha) \circ (f \circ \psi^{-1}, \mathrm{id}_\Lambda). \tag{20}$$

Note that this generating family may not be a Morse family and the set generated by it in the standard way may not be a submanifold. In the same way are one can write down the generating family for the equivalent pushforward.

5. NORMAL FORMS FOR SINGULAR PULLBACKS AND PUSHFORWARDS

It is not easy to classify, in general, the normal forms of pullbacks and pushforwards of lagrangian submanifolds. This paper deals with the problem in the case of small dimension of the manifolds X, Y and for the pairs (L, T^*f) $[(T^*f, N)]$ in which L [N respectively] and f are locally stable in the standard sense [see [24], [8]].

PROPOSITION 5.1. *Let* $\dim X = \dim Y = k \leqslant 3$, *then for the generic pullbacks* (L, T^*f), *where L is of type A_2 [fold, according to Arnold's notation* [3]] *and f is a stable mapping, the corresponding germ of generating family $H : X \times \mathbb{R} \to \mathbb{R}$ for $T^*f(L)$, at every point of X is equivalent to one in the following table*

k	$f : X \to Y$	$H : X \times \mathbb{R} \to \mathbb{R}$
1	$y = x$ $y = x^2$	$\lambda^3 + \lambda x$ $\lambda^3 \pm \lambda x^2$
	$y_1 = x_1,\quad y_2 = x_2$ $y_1 = x_1,\quad y_2 = x_2^2$ $y_1 = x_1,\quad y_2 = x_2^3 + x_1 x_2$	$\lambda^3 + \lambda x_1$ $\lambda^3 + \lambda x_1,\quad \lambda^3 + \lambda(\pm x_1^2 \pm x_2^2)$ $\lambda^3 \neq \lambda x_1$
	$y_1 = x_1,\quad y_2 = x_2,\quad y_3 = x_3$ $y_1 = x_1,\quad y_2 = x_2^2,\quad y_3 = x_3$ $y_1 = x_1,\quad y_2 = x_2^3 + x_1 x_2,\quad y_3 = x_3$ $y_1 = x_1,\quad y_2 = x_2,\quad y_3 = x_3^4 + x_1 x_3^2 +$ $+ x_1 x_3$	$\lambda^3 + \lambda x_1$ $\lambda^3 + \lambda x_1,\quad \lambda^3 + \lambda(\pm x_1^2 \pm x_2^2 \pm x_3^2)$ $\lambda^3 + \lambda x_3,\quad \lambda^3 + \lambda(\pm x_3^2 \pm x_1)$ $\lambda^3 \pm \lambda x_1$.

Proof. Since $L \subset T^*Y$ has a singularity of type A_2 at a point $p \in L$, it is generated by the Morse family $F : Y \times \mathbb{R} \to \mathbb{R}$,

(21) $$F(y, \lambda) = \lambda^3 + \lambda g(y), \qquad \text{[see [24], p. 28]},$$

where g is a smooth function, such that $g(0) = 0$, $dg(0) \neq 0$. It follows from our definition of equivalence of pairs (L, T^*f), [see Definition 4.4] and the from of the generating family for pullbacks of L [see formula (20)] we can reduce the problem of finding normal forms of such pullbacks to the problem of classification of normal forms of mapping diagrams

(22) $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\psi"'] & Y \arrow[d, "\varphi"] \arrow[dr, "g"] & \\ X \arrow[r, "f'"] & Y \arrow[r, "g'"'] & \mathbb{R} \end{tikzcd}$$

Since the function g satisfies the condition $g(0) = 0$ we can apply well known Arnold's results [2] on classification of generic mapping diagrams with respect to the so-called strong equivalence, i.e.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\psi"'] & Y \arrow[r, "g"] \arrow[d, "\varphi"] & \mathbb{R} \arrow[d, "\mathrm{id}_{\mathbb{R}} + \text{const.}"] \\ X \arrow[r, "f'"] & Y \arrow[r, "g'"] & \mathbb{R} \end{tikzcd}$$

On the basis of the classification theorem there, with $\dim Y = \dim X \leqslant 3$ [[2], Theorem 5.1, p. 572-3], we obtain the normal forms for the function g using f-liftable diffeomorphisms of $(Y, 0)$. Inserting these forms in (21) we obtain the normal forms listed above. ■

REMARK 5.2. Some of the functions in the table of Proposition 5.1 are not Morse families. However the sets $R(L)$ defined by the equations $\zeta = \partial H/\partial x(x, \lambda)$, $0 = \partial H/\partial \lambda(x, \lambda)$ are semi-algebraic sets and can be endowed with the stratification [see [9]] into isotropic submanifolds of $(T^*X, d\vartheta_X)$. The generic singularity of a pullback in the case of $n = 1$ and fold mapping f is shown in Fig. 2.

By using the methods of the proof of Proposition 5.1, we can conduct the classification of normal forms of $R(L)$ for the case when f is a fold mapping and

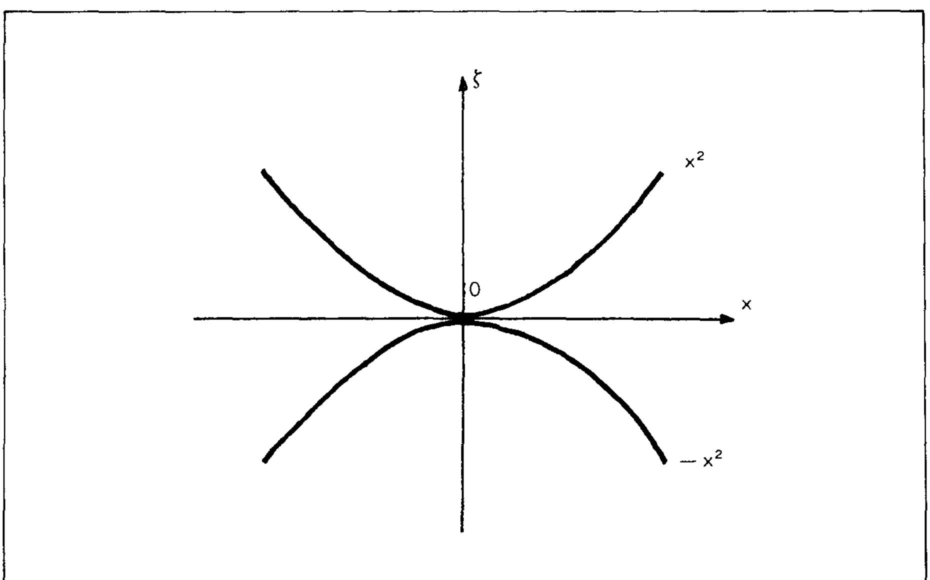

Fig. 2.

L is a lagrangian sumbanifold of type A_2 without restricting the dimension of X and Y.

PROPOSITION 5.3. *Let* $f:(X,x_0)\to(Y,y_0)$, $\dim Y \leqslant \dim X$, *be a fold singularity, i.e.* $f(x_1,\ldots,x_n)=(x_1^2,\ldots,x_m)$ *in local coordinates on* (X,x_0) *and* (Y,y_0) *respectively. Let* (L,p) *be also a fold singularity and let* $\pi_Y(p)=p_0$. *Then the normal forms for generating families of pullbacks [without modal parameters] are the following functions*

$$\tilde{A}_\mu : H(x,\lambda)=\lambda^3+\lambda(\pm x_1^2 \pm x_2^{\mu+1}+Q(x_3,\ldots,x_m)), \qquad \mu\geqslant 2,$$

$$\tilde{D}_\mu : H(x,\lambda)=\lambda^3+\lambda(\pm x_1^2 \pm x_2^2x_3 \pm x_3^{\mu-1}+Q(x_4,\ldots,x_m)), \mu\geqslant 4,$$

$$\tilde{E}_6 : H(x,\lambda)=\lambda^3+\lambda(\pm x_1^2 \pm x_2^3 \pm x_3^4+Q(x_4,\ldots,x_m)),$$

$$\tilde{E}_7 : H(x,\lambda)=\lambda^3+\lambda(\pm x_1^2 \pm x_2^3 \pm x_3^3x_2+Q(x_4,\ldots,x_m)),$$

$$\tilde{E}_8 : H(x,\lambda)=\lambda^3+\lambda(\pm x_1^2 \pm x_2^3 \pm x_3^5+Q(x_4,\ldots,x_m)),$$

where $Q(x_s,\ldots,x_m)$ *is a nondegenerate quadratic form of* $m-s+1$ *variables.*

Proof. According to Arnold's method of liftable diffeomorphisms [see [2], Lemma 3.1 adapted to the real case] the problem of finding normal forms of the composition of mappings

$$\mathbb{R}^n \xrightarrow{f\text{-fold}} \mathbb{R}^m \xrightarrow{g} \mathbb{R}$$

can be reduced to the problem of finding normal forms of functions $g:\mathbb{R}^m\to\mathbb{R}$ with respect to the right action of the group of diffeomorphisms [germs] preserving a hypersurface [apparent contour of f]. For this problem we can use the results of [3] and obtain the following facts. If the normal form of f is $f=(x_1^2, x_2,\ldots,x_m)$, then the simple normal forms of functions $g:(\mathbb{R}^m,0)\to(\mathbb{R},0)$, such that $dg(0)\neq 0$, with respect to the group of germs of diffeomorphisms preserving the hypersurface $\{x_1=0\}$ are

$$A_\mu : g=\pm y_1 \pm y_2^{\mu+1}+Q(y_3,\ldots,y_m),$$

$$D_\mu : g=\pm y_1 \pm y_2^2y_3 \pm y_3^{\mu-1}+Q(y_4,\ldots,y_m),$$

$$E_6 : g=\pm y_1 \pm y_2^3 \pm y_3^4+Q(y_4,\ldots,y_m),$$

$$E_7 : g=\pm y_1 \pm y_2^3 \pm y_2y_3^3+Q(y_4,\ldots,y_m),$$

$$E_8 : g=\pm y_1 \pm y_2^3 \pm y_3^5+Q(y_4,\ldots,y_m),$$

where Q is a nondegenerate quadratic form depending on the remaining variables [cf. [3], p. 104]. Combining these and the fact that the Morse family for L of type

A_2 has the form $\lambda^3 + g(y)\lambda$ we obtain the thesis of our proposition. ∎

Now we pass to the classification of pushforwards. Let $N \subset T^*X$ be a lagrangian submanifold generated by a Morse family, say $G : X \times \mathbb{R}^k \to \mathbb{R}$. Using the standard facts concerning generating families for symplectic relations [cf. [19], [5]] we obtain almost immediately the following assertion,

PROPOSITION 5.4. *A generating family for the pushforward (T^*f, N), say $P : Y \times \times \mathbb{R}^N \to \mathbb{R}$ can be written as follows*

$$P(y; \lambda, \mu, \nu) = \sum_{i=1}^{m} \lambda_i (y_i - f_i(\mu)) + G(\mu, \nu),$$

where $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_k)$, $N \leqslant m + n + k$.

On the basis of this proposition and section 4 we obtain that the generating family for an equivalent pushforward is the following:

$$\widetilde{P}(y; \lambda, \mu, \nu) = \sum_{i=1}^{m} \lambda_i (y_i - (\varphi \circ f \circ \psi^{-1})_i(\mu)) + \\ + \alpha \circ f \circ \psi^{-1}(\mu) + G(\psi^{-1}(\mu), \nu),$$

where α is a smooth function on Y and (φ, α, ψ) is an element of equivalence group.

PROPOSITION 5.5. *Let* $\dim X = \dim Y = k \leqslant 3$, *then the begining of the classification of the germs of pushforwards (T^*f, N), where N is of type A_2 and f is a stable mapping is given in the following table. (see below).*

Here G is a normal form for a Morse family generating the germ of lagrangian submanifold N, φ_i are smooth functions and $[\beta_i] \in \mathfrak{m}^2/f^*\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal in the local algebra of smooth function-germs in (X, x_0).

Proof. Applying the same arguments as in the proof of Proposition 5.1, and using Arnold's results [2] concerning the classification of mapping diagrams

$$\begin{tikzcd} Y \arrow[d, "\varphi"'] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\psi"] & \mathbb{R} \\ Y & X \arrow[l, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

by means of f-lowerable diffeomorphisms we obtain the classification of normal forms of functions $g : X \to \mathbb{R}$. Inserting these normal forms to the Morse family

k	$f : X \to Y$	$G : X \times \mathbb{R} \to \mathbb{R}$
1	$y = x$	$\lambda^3 + \lambda x + \beta_1(x)$
	$y = x^2$	$\lambda^3 + \lambda(\pm x + \varphi_1(x^2)) + \beta_2(x)$
2	$y_1 = x_1, \quad y_2 = x_2$	$\lambda^3 + \lambda x_1 + \beta_3(x)$
	$y_1 = x_1^2, \quad y_2 = x_2$	$\lambda^3 + \lambda(x_1 + x_2) + \beta_4(x)$, $\lambda^3 + \lambda(x_1 \pm x_2^2) + \beta_5(x)$, $\lambda^3 + \lambda(x_2 \pm x_1 x_2 + x_1^3) + \beta_6(x)$
	$y_1 = x_1^3 + x_1 x_2, \quad y_2 = x_2$	$\lambda^3 + \lambda(\pm x_1 + \varphi_2(x_1^3 + x_1 x_2, x_2) + \beta_7(x)$
3	$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3$	$\lambda^3 + \lambda x_1 + \beta_8(x)$,
	$y_1 = x_1^2, \quad y_2 = x_2, \quad y_3 = x_3$	$\lambda^3 + \lambda(x_1 + x_2) + \beta_9(x)$, $\lambda^3 + \lambda(x_1 \pm x_2^2 \pm x_3^2) +$ $+ \beta_{10}(x)$, $\lambda^3 + \lambda(x_2 + x_1 x_3) + \beta_{11}(x)$, $\lambda^3 +$ $+ \lambda(x_2 + x_1 x_2 + x_1^3 \pm x_1 x_3^2) + \beta_{12}(x)$
	$y_1 = x_1^3 + x_1 x_2, \quad y_2 = x_2, \quad y_3 = x_3$	$\lambda^3 + \lambda(x_1 + x_3) + \beta_{13}(x)$, $\lambda^3 + \lambda(x_1 \pm x_3^2 + \varphi_3(x_1^3$ $+ x_1 x_2, x_2)) + \beta_{14}(x)$, $\lambda^3 + \lambda(x_3 \pm x_1^2 + x_1 \varphi_4(x_1^3 +$ $+ x_1 x_2, x_2)) + \beta_{15}(x)$,
	$y_1 = x_1^4 + x_2 x_1^2 + x_3 x_1, \; y_2 = x_2, \; y_3 = x_3$	$\lambda^3 + \lambda(\pm x_1 + \varphi_5(x_1^4 + x_2 x_1^2 + x_3 x_1, \; x_2, x_3) +$ $+ \beta_{16}(x)$

$G(x, \lambda) = \lambda^3 + \lambda g(x) + \beta(x)$ we obtain the desired result with the additional term $\beta(x)$ modulo an element belonging to $f^* \mathfrak{m}^2$. ∎

As we see this classification is not satisfactory complete and depends heavily on the classification of mapping diagrams of more general type, hence we leave it to the forthcoming paper.

6. FINAL REMARKS AND APPLICATIONS

6.1. As a simple mechanical example of singular image with respect to the symplectic reduction we consider the finite element analogue of the Euler beam problem illustrated in Fig. 3. This system, consisting of two rigid rods of unit length connected by friction-less pins, is subjected to a compressive force $-p_q$ which is resisted by a torsion spring of unit strength. The angle φ and the force p_q are considered coordinates of a manifold X. Together with the torque p_φ and the position q they form a canonical coordinate system $(\varphi, p_q, p_\varphi, -q)$ of T^*X.

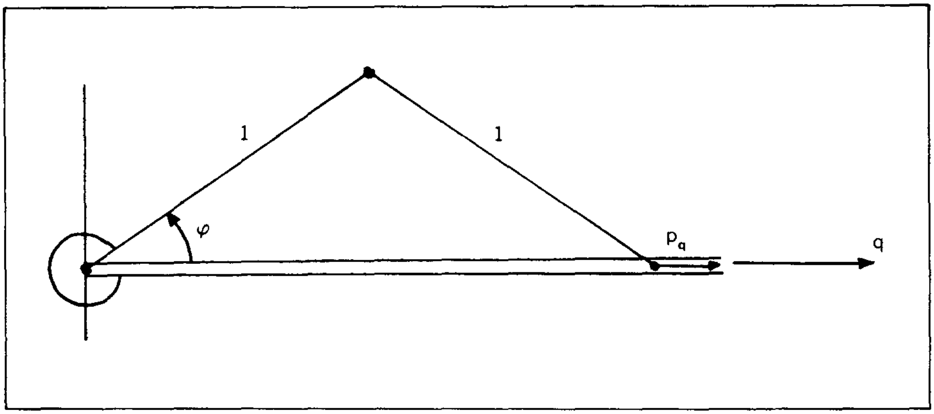

Fig. 3.

The potential energy of this system [generating function of the lagrangian submanifold $N \subset T^*X$] has the form

$$V(\varphi, p_q) = \frac{1}{2}\varphi^2 - 2p_q \cos\varphi.$$

If we take the reduced phase space T^*Y with the local coordinate system $(p_q, -q)$ and the mapping $f: X \to Y$, $f(\varphi, p_q) = p_q$ then we obtain for the image of N the following formula

$$^tT^*f(N) = \Big\{(p_q, -q) \in T^*Y; 0 = \frac{\partial V}{\partial \varphi}(\varphi, p_q) = = \varphi + 2p_q \sin\varphi, -q = \frac{\partial V}{\partial p_q}(\varphi, p_q) = -2\cos\varphi\Big\},$$

which is a space of equilibrium states in the control phase space T^*Y. A simple calculation shows that if $p_q = -\frac{1}{2}$, $\varphi = 0$; V is not Morse family and the set $^tT^*f(N)$ has a standard singularity well known in bifurcation theory [see Fig. 4]. Unfortunately that singularity is not stable, it disappears after a small deformation of V because the respective transversality condition [cf. §2] is not fulfilled. However for examples of this type we can construct the space of deformations and treat the unstable singular lagrangian submanifold as an element of a family of deformations [a kind of unfolding [9] or more precisely Wassermann's (r, s)--unfolding [21]]. The number of parameters of this family is connected to the codimension according to the above classified singularity types. This approach

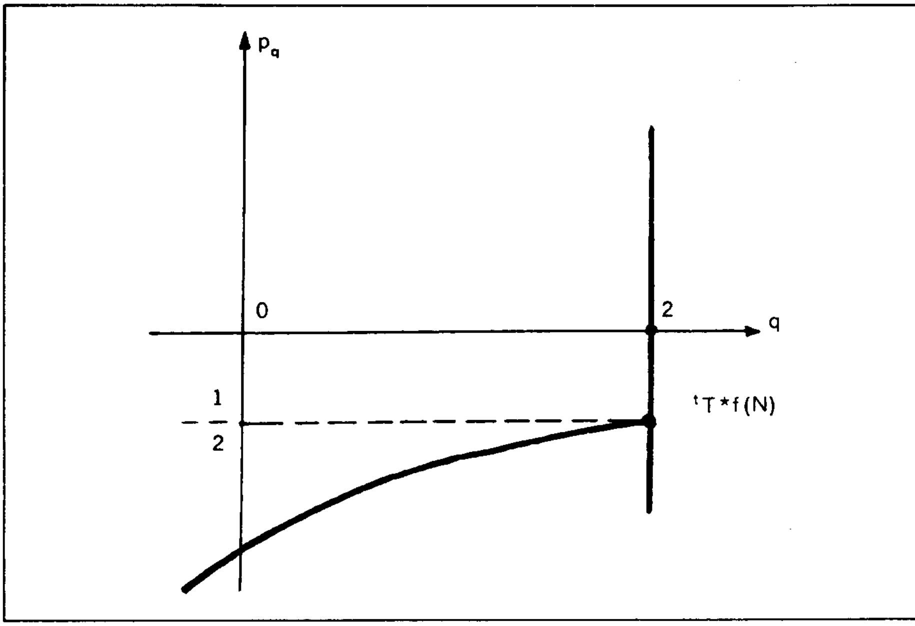

Fig. 4.

leads to the classification of stable images according to the composition of two reduction relations.

6.2. Let $\mathbb{R}^{2n} = \{(x, p)\}$ be a phase space of a particle in classical mechanics [1], let $h(x, p) = \frac{1}{2}(|p|^2 - 1)$ be a Hamilton function for this particle. Then the space of bicharacteristics in $H = \{h = 0\}$, say M, which forms a manifold of all oriented lines in $\mathbb{R}^n$ has a canonical symplectic structure. Let K be a hypersurface in $\mathbb{R}^n$ [an obstacle] and γ a geodesic flow on K [e.g. that one defined on K by the variational problem of shortest bypassing of K]. It is proved in [4] that the set of oriented lines tangent to γ on K forms a lagrangian submanifold in M which is not necessarily smooth. The appropriate local classification of these singular lagrangian submanifolds is carried out in the quoted paper. It turned out that the generic singularities of this classification, so-called open swallowtails, can be conveniently described in the $SL_2(\mathbb{R})$-invariant symplectic space of binary forms of an appropriate degree. We find, using the results of the previous sections, that the open swallowtails can be obtained as images from the regular lagrangian submanifolds by means of a canonical symplectic procedure. Now we briefly describe this resolution procedure.

Let us consider the space of polynomials of the form [cf. [4]]

$$T^*Q = \left\{ \frac{x^{2n}}{(2n)!} + q_1 \frac{x^{2n-1}}{(2n-1)!} + \ldots + q_n \frac{x^n}{n!} - \right.$$

$$\left. - p_n \frac{x^{n-1}}{(n-1)!} + \ldots + (-1)^{n-1} p_1 \right\}$$

endowed with the symplectic form $\omega = \sum_i \mathrm{d}p_i \wedge \mathrm{d}q_i$. Let a hypersurface $H \subset \subset T^*Q$ be defined by a Hamiltonian of translations along the x-axis, $h = p_1 + + q_1 p_2 + \ldots + q_{n-1} p_n + \frac{q_n^2}{2}$, $H = \{h = 0\}$. If $i_H : H \to T^*Q$ is an embedding of H then the pull-back $\omega_H = i_H^* \omega$ has rank $2n - 2$, so has kernel a one dimensional subbundle $TH^{\S} \subset TH$, the Hamilton foliation of H, the integral curves of which are the bicharacteristics on H. The space, say M, of [local] bicharacteristics on H is itself a symplectic manifold with the induced symplectic structure and the canonical projection $\pi : H \to M$ associating to each point of H the local bicharacteristic on H through it.

DEFINITION. [cf. [4]]. The system of objects $(H, L, 1)$, *where* H is a hypersurface in T^*Q, L is a lagrangian submanifold in T^*Q tangent with the first order tangency to H along submanifold $1 = H \cap L$, which has codimension 1 in L, is called a symplectic triplet in T^*Q.

Let us define L as a trivial section of T^*Q, $L = \{p_1 = \ldots = p_n = 0\}$. It is easily seen that $(H, L, H \cap L)$, where $H = \{h = 0\}$, is a symplectic triplet. An image $\tilde{L} = \pi(H \cap L)$ is called an $n - 1$-dimensional open swallowtail [4]. It is not difficult to prove the following fact

PROPOSITION. *An open k-dimensional swallowtail $\tilde{L}_k$ can be represented as a canonical pushforward of a regular lagrangian submanifold L_k, i.e.*

$$\tilde{L}_k = T^*f(L_k), \dim Q = k + 1,$$

where L_k is a lagrangian submanifold of (T^*Q, ω_Q) with the following generating function:

$$F_k(q_1, \ldots, q_{k+1}) = \sum_{i=-1}^{k-2} \sum_{s=2}^{k-i-1} D^{(k)}_{k-i,s} q_1^{k+i-s+3} q_s q_{k-i} +$$

$$+ \frac{1}{2} \sum_{i=0}^{k-2} D^{(k)}_{k-i,k-i} q_1^{2i+3} q_{k-i}^2 +$$

$$+\sum_{i=1}^{k-2} E_{k-i}^{(k)} q_1^{k+i+3} q_{k-i} + \frac{1}{2} D_{k+1,k+1}^{(k)} q_1 q_{k+1}^2 +$$

$$+ E_{k+1}^{(k)} q_1^{k+2} q_{k+1} - \frac{E_2^{(k)}}{2k+3} q_1^{2k+3},$$

where

$$D_{r,s}^{(k)} = (-1)^{k-r} \sum_{j=s}^{k+1} \frac{(-1)^{j-s}}{(j-s)!\,(2k+3-j-r)!},$$

$$E_r^{(k)} = (-1)^{k-r} \left(\frac{1}{(2k+3-r)!} - \sum_{j=2}^{k+1} \frac{(-1)^j (j-1)}{j!(2k+3-j-r)!} \right), \quad 1 \leqslant r, s \leqslant k+$$

and f is defined as follows:

$$f(q) = (f_1(q), \ldots, f_k(q)), \quad (q) = (q_1, \ldots, q_{k+1})$$

$$f_j(q) = \sum_{1=0}^{j-1} (-1)^1 \frac{1}{1!} q_1^1 q_{j+1-1} +$$

$$+ (-1)^j \frac{j}{(j+1)!} q_1^{j+1}, \qquad j = 1, \ldots, k.$$

6.3. Now we show that the regular geometric interaction between holonomic components [in the sense of Kashiwara in Micro-local calculus [13], [16]] can be resolved, i.e. obtained as an image of a regular lagrangian submanifold by a symplectic reduction relation.

Let V_1, V_2 be lagrangian submanifolds of a symplectic manifold (P, ω) [the respective holonomic components of an interaction [15]].

DEFINITION. The lagrangian subset $V_1 \cup V_2$ [or pair (V_1, V_2)] of (P, ω) is called a regular geometric interaction if the following conditions are fulfilled

a) $V_1 \cap V_2$ is a submanifold of P, $\dim V_1 \cap V_2 = \dim V_1 - 1$,

b) for every point $p \in V_1 \cap V_2$ we have

$$T_p(V_1 \cap V_2) = T_p V_1 \cap T_p V_2.$$

Let $(V_1 \cup V_2, p)$ be a germ of a regular geometric interaction in (P, ω).

PROPOSITION. *There is a symplectic manifold $(\widetilde{P}, \widetilde{\omega})$ and a symplectic reduction* *relation* $R \subset (\widetilde{P} \times P, \widetilde{\omega} \ominus \omega)$ *such that for a germ of regular geometric interaction, say* $(V_1 \cup V_2, p) \subset (P, \omega)$ *we have a canonical resolution formulae*

$$V_1 \cup V_2 = R(L),$$

for some regular lagrangian submanifold $L \subset (\widetilde{P}, \widetilde{\omega})$.

Proof. On the basis of the Kostant-Weinstein theorem [se e.g. [10], [23]] we can isomorphically represent (P, ω) by means of $(T^* V_1, \omega_{V_1})$, where V_1 is a zero-section of the bundle. Hence $V_1 = \{p_1 =, \ldots, = p_n = 0\}$ and a generating function for V_2, in $T^* V_1$, can be written as $H(q) = q_1^2 \varphi(q)$, where $\varphi(0) \neq 0$ [because of the point b) of the definition]. So we can choose local Darboux coordinates in $T^* V_1$, near p, preserving the zero section V_1 and such that the respective germ of generating function for V_2 is

$$H(q) = q_1^2.$$

Taking the new Darboux coordinates in $T^* V_1$ preserving V_1, namely

$$\Phi(q_1, \ldots, q_n, p_1, \ldots, p_n) = \left(q_1 - \frac{1}{2} p_1, q_2, \ldots, q_n, p_1, \ldots, p_n\right)$$

we obtain the following local equations for V_1 and V_2 respectively

$$V_1 : p_1 = , \ldots, = p_n = 0,$$

$$V_2 : p_2 = 0, \ldots, p_n = 0, q_1 = 0.$$

But for this germ of geometric interaction we can easily write the respective generating family:

$$F(q_1, \ldots, q_n, \lambda) = q_1 \lambda^3.$$

If T^*X is any initial, special symplectic structure of (P, ω) then using the Morse family, say $G : X \times Q \times \mathbb{R}^N \to \mathbb{R}$, for the respective symplectomorphisms in the above procedure [according to [17], [20]] we can write down the desired generating family for $V_1 \cup V_2$:

$$\widetilde{F}(x; \nu, \mu, \lambda) = G(x_1, \ldots, x_n, \nu_1, \ldots, \nu_n, \mu_1, \ldots, \mu_N) + \nu_1 \lambda^3.$$

This completes the proof. ∎

Having such analytical description of regular holonomic interaction we can formulate the appropriate stability problem and use it to determine the respective Gause-Manin systems [15], [16].

6.4. [Landau singularities]. Let us consider the motion of a free particle of mass m in space-time $\mathbb{R}^4$ endowed with the Minkowski metric tensor. The phase space is the cotangent bundle $P \cong T^*\mathbb{R}^4$. A mass surface or a first class constraint submanifold $M \subset P$ is defined by

$$\widetilde{M} = \{(x, p) \in P; p^2 = p_0^2 - \mathbf{p}^2 = m^2, p_0 > 0\},$$

where the respective Hamiltonian is defined as a zero function on $\widetilde{M}$.

In elementary particle physics the collision processes constitute one of the main subjects of interest [for the basis of the theory of multiple collisions processes see e.g. F. Pham, «Singularitiés des processus de diffusion multiple», Ann. Inst. H. Poincaré 6, 2(1967)]. Let us consider a collision process $I \to J$ described by the coisotropic submanifold $\widetilde{M}_{(I,J)}$ in $\prod_{i \in I \cup J} P_i$, namely

$$(*) \qquad \widetilde{M}_{(I,J)} = \left\{(\tilde{x}, \tilde{p}) \in \prod_{i \in I \cup J} P_i; (\tilde{x}, \tilde{p}) \in \prod_{i \in I \cup J} \widetilde{M}_i, \sum_{i \in I} p_i = \sum_{j \in J} p_j\right\},$$

where I, J are the numbering sets for the respective particles [as in Fig. 5] in the collision process (I, J). Let us consider an associated causal configuration for (I, J) corresponding to the graph G of an appropriate multiple diffusion process [see Fig. 6]. Let I resp. J denote the set of external lines incoming and resp. outgoing from G. Let $\widetilde{M}_G$ be the coisotropic submanifold defined analogously as in (*) using the conservation laws. It is easy to check that the symplectic spaces $\widetilde{M}_{(I,J)}/\sim$ and $\widetilde{M}_G/\sim$ associated canonically with $\widetilde{M}_{(I,J)}$ and $\widetilde{M}_G$ are isomorphic to $T^*M_{(I,J)}$ and T^*M_G respectively, where

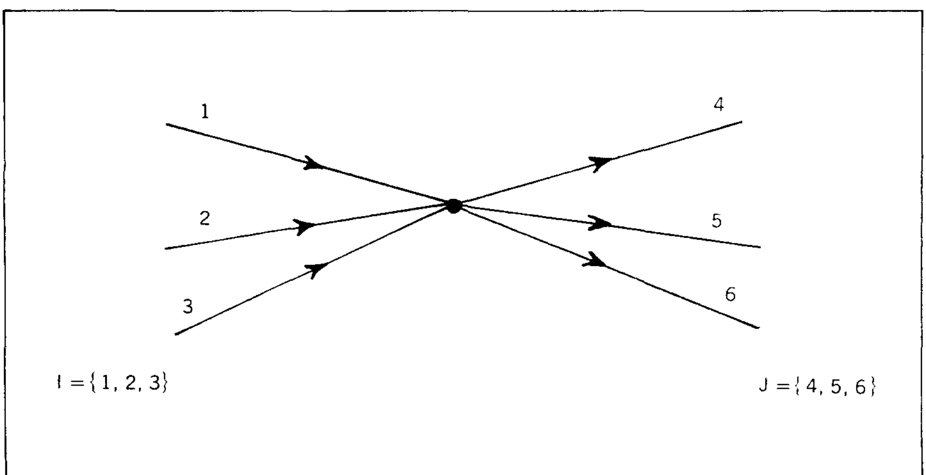

Fig. 5.

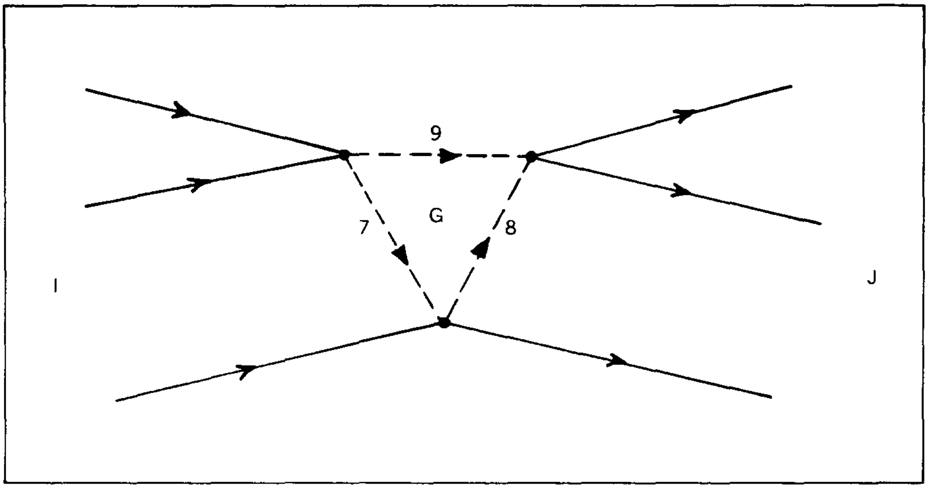

Fig. 6.

$$\mathbb{R}^{4N} \supset M_{(I,J)} = \left\{ (p_i) \in \mathbb{R}^{4N}; p_{oi}^2 - \mathbf{p}_i^2 = m_i, p_{oi} > 0, \sum_{i \in I} p_i = \sum_{j \in J} p_j \right\}$$

and analogously for M_G.

We have here the natural projection

$$f : M_G \to M_{(I,J)},$$

which defines the corresponding symplectic relation

$$T^*f \subset \widetilde{M}_{(I,J)}/\sim \quad \times \; \widetilde{M}_G/\sim$$

responsible for the geometrical properties of the collision process. The set of critical values of f, say $\Gamma f \subset M_{(I,J)}$ [an apparent contour of f] is called a Landau set corresponding to the graph G. The singularity type of f is responsible for the singularity type of the Landau set and is frequently called the Landau singularity.

COROLLARY. *The geometrical properties of a multiple diffusion process with a graph G are described by the pair:*

$$(L_{\Gamma f}, T^*f),$$

where $L_{\Gamma f}$ *is a constrained lagrangian submanifold over constraint* Γf [cf. [12]].

Hence the classification of normal forms, as in the Pham approach for the Landau singularities, can be easily derived using our classification theorems for pullbacks.

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