PHASE TRANSITIONS IN FERROMAGNETS AND SINGULARITIES

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This paper is devoted to the investigation of critical phenomena from the mathematical point of view. An interesting question in this field is universality (i.e. independence of the type of material and, to some extent, of the type of process) of the so-called critical exponents. We try to show that one can compute the critical exponents using structural stability arguments. The universality is then implied automatically as a consequence of similarity of models and of the structure of stable singularities. The considerations lead to some new results: information about the generic shape of hysteresis loops and universal form of the equation of state in the critical region.

Introduction

Critical phenomena in a homogenous and isotropic ferromagnet are similar, in a sense, to those in a gas. This paper provides a ferromagnetic version of the ideas introduced in [3], where the singularity theory (Arnold, Thom) was used as the main mathematical tool for a description of gas–liquid phase transitions. However, there are also differences between these two thermodynamical systems. Magnetic experiments exhibit the symmetry with respect to the simultaneous change of direction of magnetization M and external magnetic field H. Moreover, the hystereses for ferromagnets have quite different shape from those formed by oversaturation and overheating processes for gas–liquid systems. These and few other features cause that the present translation of [3] to the ferromagnetic case does not consist in a mere replacement of volume V and pressure P by $-M$ and H as it is indicated by the second law of thermodynamics. However, to make the paper self-contained, we give in extenso all necessary definitions, hypotheses and comments even if they are direct metaphrases of the respective parts of [3].

The main points of our approach are as follows:

1° We view the set of all thermodynamical states of a ferromagnet as consisting of two subsets: the overcritical states, i.e. those for temperatures T greater than the critical temperature t_c, and those for $T \leqslant t_c$ which we call undercritical. This paper deals mainly with the undercritical region: our ideas how to describe the overcritical region and how to combine the descriptions of both regions are sketched in Remark 16.

2° For each temperature $T < t_c$ we observe a hysteresis, i.e. two processes of remagnetization: the first called *remagnetization upward*, is realized by increase of H from $-\infty$ to $+\infty$ (in practice we do not go beyond the saturation magnetization), and the second called *remagnetization downward*, is realized by decrease of H from $+\infty$ to $-\infty$.

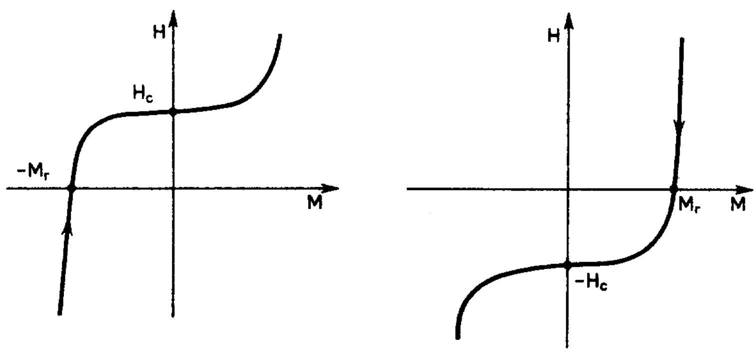

Fig. 1

3° The shape and the size of hysteresis loop of a given ferromagnetic sample at fixed $T < t_c$, may vary while the sample is subject, for instance, to a plastic treatment. Such or other technological treatments can modify friction between magnetic domains as well as influence their growth. From the macroscopic point of view this enables us to get a new sample the hysteresis loop of which is much narrower and almost rectangular: cf. Figs. 1 and 2.

Besides a single ferromagnet we shall consider a family of samples labelled by a parameter z, made of the same material but with different properties of magnetic domain structure. Namely, decrease of z corresponds to narrower and narrower hysteresis loops. Thus, for a fixed $T < t_c$, the coercivity H_c and the residual magnetization M_r are functions of z and $H_c \downarrow 0$ as $z \downarrow 0$.

4° Let us consider a ferromagnet ($z = \text{const} > 0$) in isothermal conditions $T < t_c$. Because of the hysteresis phenomenon, one thermodynamic variable, e.g. M, does not suffice to determine uniquely a state of the system. Such uniqueness is achieved if we consider only the states obtainable in the processes of remagnetiz-

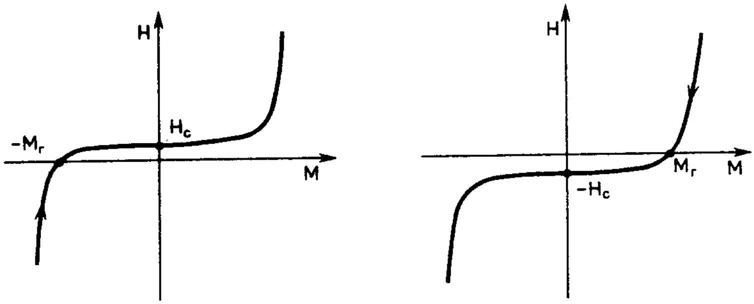

Fig. 2

ation upward and downward,[1] and if we add to M the information to which of those *two* kinds of processes the state we want to determine belongs: cf. Fig. 3. Such considerations result in the postulate (Hypothesis I) to model the space of undercritical states of the ferromagnet by the space W consisting of *two* copies of a half-plane ($M \in \boldsymbol{R}$ and $T < t_c$) glued along their boundaries, i.e. the critical isotherm

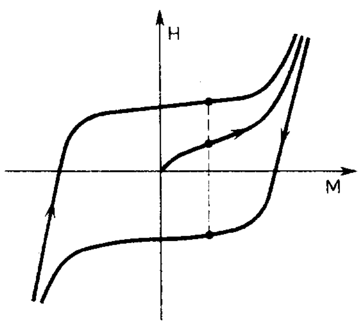

Fig. 3

($M \in \boldsymbol{R}$ and $T = t_c$). In other words $W \cong \boldsymbol{R}^2$ but we do not take $W = \boldsymbol{R}^2$ to avoid the canonical coordinates on $\boldsymbol{R}^2$. The two half-planes of W, separated by a curve L which is to play in future the role of critical isotherm, are distinguished by a two-valued index $\varepsilon = \pm 1$. We parametrize W by two coordinates x, y directly related to magnetization M and (total) energy E: see (6) and Fig. 4.

5° The magnetic field H, the next, after M and E, thermodynamic function appearing in our model, is introduced with two hypotheses. First, we assume that H restricted to the critical isotherm L is infinitesimally cubic at the critical point (Hypothesis II). Then, we require a stability of this feature, namely, the infinitesimal cubicity of H restricted to all curves obtained from L by small perturbations (Hypothesis III). As a result we get a general infinitesimal form of H

[1] We exclude other processes, e.g. the magnetization of a neutral sample, which starts from $H = 0 = M$ and proceeds with $H \uparrow +\infty$: see Fig. 3.

near the critical point $0 \in W$. Further constrains are imposed on H by next postulates.

6° Similar considerations apply to the temperature T. Its infinitesimal cubicity while restricted to the isoenergetic curve $E = e_c$ (e_c – the critical energy) is a consequence of 5° and the Maxwell identity

$$\frac{\partial}{\partial M}\frac{1}{T} = -\frac{\partial}{\partial E}\frac{H}{T}. \tag{1}$$

As above, we assume this cubicity to be stable with respect to small perturbations of the curve $E = e_c$ (Hypothesis III′). Then a general infinitesimal form of T near the critical point is deduced.

7° We require the following symmetry property of our model: the state characterized by a triple (M, H, T) is attainable in an isothermal remagnetization *upward* process iff the state characterized by the triple $(-M, -H, T)$ is attainable in an analogous downward process: see Fig. 1. We look at this symmetry as a condition to be satisfied by H and T as functions of (M, E). In our approach both of them are somehow determined by smooth functions η and ξ. We are interested in establishing which singularity types of η and ξ are *stable* among all η's and ξ's compatible with the symmetry condition as well as with the previous hypotheses. We find the answer

$$\eta \in J_{10}, \quad \xi \in D_4^+$$

in Arnold's notation [2].[2]

8° This enables us to show that for temperature $t \nearrow t_c$ the residual magnetization

$$M_r(t) \sim (1 - t/t_c)^{2/3}$$

and the magnetic susceptibility for $H = 0$

$$\chi(0, t) \sim (1 - t/t_c)^{-4/3}.$$

In other words, within our approach we have the critical exponents

$$\beta = 2/3 \quad \text{and} \quad \gamma' = 4/3.$$

An attempt to explain the only partial agreement between our critical exponents and the mostly accepted experimental data $\beta = 0.36 - 0.39$, $\gamma' = 1.2 - 1.36$ [7], is given in Remark 14.

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[2] This means that

$$\eta = u^3 \pm v^6, \quad \xi = u^3 + v^3$$

if for each of them appropriate coordinates (u, v) are chosen.

§ 1. The space of thermodynamic undercritical states

Let Σ be the space of thermodynamic states in a phenomenological theory of ferromagnetism and $T\colon \Sigma \to \boldsymbol{R}$ the temperature within this theory. If t_c is the critical temperature then Σ_- (resp. Σ_+) consists of those points for which $T \leqslant t_c$ (resp. $T > t_c$).

The classical thermodynamics of ferromagnets can be seen as a theory for which $\Sigma = \boldsymbol{R}^2$ and the points of Σ are physically interpreted by two functions: magnetization and temperature defined as follows:

$$\Sigma \ni (x, y) \to M(x, y) := x \in \boldsymbol{R},$$

$$\Sigma \ni (x, y) \to T(x, y) := t_c + y \in \boldsymbol{R}.$$

Then Σ_- and Σ_+ are, respectively, a closed and an open half-plane in $\boldsymbol{R}^2$.

Presentation of our model starts with a new definition and an interpretation of the space Σ_-; Σ_+ will be discussed in Remark 16. Let W be a real two-dimensiomal vector space and W_0 an open half-plane in W. Then $L := \bar{W}_0 \cap -\bar{W}_0$ is a line in W. We define

$$\varepsilon(w) := \begin{cases} 1, & w \in W_0, \\ -1, & w \in -W_0, \\ \pm 1, & w \in L; \end{cases} \tag{2}$$

$\varepsilon(w)$ can be seen as a function on W, which is double-valued on L.

HYPOTHESIS I. The *undercritical thermodynamic space of a ferromagnetic sample* is the pair (Σ_-, ε), where

$$\Sigma_- := W \tag{3}$$

and ε is the (double-valued) function (2) on W.[3] A complete physical interpretation of (Σ_-, ε) consists of two steps:

A. We take a smooth (curvilinear) coordinate system $\{x, y\}$ on W, vanishing at zero and such that the line L is symmetric with respect to zero, in these coordinates, i.e.

$$(w \in L) \Rightarrow (\exists\, w' \in L\colon\ x(w') = -x(w),\ y(w') = -y(w)).$$

If (x, y) denotes a unique point in W with coordinates x and y then the symmetry of L means that

$$(x, y) \in L \Leftrightarrow (-x, -y) \in L. \tag{4}$$

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[3] Later on, in § 6, we speak of a ferromagnetic material — in the sense of 3° in Introduction, i.e. a one-parameter family of samples. In that case $\Sigma_- := W \times \boldsymbol{R}_+$ and the extra variable $z \in \boldsymbol{R}_+$ parametrize the family.

We assume also that the y-axis is not tangent to L, i.e. there exists a smooth function f such that

$$L = \{(x, f(x)) \in W\colon\ x \in \mathbf{R}\}. \tag{5}$$

Moreover, without loss of generality, we can restrict ourselves to the case in which L does not intersect the set $xy < 0$.

B. Magnetization M and energy E are defined as:

$$\begin{aligned} M(x, y) &:= -\varepsilon(x, y)x, \\ E(x, y) &:= \varepsilon(x, y)y. \end{aligned} \tag{6}$$

Obviously, M and E are double-valued on L: the coordinates x and y can be viewed as their *smooth* representatives. This is discussed in Remark 1.[4]

Remark 1. Next we introduce further hypotheses. They permit to construct a function T which – if necessary after modifications analogous to the Maxwell construction – is interpreted as the temperature. Let z be fixed, which is the case when we deal with a concrete ferromagnetic sample. The level set $T(\cdot, \cdot, z) = t_c$ is the critical isotherm. By experiments, it does not seem to depend on z, which we are shall assume next. We want L of our model to be the critical isotherm, i.e. $L = \{(x, y) \in W\colon\ T(x, y, z) = t_c\}$.[5] (This can be achieved by an appropriate choice of the arbitrary parameters appearing in the construction of T and or by a diffeomorphically equivalent other choice of the coordinates $\{x, y\}$). Let us answer the question why $M(\cdot, \cdot, z)$ and $E(\cdot, \cdot, z)$ are double-valued on L in the plane W shown in Fig. 4. Whatever the future definition of the temperature T will be, undercritical isotherms (level sets $T(\cdot, \cdot, z) = \text{const} < t_c$) have to be situated somehow along L, e.g. as the dotted and dashed curves in Fig. 4. If we run over

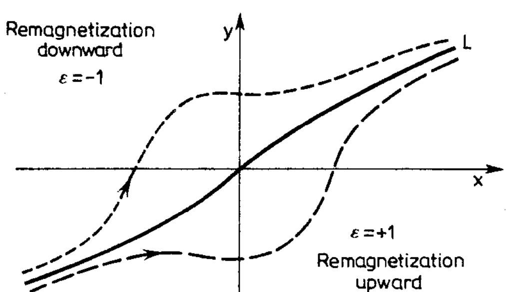

Fig. 4

[4] By (6), x corresponds to $-M$ while in the gas–liquid version of Hypothesis I (see [3]) x corresponds to the volume V. This is justified by the second law of thermodynamics if the external magnetic field H is associated with the pressure P.

[5] Admission of $T(\cdot, \cdot, z) = t_c$ dependent on z does not seem to introduce essential complications.

them from left to right then the magnetization M is decreasing on the dotted isotherm and it is increasing on the dashed one. This is why the domain where $\varepsilon = -1$ (resp. $\varepsilon = +1$) is called a *region of remagnetization downward* (resp. *remagnetization upward*). On the critical isotherm L, if we choose for the double-valued ε the value -1 (resp. $+1$) then the magnetization M decreases (resp. increases) from left to right. These two physical interpretations of L correspond to two different experiments: the critical isothermal remagnetization downward and upward. Now, the symmetry (4) of L becomes a consequence of (6).

Remark 2. Σ_- could be introduced not in the axiomatic but more constructive manner. We start with the classical description of a ferromagnetic sample, mentioned at the beginning of this section. The undercritical region in M, E plane has the shape sketched in Fig. 5. We take two copies of that region. They are shown in Fig. 5 where different directions of remagnetization processes are indicated by arrows. Then, the space Σ_- of our previous axiomatic definition can be obtained if we glue those two copies along L in such a way that the opposite directions on both copies coincide. Such an operation can be done smoothly in the sense of the coordinates M, E provided that L is symmetric with respect to the point $M = 0 = E$.

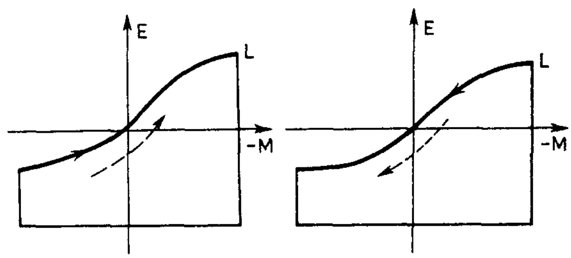

Fig. 5

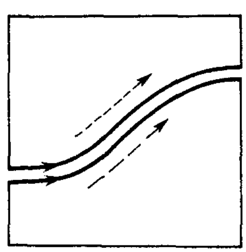
Fig. 6

§ 2. Magnetic field H

According to Remark 1, L is physically interpreted as the critical isotherm. But the interpretation is very different from that which is met in the classical approach.

The critical isothermal remagnetization upward and downward processes are not obtained by running along L in two opposite directions: we get both of them if we run along L from left to right but we have to choose the appropriate values for ε. By (6), magnetization and energy are not uniquely defined on L because ε is double-valued on L.[6] For the same reasons the (external) magnetic field H can be defined on L only up to ε, i.e.

$$H(x, y) := \varepsilon(x, y)\eta(x, y), \tag{7}$$

where η is a smooth function on Σ_-.

We know from experiments that, in the critical isothermal process, H is a monotone function of magnetization and its derivative, i.e. the inverse of susceptibility $(\partial M/\partial H)_T$, vanishes at the critical point (see e.g. [5]) as is shown in Fig. 7. The "simplest" example of such a function is a cubic parabola.[7] This encourages us to introduce the following

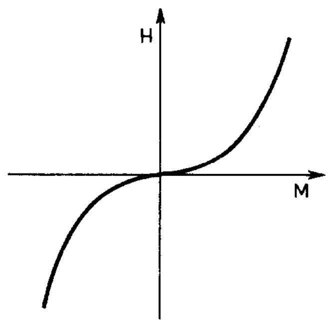

Fig. 7

HYPOTHESIS II. Along the critical isothermal process, the magnetic field is cubic (infinitesimally at the critical point) with respect to magnetization, i.e.

$$\eta(x, f(x)) = ax^3 + O(x^4), \qquad \text{cf. (3)}, \tag{8}$$

where, by (6),

$$a < 0. \tag{9}$$

The assumption of cubicity of magnetic field in the critical isothermal process could be understood as an implicit admission that this feature can be observed

[6] One may ask whether our decision to deal with the coordinates x, y instead of M, E has any justification? Have they more fundamental physical interpretation than M and E, if any? The answer is the subject of the whole paper: see also Remark 16.

[7] Here the term "the simplest" can be understood rigorously as follows: by the previous arguments, the germ at 0 of the function $H \to M(H, t_c)$ must lay in m^3 (see the footnote[9]), where the cubic germs form a unique orbit of codimension zero.

(measured)[8] and therefore it should exhibit a kind of stability with respect to small perturbations of the isotherm L in W. This suggests

HYPOTHESIS III. The function η restricted to a curve being a small zero-preserving smooth deformation of L is also of cubic type, i.e. there exists a neighbourhood O of zero in $C^\infty(\boldsymbol{R})$ endowed with the Whitney topology, such that

$$\eta(x, f(x)+g(x)) = a_g x^3 + O(x^4) \tag{10}$$

if $g \in O$ and $g(0) = 0$; $a_g \in \boldsymbol{R}$.

THEOREM 3. *Hypotheses* II *and* III *imply that*

$$\eta(x, y) = Ax^3 + Bx^2 y + Cxy^2 + Dy^3 \pmod{\mathfrak{m}^4},^9 \tag{11}$$

i.e. $\eta \in \mathfrak{m}^3$, *and*

$$A = a - l(B + Cl + Dl^2), \qquad l := f'(0), \qquad B, C, D \in \boldsymbol{R}. \tag{12}$$

Proof: Hypothesis III excludes from the Taylor expansion of η at zero all monomials of degree lower than 3. (12) results from (8). ∎

§ 3. Temperature

The same arguments as at the beginning of the previous section compel us to look for the temperature T of the form

$$T(x, y) := t_c + \varepsilon(x, y)\,\xi(x, y), \tag{13}$$

where ξ is a smooth function on W and $\xi(0) = 0$.

PROPOSITION 4. *The Maxwell identity* (1) *implies that the derivatives* $\frac{\partial \xi}{\partial x}(0)$, $\frac{\partial^2 \xi}{\partial x^2}(0)$ *vanish and therefore*

$$\xi(\cdot, 0) \in \mathfrak{m}_1^3, \tag{14}$$

where $\mathfrak{m}_1$ *is the maximal ideal in the ring of smooth germs of one variable.*

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[8] We are aware of experimental data $H \sim (\operatorname{sgn} M)|M|^\delta$ for critical isothermal processes, where $4 < \delta < 5$ depends on ferromagnets. A kind of justification for neglecting them is given in [4] and Remark 16.

[9] We use the following standard notation: $\mathscr{E}$ – the ring of germs at 0, of C^∞-functions on W, $\mathfrak{m}$ – its maximal ideal consisting of all germs vanishing at 0, $\mathfrak{m}^k$ – the k-th power of $\mathfrak{m}$. To avoid inessential rigour we speak about functions instead of their germs. (11) means that the functions on both sides are equal up to a function, the germ (at zero) of which belongs to the ideal $\mathfrak{m}^4$.

Proof: By the Maxwell identity and Theorem 3 we have that

$$\frac{\partial \xi}{\partial x} = t_c \frac{\partial \eta}{\partial x} + \varepsilon \left(\xi \frac{\partial \eta}{\partial y} - \eta \frac{\partial \xi}{\partial y} \right) \tag{15}$$

and it belongs to $\mathfrak{m}^2$. ∎

As for η, we assume the cubic type (14) of ξ to be stable.

HYPOTHESIS III'. The function ξ restricted to a curve, which is a small zero-preserving deformation of the curve $y = 0$, is also of cubic type, i.e. there exists a neighbourhood O of zero in $C^\infty(\boldsymbol{R})$ endowed with the Whitney topology, such that

$$\xi(x, g(x)) = b_g x^3 + O(x^4) \tag{16}$$

if $g \in O$ and $g(0) = 0$; $b_g \in \boldsymbol{R}$.

THEOREM 5. *The Maxwell identity and Hypothesis* III' *imply that*

$$\xi(x, y) = t_c [\tfrac{1}{3} Bx^3 + Cx^2 y + 3Dxy^2 + D' y^3] (\operatorname{mod} \mathfrak{m}^4), \tag{17}$$

where B, C, D are as in Theorem 3 *and D' is an arbitrary real number.*

Proof: Hypothesis III' excludes from the Taylor expansion of ξ at zero all monomials of degree lower than 3. The interrelation between coefficients of ξ and η follows from (15).

§ 4. Symmetry

The fundamental symmetry of magnetic phenomena – see 7° of the Introduction – can be expressed in terms of our model as follows: For every $(x_i, y_i) \in \Sigma_-$, $i = 1, 2$ such that:

$$T(x_1, y_1) = T(x_2, y_2), \tag{*}$$

they are attainable in the opposite remagnetization processes, (**)

$$M(x_1, y_1) = -M(x_2, y_2), \tag{*{*}*}$$

one has

$$H(x_1, y_1) = -H(x_2, y_2).$$

PROPOSITION 6. *The above symmetry condition is equivalent to*:

$$\forall_{x, y_1, y_2}\, \xi(x, y_1) = -\xi(x, y_2) \Rightarrow \eta(x, y_1) = \eta(x, y_2). \tag{18}$$

Proof: It is sufficient to show that the conditions (*), (**) and (***) are equivalent to

$$\varepsilon(x_1, y_1) = -\varepsilon(x_2, y_2), \tag{19}$$

$$\xi(x_1, y_1) = -\xi(x_2, y_2), \tag{20}$$

$$x_1 = x_2. \tag{21}$$

Indeed, it follows from (**) that (x_i, y_i) are on the opposite sides of L, i.e. (19). From (*), (13) and (19) we get (20). Similarly, from (***) (6) and (19) we get (21). At last, the fact that (19)–(21) imply (*)–(***) is obvious.

Let us denote

$$\mathscr{S} := \{(\eta, \xi) \in \mathfrak{m}^3 \times \mathfrak{m}^3 \colon \text{(18) is fulfilled}\}. \tag{22}$$

Obviously, if $g(\cdot, \cdot)$ is smooth and cubic in the first variable and we put $\eta(x, y) := g(x, \xi^2(x, y))$ then $(\eta, \xi) \in \mathscr{S}$. The following theorem tells us that this case is stable.

THEOREM 7. *There exists a subset $O \in \mathfrak{m}^3$ which is open in the set of all germs from $\mathfrak{m}^3$ vanishing on L, and such that*

$$\begin{pmatrix} (\eta, \xi) \in \mathscr{S} \\ \xi \in O \end{pmatrix} \Rightarrow \big(\exists_{g \in \mathfrak{m}} \ \eta(x, y) = g(x, \xi^2(x, y)) \big).$$

Proof: It is easily seen that if we put

$$\begin{aligned} \xi_1(x, y) &:= \xi(x, f(x)+y), \\ \eta_1(x, y) &:= \eta(x, f(x)+y), \end{aligned} \tag{23}$$

where f is that of (5), then ξ_1 and η_1 satisfy the assumptions of Theorem 17 given in the Appendix.

Remark 8. From Theorem 17 we get an explicit description of the set O. It consists of such $\xi \in \mathfrak{m}^3 \setminus \mathfrak{m}^\infty$ that:

$$\xi(x, f(x)) = 0,$$

$$\frac{\partial \xi}{\partial y}(x, y) > 0 \quad \text{for } (x, y) \neq 0,$$

the Taylor series $T_0 \xi_1(x, y)$ has no multiple factors in $\boldsymbol{R}[[x, y]]$,

the series $T_0 \xi_1(x, y) + T_0 \xi_1(x, y')$ is irreducible in $\boldsymbol{R}[[x, y, y']]$:

the germ ξ_1 is defined in (23).

Besides the symmetry we want η and ξ to satisfy the Maxwell identity. Let

$$\mathscr{M} := \{(\eta, \xi) \in \mathfrak{m}^3 \times \mathfrak{m}^3 \colon \text{(15) is fulfilled}\}. \tag{24}$$

So, we are interested in $\mathscr{S} \cap \mathscr{M}$.

Remark 9. Some preliminary calculations suggest that the orbit structure of $\mathscr{S}$ is, roughly speaking, the same as that of $\mathscr{S} \cap \mathscr{M}$. Of course, such a statement should be first made precise and then proved; this we leave to another paper. Here we mean that by looking for stable singularities of our η and ξ we can remain in the whole $\mathscr{S}$ instead of $\mathscr{S} \cap \mathscr{M}$, i.e. we can neglect, but not violate, the Maxwell identity. The next is based on this idea.

§ 5. Critical exponents

In this paragraph we find out physical consequences implied by our η and ξ satisfying the symmetry condition (the symmetry of hysteresis). Theorem 7 distinguishes as stable the following case:

$$\begin{gathered} \xi \in O \subset \mathfrak{m}^3 \quad \text{(see Remark 8)}, \\ \eta(x, y) = g(x, \xi^2(x, y)), \end{gathered} \tag{25}$$

where $g \in \mathfrak{m}$, $g(\cdot, 0) \in \mathfrak{m}_1^3$ cf. (14). Therefore we take

$$\begin{aligned} g(x, y) = {} & x^3(a + a_1 x + a_2 x^2 + a_3 x^3 + \text{higher order terms in } x) + \\ & + y(b + b_1 x + b_2 y + \text{higher order terms in } x \text{ and } y) \end{aligned} \tag{26}$$

– here a is the same as that in (8) – and

$$\xi(x, y) = cy^3 + c_1 xy^2 + c_2 x^2 y + c_3 x^3 (\text{mod } \mathfrak{m}^4). \tag{27}$$

PROPOSITION 10. *If* $a, b, c \neq 0$[10] *then there exists a diffeomorphism* ψ: $\boldsymbol{R}^2, 0 \to \boldsymbol{R}^2, 0$ *such that for* η *given by* (25)

$$\eta \circ \psi(x, y) = x^3 + (\operatorname{sgn} b) y^6, \tag{28}$$

i.e. η *is a singularity of type* J_{10} *according to Arnold's classification* [2].

For the proof see Appendix. ■

Let us switch off the external magnetic field, i.e. we pass to $H = 0$. The remaining magnetization M_r is called a residual magnetization. By (7), $H = 0$ is equivalent to $\eta = 0$. This is why we are interested in the set $\eta^{-1}(0)$.

COROLLARY 11. *The set* $\eta^{-1}(0)$ *is the curve*

$$s \to p(s) := \begin{pmatrix} -2\sqrt[3]{bc^2/a}\, s^2 + O(s^3) \\ s \end{pmatrix}. \tag{29}$$

For the proof see Appendix. ■

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[10] $a \neq 0$ was already assumed in Hypothesis II. The assumptions $b, c \neq 0$ imposed on $(\eta, \xi) \in \mathscr{S} \cap (\mathfrak{m}^3 \times O)$, see Theorem 7, introduce a restriction to an open and dense (generic) subset of $\mathscr{S} \cap (\mathfrak{m}^3 \times O)$. One of the principles of our approach is to neglect such restrictions.

Now, we want to calculate at the critical point $x = 0 = y$ the infinitesimal dependence of the residual magnetization $M_r(t)$ and the inverse of the isothermal magnetic susceptibility (i.m.s.) $1/\chi(h, t)$ as functions of temperature. Let h be fixed a magnetic field and t a fixed temperature. What corresponds in our approach to $1/\chi(h, t)$ which in thermodynamics is introduced as $(\partial H/\partial M)_T$? Let $\varphi(x, t)$ be defined by the equation

$$\varepsilon(x, \varphi(x, t))\xi(x, \varphi(x, t)) = t - t_c. \tag{30}$$

Since

$$H(x, y) = \varepsilon(x, y)\eta(x, y),$$
$$M(x, y) = -\varepsilon(x, y)x,$$

the inverse of i.m.s. is the derivative of the mapping assigning $\varepsilon(x, \varphi(x, t)) \times \times \eta(x, \varphi(x, t))$ to each $-\varepsilon(x, \varphi(x, t))x$.

This results in differentiation of the mapping $x \to -\eta(x, \varphi(x, t))$. Thus we define

$$\frac{1}{\chi(h, t)} := -\frac{d}{dx}\eta(\cdot, \varphi(\cdot, t))\Big|_{x:\, H(x,\varphi(x,t)) = h}. \tag{31}$$

As concerns the residual magnetization, we are interested only in its absolute value and therefore we put

$$M_r(t) := \left|M(x, \varphi(x, t))|_{x:\, H(x,\varphi(x,t)) = 0}\right|. \tag{32}$$

By the infinitesimal dependence of the above two functions on temperature we mean the asymptotic proportionalities

$$M_r(t) \sim (1 - t/t_c)^{\beta},$$
$$\frac{1}{\chi(0, t)} \sim (1 - t/t_c)^{\gamma'}, \tag{33}$$

for $t \nearrow t_c$. Their occurrence in our model is expressed in

PROPOSITION 12. *For our ξ and η, we have*

$$M_r(t) = 2\sqrt[3]{bc^2/a}\,(t_c - t)^{2/3} + O(t_c - t),$$
$$\frac{1}{\chi(0, t)} = -12\sqrt[3]{b^2 a}\,(t_c - t)^{4/3} + O((t - t_c)^{5/3}).$$

Proof: For a given t the corresponding parameter s_t on the curve (29) is given by the equation

$$t = T(p(s)) = t_c + cs_t^3 + O(s_t^4).^{11}$$

[11] Without loss of generality, we restrict our considerations to $W_0 \subset \Sigma_-$, where $\varepsilon = +1$, cf. (2).

Thus

$$s_t = -(t_c/c)^{1/3}(1-t/t_c)^{1/3} + O\big((1-t/t_c{}^{2/3})\big)$$

and

$$M_r(t) = M\big(p(s_t)\big) = 2\sqrt[3]{bc^2/a}\,(1-t/t_c)^{2/3} + O(1-t/t_c).$$

Passing to the other critical exponent we see that, by (25) and (30),

$$\eta\big(x, \varphi(x, t)\big) = g\big(x, \xi^2(x, \varphi(x, t))\big) = g\big(x, (t_c - t)^2\big).$$

Let us denote by p_1, p_2 the coordinates of the curve (29). By (31) and (26)

$$\begin{aligned}\frac{1}{\chi(0, t)} &= -\frac{\partial}{\partial x} g\big(x, (t_c - t)^2\big)\big|_{x:\, H(x,\varphi(x,t))=0}\\ &= -3ax^2 - (t_c - t)^2(b_1 + \text{terms of higher order in } x \text{ and/or } t - t_c)|_{x = p_1(s_t)}\\ &= -12(ab^2c^4)^{1/3}(1-t/t_c)^{4/3} + O\big((1-t/t_c)^{5/3}\big). \ \blacksquare\end{aligned}$$

Remark 13. It is known that the singularities of types $D_4^{\pm}$, i.e. equal $u(u^2 \pm v^2)$ in appropriate coordinates, form an open and dense set m^3. Their zero sets are: a curve for D_4^+ and three curves intersecting transversally at 0 for D_4^-. Since $\xi^{-1}(0)$ is interpreted in our model as the critical isotherm, we postulate that ξ belong to D_4^+. However, not whole orbit D_4^+ is admissible in the light of the assumption: $\partial\xi/\partial y > 0$ outside zero, see Remark 8. Nevertheless, there are in D_4^+ singularities satisfying that assumption, e.g. $\xi(x, y) = y(x^2 + y^2)$, and they form an open set in D_4^+.

§ 6. Final remarks

Remark 14. The parameter z was introduced in Introduction 3°. Roughly speaking, the loop of hysteresis becomes narrower as $z \to 0$. In this sense, by passing with z to zero we approximate an ideal ferromagnet consisting of one magnetic domain. Up to now we have considered the case of a fixed $z \neq 0$. Let us imagine that all objects of our approach, from the coordinates x, y beginning and up to the functions η and ξ, depend on z. Then for very small z the residual magnetization $M_r(t)$ could be interpreted as the spontaneous magnetization $M_s(t)$. If we assume a regular (non-singular) dependence of our model on z then we obtain $\beta = 2/3$ as the critical exponent for M_s. It seems to disagree with the values $\beta = 0.36-0.39$ given by experimental physics. On the other hand, our $\gamma' = 4/3$ seems to fit very well the experimental data $\gamma' = 1.2-1.36$.

Remark 15. Measurements of the critical exponents consist in fitting the experimentally obtained points by a "monomial" $a_0(1-t/t_c)^{\alpha_0}$, where $a_0, \alpha_0 \in \boldsymbol{R}$. The fitting is done on an interval $[t_1, t_2]$ strictly below t_c, i.e. $t_2 < t_c$. Such a method assumes implicitly that the coefficients a_i in a rational power expansion

$$\sum_{i=0}^{\infty} a_i (1-t/t_c)^{\alpha_i} \tag{34}$$

of the thermodynamical function whose critical exponent is to be measured, are such that the first term $a_0(1-t/t_c)^{\alpha_0}$ dominates on the interval $[t_1, t_2]$. Undoubtedly, this cannot be true in general. Therefore we propose another point of view mentioned already in [4]. According to it

1° This is only a theory which is able to provide an expansion of the type (34), i.e. the exponents α_i, $i = 0, 1, \ldots$

2° Experimental data provide information about the coefficients a_i, $i = 0, 1, \ldots$

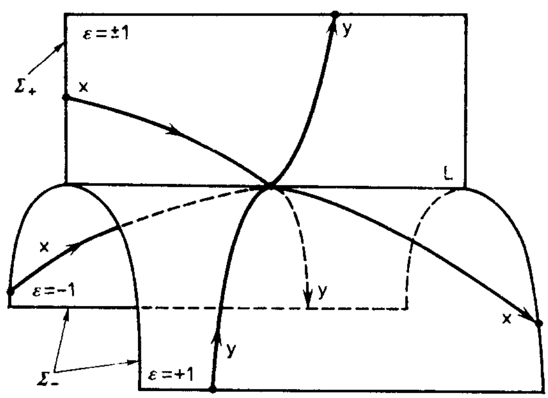

Fig. 8

Remark 16. To anybody interested in our ideas about Σ_+ and the whole $\Sigma = \Sigma_- \cup \Sigma_+$ we recommend Remark 19 of [3]. They are drawn in Fig. 8 which augments the illustrations given there: two leaves of Σ_+, as presented in [3], are now identified and ε becomes a double-valued function on the whole Σ_+. The thermodynamic space Σ is the product of a spur $\supset\!\!-$ and a line. This is, of course, a neighbourhood in Σ of the critical point.

Appendix. Proofs

THEOREM 17. *Let* $\xi \in \mathfrak{m}^2 \setminus \mathfrak{m}^\infty$ *be finite-determined*, $\xi(x, 0) = 0$,

$$\frac{\partial \xi}{\partial y}(x, y) > 0 \quad \textit{for } (x, y) \neq 0, \tag{35}$$

the Taylor series $T_0\xi(x, y) \in \boldsymbol{R}[[x, y]]$ *have no multiple factors, the formal power series* $T_0\xi(x, y) + T_0\xi(x, y') \in \boldsymbol{R}[[x, y, y']]$ *be irreducible. If we are given* $\eta \in \mathfrak{m}$

compatible with ξ, *i.e. for every* (x, y, y') $(\xi(x, y) = -\xi(x, y')) \Rightarrow (\eta(x, y) = \eta(x, y'))$, *then there exists* $g \in \mathfrak{m}$ *such that*

$$\eta(x, y) = g(x, \xi^2(x, y)). \tag{36}$$

Proof: Since x and ξ are smooth coordinates on an annular neighbourhood of zero, there exists a smooth f_0 such that $\eta(x, y) = f_0(x, \xi(x, y))$. Decomposing f_0 into its even and odd parts:

$$f_0(x, t) = f_1(x, t^2) + t f_2(x, t^2)$$

and taking into account the compatibility of η and ξ, we get $f_2(x, \xi^2(x, y)) = 0$. Thus, outside zero we have f_1 such that

$$\eta(x, y) = f_1(x, \xi^2(x, y)). \tag{37}$$

LEMMA 1. *If there exists a formal power series* $S \in \boldsymbol{R}[[x, y]]$ *such that*

$$T_0 \eta(x, y) = S(x, (T_0 \xi)^2(x, y)) \tag{38}$$

then f_1 *can be smoothly extended to zero.*

So we have only to show the existence of such S. Let

$$\begin{aligned} F(x, y, y') &:= \xi(x, y) + \xi(x, y') \\ G(x, y, y') &:= \eta(x, y) - \eta(x, y'). \end{aligned} \tag{39}$$

The compatibility of ξ and η means that for any (x, y, y') close to $0 \in \boldsymbol{R}^3$

$$F(x, y, y') = 0 \Rightarrow G(x, y, y') = 0. \tag{40}$$

Let us check the following analogous implication for their Taylor series: for each formal curve $\bar{z}(\lambda) \in \boldsymbol{R}[[\lambda]]^3$ close to $0 \in \boldsymbol{R}[[\lambda]]^3$

$$(T_0 F)(\bar{z}(\lambda)) = 0 \Rightarrow (T_0 G)(\bar{z}(\lambda)) = 0. \tag{41}$$

By Borel's theorem there exists a smooth curve $\lambda \to z_0(\lambda) \in \boldsymbol{R}^3$ such that $(T_0 z_0)(\lambda) = \bar{z}(\lambda)$ and $F(z_0(\cdot))$ is flat. Then, using Tougeron's implicit function theorem we infer the existence of a smooth $\lambda \to z(\lambda) \in \boldsymbol{R}^3$ such that $z - z_0$ is flat and $F(z(\lambda)) = 0$. But we have to check the hypothesis of the theorem. To this end let us notice that by the finite determinacy of ξ, an appropriate change of coordinates in $\boldsymbol{R}^3$ allows us to assume analyticity of F. Then the components of its gradient satisfy the Łojasiewicz inequality

$$\left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial y'}\right)^2\right](x_1, x_2, x_3) \geqslant C\left(\sum_{i=1}^{3} x_i^2\right)^{\alpha}$$

for some positive C and α; see [6], Corollary 1.6, p. 119 and Definitions 4.1, 4.2, p. 102. Thus

$$\left[\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial y'}\right)^2\right](z_0(\lambda)) \geqslant C_1\lambda^{\alpha_1}$$

for $C_1, \alpha_1 > 0$: we consider only $\bar{z}(\lambda) \neq 0$. Hence the ideal I generated by

$$\frac{\partial F}{\partial x}(z_0(\cdot)) \quad \text{and} \quad \frac{\partial F}{\partial y}(z_0(\cdot)) = \frac{\partial F}{\partial y'}(z_0(\cdot))$$

is not flat and, in particular, $\mathfrak{m}^\infty = \mathfrak{m}^\infty I^2$. Therefore $F(z_0(\cdot)) \in \mathfrak{m}^\infty I^2$ and Tougeron's theorem ensures the existence of smooth z for which $F(z(\cdot)) = 0$ and $z - z_0 \in \mathfrak{m}^\infty I = \mathfrak{m}^\infty$.

Now, by (40) we obtain the right-hand side of (41). It is easily seen that in the light of our assumptions about ξ $(\nabla T_0\xi)(\bar{x}_0(\lambda), \bar{y}_0(\lambda)) \neq 0$ for almost all formal curves $(\bar{x}_0(\lambda), \bar{y}_0(\lambda)) \in \boldsymbol{R}[[\lambda]]^2$, i.e. except those for which $\bar{y}_0(\lambda) = 0$. Now, if we denote

$$\bar{F} := T_0 F, \qquad \bar{G} := T_0 G,$$

the following lemma can be applied:

LEMMA 2. *Let* $F, G \in \boldsymbol{R}[[x_1, \ldots, x_n]]$, *$F$ be irreducible. If* $\bar{x}(\lambda) \in \boldsymbol{R}[[\lambda]]^n$, $\bar{x}(0) = 0$, *is such that*

$$\bar{F}(\bar{x}(\lambda)) = 0, \qquad (\nabla\bar{F})(\bar{x}(\lambda)) \neq 0$$

and for every $\bar{y}(\lambda)$ *from a neighbourhood of* $\bar{x}(\lambda)$ *in* $\boldsymbol{R}[[\lambda]]^n$ *(endowed with the Krull topology)*

$$\bar{F}(\bar{y}(\lambda)) = 0 \Rightarrow \bar{G}(\bar{y}(\lambda)) = 0,$$

then $\bar{F}$ *divides* $\bar{G}$.

Hence we have

$$(T_0 F)(\bar{z}(\lambda)) = 0 \Rightarrow (T_0 G)(\bar{z}(\lambda)) = 0 \tag{42}$$

for all complex formal curves $\bar{z}(\lambda) \in \boldsymbol{C}[[\lambda]]^3$.

So we have stated our problem in the complex domain and now we shall investigate $T_0\eta$ on the set of complex formal curves which are zeros of $T_0\xi$. The Puiseux decomposition for $T_0\xi \in \boldsymbol{C}[[x, y]]$ has the form

$$(T_0\xi)(x, y) = x^q \prod_{i=1}^{k} (y - \varphi_i(x^{1/p_i}))\, a(x, y) \tag{43}$$

where $q = 0, 1, \ldots$, $\varphi_i(z) \in \boldsymbol{C}[[z]]$, $p_i = 1, 2, \ldots$, and $a(x, y) \in \boldsymbol{C}[[x, y]]$ is invertible; $\varphi_i(x^{1/p_i})$ are fractional formal power series. Let us show that $T_0\eta(x, \varphi_i(x^{1/p_i}))$

do not depend on i. Thus, we take $p = p_1 \dots p_k$ and two complex formal curves $y_i(x)$ such that

$$(T_0\xi)(x^p, y_i(x)) = 0, \quad i = 1, 2. \tag{44}$$

Hence, $T_0 F$ vanishes on the formal curve $(x^p, y_1(x), y_2(x))$. By (42), we have

$$(T_0\eta)(x^p, y_1(x)) = (T_0\eta)(x^p, y_2(x))$$

i.e.

$$(T_0\eta)(x, y_1(x^{1/p})) = (T_0\eta)(x, y_2(x^{1/p})).$$

So the complex fractional power series

$$\psi_0(x^{1/p}) := T_0\eta(x, y_i(x^{1/p})),$$

where $(x, y_i(x^{1/p})) \in \boldsymbol{C}[[x^{1/p}]]^2$ runs over all zeros of $T_0\xi$, does not depend on i.

Let ε be any of the complex p-th roots of 1, i.e. $\varepsilon^p = 1$. If we substitute $\varepsilon x^{1/p}$ into (44) in the place of x, we see that $(T_0\xi)(x, y_1(\varepsilon x^{1/p})) = 0$, i.e. the curve $(x, y_1(\varepsilon x^{1/p})) \in \boldsymbol{C}[[x^{1/p}]]^2$ coincide with one of the curves $(x, y_i(x^{1/p}))$, $i = 1, 2, \dots, k$. Thus $\psi_0(x^{1/p}) = T_0\eta(x, y_1(\varepsilon x^{1/p})) = (T_0\eta)((\varepsilon x^{1/p})^p, y_1(\varepsilon x^{1/p})) = \psi_0(\varepsilon x^{1/p})$. Hence, all terms of the series $\psi_0(x^{1/p})$, containing fractional (non-natural) powers of x must vanish. In other words,

$$(T_0\eta)(x, y_i(x^{1/p})) =: \psi(x) \in \boldsymbol{C}[[x]].$$

Since the formal series $T_0\xi$ has real coefficients, there exists such $i = 1, \dots, k$ that the complex conjugate

$$\overline{y_1(x^{1/p})} = y_i(x^{1/p}).$$

This and the fact that $T_0\eta$ is a real formal series tells us that ψ is also real.

Let us denote

$$\theta(x, y) := T_0\eta(x, y) - \psi(x) \in \boldsymbol{R}[[x, y]].$$

Obviously, θ vanishes on all complex formal curves which are zeros of $T_0\xi(x, y)$. Since $T_0\xi$ has no multiple factors, there exists $\bar{\eta}_1(x, y) \in \boldsymbol{R}[[x, y]]$ such that

$$T_0\eta(x, y) = \psi(x) + \bar{\eta}_1(x, y)\, T_0\xi(x, y). \tag{45}$$

Let us define

$$\bar{G}_1(x, y, y') := \bar{\eta}_1(x, y) + \bar{\eta}_1(x, y') \in \boldsymbol{R}[[x, y, y']].$$

It follows from (41) and (45) that

$$(T_0 F)(\bar{z}(\lambda)) = 0 \Rightarrow \bar{G}_1(\bar{z}(\lambda)) = 0. \tag{46}$$

It is easily seen that the part of our proof contained between (41) and (45) can be repeated with $T_0\eta$, $T_0 G$ replaced by $\bar{\eta}_1$, $\bar{G}_1$ (respectively). Then (46) is an analog of (41) and as an analog of (45) we get

$$\bar{\eta}_1(x, y) = \psi_1(x) + \bar{\eta}_2(x, y)\, T_0\xi(x, y).$$

An infinite sequence of such steps yields

$$T_0\eta(x, y) = \psi(x) + \sum_{k=1}^{\infty} \psi_k(x)\big(T_0\xi(x, y)\big)^k,$$

where, by the compatibility of η and ξ, $\psi_k(x) \equiv 0$ for odd k. Finally we put

$$S(x, y) = \psi(x) + \sum_{k=1}^{\infty} \psi_{2k}(x)\, y^k,$$

which completes the proof. ■

Proof of Lemma 1: Let f_3 be a smooth function (germ at 0) the Taylor series of which $T_0 f_3 = S$. Then the function

$$F(x, y) = \eta(x, y) - f_3\big(x, \xi^2(x, y)\big)$$

is flat at 0, i.e. for any N and $\alpha = (\alpha_1, \alpha_2)$ there exists a constant $C_{\alpha,N}$ such that

$$|D^\alpha F(x, y)| \leqslant C_{\alpha,N}(|x|+|y|)^N. \tag{47}$$

Since $\xi(0,\cdot) \notin \mathfrak{m}^\infty$, there exists the smallest l such that

$$\frac{\partial^l \xi}{\partial y^l} \neq 0 \text{ and } \frac{\partial^{l-1}\xi}{\partial y^{l-1}}(0) = 0,$$

and so forth. Therefore

$$(|x|+|y|)^{2l} \leqslant (x^2+y^2)^l \in \mathfrak{m}^{2l} \subset \langle x, \xi\rangle,$$

where $\langle x, \xi\rangle$ is the ideal in ξ generated by x and ξ. Thus

$$(|x|+|y|)^{2l} \leqslant |a(x, y) + b(x, y)\,\xi(x, y)| \leqslant C\big(|x| + |\xi(x, y)|\big). \tag{48}$$

By (47) and (48) we have

$$|F(x, y)| \leqslant C^N C_{0,2lN}\big(|x| + |\xi(x, y)|\big)^N \quad \text{for any } N. \tag{49}$$

Let us define

$$f(x, t) := f_1(x, t^2) - f_3(x, t^2) \quad \text{for } (x, t) \neq 0.$$

Obviously, if $\big(x, \xi(x, y)\big) \neq 0$ then

$$f\big(x, \xi(x, y)\big) = F(x, y). \tag{50}$$

Thus, by (49)

$$|f(x, t)| \leqslant C^N C_{0,2lN}(|x|+|t|)^N \tag{51}$$

for any N and $(x, t) \neq 0$. We shall prove an analogous estimate for all derivatives of f. We use the notation

$$\partial_x^p := \frac{\partial^p}{\partial x^p}, \quad \partial_{x,y}^\alpha := \frac{\partial^{\alpha_1+\alpha_2}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}, \quad \text{where } \alpha = (\alpha_1, \alpha_2)$$

and similarly for other variables. Let $(x, \xi(x, y)) \neq 0$, then

$$\partial_y^q F(x, y) = \partial_t^q f(x, \xi(x, y))(\partial_y \xi(x, y))^q + \text{terms involving lower order derivatives of } f,$$

$$\partial_x^p \partial_y^q F(x, y) = \Big[\sum_{k=0}^{p} \partial_x^{p-k}\partial_t^{q+k} f(x, \xi(x, y))(\partial_x \xi(x, y))^k\Big] \times$$
$$\times (\partial_y \xi(x, y))^q + \text{terms involving lower order derivatives of } f,$$

$$\partial_{x,y}^\alpha F(x, y) = (\partial_y \xi(x, y))^{\alpha_2} \partial_{x,t}^\alpha f(x, \xi(x, y)) +$$
$$+ \sum_{\substack{|\beta|=r\\ \beta<\alpha}} \Phi_\beta^\alpha(\partial_x \xi(x, y), \partial_y \xi(x, y))\partial_{x,t}^\beta f(x, \xi(x, y)) +$$
$$+ \text{terms involving lower order derivatives of } f, \tag{52}$$

where Φ_β^α are monomials of two variables and $<$ is the lexicographic order in 2-indices of the norm r, i.e. if $|\beta| = |\alpha| = r$ then

$$(\beta < \alpha) \Leftrightarrow (\beta_1 < \alpha_1 \vee (\beta_1 = \alpha_1 \wedge \beta_2 < \alpha_2)).$$

For a given r we take the system of equations (52) for all α with $|\alpha| = r$. Let us solve it with respect to $\partial_{x,t} f(x, \xi(x, y))$ as unknowns. If the equations and unknowns are ordered lexicographically then the matrix of the system is triangular with powers of non-zero $\partial_y \xi(x, y)$ at the diagonal. Its determinant equals $(\partial_y \xi(x, y))^s$ where $s = r(r+1)/2$. We solve these systems subsequently starting with $r = 1$. In this way we get

$$(\partial_y \xi(x, y))^s \partial_{x,t}^\alpha f(x, \xi(x, y)) = \sum_{|\beta|=|\alpha|} \Psi_\beta^\alpha(\partial_x \xi(x, y), \partial_y \xi(x, y))\partial_{x,y}^\beta F(x, y) +$$
$$+ \text{terms involving lower order derivatives of } F.$$

Now, by (47) and (48), for any N there exists a constant $M_{\alpha,N}$ such that

$$|\partial_y \xi(x, y)|^s |\partial_{x,t} f(x, \xi(x, y))| \leqslant M_{\alpha,N}(|x|+|y|)^N. \tag{53}$$

Being finitely determined, ξ as well as $\partial_y \xi$ are analytical up to a diffeomorphism. In particular, the Łojasiewicz inequality occurs for $\partial_y \xi$, i.e. there exist M and k

such that

$$|\partial_y \xi(x, y)| \geqslant M(|x|+|y|)^k.$$

Hence, by (53) and (48),

$$\begin{aligned}\partial_{x,t} f(x, \xi(x, y)) &\leqslant MM_{\alpha,N}(|x|+|y|)^{N-sk} \\ &\leqslant MM_{\alpha,N} C^{1/2l}(|x|+|\xi(x, y)|)^{(N-sk)/2l}.\end{aligned}$$

In other words

$$|\partial_{x,t} f(x, t)| \leqslant MM_{\alpha,N} C^{1/2l}(|x|+|t|)^{(N-sk)/2l}.$$

Therefore, if we extend f to zero, i.e. we put $f(0, 0) = 0$, f becomes a smooth and flat function at zero. So, f_1 can be extended to zero by the formula

$$f_1(x, t) = f_3(x, t) + f(x, \sqrt{|t|})$$

which is smooth at zero. ∎

Proof of Lemma 2: 1°. Without loss of generality we can assume that

$$\frac{\partial \bar{F}}{\partial x_n}(0, \ldots, 0, x_n) \neq 0$$

and

$$\frac{\partial \bar{F}}{\partial x_n}(\bar{x}(\lambda)) \neq 0.$$

By the preparation theorem [6], p. 189, $\bar{F}$ is equivalent to a Weierstrass polynomial of x_n, i.e. there exists $\bar{Q} \in \boldsymbol{R}[[x]]$, $\bar{Q}(0) \neq 0$ (i.e. Q invertible), and

$$\bar{F}_1(x) = x_n^p + \sum_{i=0}^{p-1} \bar{a}_i(x') x_n^i,$$

where $x' := (x_1, \ldots, x_{n-1})$, $\bar{a}_i(x') \in \boldsymbol{R}[[x']]$, such that

$$\bar{F} = \bar{Q}\bar{F}_1.$$

Let $\bar{G}_1(x) = \sum_{i=0}^{p-1} \bar{b}_i(x') x_n^i$ be the rest from dividing $\bar{G}$ by $\bar{F}_1$ i.e.

$$\bar{G} = \bar{R}\bar{F}_1 + \bar{G}_1.$$

We want to show that $\bar{G}_1 = 0$.

Let us notice that $\bar{F}_1$ and $\bar{G}_1$ satisfy the assumptions made in our lemma about $\bar{F}$ and $\bar{G}$, respectively. Moreover, $\deg_{x_n} \bar{G}_1 < \deg_{x_n} \bar{F}_1$. Therefore it is enough to prove that if $\bar{F}$, $\bar{G}$ of our lemma are polynomials with respect to x_n and

$$\deg_{x_n}\bar{G} < \deg_{x_n}\bar{F}$$

then $\bar{G} = 0$. To this end, let us suppose that $\bar{G} \neq 0$.

Let N be the degree of $\frac{\partial\bar{F}}{\partial x_n}(\bar{x}(\lambda)) \in \boldsymbol{R}[[\lambda]]$, i.e. there exists $\varphi(\lambda) \in \boldsymbol{R}[[\lambda]]$ such that $\varphi(0) \neq 0$ and $\frac{\partial\bar{F}}{\partial x_n}(\bar{x}(\lambda)) = \lambda^N \varphi(\lambda)$.

2°. We introduce new variables $a_1, \ldots, a_{n-1}$ and denote $a' := (a_1, \ldots, a_{n-1})$. By the previous notation $\bar{x}'(\lambda) \in \boldsymbol{R}[[\lambda]]^{n-1}$.[12]

Let us define

$$\xi'(a', \lambda) := \bar{x}'(\lambda) + a'\lambda^{2N+1} \in \boldsymbol{R}[[a', \lambda]]^{n-1}.$$

We shall prove that there exists a unique series $\xi_n(a', \lambda) \in \boldsymbol{C}[[a', \lambda]]$ such that

$$\bar{F}(\xi'(a', \lambda), \xi_n(a', \lambda)) = 0$$

and

$$\xi_n(a', \lambda) - \bar{x}_n(\lambda) \in \lambda^{N+1}\boldsymbol{C}[[a', \lambda]].$$

In fact, we have (exactly as above)

$$\bar{F}(\xi'(a', \lambda), \bar{x}_n(\lambda)) \underset{\bmod \lambda^{2N+1}}{\equiv} \bar{F}(x'(\lambda), \bar{x}_n(\lambda)) = 0.$$

Hence

$$\bar{F}(\xi'(a', \lambda), \bar{x}_n(\lambda)) \in \lambda^{2N+1}\boldsymbol{R}[[a', \lambda]] = (\lambda\boldsymbol{C}[[a', \lambda]])(\lambda^N\boldsymbol{C}[[a', \lambda]])^2, \tag{$*$}$$

$$\bar{F}(\xi'(a', \lambda), \bar{x}_n(\lambda)) = \lambda^{2N+1}\Phi(a', \lambda)$$

for some $\Phi(a', \lambda) \in \boldsymbol{R}[[a', \lambda]]$, and moreover

$$\frac{\partial\bar{F}}{\partial x_n}(\xi'(a', \lambda), \bar{x}_n(\lambda)) \underset{\bmod \lambda^{2N+1}}{\equiv} \frac{\partial\bar{F}}{\partial x_n}(x'(\lambda), \bar{x}_n(\lambda)) = \lambda^N\varphi(\lambda).$$

Thus $\lambda^N\varphi(\lambda) = \frac{\partial\bar{F}}{\partial x_n}(\xi'(a', \lambda), \bar{x}_n(\lambda)) + \lambda^{2N+1}\psi(a', \lambda)$ for some $\psi \in \boldsymbol{R}[[a', \lambda]]$. Since $\varphi(0) \neq 0$,

$$\lambda^N = \frac{\partial\bar{F}}{\partial x_n}(\xi'(a', \lambda), \bar{x}_n(\lambda))/(\varphi(\lambda) - \lambda^{N+1}\psi(a', \lambda)),$$

---

[12] i.e. $\bar{x}(\lambda) = (\bar{x}'(\lambda), \bar{x}_n(\lambda))$.

which shows that λ^N belongs to the ideal in $\boldsymbol{C}[[a', \lambda]]$ generated by $\frac{\partial \bar{F}}{\partial x_n}(\xi'(a', \lambda), \bar{x}_n(\lambda))$. Therefore

$$\frac{\partial \bar{F}}{\partial x_n}(\xi'(a', \lambda), \bar{x}_n(\lambda))\boldsymbol{C}[[a', \lambda]] = \lambda^N \boldsymbol{C}[[a', \lambda]].$$

This and (*) permits us to use Tougeron's theorem by which the existence and uniqueness of our $\xi_n(a', \lambda)$ is ensured.

3°. Let $A := \boldsymbol{R}[[x']]$ and

$$P(x_n) := F(x', x_n) \in A[x_n],$$
$$Q(x_n) := G(x', x_n) \in A[x_n].$$

The discriminant Δ of P and Q is a polynomial (with integer coefficients) in coefficients of polynomial P and Q. Thus Δ is an element of A and to stress this fact we write $\Delta = \Delta(x')$. We shall prove that $\Delta(x') = 0$.

Let $B := \boldsymbol{R}[[a', \lambda]]$ and

$$P_1(x_n) := \bar{F}(\xi'(a', \lambda), x_n) \in B[x_n],$$
$$Q_1(x_n) := \bar{G}(\xi'(a', \lambda), x_n) \in B[x_n].$$

As above, the discriminant $\Delta_1 = \Delta_1(a', \lambda)$ of polynomials P_1 and Q_1 is an element of B. Obviously,

$$\Delta_1(a', \lambda) = \Delta(\xi'(a', \lambda)).$$

It was shown in part 2° of the present proof that P_1 and Q_1 have a common root, namely, $\xi_n(a', \lambda) \in B$. This implies $\Delta_1(a', \lambda) = 0$. Let $a'(\lambda)$ run over a neighbourhood of zero in $\boldsymbol{R}[[\lambda]]^n$, then $\xi'(a'(\lambda), \lambda)$ runs over a neighbourhood of $\bar{x}'(\lambda)$ in $\boldsymbol{R}[[\lambda]]$. Since $\Delta(\xi'(a'(\lambda), \lambda)) \underset{a'(\lambda)}{\equiv} 0$, we have $\Delta = 0$.

So, P and Q have a common root α belonging to the field $\boldsymbol{K}$ which is the algebraic closure of the field of quotients of A. Let $M(x_n) \in \boldsymbol{K}[x_n]$ be the minimal polynomial of α. M divides in $\boldsymbol{K}[x_n]$ both Q and P. Thus P is reducible in $\boldsymbol{K}[x_n]$. Since $P(\alpha) = 0$ and $P(x_n) = x_n^p +$ lower order terms, α is an integral element over $A[x_n]$. So its minimal polynomial $M \in A[x_n]$. Therefore $P(x_n)$ is reducible in $A[x_n] = \boldsymbol{R}[[x']][x_n]$, which implies reducibility in $\boldsymbol{R}[[x', x_n]]$ of $\bar{F}(x', x_n) = P(x_n)$. This contradiction is a consequence of the assumption $\bar{G} \neq 0$. Thus $\bar{G} = 0$, which completes the proof. ∎

Proof of Proposition 10: By (25)–(27)

$$\begin{aligned}\eta(x, y) &= x^3(a+a_1 x+a_2 x^2+a_3 x^3)+\\ &\quad +b(cy^3+c_1 xy^2+c_2 x^2 y+c_3 x^3)^2 (\text{mod } \mathfrak{m}^7)\\ &= x^3[a+a_1 x+a_2 x^2+(a_3+bc_3^2)x^3]+bc^2 y^6+2bcc_1 xy^5+\\ &\quad +x^2 P_4(x, y)(\text{mod } \mathfrak{m}^7),\end{aligned} \tag{$*$}$$

where P_4 is a homogeneous polynomial of degree 4.

Let us define

$$X(x, y) := [a+a_1 x+a_2 x^2+(a_3+bc_3^2)x^3]^{1/3}+\frac{1}{3a^{2/3}} P_4(x, y),$$

$$Y(x, y) := (|b|\, c^2)^{1/6}(y+\tfrac{1}{3} C^{1/3} C_1 x).$$

It is easily seen that

$$X^3(x, y)+(\operatorname{sgn} b)\, Y^6(x, y) = \eta(x, y)+x^2 Q_4(x, y)(\text{mod } \mathfrak{m}^7),$$

where Q_4 is a homogeneous polynomial of degree 4. Obviously, the mapping $(x, y) \to (X(x, y), Y(x, y))$ is a diffeomorphism $\boldsymbol{R}^2, 0 \to \boldsymbol{R}^2, 0$. If we calculate the linear part (derivative) of its inverse we see that $x^2 Q_4(x, y) = X^2(x, y) R_4(X(x, y), Y(x, y))(\text{mod } \mathfrak{m}^7)$, for some homogeneous polynomial R_4 of degree 4. Hence

$$\eta = X^3+(\operatorname{sgn} b)\, Y^6 - X^2 R_4(X, Y)(\text{mod } \mathfrak{m}^7).$$

If we put $X_1 := X-\frac{1}{3} R_4(X, Y)$ and $Y_1 := Y$, then

$$\eta = X_1^3+(\operatorname{sgn} b)\, Y_1^6 (\text{mod } \mathfrak{m}^7).$$

Obviously, $\psi_1 := (X_1, Y_1)$ is a diffeomorphism $\boldsymbol{R}^2, 0 \to \boldsymbol{R}^2, 0$ and

$$\eta \circ \psi_1^{-1}(x, y) = x^3+(\operatorname{sgn} b)\, y^6 (\text{mod } \mathfrak{m}^7).$$

Using the well known sufficient condition for k-determinacy (see, for instance, [1], Lemma 2, or [8]) we see that our $\eta \circ \psi_1^{-1}$ is 6-determined, which completes the proof. ∎

Proof of Corollary 11: Let ψ be as in Proposition 10. We denote $\psi^{-1} = (\psi_1, \psi_2)$ and we put

$$\psi_1(x, y) = \alpha x+\alpha_1 y+\alpha_2 x^2+\alpha_3 xy+\alpha_4 y^2 (\text{mod } \mathfrak{m}^3),$$

$$\psi_2(x, y) = \beta x+\beta_1 y+\beta_2 x^2+\beta_3 xy+\beta_4 y^2 (\text{mod } \mathfrak{m}^3).$$

Then, by (28) and (*) of the proof of Proposition 10

$$\eta(x, y) = \psi_1^3(x, y) + (\operatorname{sgn} b)\psi_2^6(x, y)$$
$$= x^3[a + a_1 x + a_2 x^2 + (a_3 + bc_3^2)x^3] + bc^2 y^6 + 2bcc_1 xy^5 \pmod{\mathfrak{m}^7}.$$

This implies $\alpha_1 = \alpha_3 = \alpha_4 = 0$, $\alpha = a^{1/3}$, $\beta = |b|^{1/6} c_1/3c^{2/3}$, $\alpha_2 = a_1/3a^{2/3}$, $\beta_1 = |b|^{1/6} c^{1/3}$.

On the other hand, $\eta^{-1}(0)$ is given by the equation $\psi_1(x, y) + (\operatorname{sgn} b)\psi_2^2(x, y) = 0$ which leads to

$$\alpha x + \alpha_2 x^2 + (\beta x + \beta_1 y)^2 = 0 \pmod{\mathfrak{m}^3}.$$

Since $\alpha = a^{1/3} \neq 0$, we can find x as an implicit function of y and we get

$$x = \frac{2\beta_1}{\alpha^2}(\beta - \alpha) y^2 + O(y^3).$$

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