# HAMILTONIAN GEODESICS IN NONHOLONOMIC DIFFERENTIAL SYSTEMS

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We derive the general form of a sub-Riemannian Hamiltonian on a Riemannian manifold endowed with the nonholonomic distribution. The generic properties of sub-Riemannian exponential map and horizontal curves are investigated.

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## 1. Introduction

Let a smooth manifold $M$ be equipped with a smoothly varying positive definite quadratic form on a distribution-subbundle $V$ of the tangent bundle $TM$. One assumes that $V$ is completely nonintegrable, equivalently all sections of $V$ together with all brackets between them generate $TM$ at each point of $M$. Then by the well-known theorem of Rashevski–Chow [3] it is possible to connect any two points (in a connected component of $M$) by a piecewise smooth curve which is tangent to $V$ playing the role of nonholonomic constraint [15]. Having any Riemannian metric on $M$, we can endow $M$ with a metric $d_V$ defined to be the infimum of the lengths of all such curves joining two points. This metric is called the Carnot–Carathéodory metric [9] or sub-Riemannian metric. It is expressed by a contravariant metric tensor $g^{ij}(x) : T^*M \to TM$, which is nonnegative definite and its image defines the distribution $V$.

Variational problems with nonholonomic constraints offer questions of great mathematical and physical interest [15, 2]. One of such questions concerns the local structure of Exp-map and the generic properties of systems of geodesics rays forming the corresponding wavefronts. The problem of the shortest paths in sub-Riemannian manifold with boundary provides new singular Lagrangian varieties of geodesics analogously to the classification of systems of gliding rays in the Riemannian case (cf. [7]).

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## 2. Nonholonomic differential systems

Let $M$ be a connected differentiable manifold, $\dim M = n$. Let $V$ be a smooth $m$-dimensional distribution on $M$ equipped with a Riemannian metric $\langle , \rangle$. A continuous $C^2$-curve $c : [a, b] \to M$ will be called *horizontal* if $dc/dt(t) \in V(c(t))$ for almost all $t \in [a, b]$.

Let $X_1, \ldots, X_m$ be a local basis of vector fields generating the distribution $V$ near $q \in M$. $V$ is said to satisfy the *Hörmander condition* at $q$ if these vector fields together with all their commutators span $T_qM$. If this condition is fulfilled at every point of $M$, then $V$ is called *nonholonomic distribution* [15]. Let $C_{x,q}$ be the set of all horizontal curves joining $x$ and $q$. We can define the Carnot–Carathéodory (cf. [9]) distance function between the points $x$ and $q$ by $d_V(x, q) = \inf\{L(c); c \in C_{x,q}\}$. For the horizontal curves, the length and energy are defined as follows

$$L(c) = \int_I \langle \dot{c}(t), \dot{c}(t)\rangle^{1/2} dt, \qquad E(c) = \frac{1}{2} \int_I \langle \dot{c}(t), \dot{c}(t)\rangle dt.$$

If $V$ is a nonholonomic distribution, then we know [3] that $d_V$ is a finite metric on $M$. In what follows, the pair $(M, d_V)$ we will call also the sub-Riemannian space [12]. An alternative way to obtain the sub-Riemannian structures is given by means of the symmetric positive semi-definite bilinear form $\langle , \rangle_q$ on the cotangent bundle $T^*M$ depending smoothly on the base point. Let $h_g : T^*M \to TM$ be the vector bundle homomorphism: $T_q^*M \ni \xi \to v \in T_qM$, where for the unique $v$ we have $\langle \xi, \eta \rangle_g = \eta(v)$, for all $\eta \in T_q^*M$. If $h_g$ is a constant rank map, then $V(q) = h_g(T_q^*M) \subset T_qM$ forms a smooth distribution. The corresponding Riemannian metric on this distribution is defined by $\langle v, u \rangle = \langle h_g\xi, h_g\eta \rangle_g = \langle \xi, \eta \rangle_g$. In local coordinates, $\langle \cdot, \cdot \rangle_g$ is defined by the tensor $(g^{ij})$ and the corresponding sub-Riemannian geodesics are described by the Hamiltonian equations $\dot{q}^i = \frac{\partial H}{\partial p_i}(q, p)$, $\dot{p}_j = -\frac{\partial H}{\partial q^j}(q, p)$, $1 \leq i, j \leq n$, where $H(q, p) = \frac{1}{2}\sum_{ij} g^{ij}(q)p_ip_j$. The only difference from the Riemannian case is that $h_g$ may have a nontrivial kernel, so that some different initial covectors which give rise to different geodesics will be mapped to the same vector.

The space $X$ of sub-Riemannian geodesics on $M$ is constructed by the cylindrical subbundle of $T^*M$ defined by the coisotropic hypersurface $W = H^{-1}(1/2)$. By the canonical symplectic reduction procedure we have the symplectic space of geodesics $X$ equipped with the canonical symplectic form $\nu$. Let $I$ be a submanifold of $X$. If $I$ is isotropic for $\nu$, i.e. $\nu|_I = 0$, then $I$ is called a *system of rays* in $(X, \nu)$. An especially distinguished system of rays defining the corresponding wave-front in $M$ is formed by a Lagrangian, i.e. maximal isotropic submanifold of $(X, \nu)$.

Now we introduce the symplectic formulation of nonholonomic systems. We consider the symplectic manifold $(T^*M, \omega_M)$. Then the tangent bundle $T(T^*M, \omega_M)$ is isomorphic to the cotangent bundle $T^*(T^*M, \omega_M)$. The isomorphism is defined by the vector bundle morphism $\beta : T(T^*M, \omega_M) \to T^*(T^*M, \omega_M)$, $\beta(u) = u \rfloor \omega_M$. Now we can pull-back all objects from $T^*(T^*M)$ to $T(T^*M)$. In this way we obtain the canonical symplectic structure on $T(T^*M)$, namely $\hat{\omega} = \beta^*\omega_{T^*M} = d(\beta^*\theta_{T^*M})$, where

$\theta_{T^*M}$ is the Liouville form. The symplectic manifold $(T(T^*M), \hat{\omega})$ is the underlying symplectic manifold of two special Lagrangian fiberings on $T(T^*M)$,

$$\mathcal{H} = (T(T^*M), T^*M, \tau_{T^*M}, \kappa), \quad \text{and} \quad \mathcal{L} = (T(T^*M), TM, T\pi_M, \mu),$$

where $\tau_{T^*M}$ is the tangent bundle projection, $\kappa$ and $\mu$ denote the corresponding 1-forms on $T(T^*M)$ such that $d\mu = d\kappa = \hat{\omega}$. *Differential system* $D$ is defined as a Lagrangian submanifold of the phase space $(T(T^*M), \hat{\omega})$. Representation of the differential system $D$ by generating families in $\mathcal{H}$ and $\mathcal{L}$ gives the Hamiltonian and Lagrangian formulations of the system. The Legendre transformation from the $\mathcal{L}$ to $\mathcal{H}$ representations of $D$ is a symplectomorphism $\alpha : \mathcal{L} \to \mathcal{H}$ whose graph in the product symplectic space [8] $(\mathcal{L} \times \mathcal{H}, \pi_2^*\kappa - \pi_1^*\mu)$ is generated by the function $L$ defined on the Whitney sum $TM \times_M T^*M$, $L(v,p) = -\langle p, v\rangle$.

Let $V$ denote a nonholonomic distribution of dimension $n-k$ on $M$ defined by the basic (annihilating) 1-forms $\omega_1, \dots, \omega_k$ of the corresponding codistribution in $T^*M$. Let $\hat{L} : V \to \mathbb{R}$ be a positive definite quadratic form on $V$ and $L : TM \to \mathbb{R}$ denotes its extension on $TM$.

DEFINITION 2.1. *Nonholonomic differential system on $M$* is defined as a constrained Lagrangian submanifold of $(T(T^*M), \hat{\omega})$ over $V$ with a generating function $\hat{L}$ in the special symplectic structure $\mathcal{L}$. We denote this system by $N_V \subset T(T^*M)$.

Let $L$ be an extension of $\hat{L}$, then we can write

$$\begin{aligned} N_V = \{p \in T(T^*M) : T\pi_M(p) \in V \quad &\text{and} \quad \langle \alpha(p), v\rangle = \langle dL, v\rangle \\ \text{for all } v \in TV \subset T(TM), \text{ such that } &T\pi_M(p) = \tau_{TM}(v)\}, \end{aligned}$$

where $\alpha : T(T^*M) \to T^*(TM)$ is a uniquely defined symplectomorphism defining the special symplectic structure $\mathcal{L}$, i.e. $\pi_{TM} \circ \alpha = T\pi_M$ and $\alpha^*\theta_{TM} = \mu$.

Now we write these formulae in local coordinates $(x^i)$ on $M$, $(x^i, y_j)$ on $T^*M$, $(x^i, \dot{x}^j)$ on $TM$, $(x^i, y_j, \dot{x}^k, \dot{y}_l)$ on $T(T^*M)$ such that $\theta_M = \sum_i y_i dx^i$. Then in local coordinates we have

$$\mu = \sum_i (\dot{y}_i dx^i + y_i d\dot{x}^i), \qquad \kappa = \sum_i (\dot{y}_i dx^i - \dot{x}^i dy_i).$$

The Lagrangian function $L$ is defined by some Riemannian structure on $M$, $L(x, \dot{x}) = \sum_{ij} g_{ij}(x)\dot{x}^i\dot{x}^j$, which is an extension of $\hat{L}$ to a Riemannian metric on $M$. Thus the generating family for the nonholonomic differential system has the form

$$\mathcal{L}(x, \dot{x}, \lambda) = \sum_{ij} g_{ij}(x)\dot{x}^i\dot{x}^j + \sum_{r=1}^{k} \lambda_r \langle \omega_r, \dot{x}\rangle,$$

where $\lambda_r$ are the corresponding Lagrange multipliers—Morse parameters. $N_V \subset T(T^*M)$ is described by the equations

$$y_i = \frac{\partial L}{\partial \dot{x}^i} + \sum_r \lambda_r \frac{\partial}{\partial \dot{x}^i}\langle \omega_r(x), \dot{x}\rangle, \quad \dot{y}_i = \frac{\partial L}{\partial x^i} + \sum_r \lambda_r \frac{\partial}{\partial x^i}\langle \omega_r(x), \dot{x}\rangle,$$

where $1 \le i, j \le n$, $1 \le r \le k$. We easily see that the above introduced system is equivalent to the system of Euler–Lagrange equations describing the nonholonomic geodesics

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^j}\right) - \frac{\partial L}{\partial x^j} = -\sum_r \lambda_r \omega_{rj} - \sum_r \lambda_r (\dot{x} \rfloor d\omega_r)_j.$$

By the Legendre transformation we pass from the $\mathcal{L}$-representation of $N_V$ to its Hamiltonian representation in $\mathcal{H}$. Now, $N_V$ is generated by the function (Hamiltonian) $H : T^*M \to \mathbb{R}$ in the following form

$$H(x^i, y_j) = \sum_i y_i \dot{x}^i - \mathcal{L}(x, \dot{x}, \lambda)\,|_{y_j = \frac{\partial \mathcal{L}}{\partial \dot{x}^j}(x,\dot{x},\lambda), 0 = \frac{\partial \mathcal{L}}{\partial \lambda}(x,\dot{x},\lambda)},$$

and we can write the identification on $N_V$, $\sum_i (\dot{y}_j dx^i - \dot{x}^i dy_i) = -dH(x, y)$.

EXAMPLE 2.1. Consider the Heisenberg group $H^n = \mathbb{R} \times \mathbb{C}^n$ endowed with the nonholonomic distribution $V$ (cf. [10]), $\omega = dz + \sum_{i=1}^{2n} A_i(x)dx^i = 0$, spanned by the system of generating vector fields $\{X_i = \frac{\partial}{\partial x^i} - A_i(x)\frac{\partial}{\partial z}\}$. The positive quadratic form on $V$ we choose in the simplest form

$$\hat{L}(x, \dot{x}) = \sum_{ij} g_{ij}(x)\dot{x}^i \dot{x}^j, \qquad g_{ij} = \delta_{ij}.$$

$V$ is transversal to the $z$-direction, so we can parametrize the leafs of $V$ by $(x, \dot{x})$. Thus the corresponding Lagrangian has the form

$$\mathcal{L}(x, z, \dot{x}, \dot{z}, \lambda) = \sum_{ij} g_{ij}(x)\dot{x}^i \dot{x}^j + \lambda\Big(\dot{z} + \sum_i A_i(x)\dot{x}^i\Big).$$

Now we derive the system of Hamiltonian equations defining the sub-Riemannian geodesics. At first we have the corresponding Morse family

$$\hat{H}(x, z, y, w, \dot{x}, \dot{z}, \lambda) = \sum_i y_i \dot{x}^i + w\dot{z} - \sum_{ij} g_{ij}(x)\dot{x}^i \dot{x}^j - \lambda\Big(\dot{z} + \sum_i A_i(x)\dot{x}^i\Big),$$

and the defining equations $\frac{\partial \hat{H}}{\partial \dot{x}^i} = y_i - 2\sum_j g_{ij}(x)\dot{x}^j - \lambda A_i(x) = 0$, and

$$\frac{\partial \hat{H}}{\partial \lambda} = -\dot{z} - \sum_i A_i(x)\dot{x}^i = 0, \quad \frac{\partial \hat{H}}{\partial \dot{z}} = w - \lambda = 0.$$

Thus, after elimination of variables, we obtain

$$\begin{aligned} H(x, z, y, w) &= \tfrac{1}{2}\sum_{ij}(y_i - wA_i(x))g^{ij}(y_j - wA_j(x)) \\ &\quad -\tfrac{1}{4}\sum_{ijkl} g_{ij}g^{ik}(y_k - wA_k(x)g^{jl}(y_l - wA_l(x)) \\ &= \tfrac{1}{4}\sum_{ij} g^{ij}(y_i - wA_i(x))(y_j - wA_j(x)), \end{aligned}$$

which gives the general form of sub-Riemannian Hamiltonian in the case of codimension one distribution.

In the more general case we obtain the following formulae.

THEOREM 2.1. *Let* $(g_{ij})$ *be a Riemannian structure on* $M$, *and*

$$\{\omega_r = \sum_{i=1}^{n} A_i^r(x)dx^i = 0\}_{r=1}^{k}$$

*define a nonholonomic distribution on* $M$. *Then the corresponding sub-Riemannian Hamiltonian has the form*

$$H(x,y) = \frac{1}{2}\sum_{ij} g^{ij}\Big(y_i + \sum_{r=1}^{k} \lambda_r A_i^r(x)\Big)\Big(y_j + \sum_{r=1}^{k} A_j^r(x)\Big),$$

*where the parameters* $\lambda$ *are determined by the system of linear algebraic equations*

$$\sum_{u,i=1}^{n} g^{iu} A_u^s(x)\Big(y_i + \sum_{r=1}^{k} \lambda_r A_i^r(x)\Big) = 0, \quad s = 1,\ldots,k.$$

## 3. Lagrange projections with nonisolated singularities

Let $L$ be a Lagrangian submanifold of $(T^*M, \omega_M)$ and let $\pi_M : T^*M \to M$ be the canonical projection. By $\rho_L = \pi_X|_L : L \to M$ we denote the Lagrange projection.

DEFINITION 3.1. We say that the Lagrange projection $\rho_L$ has *a nonisolated singularity at* $q_0 \in M$ if there exists a (compact) submanifold $N \subset T^*_{q_0}M$ and a neighbourhood $U$ of $N$ in $T^*M$ such that $\rho_L^{-1}(q_0) = N$ and $\rho_L|_{U-N}$ is finite to one.

Locally all Lagrangian submanifolds of $T^*M$ can be generated by Morse families (cf. [14]). In the case of nonisolated singularities we have the following result.

PROPOSITION 3.1. *Let* $(L, p_0)$ *be a germ of Lagrangian submanifold of* $T^*M$ *with* $\rho_L$ *having a nonisolated singularity at* $q_0 = \pi_M(p_0)$ *along the submanifold* $N^r$. *Then* $(L, p_0)$ *is symplectically equivalent (preserving the fibering* $\pi_M$*) to* $(\tilde{L}, 0)$ *with the generating family* $F : M \times \mathbb{R}^n \to \mathbb{R}$,

$$F(q,\lambda) = \sum_{ij>r}^{n} a_{ij}(\lambda)\lambda_i\lambda_j + \sum_{i=1}^{n} q_i\lambda_i,$$

*where* $a_{ij}(\lambda)$ *are smooth functions on* $(\mathbb{R}^n, 0)$.

Let $\Phi_t : T^*M \to T^*M$ be a Hamiltonian flow. We define the Exp-map at $q \in M$,

$$\mathrm{Exp}_q = \Phi_1|_{T_q^*M} : T_q^*M \to T^*M \xrightarrow{\pi_M} M.$$

It is the Lagrange immersion of $T_q^*M$ into $T^*M$. Let $H$ be a sub-Riemannian Hamiltonian. Then the germ $(L_q = \mathrm{Exp}_q(T_q^*M), \bar{p})$, $\bar{p} \in \mathrm{Ker}\, h_g|_{T_q^*M}$ is Lagrange equivalent to the Lagrange projection with nonisolated singularities along the space $\mathrm{Ker}\, h_g|_{T_q^*M} \subset T_q^*M$.

Let $H_r(T^*M)$ denote the space of corank $r$ of sub-Riemannian vector fields on $T^*M$. Let $W$ denote a subbundle of $T^*M$, $\dim W_q = n - r$, with $W_q$ transversal to $\mathrm{Ker}\, h_g|_{T_q^*M}$ at each $q \in M$. We define $\exp|_W^s : H_r(T^*M) \times W \to J^s(W, T^*M)$, the $s$-jet extension of $\Phi_1$ restricted to $W$. By the transversality theorem we obtain the analogue of the standard genericity theorem (in the Riemannian case [13] ) for $\exp|_W$ in sub-Riemannian geometry. By $(I_q, \bar{p})$ we denote the germ of an isotropic submanifold $\Phi_1|_W(W_q) \subset T^*M$.

THEOREM 3.1 (cf. [6]). *We assume that $\bar{p}$ does not belong to the zero section of $T^*M$. Then, generically, the family of germs of isotropic projections $(I_q, \bar{p})$, $\pi_M|_{I_q} : I_q \to M$ has only generic singularities appearing in $n$-parameter families of the isotropic submanifolds in $T^*M$.*

EXAMPLE 3.1. We consider the exponential map for sub-Riemannian Heisenberg group $X = H^3$ endowed with the distribution $\{dz + \frac{1}{2}(ydx - xdy) = 0\}$. Then $\rho_L = \pi_X \circ \mathrm{Exp}|_{T^*X}$ has a nonisolated singularity along $N = \{(p_1, p_2, p_3) \in T_0^*X; p_1 = 0, p_2 = 0\}$. The generating family for $L$ has the following form

$$F(x, y, z, \lambda)$$
$$= x\lambda_1 \cos\lambda_3 + y\lambda_2 \cos\lambda_3 + z\lambda_3 - \tfrac{1}{2}(\lambda_1^2 + \lambda_2^2)\frac{\sin 2\lambda_3}{2\lambda_3} + \lambda_1\lambda_2 \frac{\cos\lambda_3(\cos\lambda_3 - 1)}{\lambda_3}.$$

The corresponding caustic of $L$ is a family of rotationally invariant paraboloids $z = (x^2 + y^2)\frac{2\delta - \sin 2\delta}{4(1 - \cos 2\delta)}$, where $\delta$ is a solution of the equation $\mathrm{tg}\,\delta = \delta$.

EXAMPLE 3.2. Consider the rotationally invariant Lagrangian submanifold of $T^*\mathbb{R}^2$, generated by the stable family

$$F(q, \lambda) = (\lambda_1^2 + \lambda_2^2)^2 - \alpha(\lambda_1^2 + \lambda_2^2) + q_1\lambda_1 + q_2\lambda_2, \qquad \alpha > 0.$$

We see that $\rho_L$ has a nonisolated singularity at 0, and $N = \{(p_1, p_2) \in T_0^*\mathbb{R}^2\colon p_1^2 + p_2^2 = \frac{\alpha}{2}\}$,

$$\rho_L(p_1, p_2) = (-2p_1(2(p_1^2 + p_2^2) - \alpha), -2p_2(2(p_1^2 + p_2^2) - \alpha)).$$

The caustics of this projection $C_L$ has a highly degenerate point $q = 0$ and the circle $q_1^2 + q_2^2 = \left(\frac{2\alpha}{3}\right)^3$.

## 4. Local properties of the horizontal curves

Now we work with the Heisenberg group $H^3$ endowed with the contact distribution $\mathcal{D} = \{dz - xdy = 0\}$, spanned by $X_1 = \frac{\partial}{\partial x}$, $X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$. We consider the horizontal curves $\beta : t \to (x(t), y(t), z(t))$, transversal to the plane $\{x = 0\}$. $\beta$ is a solution of the system of differential equations $\dot{x} = 1$, $\dot{z} = x\dot{y}$. We integrate these equations and get $x(t) = t$, $y(t) = \varphi(t)$, $z(t) = \int_0^t s\varphi'(s)\,ds$, $\varphi \in C^\infty(\mathbb{R})$. Using the local Darboux coordinates of [4] we calculate explicitely the families of gliding geodesics. To each tangent vector $v(t) = \frac{\partial}{\partial x} + \varphi'(t)\frac{\partial}{\partial y} + t\varphi'(t)\frac{\partial}{\partial z}$ to the horizontal curve $\beta$ there corresponds a one-parameter family of local geodesics

$$x(\tau) = \tau + t, \quad y(\tau) = \varphi(t) + \frac{1}{p_3}[(1 - (p_2 + tp_3)^2)^{1/2} - (1 - (p_2 + (t+\tau)p_3)^2)^{1/2}],$$

$$\begin{aligned} z(\tau) = &\int_0^t s\varphi'(s)ds + \frac{t}{p_3}(1 - (p_2 + tp_3)^2)^{1/2} - \frac{(t+\tau)}{p_3}(1 - (p_2 + (t+\tau)p_3)^2)^{1/2} \\ &- \frac{1}{2p_3^2}(\arcsin(p_2 + tp_3) + (p_2 + tp_3)(1 - (p_2 + tp_3)^2)^{1/2}) \\ &+ \frac{1}{2p_3^2}(\arcsin(p_2 + (t+\tau)p_3) + (p_2 + (t+\tau)p_3)(1 - (p_2 + (t+\tau)p_3)^2)^{1/2}, \end{aligned}$$

where $(p_2, p_3, t)$ satisfy the following equation

$$\varphi'(t) = \frac{(p_2 + tp_3)}{(1 - (p_2 + tp_3)^2)^{1/2}}.$$

PROPOSITION 4.1. *The system of geodesics gliding along the horizontal curve $\beta$ in the sub-Riemannian space $(M, N_V)$ is an isotropic variety $I_\beta$ of the symplectic space of geodesics $(X, \nu)$.*

To prove this result we need to show that $(\omega_M|_{H^{-1}(0)})\,|_{I_\beta} = 0$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be the Morse parameters in a generating family of $N_V$, and let $(t, \lambda) \to (\phi^i(t), \psi_j(t, \lambda))$ be a parametrization of the space of geodesics along the horizontal curve $t \to (\phi^i(t))$. Now we have to check that all Lagrange brackets $\{t, \lambda_i\}$, $\{\lambda_i, \lambda_j\}$ vanish. Indeed, $\{t, \lambda_j\} = \sum_{i=1}^n \dot{x}^i \frac{\partial p_i}{\partial \lambda_j}$ because for the gliding geodesics the tangent $(\dot{x}^i)$ to the geodesics at $\tau = 0$ is equal to the tangent to the horizontal curve $\beta$ at $t$. However, $(\dot{x}^i)$ fulfills the Hamilton's equations $\dot{x}^i = \partial H/\partial p_i$, so

$$\{t, \lambda_j\} = \sum_{i=1}^n \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \lambda_j} = \frac{d}{d\lambda_j}(H(x(t), p(\lambda, t))) = 0.$$

But $\phi^i$ does not depend on $\lambda$, so we have immediately that $\{\lambda_i, \lambda_j\} = 0$, which proves the above result.

Let $\gamma$ be a horizontal curve in $(H^3, \mathcal{D})$. We see that $\gamma$ is uniquely defined by its projection plane curve $\hat{\gamma}$ on the $xy$-plane and an initial point at $t = 0$. Now we seek for the generic properties of the horizontal curves. Let $T(t)$ be a unit tangent vector to $\hat{\gamma}(t)$, and $V(t)$ its perpendicular at $\hat{\gamma}(t)$. Now we use the perpendicular lines spanned by $T(t)$ and $V(t)$ as axes at $\hat{\gamma}(t)$. Clearly at each point $\hat{\gamma}(t)$ we can locally write $\hat{\gamma}(I)$ as the graph $(\xi, f_t(\xi))$, with $f_t(\xi)$ smooth and $j^1 f_t(0) = 0$. In coordinates with the axes $T(t)$, $V(t)$ and $z$ defined at $\gamma(t)$, $\gamma(I)$ is locally represented as a graph

$(\xi, f_t(\xi), g_t(\xi)) \in H$, where $g_t(\xi)$ is uniquely defined by $f_t(\xi)$ and the coordinate system. Let $P_k$ denote the space of polynomials in $\xi$ of degree $\geq 2$ and $\leq k$. Then we have a Monge–Taylor map $\mu_\gamma$ for the horizontal curve $\gamma$, $\mu_\gamma : I \to P_k$, given by $\mu_\gamma(t) = (j^k f_t)(0)$. Next we formulate the transversality result using the space $Z_k$ of the polynomial maps $H \to H$ of degree $\leq k$ and preserving the distribution $\mathcal{D}$. A deformation of $\gamma$ is obtained by composing with a polynomial map $\Phi : H \to H$, which is a diffeomorphism on some open set containing $\gamma(I)$. We assume $I = S^1$, and let us choose an open neighbourhood $U$ of id $\in Z_k$ consisting of polynomial maps which map an open set, containing $\gamma(S^1)$, diffeomorphically to its horizontal image. Thus we have a smooth map $\mu : S^1 \times U \to P_k$, which is a Monge–Taylor map for the curve $\Phi \circ \gamma$.

PROPOSITION 4.2. *There exists an open neighbourhood $U_1 \subset U$ of* id $\in Z_k$ *such that the map $\mu : S^1 \times U \to P_k$ is a submersion.*

We endow the space of smooth horizontal curves $\gamma$ with the $C^\infty$-Whitney's topology.

COROLLARY 4.1. *An open and dense set of regular horizontal curves $\gamma : S^1 \to H$ contains those curves whose projections $\hat{\gamma}$ have only finitely many ordinary inflections and vertices.*

Indeed, let $Q$ be a manifold in $P_k = R^{k-1}$. By the Proposition 4.2 for some open set $U_1 \ni$ id, the map $\mu : S^1 \times U_1 \to P_k$ is transverse to $Q$. Taking $f_t(\xi) = a_2\xi^2 + a_3\xi^3 + \ldots + a_k\xi^k \in P_k$ and $Q_1 = \{a_2 = 0\}$ or $Q_2 = \{a_3 = 0\}$, we obtain the result.

*Remark* 4.1: There is a question: how does it look like, generically, the variety of gliding rays along the horizontal curve? Is it singular or smooth only? We conjecture that for the generic horizontal curve in the Heisenberg group the Lagrangian variety of gliding rays has only open Whitney's umbrella singularities in isolated points.

REFERENCES

[1] V. I. Arnold: *Functional Anal. Appl.* **15** (1981), 235.
[2] R. W. Brockett: Control theory and singular Riemannian geometry, in *New Directions in Applied Mathematics*, eds. P. J. Hilton and G. S. Young, Springer, Berlin, 1981.
[3] W. L. Chow: *Math. Ann.* **117** (1939), 98.
[4] S. Janeczko: *Banach Center Publ.* **32** (1995), 245.
[5] S. Janeczko: *Ann. Inst. H. Poincaré* **46** (1987), 1.
[6] S. Janeczko and T. Mostowski: *Compositio Mathematica* **96** (1995), 345.
[7] R. B. Melrose: *Commun. Pure Appl. Math.* **33** (1980), 461.
[8] M. R. Menzio and W. M. Tulczyjew: *Ann. Inst. H. Poincaré* **28** (1978), 349.
[9] J. Mitchell: *J. Diff. Geom.* **21** (1985), 35.
[10] A. I. Nachman: *Commun. Partial Diff. Equations* 7(6) (1982), 675.
[11] C. B. Rayner: *Phil. Trans. Royal Soc. London* **A262** (1967), 299.
[12] R. S. Strichartz: *J. Diff. Geometry* **24** (1986), 221.
[13] C. T. C. Wall: *Springer Lecture Notes in Math.* **597** (1976), 707.
[14] A. Weinstein: *Lectures on Symplectic Manifolds*, C.B.M.S. Conf. Series, A.M.S. **29** (1977).
[15] A. M. Vershik and V. Ya Gershkovich: Non-holonomic Riemannian manifolds, in: *Dynamical Systems* **7**, *Mathematical Encyclopaedia*, Vol. **16**, Mir, Moscow 1987.