# STABILITY CRITERIA AND CLASSIFICATION OF SINGULARITIES FOR EQUIVARIANT LAGRANGIAN SUBMANIFOLDS

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(*Received December 16, 1986*)

One of the useful methods of mathematical physics is the one arising from symplectic geometry and associating the singularities of Lagrangian submanifolds with the optical caustics, phase transitions, bifurcation patterns, obstacle geometry etc. In this paper we derive the stability criteria for singularities of equivariant Lagrangian submanifolds with a compact Lie group action determined by a system with symmetry. The recognition problem and classification list for stable $(Z_2)^q$-equivariant singularities is proved. We find that the classified stable local models occur as possible realizations for the equilibrium states in the breaking of symmetry and structural phase transitions. Additionally, the connection between two technically different (by generating functions, by Morse families) infinitesimal $G$-stability conditions for equivariant Lagrangian submanifols is studied and an alternative approach is proposed.

## 1. Introduction

Singularities of Lagrangian submanifolds appeared as natural objects in the study of the wave pattern with high-frequency waves coming from a point source and moving through a medium (cf. [14], [12]). The corresponding intensity of radiation is described by the asymptotics of the so-called rapidly oscillating integrals (cf. [7], [3]). Asymptotically (with high frequency) this intensity is infinite around the singularities (coustics) of Lagrangian submanifolds generated by the appropriate phase functions (cf. [19], [7]). Thus the Lagrangian submanifolds appeared initially as the spaces which model the systems of rays in geometrical optics [3]. In the case of symmetries of the sources of radiation, as well as when

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the boundary conditions (mirrors) exhibit some symmetry properties, then the corresponding Lagrangian submanifold describing the respective optical geometry of the system possesses also some symmetry properties (cf. [14], [17]). Similar problems with symmetric Lagrangian submanifolds appeared also in the variational calculus, nonlinear partial differential equations, and optimization (cf. [14], [24]).

Another domain where the singularities of Lagrangian submanifolds play an important role is the symplectic bifurcation theory (cf. [23], [11], [10]) and the breaking of symmetry in mechanics and the structural phase transitions (cf. [17], [8], [9]). It was observed in [10] that the Lagrangian submanifolds model the space of equilibrium states of thermodynamical systems. In most thermodynamic phase transitions in crystals (cf. [12]) the whole bifurcation picture can be described by an appropriate $G$-equivariant Lagrangian submanifold in the corresponding phase space with the compact Lie group $G$ of symmetry (cf. [11], [9]). The first step in the study of typical properties of constitutive sets in structural physics is the recognition and classification of stable $G$-equivariant germs of Lagrangian submanifolds, which is the aim of the present paper.

In this paper we will study the infinitesimal stability and local stability criteria for the germs of equivariant Lagrangian submanifolds near the fix-point of the symplectic action of the compact Lie group. Our purpose is twofold. First, we want to write down the algebraic criteria for the local $G$-stability. Secondly, we want to use this general method to investigate the normal forms of the stable $G$-equivariant Lagrangian germs.

In [2], [22] there is a study of stable singularities of Lagrangian submanifolds in the nonsymmetric case, and we will follow the notations and terminology used there. In Section 2 of our paper we present the basic results and notation. In Section 3 we construct the infinitesimal stability conditions for $G$-invariant generating functions of $G$-equivariant Lagrangian germs and show their effectiveness in calculations with the trivial $\boldsymbol{Z}_2$ and $D_m$ symplectic group actions. Section 4 is devoted to the complete calculation of stability criteria and classification of stable normal forms of equivariant Lagrangian germs in the concrete $(\boldsymbol{Z}_2)^q$ group action. This action is motivated by the theory of phase transitions in uniaxial ferromagnets as well as in all types of ferroelectrics. In Section 5, 6 we present the stability criteria in the Morse family (cf. [19]) approach. Here we derive the so-called linear infinitesimal stability condition and show its usefulness in some concrete symmetric problems. Following [9] we also present there an alternative approach to the study of $G$-equivariant Lagrangian singularities in physical applications.

## 2. Preliminaries

Let $\nu\colon G \to 0(n)$ be an orthogonal representation of $G$ in $\boldsymbol{R}^n$. By $C^\infty_\nu(n)$ we denote the set of smooth $\nu$-invariant functions on $\boldsymbol{R}^n$ and by $\mathfrak{E}_\nu(n)$ the set of all their germs at $0 \in \boldsymbol{R}^n$ (cf. [13]). We denote $\mathfrak{M}_\nu^k(n) = \mathfrak{M}^k(n) \cap \mathfrak{E}_\nu(n)$, where $\mathfrak{M}^k(n)$ denotes the $k$-th power of the maximal ideal $\mathfrak{M}(n) \subset \mathfrak{E}(n)$ (cf. [21]). For convenience we shall write also $\mathfrak{E}_\nu(z)$, $\mathfrak{M}_\nu(z)$ etc. instead of $\mathfrak{E}_\nu(n)$, $\mathfrak{M}_\nu(n)$, etc., where $z = (z_1, \ldots, z_n)$ denote the corresponding coordinates of $\boldsymbol{R}^n$. By $\mathfrak{E}(n, \nu; m, \delta)$, where $\delta$ is an orthogonal representation of $G$ in $\boldsymbol{R}^m$, we shall denote the set of germs (at $0 \in \boldsymbol{R}^n$) of equivariant mappings $\boldsymbol{R}^n \to \boldsymbol{R}^m$.

The foundational theory of equivariant singularities may be found in [13], [21]. Now we recall some of the basic facts needed for the development of the theory of equivariant Lagrangian submanifolds.

PROPOSITION 2.1 ([15], [21]). *Let $\nu$ be an orthogonal representation of the compact Lie group $G$ in $\boldsymbol{R}^n$.*

(a) *There exists a polynomial mapping $\varrho\colon \boldsymbol{R}^n \to \boldsymbol{R}^k$, called a Hilbert map, such that*

$$\mathfrak{E}_\nu(n) = \varrho^* \mathfrak{E}(k).$$

*The set $\varrho(\boldsymbol{R}^n) \subset \boldsymbol{R}^k$ is semialgebraic.*

(b) *If $\delta\colon G \to O(n)$ is an orthogonal representation of $G$ in $\boldsymbol{R}^m$ and $\boldsymbol{R}^{n+m} \ni (x, y) \to \mu(x, y) \in \boldsymbol{R}^r$ is the corresponding Hilbert map for $\nu \oplus \delta$, then the germs $\boldsymbol{R}^n \ni (x) \to \dfrac{\partial \mu_i}{\partial y}(x, 0)$, $1 \leqslant i \leqslant r$ generate the module $\mathfrak{E}(n, \nu; m, \delta)$ over $\mathfrak{E}_\nu(n)$.*

Let us consider the cotangent bundle $T^* \boldsymbol{R}^n$ endowed with the standard symplectic structure (see [1]). We identify it with the Lagrangian fibre bundle $\pi\colon \boldsymbol{R}^{2n} \to \boldsymbol{R}^n$, $\pi'\colon (x, \xi) \to (x)$ endowed with the canonical symplectic structure $\omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i$. The action $\nu$ of $G$ on $\boldsymbol{R}^n$ can be canonically lifted to the symplectic action of $G$ on $\boldsymbol{R}^{2n} \cong T^* \boldsymbol{R}^n$, say $T^* \nu\colon G \times \boldsymbol{R}^{2n} \to \boldsymbol{R}^{2n}$. One can easily see that $T^* \nu \equiv \nu \oplus \nu$, where $(\nu \oplus \nu)_g(x, \xi) = (\nu_g x, \nu_g \xi)$ for $g \in G$, $(x, \xi) \in \boldsymbol{R}^{2n}$. An equivariant symplectomorphism $\Phi\colon \boldsymbol{R}^{2n} \to \boldsymbol{R}^{2n}$ which preserves the fibre bundle structure $\pi'\colon \boldsymbol{R}^{2n} \to \boldsymbol{R}^n$ will be called an equivariant Lagrangian equivalence ($\nu$-L-equivalence for short). By direct generalization of well-known results [19], [22] concerning of the nonequivariant case we obtain

PROPOSITION 2.2. *Let $\Phi\colon (\boldsymbol{R}^{2n}, 0) \to (\boldsymbol{R}^{2n}, 0)$ be a germ of $\nu$-L-equivalence; then there exists a diffeomorphism $\varphi \in \xi(n, \nu; n, \nu)$ and a smooth function $S \in \mathfrak{E}_\nu(n)$ such that*

$$\Phi(x, \xi) = \varphi^*(x)(\xi + dS(x)).$$

Let $p$ be the $\nu$-invariant point of $\boldsymbol{R}^{2n}$. By $(L^G, p)$ we denote the germ of $\nu$-invariant Lagrangian submanifold in $(\boldsymbol{R}^{2n}, \omega)$ ($\nu$-L-germ for short). As we know by [9], any $\nu$-L-germ $(L^G, p = (x_0, \xi_0))$ can be generated by the germ of the so-called

Morse family $F\colon (\boldsymbol{R}^n \times \boldsymbol{R}^l, (x_0, 0)) \to \boldsymbol{R}$, $F \in \mathfrak{E}_{\nu\oplus\delta}(n+l)$. Locally $(L^G, (x_0, \xi_0))$ can be written by the following equations:

$$\xi = \frac{\partial F}{\partial x}(x, \lambda), \qquad 0 = \frac{\partial F}{\partial \lambda}(x, \lambda), \tag{2.1}$$

where

$$\operatorname{rank}\left(\frac{\partial^2 F}{\partial x \partial \lambda}, \frac{\partial^2 F}{\partial \lambda \partial \lambda}\right)(x_0, 0) = l. \tag{2.2}$$

Conversely, any germ $F \in \mathfrak{E}_{\nu\oplus\delta}(n+l)$ satisfying (2.2) ($G$-Mf-germ for short) defines the $\nu$-L-germ via equations (2.1). An G-Mf-germ, generating $(L, p)$, with minimal number of parameters $l$ is called a minimal $G$-Mf-germ (cf. [2], [7]). A minimal $G$-Mf-germ can be equivalently characterized by the requirements

$$\left(\frac{\partial^2 F}{\partial \lambda \partial \lambda}\right)(x_0, 0) = 0.$$

The two $G$-Mf-germs $F' \in \mathfrak{E}_{\nu\oplus\delta'}(n+l)$, $F \in \mathfrak{E}_{\nu\oplus\delta}(n+l)$ are called $G$-L-equivalent if

$$F(x, \lambda) = F'(\varphi(x), \Lambda(x, \lambda)) + f(x),$$

where $(\Lambda, \varphi)\colon \boldsymbol{R}^{n+l} \to \boldsymbol{R}^{n+l}$ is a diffeomorphism, and $\Lambda \in \mathfrak{E}(n+l, \nu\oplus\delta; l, \delta')$, $\varphi \in \mathfrak{E}(n, \nu; n, \nu)$, $f \in \mathfrak{E}_\nu(n)$. To be able to compare the various $G$-Mf-germs with different dimensions of parameter spaces we introduce the notion of stable $G$-L-equivalence. We say that two $G$-Mf-germs $F_1 \in \mathfrak{E}_{\nu\oplus\delta'}(n+l_1)$, $F_2 \in \mathfrak{E}_{\nu\oplus\delta}(n+l_2)$ are stable $G$-L-equivalent if the corresponding $G$-Mf-germs $F_1 + Q_1 \in \mathfrak{E}_{\nu\oplus\delta'\oplus id}(n+l_1+r_1)$, $F_2 + Q_2 \in \mathfrak{E}_{\nu\oplus\delta\oplus id}(n+l_2+r_2)$, where $Q_1$, $Q_2$ are the nondegenerate quadratic forms of the additional variables, are $G$-L-equivalent (cf. [2]). By straightforward generalization of [2], [22], [9] we obtain

PROPOSITION 2.3. *Let $(L_1^G, p_1)$, $(L_2^G, p_2)$ be two $\nu$-L-germs of $(T^*\boldsymbol{R}^n, \omega)$. They are $\nu$-L-equivalent, i.e. there exists an $\nu$-L-equivalence $\Phi\colon T^*\boldsymbol{R}^n \to T^*\boldsymbol{R}^n$, such that $\Phi(p_1) = p_2$ and $\Phi(L_1^G) = L_2^G$, if and only if their $G$-Mf-germs are stable $G$-L-equivalent.*

For the corresponding minimal $G$-Mf-germs we have the stronger result.

PROPOSITION 2.4. *Two $\nu$-L-germs of $(T^*\boldsymbol{R}^n, \omega)$ are $\nu$-L-equivalent if and only if their minimal $G$-Mf-germs are $G$-L-equivalent.*

Correctness of these two equivalences is assured by the easily seen fact that any two $G$-Mf-germs generating the $\nu$-L-germ $(L^G, p) \subset T^*\boldsymbol{R}^n$ are stable $G$-L-equivalent. Let $(L^G, 0) \subseteq (\boldsymbol{R}^{2n}, \omega)$ be a $\nu$-L-germ. Let $k = \dim\ker D(\pi | L^G)(0)$, then the representation $\nu$ is reducible and can be written as the direct sum, at least, of two components $\nu = \nu_1 \oplus \nu_2$. The corresponding invariant subspaces for $\nu_1$ and $\nu_2$

respectively are given by Arnold's results (cf. [2], Theorem 10.6.), namely we can choose the numeration of coordinates in a neighbourhood of $0 \in \boldsymbol{R}^{2n}$ in such a way that $(x_I)$, $(x_J)$ parametrize the invariant subspaces corresponding to the representations $\nu_1$ and $\nu_2$ respectively, $I = (i_1, \ldots, i_k)$, $J = \{1, \ldots, n\} - I$. The lifted representation $T^* \nu$ has the form $\nu_1 \oplus \nu_2 \oplus \nu_1 \oplus \nu_2$. Thus we can consider $(\xi_I, x_J)$ as the new parametrization of the representation space for $\nu$. By [9], [2], [22] there exists a generating function, say $(\xi_I, x_J) \to S(\xi_I, x_J)$, for $(L^G, 0)$ and $S \in \mathfrak{E}_\nu(n)$. We will call this function a $\nu$-*IJ*-germ generating the $\nu$-L-germ $(L^G, 0)$ if $L^G$ is defined near $0 \in \boldsymbol{R}^{2n}$ by the equations:

$$\xi_J = \frac{\partial S}{\partial x_J}(\xi_I, x_J), \qquad x_I = -\frac{\partial S}{\partial \xi_I}(\xi_I, x_J). \tag{2.3}$$

If $k = \dim\ker D(\pi | L^G)(0)$, then we have

$$\frac{\partial^2 S}{\partial \xi_I \, \partial \xi_I}(0) = 0$$

and the germ

$$F\colon \boldsymbol{R}^{n+k} \ni (x, \lambda) \to S(\lambda, x_J) - \sum_{\alpha=1}^{k} \lambda_\alpha x_{i_\alpha} \tag{2.4}$$

is a minimal $G$-Mf-germ for $(L^G, 0)$, where the corresponding representation $\delta$ in the parameter space can be chosen as $\delta \cong \nu|_{\{x_J = 0\}}$ (cf. [22]). Summarizing the above properties of $(L^G, 0)$ and repeating the genericity argument of [2] (Proposition 10.11), we obtain

PROPOSITION 2.5. *Generically, any* $\nu$-L-*germ* $(L^G, 0) \subseteq (T^* \boldsymbol{R}^n, \omega)$ *has a* $\nu$-*IJ-germ of generating function* $S$ *with* $J = \emptyset$, i.e. $\xi \to S(\xi)$, $S \in \mathfrak{M}_\nu^2(n)$.

Now we introduce the fundamental notions necessary to obtain the finite classification of $\nu$-invariant Lagrangian submanifolds.

DEFINITION 2.6. Let $L^G \subseteq (T^* \boldsymbol{R}^n, \omega)$ be a $\nu$-invariant Lagrangian submanifold. A $\nu$-L-*germ* $(L^G, p)$ is called *stable* if for an open $\nu$-inv. neighbourhood $U$ of $p$ in $T^* \boldsymbol{R}^n$ and every smooth family $L_t^G$, $|t| < \varepsilon$, $(L_0^G, p) = (L^G, p)$ of $\nu$-invariant Lagrangian submanifolds there exist a smooth family $\Phi_t$ of $\nu$-L-equivalences such that $\Phi_t(L_t^G \cap U) \supset L^G \cap V$, for some open $\nu$-inv. neighbourhood $V$ of $p$ and sufficiently small $t$.

As was shown in [9] (cf. [2]) the standard notion of unfolding of a singularity [20] can be adapted to represent the $G$-Mf-germs generating the germs of Lagrangian submanifolds. Let $F \in \mathfrak{E}_{\nu \oplus \delta}(n+l)$. We will call $F$ the $\nu$-unfolding of $f = F|_{\{0\} \times \boldsymbol{R}^l} \in \mathfrak{E}_\delta(l)$ (cf. [9], [16]).

DEFINITION 2.7. Let $\tilde{F} \in C^{\infty}_{\nu\oplus\delta}(n+l)$ be a representative of the germ of the $\nu$-unfolding $F \in \mathfrak{E}_{\nu\oplus\delta}(n+l)$. We say that $F$ is *stable* if for any smooth family of functions $\tilde{F}_t \in C^{\infty}_{\nu\oplus\delta}(n+l)$, $|t| < \varepsilon$, $\tilde{F}_0 = \tilde{F}$, there exists a neighbourhood $U$ of 0 in $\boldsymbol{R}^{n+l}$, a family of diffeomorphisms $(\varphi_t, \Lambda_t) \in C^{\infty}(n, \nu; n, \nu) \oplus C^{\infty}(n+l, \nu\oplus\delta; l, \delta)$ and family of functions $f_t \in C^{\infty}_{\nu}(n)$ such that

$$F(x, \lambda) = F_t(\varphi_t(x), \Lambda_t(x, \lambda)) + f_t(x),$$

for $(x, \lambda) \in U$ and sufficiently small $t$.

According to the standard results of the theory of stable singularities we can at first characterize the stable germs by the necessary infinitesimal condition, the so-called versality condition.

DEFINITION 2.8 (cf. [16]). Let $F \in \mathfrak{E}_{\gamma\oplus\delta}(m+k)$ be a $\gamma$-unfolding of $f \in \mathfrak{E}_{\delta}(k)$. $F$ is called the *G-versal unfolding of f* if for any orthogonal representation $\nu$ of $G$ in $\boldsymbol{R}^n$ any $\nu$-unfolding $\bar{F} \in \mathfrak{E}_{\nu\oplus\delta}(n+k)$ of $f$ has the form

$$\bar{F}(x, \lambda) = F(\varphi(x), \Lambda(x, \lambda)) + \alpha(x),$$

where $\Lambda \in \mathfrak{E}(n+k, \nu\oplus\delta; k, \delta)$, $\varphi \in \mathfrak{E}(n, \nu; m, \gamma)$, $\alpha \in \mathfrak{E}_{\nu}(n)$.

On the basis of [7], [9], [18], [2] we know that the stable $\nu$-L-germs $(L^G, p)$ are effectively represented by the corresponding stable germs of $\nu$-unfoldings. Our notion of $\nu$-unfolding reduces to the standard notion of unfolding if we assume the trivial action of the group $G$. The corresponding theory is exhaustively presented in [24], [14]. For the symmetric case, following [2], [22], [7], we have the following *equivalent*:

PROPOSITION 2.9. *Let $(L^G, p)$ be a $\nu$-L-germ contained in $(T^* \boldsymbol{R}^n, \omega)$, let $F \in \mathfrak{E}_{\nu\oplus\delta}(n+k)$ be the corresponding G-Mf-germ, then the following properties are equivalent*:

(a) *$(L^G, p)$ is stable $\nu$-L-germ.*

(b) *The G-Mf-germ $F$ is stable as a $\nu$-unfolding of $f = F|_{\{0\}\times\boldsymbol{R}^k} \in \mathfrak{E}_{\delta}(k)$.*

Having the analytical representation of stable $\nu$-L-germs, given in Proposition 2.9, we can characterize them by the infinitesimal stability property, i.e. versality of the corresponding $G$-Mf-germs as $\nu$-unfoldings.

## 3. Infinitesimal stability conditions for G-invariant generating functions

Let $(L^G_t, 0)$ be a germ of the smooth family of $\nu$-L-germs $L^G_t \subseteq T^* \boldsymbol{R}^n$, $|t| < \varepsilon$. Up to the $\nu$-L-equivalence (cf. [2], Proposition 10.11) we can represent this family in the following form:

$$L^G_t = \left\{(x, \xi) \in T^* \boldsymbol{R}^n;\ x = \frac{\partial S_t}{\partial \xi}(\xi)\right\}, \tag{3.1}$$

where $t \to S_t(\xi) \in \mathfrak{E}_\nu^\infty(n)$ is an appropriate family of generating functions (deformation of $S_0$). So we can reformulate the local stability of $(L_0^G, 0)$ in terms of the smooth deformations $S_t$. If $(L_0^G, 0)$ is stable and $\varepsilon$ sufficiently small, then there exists a smooth family $\Phi_t$ of $\nu$-L-equivalences and an open neighbourhood $U$ of $0 \in T^* \mathbf{R}^n$ such that

$$\Phi_t(L_0 \cap U) \subset L_t. \tag{3.2}$$

Let us consider the vector field $X = \frac{d}{dt}\Phi_t|_{t=0}$ on $T^* \mathbf{R}^n$. Since each $\Phi_t$ ($|t| < \varepsilon$) is an equivariant symplectomorphism preserving the canonical fibration $T^* \mathbf{R}^n \to \mathbf{R}^n$, therefore $X$ must be the equivariant Hamiltonian vector field constant along the fibers of $T^* \mathbf{R}^n$, i.e. $X = -\frac{\partial H}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial H}{\partial \xi}\frac{\partial}{\partial x}$, where for $H \in C^\infty_{\nu\oplus\nu}(T^* \mathbf{R}^n)$ we can write

$$H(x, \xi) = \big(A(x) | \xi\big) + B(x), \tag{3.3}$$

where $(\cdot|\cdot)$ denotes the canonical scalar product on $\mathbf{R}^n$, and $\nu$-invariance of $H$ implies $A \in C^\infty(n, \nu; n, \nu)$, $B \in C_\nu^\infty(n)$. Now, using the Hamilton–Jacobi theorem [1] for the family $L_t^G$ we can write the equation

$$\frac{\partial S_t}{\partial t}(\xi)|_{t=0} = H\left(\frac{\partial S_t}{\partial \xi}(\xi), \xi\right)\Bigg|_{t=0} \tag{3.4}$$

near $0 \in \mathbf{R}^{n+1}$.

Note that to assure stability of $(L_0^G, 0)$ the left-hand side of (3.4) can be an arbitrary element of $C_\nu^\infty(n)$ satisfying the equation (3.4) with some $\nu$-equivariant Hamiltonian $H$ of the form (3.3).

Let us denote by $H_\nu$ the space of germs at $0 \in T^* \mathbf{R}^n$ of $\nu$-invariant Hamiltonians $H$: $T^* \mathbf{R}^n \to \mathbf{R}$ of the form (3.3). Let $i_{L^G} \in \mathfrak{E}(n, \nu; T^* \mathbf{R}^n, T^* \nu)$ be the Lagrangian immersion $\xi \to \left(\frac{\partial S_0}{\partial \xi}, \xi\right)$ corresponding to $(L^G, 0)$.

LEMMA 3.1. *Let $(L_0^G, 0)$ be a stable $\nu$-L-germ, with a generating function $S_0 \in \mathfrak{E}_\nu(n)$. Then we have*

$$\mathfrak{E}_\nu(n) = i^*_{L_0^G} H_\nu. \tag{3.5}$$

The proof of this lemma follows immediately from Definition 2.6 and [2] p. 21.

Let $\pi$ be the projection, $\pi(x, \xi) = x$, we denote $V_j(x, \xi) = \big(\xi | \varphi_j(x)\big)$, where $\varphi_j(x) := \frac{\partial \mu_j}{\partial y}(x, 0)$ and $\mu = (\mu_1, \ldots, \mu_b)$: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^b$ is the Hilbert map for the $\nu\oplus\nu$ action of $G$ on $\mathbf{R}^n \times \mathbf{R}^n$.

PROPOSITION 3.2. *Let $(L_0^G, 0)$ be a stable v-L-germ, with a generating function $S_0 \in \mathfrak{E}_v(n)$. Then the following infinitesimal stability condition is fulfilled:*

$$\mathfrak{E}_v(n) = i_L^* G \langle V_1, \ldots, V_b, 1 \rangle_{\pi^* \mathfrak{E}_v(n)}, \tag{3.6}$$

*where $\langle V_1, \ldots, V_b, 1 \rangle_{\pi^* \mathfrak{E}_v(n)}$ is the submodule of $\mathfrak{E}_{v \oplus v}(n+n)$ generated by $V_1, \ldots, V_b$, 1 over $\pi^* \mathfrak{E}_v(n)$.*

*Proof*: We know that $\mathfrak{E}(n, v; n, v)$ is finitely generated over $\mathfrak{E}_v(n)$ with generators $\varphi_j(x) = \frac{\partial \mu_j}{\partial y}(x, 0)$ (see Proposition 2.1, b). Thus the right-hand side of (3.5) can be written in the following way. Let $f \in \mathfrak{E}_v(n)$, so by Lemma 3.1 we have

$$f(\xi) = \left(\xi \,|\, \sum_{i=1}^{b} c_i\left(\frac{\partial S_0}{\partial \xi}(\xi)\right)\varphi_i\left(\frac{\partial S_0}{\partial \xi}(\xi)\right)\right) + B\left(\frac{\partial S_0}{\partial \xi}(\xi)\right)$$

for some $c_i(x) \in \mathfrak{E}_v(n)$, $B(x) \in \mathfrak{E}_v(n)$, which gives exactly the infinitesimal stability condition (3.6).

Let $F \in \mathfrak{E}_v(n)$, $(\xi_I, x_J) \to F(\xi_I, x_J)$ be the $v$-$IJ$-germ generating for the v-L-germ $(L^G, 0) \subseteq (T^* \boldsymbol{R}^n, \omega)$ (cf. § 2). The corresponding immersion of $L^G$, $i_{IJ}$: $\boldsymbol{R}^n \to T^* \boldsymbol{R}^n$, has the form

$$i_{IJ}(\xi_I, x_J) = \left(\frac{\partial F}{\partial \xi_I}(\xi_I, x_J), x_J, \xi_I, -\frac{\partial F}{\partial x_J}(\xi_I, x_J)\right). \tag{3.7}$$

Let us define for the $v$-invariant germs $\varrho \circ \pi \circ i_{IJ}$, $V_j \circ i_{IJ}$ $(j = 1, \ldots, b)$ the following smooth mappings $\tilde{U} \in \mathfrak{E}(a, a)$, $\tilde{V}_j \in \mathfrak{E}(a)$

$$\tilde{U} \circ \varrho = \varrho \circ \pi \circ i_{IJ}, \qquad \tilde{V}_j \circ \varrho = V_j \circ i_{IJ} \qquad (j = 1, \ldots, b), \tag{3.8}$$

where $\varrho$: $\boldsymbol{R}^n \to \boldsymbol{R}^a$ is the Hilbert map corresponding to the $v$-action of $G$ on $\boldsymbol{R}^n$.

PROPOSITION 3.3. *For a stable v-L-germ $(L^G, 0) \subseteq T^* \boldsymbol{R}^n$ and for its corresponding v-IJ-germ $F \in \mathfrak{E}_v(n)$ of generating function we have the following equivalent infinitesimal stability conditions*

$$\mathfrak{E}_v(n) = \langle \varrho^* \tilde{V}_1, \ldots, \varrho^* \tilde{V}_b, 1)_{(\pi \circ i_{IJ})^* \mathfrak{E}_v(n)}, \tag{3.9}$$

$$\mathfrak{E}_v(n) = \langle \varrho^* \tilde{V}_1, \ldots, \varrho^* \tilde{V}_b, 1 \rangle_{(\tilde{U} \circ \varrho)^* \mathfrak{E}(a)}, \tag{3.10}$$

$$\mathfrak{E}_v(n) = \langle \varrho^* \tilde{V}_1, \ldots, \varrho^* \tilde{V}_b, 1 \rangle_{\boldsymbol{R}} + ((\pi \circ i_{IJ})^* \mathfrak{M}_v(n))\, \mathfrak{E}_v(n), \tag{3.11}$$

$$\mathfrak{E}(a) = \langle \tilde{V}_1, \ldots, \tilde{V}_b, 1 \rangle_{\boldsymbol{R}} + \langle \tilde{U}_1, \ldots, \tilde{U}_a)_{\mathfrak{E}(a)} + M_\varrho(a), \tag{3.12}$$

*where by $M_\varrho(a) \subset \mathfrak{E}(a)$ we denote all germs vanishing on $\varrho(\boldsymbol{R}^n)$.*

*Proof*: One can easily see that (3.9) results from (3.5), (3.6) and (3.7). By (3.8), conditions (3.9) and (3.10) are equivalent. Equivalence of (3.9) and (3.11) is a consequence of the Equivariant Preparation Theorem (see [13] p. 116). In fact $\mathfrak{E}_v(n)/((\pi \circ i_{IJ})^* \mathfrak{M}_v(n))\,\mathfrak{E}_v(n)$ is a finite-dimensional vector space and for its generators we can choose $\varrho^* \tilde{V}_1, \ldots, \varrho^* \tilde{V}_b, 1$. Taking into account the equation $\tilde{U} \circ \varrho = \varrho \circ \pi \circ i_{IJ}$ we can rewrite (3.11) in the form (3.12). We need here only the fact that the equality $g \circ \varrho = g' \circ \varrho$, for some functions $g, g' \in \mathfrak{E}(a)$, implies $g - g' \in M_\varrho(a)$.

*Remark* 3.4. Assume that $v$ is trivial, thus $\varrho = id_{\mathbf{R}^n}$ and $\varrho^* \tilde{V}_I(\xi_I, x_J) = \xi_I$, $\varrho^* \tilde{V}_J(\xi_I, x_J) = \frac{\partial F}{\partial x_J}(\xi_I, x_J)$, $\mathfrak{E}_v(n) = \mathfrak{E}(n)$, $M_\varrho(n) = \{0\}$, $\tilde{U} = id_{\mathbf{R}^n}$. Finally (3.9), (3.10) take the form

$$\mathfrak{E}(n) = \left\langle \frac{\partial F}{\partial x_J}, \xi_I, 1 \right\rangle_{(\pi \circ i_{IJ})^* \mathfrak{E}(n)},$$

where

$$(\pi \circ i_{IJ})(\xi_I, x_J) = \left( \frac{\partial F}{\partial \xi_I}(\xi_I, x_J), x_J \right),$$

and for (3.11), (3.12) we have

$$\mathfrak{E}(n) = \left\langle \frac{\partial F}{\partial \xi_I}, x_J \right\rangle_{\mathfrak{E}(n)} + \left\langle \frac{\partial F}{\partial x_J}, \xi_I, 1 \right\rangle_{\mathbf{R}}.$$

Eliminating the variables $x_J$ by the Preparation Theorem [6], we obtain

$$\mathfrak{E}(k) = \left\langle \frac{\partial F}{\partial \xi_I}\bigg|_{x_J = 0} \right\rangle_{\mathfrak{E}(k)} + \left\langle \frac{\partial F}{\partial x_J}\bigg|_{x_J = 0} \xi_I, 1 \right\rangle_{\mathbf{R}}, \quad k = \# I,$$

which is exactly the standard versality condition for versal deformations [20], used by Arnold [2] in the classification theory of stable Lagrangian singularities.

EXAMPLE 3.5. (Infinitesimal stability condition for $D_m$-action.) In many applications of equivariant singularity theory [8] we find the following irreducible representation of the group $D_m$:

$$\mu(g_1)\colon (x_1, x_2) \to (x_1, -x_2),$$

$$\mu(g_2)\colon (x_1, x_2) \to \left( x_1 \cos\frac{2\pi}{m} - x_2 \sin\frac{2\pi}{m}, x_1 \sin\frac{2\pi}{m} + x_2 \cos\frac{2\pi}{m} \right),$$

where, $g_1, g_2$ are generators of $D_m$. Let us write the corresponding infinitesimal stability conditions for $D_m$-equivariant singularities with corank at most two. In this case we consider the action

$$\nu\colon D_m \times \mathbf{R}^n \to \mathbf{R}^n, \big(g, (x_1, \ldots, x_n)\big) \to \big(\mu(g)(x_1, x_2), x_3, \ldots, x_n\big)$$

and the generating function

$$F(\xi_1, \xi_2, x_3, \ldots, x_n) = \tilde{F} \circ \varrho(\xi_1, \xi_2, x_3, \ldots, x_n),$$

where the corresponding Hilbert map

$$\varrho(x_1, \ldots, x_n) = (z\bar{z}, z^m + \bar{z}^m, x_3, \ldots, x_n), \qquad z = x_1 + ix_2.$$

Here $I = \{1, 2\}$, $J = \{3, 4, \ldots, n\}$ and

$$i_{IJ}(\xi_I, x_J) = \left(\frac{\partial F}{\partial \xi_I}(\xi_I, x_J), x_J, \xi_I, -\frac{\partial F}{\partial x_J}(\xi_I, x_J)\right).$$

We easily calculate

$$V_1(x, \xi) = \tfrac{1}{2}(\xi_I \bar{z} + \bar{\xi}_I z), \qquad V_2(x, \xi) = \tfrac{1}{2}(\xi_I z^{m-1} + \bar{\xi}_I \bar{z}^{m-1}),$$
$$V_i(x, \xi) = \xi_i, \qquad i = 3, \ldots, n,$$

where we also denote

$$\xi_I = \xi_1 + i\xi_2.$$

From (3.8) after straightforward calculations we obtain

$$\tilde{V}_1(u) = 2u_1 \tilde{F}_{,1}(u) + mu_2 \tilde{F}_{,2}(u), \qquad u = (u_1, u_2, \ldots, u_n),$$
$$\tilde{V}_2(u) = 2^{m-2} \sum_{j=0}^{m-1} \binom{m-1}{j} m^j u_1^{m-j} \tilde{F}_{,1}(u)^{m-j-1} \tilde{F}_{,2}(u)^j w_{j-1}(u),$$
$$\tilde{V}_i(u) = -\tilde{F}_{,i}(u), \qquad 3 \leqslant i \leqslant n,$$

where $w_{-1}(n) = u_2/u_1^m$ and the polynomials (of $(j-1)$-degree) $w_{j-1}(u) = (\bar{\xi}^m)^{j-1} + (\xi^m)^{j-1}$ are determined by the following recurrent formula:

$$(\bar{\xi}^m)^k + (\xi^m)^k = u_2^k - \sum_{i=1}^{[\frac{k-1}{2}]} \binom{k}{i} (u_1)^i \big((\bar{\xi}^m)^{k-2i} + (\xi^m)^{k-2i}\big) - \frac{1}{2}\big(1 - (-1)^{k+1}\big) \binom{k}{\frac{1}{2}k} u_1^{[k/2]}.$$

Also for $\tilde{U}_i (i = 1, \ldots, n)$ we obtain

$$\tilde{U}_1(u) = 4u_1 \tilde{F}_{,1}^2(u) + 4mu_2 \tilde{F}_{,1}(u) \tilde{F}_{,2}(u) + 4m^2 u_1^{m-1} \tilde{F}_{,2}^2(u),$$
$$\tilde{U}_2(u) = 2^m \sum_{j=0}^{m} \binom{m}{j} m^j F_{,1}(u)^{m-j} F_{,2}(u)^j u_1^{m-j} w_{j-1}(u),$$
$$\tilde{U}_i(u) = u_i, \qquad 3 \leqslant i \leqslant n.$$

Using the Malgrange preparation theorem, we find that (3.12) is equivalent to the following condition:

$$\mathfrak{E}(2) = \langle u_1 \bar{F}_{,1}^2 + mu_2 \bar{F}_{,1} \bar{F}_{,2} + m^2 u_1^{m-1} \bar{F}_{,2}^2, \sum_{j=0}^{m} \binom{m}{j} m^j \bar{F}_{,1}^{m-j} \bar{F}_{,2}^j u_1^{m-j} \bar{w}_{j-1} \rangle_{\mathfrak{E}(2)} +$$

$$+ \langle 2u_1 \bar{F}_{,1} + mu_2 \bar{F}_{,2}, \sum_{j=0}^{m-1} \binom{m-1}{j} m^j u_j^{m-j} \bar{F}_{,1}^{m-j-1} \bar{F}_{,2}^j \bar{w}_{j-1}, \bar{F}_{,3}, \ldots, \bar{F}_{,n}, 1 \rangle_{\mathbf{R}} +$$

$$+ M_{\bar{\varrho}}(2),$$

where $\bar{F}_{,i}(u_1, u_2) = \frac{\partial \bar{F}}{\partial u_k}(u_1, u_2, 0)$, $\bar{w}_{j-1}(u_1, u_2) = w_{j-1}(u_1, u_2, 0)$, and $M_{\bar{\varrho}}(2)$ denotes the ideal of smooth function-germs vanishing on the set;

$$\{(u_1, u_2)\colon\ 4u_1^m - u_2^2 \geqslant 0,\ u_1 \geqslant 0\}.$$

This reduced formula for infinitesimal stability provides us with the first step in classifying stable classes of $\nu$-L-germs. We postpone a detailed analysis of this case to a forthcoming paper. The classifying methods are the same as the ones presented in Section 4 for the $(\mathbf{Z}_2)^q$-action.

*Remark* 3.6. Let $\varrho\colon \mathbf{R}^n \to \mathbf{R}^k$ be a Hilbert map for the $\nu$-action of $G$ on $\mathbf{R}^n$, so $\varrho(\mathbf{R}^n) \subset \mathbf{R}^k$ is the semialgebraic set defined, say, by the equations $f_1(u) = 0, \ldots, f_r(u) = 0$ and inequalities $h_1(u) \geqslant 0, \ldots, h_s(u) \geqslant 0$, where $f_i,\ h_j \in \mathbf{R}[u]$, $u \in \mathbf{R}^k$ are irreducible. Let us denote by $M_\varrho^*(k) = \langle f_1, \ldots, f_r \rangle_{\mathfrak{E}(k)}$ the ideal in $\mathfrak{E}(k)$ generated by $f_1, \ldots, f_r$. Obviously we have

$$M_\varrho^*(k) \subset M_\varrho(k). \tag{3.13}$$

However, the equality in (3.13) usually does not hold, so we cannot replace $M_\varrho(k)$ by $M_\varrho^*(k)$ in the condition (3.12). Nevertheless, by Nakayama's Lemma (cf. [6]), we can make such replecement if

$$M_\varrho(k) - M_\varrho^*(k) \subset \mathfrak{M}^\infty(k). \tag{3.14}$$

Let us assume that (3.14) is fulfilled.

DEFINITION 3.7. The equality

$$\mathfrak{E}(a) = \langle \tilde{U}_1, \ldots, \tilde{U}_a \rangle_{\mathfrak{E}(a)} + \langle \tilde{V}_1, \ldots, \tilde{V}_b, 1 \rangle_{\mathbf{R}} + M_\varrho^*(a) \tag{3.15}$$

is called the *reduced condition for infinitesimal* $\nu$-L-*stability*.

*Remark* 3.8. Let us notice that the dependence of $\tilde{V}_i$, $\tilde{U}_i$ on $\frac{\partial \tilde{F}}{\partial u_j}$, in general, is not linear. In what follows we propose an equivalent approach to the classification problem of stable $\nu$-L-germs using the Morse family notion. In that approach we derive the corresponding linear infinitesimal $\nu$-L-stability condition. Equivalence of these two conditions results from the equivariant version of the Malgrange preparation theorem (cf. [13]).

EXAMPLE 3.9. Assume the representation $\nu$ of $G = \mathbf{Z}_2$ on $\mathbf{R}^n$ has the form

$$\mathbf{R}^n \ni (x_1, \ldots, x_n) \to (\varepsilon x_1, x_2, \ldots, x_n), \quad \varepsilon \in G.$$

Let the $\nu$-L-germ $(L^G, 0) \subset T^* \mathbf{R}^n$ have a $\nu$-$IJ$-germ $S(\xi_1, x_2, \ldots, x_n) = \tilde{S} \circ \varrho(\xi_1, x_2, \ldots, x_n)$, where $\varrho$: $\mathbf{R}^n \to \mathbf{R}^n$, $\varrho(\xi_1, x_2, \ldots, x_n) = (\xi_1^2, x_2, \ldots, x_n)$. In this case $M_\varrho(n) \subset \mathfrak{M}^\infty(n)$, $M_\varrho^*(n) = \{0\}$,

$$\tilde{V}_1(u) = -2u_1 \tilde{S}_{,1}(u),$$
$$\tilde{V}_i(u) = \tilde{S}_{,i}(u), \quad 2 \leqslant i \leqslant n,$$
$$\tilde{U}_1(u) = u_1 \tilde{S}_{,1}^2(u),$$
$$\tilde{U}_j(u) = u_j, \quad 2 \leqslant j \leqslant n.$$

Thus we see that (3.12) is equivalent to the following condition:

$$\mathfrak{E}(n) = \langle u_1 \tilde{S}_{,1}^2(u), u_2, \ldots, u_n \rangle_{\mathfrak{E}(n)} + \langle u_1 \tilde{S}_{,1}(u), \tilde{S}_{,2}(u), \ldots, \tilde{S}_{,n}(u), 1 \rangle_{\mathbf{R}}. \quad (3.16)$$

Using the Malgrange preparation theorem we obtain the following, suitable for further calculation, equivalent form of (3.16):

$$\mathfrak{E}(1) = \langle u_1 \bar{S}_{,1}^2(u_1) \rangle_{\mathfrak{E}(1)} + \langle u_1 \bar{S}_{,1}(u_1), \bar{S}_{,2}(u_1), \ldots, \bar{S}_{,n}(u_1), 1 \rangle_{\mathbf{R}},$$

where

$$\mathfrak{E}(1) \ni \bar{S}_{,i}(u_1) = \frac{\partial \tilde{S}}{\partial u_i}(u_1, 0), \quad i = 1, \ldots, n.$$

## 4. Stable $\nu$-L-germs with respect to the $(\mathbf{Z}_2)^q$ action

Now for the purposes of applications (cf. [11], [8]) we consider the following action of $G = (\mathbf{Z}_2)^q$

$$\nu\colon (\mathbf{Z}_2)^q \times \mathbf{R}^n \ni (\varepsilon_1, \ldots, \varepsilon_q, x) \to (x_1, \ldots, x_{n-q}, \varepsilon_1 x_{n-q+1}, \ldots, \varepsilon_q x_n) \in \mathbf{R}^n.$$

The corresponding Hilbert map (orbit mapping) for $\nu$ is defined by

$$\varrho(x) = (x_1, \ldots, x_{n-q}, x_{n-q+1}^2, \ldots, x_n^2).$$

Any $\nu$-L-germ $(L^G, 0) \subseteq T^* \mathbf{R}^n$ is $\nu$-L-equivalent to the $\nu$-L-germ, say $(L_1^G, 0) \subseteq T^* \mathbf{R}^n$, which has the following generating function (see § 2):

$$\mathfrak{E}_\nu(n) \ni S(\xi) = \tilde{S} \circ \varrho(\xi), \quad (4.1)$$

where $\tilde{S} \in \mathfrak{E}(n)$.

Let us denote the partial derivatives $\frac{\partial S}{\partial \xi_i}$, $\frac{\partial^2 S}{\partial \xi_i \partial \xi_j}$, etc. of the function $S$ by $S_{,i}$, $S_{,ij}$, etc. and their values at 0 by $a_i$, $a_{ij}$, etc. Using Proposition 3.3, after straightforward calculations we obtain immediately

PROPOSITION 4.1. *The* v-L-*germ* $(L^G, 0) \subset T^* \boldsymbol{R}^n$, *generated by the function* $S = \tilde{S} \circ \varrho$ *is infinitesimally* v-L-*stable if for every germ* $\alpha \in \mathfrak{E}(n)$ *there exists the decomposition*

$$\alpha(z) = \sum_{i=1}^{n-q} \tilde{S}_{,i}(z) h_i(z) + c_0 + \sum_{i=1}^{n-q} c_i z_i + \sum_{j=n-q+1}^{n} (z_j \tilde{S}_{,j}^2(z) h_j(z) + z_j \tilde{S}_{,j}(z) c_j), \tag{4.2}$$

*where* $h_k \in \mathfrak{E}(n)$ *and* $c_l \in \boldsymbol{R}$.

To be more concrete and useful in some physical applications (cf. [9], [11]), without loss of generality we concentrate now on the case $q = 2$, $n = 3$. The general case can be treated exactly in the same way, so we omit it here.

DEFINITION 4.2. The function germ $\tilde{S} \in \mathfrak{E}(n)$, introduced in Proposition 4.1 and such that (4.2) is fulfilled, is called an *infinitesimally* v-L-*stable germ*.

PROPOSITION 4.3. *A function-germ* $\tilde{S} \in \mathfrak{E}(3)$ *is infinitesimally* v-L-*stable if and only if the following conditions are satisfied*:

$(A_0)$ $a_1 \neq 0$ *(trivial case) or* $a_1 = 0$, *and*
$(A_1)$ $a_2 a_3 a_{11} \neq 0$ *or*
$(A_2)$ $a_{11} = 0$ *and* $a_2 a_3 a_{111} \neq 0$ *or*
$(A_3')$ $a_2 = 0$ *and* $a_3 a_{11} a_{12} (a_{12}^2 - a_{11} a_{22}) \neq 0$ *or*
$(A_3'')$ $a_3 = 0$ *and* $a_2 a_{11} a_{13} (a_{13}^2 - a_{11} a_{33}) \neq 0$.

*Proof*: (Necessity) The above conditions arise as necessary for the decomposition (4.2) mod $\mathfrak{M}^3(3)$.

(Sufficiency) For $\alpha \in \mathfrak{E}(3)$ we show how to define germs $h_i$ and constants $c_i$ satisfying (4.2) in the respective cases:

$(A_0)$: It is enough to take $h_1 = \alpha/\tilde{S}_{,1}$, $h_2 = h_3 = 0$, $c_i = 0$ for $i = 0, 1, 2, 3$.

$(A_1)$: Now $a_1 = 0$ and let $a_2 a_3 a_{11} \neq 0$. We define $\alpha_i$, $u_i \in \mathfrak{E}(3)$ as follows: $\alpha(x) = \alpha(0) + \sum_{i=1}^{3} x_i \alpha_i(x)$, $\tilde{S}_{,1}(x) = \sum_{i=1}^{3} x_i U_i(x)$. Then $U_1(0) = a_{11} \neq 0$, $\tilde{S}_{,2}(0) = a_2 \neq 0$, $\tilde{S}_{,3}(0) = a_3 \neq 0$. Hence we can take $c_0 = \alpha(0)$, $c_1 = c_2 = c_3 = 0$, $h_1 = \alpha_1/u_1$, $h_2 = (\alpha_2 - U_2 h_1)/\tilde{S}_{,2}^2$, $h_3 = (\alpha_3 - U_3 h_1)/\tilde{S}_{,3}^2$, which satisfy (4.2).

$(A_2)$: Let $a_1 = 0$, $a_{11} = 0$, $a_2 a_3 a_{111} \neq 0$. We define the new germs $\alpha_i$, $U_i \in \mathfrak{E}(3)$ $(i = 1, 2, 3)$, $\tilde{S}_{,1}(x) = x_1^2 U_1(x) + x_2 U_2(x) + x_3 U_3(x)$, $\alpha(x) = \alpha(0) + x_1 \alpha_{,1}(0) + x_1^2 \beta_1(x) + x_2 \alpha_2(x) + x_3 \alpha_3(x)$. In this case $U_1(0) = a_{111} \neq 0$, $\tilde{S}_{,2}(0) = a_2 \neq 0$,

$\tilde{S}_{,3}(0) = a_3 \neq 0$ thus it suffices to put $c_0 = \alpha(0)$, $c_1 = \alpha_{,1}(0)$, $h_1 = \beta_1/U_1$, $h_2 = (\alpha_2 - U_2 h_1)/\tilde{S}_{,2}^2$, $h_3 = (\alpha_3 - U_3 h_1)/\tilde{S}_{,3}^2$.

($A_3'$) (For $A_3''$ we have the same procedure.): Assume $a_1 = 0$, $a_2 = 0$ and $a_3 a_{11} a_{12}(a_{12}^2 - a_{11} a_{22}) \neq 0$. We see that the germ

$$\beta(x) = \alpha(x) - c_0 - c_1 x_1 - c_2 \tilde{S}_{,2}(x) x_2 + \tilde{S}_{,1}(x)(g_0 + g_1 x_1 + g_2 x_2)$$

belongs to the ideal $\langle x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_3\rangle$ provided that $c_0 = \alpha(0)$ and the constants $c_1, c_2, g_0, g_1, g_2$ satisfy the following system of linear equations (solvable iff $a_{11} a_{12}(a_{12}^2 - a_{11} a_{22}) \neq 0$):

$$\alpha_{,1}(0) = a_{11} g_0 + c_1,$$

$$\alpha_{,2}(0) = a_{12} g_0,$$

$$\alpha_{,11}(0) = a_{111} g_0 + 2a_{11} g_{11},$$

$$\alpha_{,12}(0) = a_{112} g_0 + a_{12} g_1 + a_{11} g_2 + a_{12} c_2,$$

$$\tfrac{1}{2}\alpha_{,22}(0) = a_{112} g_0 + a_{12} g_2 + a_{22} c_2.$$

Now consider the germs $U_{ij}$, $\beta_i$ satisfying the following decompositions:

$$\tilde{S}_{,i}(x) = x_1 U_{i1}(x_1) + x_2 U_{i2}(x_1, x_2) + x_3 U_{i3}(x_1, x_2, x_3)$$

for $i = 1, 2$ and

$$\beta(x) = x_1^3 \beta_1(x) + x_1^2 x_2 \beta_2(x) + x_1 x_2^2 \beta_3(x) + x_2^3 \beta_4(x) + x_3 \beta_5(x).$$

Let the germs $k_1, k_2, k_3, h_2 \in \mathfrak{E}(3)$ be the solutions to the following system of linear equations:

$$\beta_1 = U_{11} k_1,$$

$$\beta_2 = U_{12} k_1 + U_{11} k_3 + U_{21}^2 h_2,$$

$$\beta_3 = U_{11} k_2 + U_{12} k_3 + 2U_{21} U_{22} h_2,$$

$$\beta_4 = U_{12} k_2 + U_{22}^2 h_2.$$

The above system is solvable since the system determinant at 0 is equal to $a_{11}(a_{12}^2 - a_{11} a_{22})^2 \neq 0$. One can easily check that the germ

$$\gamma(x) := \beta(x) - \tilde{S}_{,1}(x_1)\big(x_1^2 k_1(x) + x_2^2 k_2(x) + x_1 x_2 k_3(x)\big) - x_2 \tilde{S}_{,2}^2(x) h_2(x)$$

belongs to the ideal $\langle x_3\rangle$ in $\mathfrak{E}(3)$, i.e. $\gamma$ has the form $\gamma(x) = x_3 \gamma'(x)$, where $\gamma' \in \mathfrak{E}(3)$. Finally, we observe that $c_0, c_1, c_3, h_2(x)$ defined as above, $c_3 := 0$ and

$$h_1(x) := g_0 + g_1 x_1 + g_2 x_2 + x_1^2 k_1(x) + x_2^2 k_2(x) + x_1 x_2 k_3(x),$$

$$h_3(x) := \gamma'(x)/\bar{S}_{,3}^2(x)$$

satisfy (4.2). This completes the proof of Proposition 4.3.

Now we consider the recognition problem for the stable $\nu$-L-germs. Let $J^2(\boldsymbol{R}^3, \boldsymbol{R}) \cong \boldsymbol{R}^3 \times J_0^2(\boldsymbol{R}^3, \boldsymbol{R})$ be the space of 2-jets of $C^\infty(3)$-functions (cf. [6]) with a coordinate system $(x_i; y, y_i, y_{ij})$. Let $M_1, M_2, M_3, M_4$ be submanifolds of $J_0^3(\boldsymbol{R}^3, \boldsymbol{R})$ defined by the following conditions:

($A_1$): $M_1 = \{y_2 y_3 y_{11} \neq 0\}$,

($A_2$): $M_2 = \{y_{11} = 0, \; y_2 y_3 \neq 0\}$,

($A_3'$): $M_3 = \{y_2 = 0, \; y_3 y_{11} y_{12}(y_{12}^2 - y_{11} y_{22}) \neq 0\}$,

($A_3''$): $M_4 = \{y_3 = 0, \; y_2 y_{11} y_{13}(y_{13}^2 - y_{11} y_{33}) \neq 0\}$.

Their codimensions in $J_0^2(\boldsymbol{R}^3, \boldsymbol{R})$ are 0, 1, 1 and 1 respectively. The subset of those 2-jets, say at $x = (x_1, 0, 0)$, $x_1 \in \boldsymbol{R}$, which do not belong to $\bigcup_i M_i$ has codimension 2, i.e. it is a finite union of submanifolds of $J_0^2(\boldsymbol{R}^3, \boldsymbol{R})$ of codimension 2. Given $F \in C^\infty(3)$, let $j^2 F\colon \boldsymbol{R}^3 \to J_0^2(\boldsymbol{R}^3, \boldsymbol{R})$ denote the 2-jet extension of $F$ (see e.g. [24], [20]). Thus on the basis of Thom's transversality theorem [13], [24] we obtain immediately

PROPOSITION 4.4. *All germs $(j^2 F)(x_1, 0, 0)$ of a generic function $F \in C^\infty(3)$ belong to $\bigcup_{i=1}^{4} M_i$.*

Let us denote by $E_i$, $i = 1, 2, 3, 4$ the subsets of all germs $F \in \mathfrak{E}(3)$ satisfying conditions ($A_1$), ($A_2$), ($A_3'$), ($A_3''$) of Proposition 4.3 respectively; together with $F(0) = F_{,1}(0) = 0$. These germs generate the corresponding $\nu$-L-germs $\left(\left\{-\dfrac{\partial(F \circ \varrho)}{\partial \xi}(\xi), \xi\right\}, 0\right)$. Using the appropriate canonical transformations we easily obtain

PROPOSITION 4.5. *Let $F \in C^\infty(3)$, $x_0 = (x_1, 0, 0)$. If*

$$(j^2 F)(x_0) \in M_i, \qquad i = 1, 2, 3, 4,$$

*then the germ $(F, x_0)$ is $\nu$-L-equivalent to a germ belonging to $E_i$.*

Let us recall that two $\nu$-inv. germs of generating functions are $\nu$-L-equivalent iff the corresponding $\nu$-L-germs are $\nu$-L-equivalent (see Section 2).

Now we try to find classes of $\nu$-L-equivalent germs in $E_i$. For this purpose we introduce

DEFINITION 4.6. Let $F(x, t) = F_t(x)$ be a smooth function on $\boldsymbol{R}^3 \times J$, where $J$ is an open interval in $\boldsymbol{R}$. $F$ is called *inf-homotopy* (and the germs $(F_a, 0)$, $(F_b, 0)$, $a, b \in J$ are called inf-homotopic) if all germs $(F_t, 0)$ belong to the same class $E_i$ (we assume $F(0, t) = \partial F(0, t)/\partial t = 0$ for any inf-homotopy $F(x, t)$).

PROPOSITION 4.7. *Any germ belonging to $E_i$ ($i = 1, \ldots, 4$) is inf-homotopic to one from the following list:*

($E_1$) $F(x_1, x_2, x_3) = \pm x_1^2 \pm x_2 \pm x_3$,
($E_2$) $F(x_1, x_2, x_3) = \pm x_1^3 \pm x_2 \pm x_3$,
($E_3$) $F(x_1, x_2, x_3) = \pm x_1^2 \pm (x_1 \pm x_2)^2 \pm x_3$,
($E_4$) $F(x_1, x_2, x_3) = \pm x_3^2 \pm (x_1 \pm x_3)^2 \pm x_2$.

Let us remark that the generating functions $F \circ \varrho$, for $F$ belonging to the respective classes ($E_i$), correspond to the classification proved by Arnold in [2]. Hence this coincidence justifies our notation ($A_1$), ($A_2$), ($A_3'$), ($A_3''$).

*Proof of Proposition 4.7*: We consider only the case ($E_3$). The conditions $\operatorname{sgn} a_{11} = \pm 1$, $\operatorname{sgn} a_2 = \pm 1$, $\operatorname{sgn} a_3 = \pm 1$, $\operatorname{sgn}(a_{12}^2 - a_{11} a_{22}) = \pm 1$, distinguish in the 4-dimensional space of coefficients $(a_{11}, a_{12}, a_3, a_{22}) = (F_{,11}, F_{,12}, F_{,3}, F_{,22})$ (0) sixteen open convex regions. So, if germs $F', F'' \in E_3$ correspond to the same region, the following function:

$$F(x, t) = tF'(x) + (1-t) F''(x)$$

is an inf-homotopy between them. The observation that the above forms of $E_3$ correspond to each of these regions completes the proof.

PROPOSITION 4.8. *Let $F(x, t)$, $(x, t) \in \boldsymbol{R}^3 \times J$ be an inf-homotopy, $S(x, t) := F(\varrho(x), t)$ and $t_0 \in J$ be a fixed point. Then there exists an open neighbourhood $U \times I$ of $(0, t_0)$ and smooth functions $a_i(x, t)$, $b(x, t)$ on $\boldsymbol{R}^3 \times \boldsymbol{R}$, with compact supports, such that*

(i) $a_1(x, t) = \dfrac{\partial b}{\partial x_1}(0, t)$ *for* $t \in I$,

*and*

(ii) $-\dfrac{\partial S}{\partial t}(x, t) = H\left(x, \dfrac{\partial S}{\partial x}(x, t), t\right)$ *for* $(x, t) \in U \times I$,

*where*

$$H(x, y, t) = a_1(\varrho(y), t) x_1 + \Sigma_2^3 a_i(\varrho(y), t) x_i y_i + b(\varrho(y), t)$$

*for* $(x, y, t) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3 \times \boldsymbol{R}$.

*Proof*: Assume $t_0 = 0$. From the proof of Proposition 4.3 it follows that for any germ $\alpha \in \mathfrak{E}(4)$ there exists the decomposition

$$\alpha(x, t) = F_{,1}(x, t) h_1(x, t) + c_1(t) x_1 + c_0(t) + \Sigma_2^3 (x_i F_{,i}^2(x, t) h_i(x, t) + \\ + x_i F_{,i}(x, t) c_i(t)),$$

with $c_i \in \mathfrak{E}(1)$, $h_i \in \mathfrak{E}(4)$. Substituting $c_i(t) = c_i(0) + \bar{c}_i(t)$, for $i = 0, 1, 2, 3$ and

$$h(x, t) = \bar{c}_0(t) + \bar{c}_1(t)x_1 + \Sigma_2^3 x_i F_{,i}(x, t)\bar{c}_i(t),$$

we obtain

$$\alpha(x, t) = F_{,1}(x, t)h_1(x, t) + c_1(0)x_1 + c_0(0) + \Sigma_2^3\big(x_i F_{,i}^2(x, t)h_i(x, t) + \\ + x_i F_{,i}(x, t)c_i(0)\big) + th(x, t).$$

From the Malgrange preparation theorem [20] applied to the germ $g\colon (\boldsymbol{R}^4, 0) \to (\boldsymbol{R}^4, 0)$,

$$g(x, t) = \big(F_{,1}(x, t), 4x_2 F_{,2}^2(x, t), 4x_3 F_{,3}^2(x, t), t\big) \quad \text{for } (x, t) \in \boldsymbol{R}^3 \times \boldsymbol{R}$$

we obtain the following decomposition:

$$\text{(iii)} \quad -\frac{\partial F}{\partial t}(x, t) = x_1 a_1 \circ g(x, t) + \Sigma_2^3 2x_i F_{,i}(x, t) a_i \circ g(x, t) + b \circ g(x, t),$$

with $a_i, b \in \mathfrak{E}(4)$ (we can take the representatives of these germs with compact supports).

Now if we consider (iii) at $(\varrho(x), t)$ and such that $g(\varrho(x), t) = \left(\varrho\left(\frac{\partial S}{\partial x}(x, t), t\right)\right)$, we easily get (ii).

In order to show (i) we have to consider the respective cases: In the case $E_2$ we have $F_{,1}(0, t) = F_{,11}(0, t) = 0 \neq F_{,111}(0, t)$. So, taking $\partial/\partial x_1$ and $\partial^2/\partial x_1^2$ of (iii) at $(0, t)$, we obtain $0 = a_1(0, t)$ and $0 = b_{,1}(0, t)F_{,111}(0, t)$. Thus (i) results. In the case $E_3$ we have $F_{,1}(0, t) = F_{,2}(0, t) = 0 \neq F_{,12}(0, t)$. Taking $\partial/\partial x_2$ of (ii) at $(0, t)$ we have $0 = b_{,1}(0, t)F_{,12}(0, t)$, so $b_{,1}(0, t) = 0$. Now by differentiation of (iii) with respect to $x_1$ at $(0, t)$ we obtain $0 = a_1(0, t)$. For the case $E_1$ we have $F_{,1}(0, t) = 0 \neq F_{,11}(0, t)$, so taking $\partial/\partial x_1$ of (iii) at $(0, t)$ we get

$$0 = a_1(0, t) + b_{,1}(0, t)F_{,11}(0, t).$$

Hence, if $a_1(0, t) = 0$, then $b_{,1}(0, t) = 0$. Thus it is enough to show that decomposition (iii) with $a_1(0, t) = 0$ is always possible. In fact as the Jacobian $(\partial g)/\partial(x, t) \neq 0$ at $(x, t) = (0, 0)$, there exists $X_1 \in \mathfrak{E}(4)$ such that $x_1 = X_1 \circ g(x, t)$. If we set $\bar{a}_1(z, t) := a_1(z, t) - a_1(0, t)$ and $\bar{b}(z, t) := b(z, t) + a_1(0, t)X_1(z, t)$, we can substitute $\bar{a}_1, \bar{b}$ into (iii) to a place $a_1$ and $b$ respectively. But $\bar{a}_1(0, t) = 0$, which completes the proof of Proposition 4.8.

Let $F(x, t)$, $S(x, t)$, $H(x, y, t) = H_t(x, y)$ be as in Proposition 4.8. We assume $t_0 = 0$, $I = (-\varepsilon, \varepsilon)$ for simplicity. Let us consider the time dependent Hamiltonian vector field on $T^*\boldsymbol{R}^3$

$$X_{H_t} = \Sigma_1^3\left(\frac{\partial H}{\partial Y_i}(x, y, t)\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}(x, y, t)\frac{\partial}{\partial Y_i}\right)$$

as well as the vector field $\tilde{X}_H = \frac{\partial}{\partial t} + X_{H_t}$ on $T^* \boldsymbol{R}^3 \times \boldsymbol{R}$. $X_{H_t}$ has the global flow $g_t$, $t \in \boldsymbol{R}$ (i.e. there exists a smooth mapping $\boldsymbol{R}^3 \times \boldsymbol{R}^3 \times \boldsymbol{R} \ni (x, y, t) \to g_t(x, y, t) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3$ such that $\frac{d}{dt} g_t(x, y) = X_{H_t}(g_t(x, y))$ and $g_0(x, y) = (x, y)$, for $(x, y) \in \boldsymbol{R}^3 \times \boldsymbol{R}^3$). This results from: (i) compactness of supports of $a_i$ and $b$, (ii) the independence of the "$y$"-component of $X_{H_t}$ on $x$ (so $y(t)$ can be found independently on $x$), (iii) linearity of the "$x$"-componenet of $X_{H_t}$ with respect to $x$.

LEMMA 4.9. *$g_t$ is a v-L-equivalence for every $t \in \boldsymbol{R}$.*

*Proof*: Since $\omega_H = \omega + dH \wedge dt$ is the invariant form of $\tilde{X}_H$ (see [1]), therefore $g_t$ is a symplectomorphism for every $t$. Take $\sigma \in G$. As $H_t(x, y) = H_t \circ \sigma(x, y)$, $\sigma \circ X_{H_t} = X_{H_t} \circ \sigma$ and $\frac{d}{dt}(\sigma \circ g_t - g_t \circ \sigma) = \sigma \circ X_{H_t} - X_{H_t} \circ \sigma = 0$ for every $t \in \boldsymbol{R}$. Hence $\sigma \circ g_t = g_t \circ \sigma$ holds for every $t \in \boldsymbol{R}$ since $g_0 = id_{T^*\boldsymbol{R}^3}$. Finally, $g_t$ preserves the fibration $\pi$ (see Section 2) because the "$y$"-component of $X_{H_t}$ is independent of $x$. Thus the proof is completed.

Let us define the mapping $\Phi$: $\boldsymbol{R}^3 \times (-\varepsilon, \varepsilon) \to T^* \boldsymbol{R}^3$ as $\Phi(x, t) = \Phi_t(x) = \left(x, \frac{\partial S}{\partial x}(x, t)\right)$ and let the v-L-germ $\Phi(\boldsymbol{R}^3 \times \{t\}) = \left\{\left(x, \frac{\partial S}{\partial x}(x, t)\right)\right\}$ be denoted by $L_t^G$.

LEMMA 4.10. *The global flow $g_t$ forms the v-L-equivalence of the v-L-germs $(L_0^G, 0)$ and $(L_t^G, 0)$ for $|t| < \varepsilon$.*

*Proof*: First we show that $g_t(L_0^G) = L_t^G$. By straightforward calculations it can be checked that the vector field

$$A(x, t) := \frac{d}{dt} \Phi_t(x) - X_{H_t}(\Phi_t(x)) = \sum_{i,j} \frac{\partial H_t}{\partial y_i}(\Phi_t(x)) \left(\frac{\partial}{\partial x_i} + \frac{\partial^2 S_t}{\partial x_i \partial x_j}(x) \frac{\partial}{\partial y_j}\right)$$

is tangent to $L_t^G$ at the point $\Phi_t(x)$ for every $(x, t) \in \boldsymbol{R}^3 \times \boldsymbol{R}$. Let $B(x, t)$ be a smooth vector field on $\boldsymbol{R}^3 \times \boldsymbol{R}$ and $\varepsilon' \in \boldsymbol{R}$, $0 < \varepsilon' < \varepsilon$, be such that

$$\Phi_*(B, (x, t)) = A(x, t) \qquad \text{for } (x, t) \in \boldsymbol{R}^3 \times (-\varepsilon', \varepsilon'),$$

where $\Phi_*$ denotes the corresponding tangent map (cf. [1]). Denote by $h_s$ the flow of $-B(x, t) + \frac{\partial}{\partial t}$ on $\boldsymbol{R}^3 \times \boldsymbol{R}$ (assumed to be defined globally, for simplicity). Then

$$h_s(\boldsymbol{R}^3 \times \{t\}) = \boldsymbol{R}^3 \times \{t + s\} \qquad \text{for } s, t \in \boldsymbol{R}.$$

Let us define $k_t$: $\boldsymbol{R}^3 \to \boldsymbol{R}^3$, $t \in \boldsymbol{R}$, by the formula

$$k_t(x) = \Phi(h_t(x, 0)).$$

It is easily seen that $\frac{d}{dt} k_t(x) = X_{H_t}(k_t(x))$ and $k_0(\boldsymbol{R}^3) = L_0^G$. Hence, by the uniqueness theorem for the first order differential equations we obtain $k_t = g_t$ and $g_t(L_0^G) = k_t(\boldsymbol{R}^3) = L_t^G$ for $|t| < \varepsilon'$. To complete the proof it suffices to notice that $g_t(0, 0) = (0, 0)$ since $X_{H_t}(0, 0) = 0$, by (i) and (ii) of Proposition 4.8 and $g_0(0, 0) = (0, 0)$ which completes the proof.

By the above two lemmas we obtain immediately

PROPOSITION 4.11. *Any two inf-homotopic germs belonging to $\mathfrak{E}(3)$ are ν-L-equivalent.*

It is easily verified that for any $F \in C^\infty(3)$ the mapping $j^2 F$: $\boldsymbol{R}^3 \to J_0^2(\boldsymbol{R}^3, \boldsymbol{R})$ is transversal to $M_i$, $(i = 1, 2, 3, 4)$. Hence if $j^2 F(x) \in M_i$ for every function $F_0 \in C^\infty(3)$ sufficiently close to $F$, there exists a point $x_0 \in \boldsymbol{R}^3$ close to $x$ such that $j^2 F_0(x_0) \in M_i$. Hence $(F, x)$ and $(F_0, x_0)$ are ν-L-equivalent to two inf-homotopic germs from $E_i$, so they are ν-L-equivalent. Thus we obtain

PROPOSITION 4.12. *Let $F \in C^\infty(3)$. Any germ $(F, x)$ where $x = (x_1, 0, 0)$ and $j^2 F(x) \in \cup M_i$ is a ν-L-stable germ.*

Now we can formulate the classification theorem for the normal forms of ν-L-stable germs of generating functions.

PROPOSITION 4.13. *Any ν-L-stable germ $(F, x_0)$, where $F \in C^\infty(3)$ and $x_0 = (x_{01}, 0, 0)$, is ν-L-equivalent to the germ at $0 \in \boldsymbol{R}^3$ of one of the following normal forms:*

($A_1$) $F(x_1, x_2, x_3) = x_1^2 + x_2 + x_3$,
($A_2$) $F(x_1, x_2, x_3) = x_1^3 + x_2 + x_3$,
($A_3$) $F(x_1, x_2, x_3) = \pm x_2^2 \pm (x_2 + x_1)^2 + x_3$.

*Proof*: By Propositions 4.4, 4.5, 4.7, 4.11, 4.12, it suffices to construct the ν-L-equivalences which reduces the normal forms of Proposition 4.7 to the normal forms listed above. But this is easily achieved by the ν-L-equivalences of the form $(x, y) \to (\alpha_i x_i + \beta_i y_i, y_i)$ for appropriate $\alpha_i, \beta_i \in \{-1, 0, 1\}$. This completes the proof of Proposition 4.13.

## 5. Stability conditions for $G$-invariant Morse families

Now using the Morse families local formalism (cf. [19]) we derive the corresponding linear infinitesimal stability conditions for ν-L-germs. Consider a smooth family $(L_t^G, 0)$, $|t| < \varepsilon$ of $\nu$-L-germs with the corresponding smooth family $F_t$, $|t| < \varepsilon$ of $G$-Mf-germs. For simplicity we denote $F_0$, $(L_0^G, 0)$ by $F$, $(L^G, 0)$ resp. and assume that all Morse families of the family $F_t$ are minimal (see § 2). Let $(L^G, 0)$ be the stable $\nu$-L-germ. Thus for sufficiently small $\varepsilon_1$, by Propostition 2.4, $F_t(|t| < \varepsilon_1)$ is locally trivial, i.e.

$$F_t(x, \lambda) = F(\varphi_t(x), \Lambda_t(x, \lambda)) + f_t(x), \tag{5.1}$$

where $\Lambda_t \in \mathfrak{E}(n+l, \nu\oplus\sigma; l, \sigma)$, $\varphi_t \in \mathfrak{E}(n, \nu; n, \nu)$, $f_t \in \mathfrak{E}_\nu(n)$ and $(\varphi_t, \Lambda_t) \in \mathfrak{E}(n+l, \nu\oplus\sigma; n+l, \nu\oplus\sigma)$ is the local family of diffeomorphisms.

By $M$ we denote the space of minimal $G$-Mf-germs

$$M = \{F \in \mathfrak{E}_{\nu\oplus\sigma}(n+l); (\partial^2 F/\partial\lambda_i\, \partial\lambda_j)(0) = 0\}.$$

According to (5.1) and theorems of Section 2 we have

PROPOSITION 5.1. *Let $(L^G, 0)$ be a stable $\nu$-L-germ. Then the necessary condition for the restricted local $G$-L-stability of the corresponding $G$-Mf-germ, $F$, is following:*

$$M \subset \left(\frac{\partial F}{\partial \lambda}\middle|\mathfrak{M}(n+l)\,\mathfrak{E}(n+l, \nu\oplus\sigma; l, \sigma)\right) + \left(\frac{\partial F}{\partial x}\middle|\pi_n^*\,\mathfrak{E}(n, \nu; n, \nu)\right) + \pi_n^*\,\mathfrak{E}_\nu(n), \tag{5.2}$$

*where the first and second terms are submodules of $\mathfrak{E}_{\nu\oplus\sigma}(n+l)$ defined by the standard scalar products $(\cdot|\cdot)$ on $\mathbf{R}^l$ and $\mathbf{R}^n$ respectively, $\pi_n$: $\mathbf{R}^n \times \mathbf{R}^l \to \mathbf{R}^n$.*

Let $\mu'$: $\mathbf{R}^{n+l} \times \mathbf{R}^l \to \mathbf{R}^b$, $\varrho'$: $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^a$ be the Hilbert maps for $\nu\oplus\sigma\oplus\sigma$ and $\nu\oplus\nu$ respectively. Let us denote

$$\varphi_i(x, \lambda) = \frac{\partial \mu_i'}{\partial \lambda'}(x, \lambda, 0) \qquad (i = 1, \ldots, b),$$

$$\psi_j(x) = \frac{\partial \varrho_j'}{\partial x'}(x, 0) \qquad (j = 1, \ldots, a).$$

Thus from Proposition 2.1 and condition (5.2) we infer immediately

COROLLARY 5.2. *In terms of the generators of the modules $\mathfrak{E}(n+l, \nu\oplus\sigma; l, \sigma)$, $\mathfrak{E}(n, \nu; n, \nu)$, the condition (5.2) of Proposition 5.1, can be rewritten in the following form:*

$$M \subset \left\langle \left(\frac{\partial F}{\partial \lambda}\middle|\varphi_1\right), \ldots, \left(\frac{\partial F}{\partial \lambda}\middle|\varphi_b\right) \right\rangle \mathfrak{E}_{\nu\oplus\sigma}(n+l) +$$

$$+ \left\langle \left(\frac{\partial F}{\partial x}\middle|\pi_n^*\psi_1\right), \ldots, \left(\frac{\partial F}{\partial x}\middle|\pi_n^*\psi_a\right), 1 \right\rangle_{\pi_n^*\mathfrak{E}_\nu(n)}. \tag{5.3}$$

When a physical system with symmetry exhibits the structural phase transitions then the notion of "order parameter" is well established (cf. [12]) and its dimensionality is a rather stable feature of the system. This is the reason for the restricted stability condition introduced in Proposition 5.1. However, from the point a view of the standard singularity theory of Lagrangian submanifolds [7] the corresponding deformation space is $\mathfrak{E}_{\nu\oplus\sigma}(n+l)$. Thus, at first, we consider the stronger condition of infinitesimal $G$-L-stability

$$\mathfrak{E}_{\nu\oplus\sigma}(n+l) = \left\langle \left(\frac{\partial F}{\partial \lambda}\middle|\varphi_1\right), \ldots, \left(\frac{\partial F}{\partial \lambda}\middle|\varphi_b\right)\right\rangle_{\mathfrak{E}_{\nu\oplus\sigma}(n+l)} + \left\langle \left(\frac{\partial F}{\partial x}\middle|\pi_n^*\psi_1\right), \ldots \right.$$
$$\left. \ldots, \left(\frac{\partial F}{\partial x}\middle|\pi_n^*\psi_a\right), 1 \right\rangle \pi_n^*\,\mathfrak{E}_\nu(n). \qquad (5.4)$$

This condition immediately follows from the $\nu$-stability of the corresponding $\nu$-L-germ $(L_0^G, 0)$, introduced in Section 2. Let $\bar{\mu}$: $\boldsymbol{R}^{n+l} \to \boldsymbol{R}^k$ and $\bar{\varrho}$: $\boldsymbol{R}^n \to \boldsymbol{R}^r$ be the Hilbert mappings for $\nu\oplus\sigma$ and $\nu$ actions respectively. For further use we define the new Hilbert map for the $\nu\oplus\sigma$-action,

$$\mu = (\bar{\mu}, \bar{\varrho}\circ\pi_n)\colon \boldsymbol{R}^{n+l} \to \boldsymbol{R}^k \times \boldsymbol{R}^r.$$

As we know, the germs $\left(\frac{\partial F}{\partial \lambda}\middle|\varphi_i\right)$, $\left(\frac{\partial F}{\partial x}\middle|\psi_j\right)$, $1 \leqslant i \leqslant b$, $1 \leqslant j \leqslant a$ are $\nu\oplus\sigma$-invariant, thus we can obtain their smooth preimages by the Schwarz [15] homomorphism:

$$\begin{aligned} \tilde{H}_i \circ \mu &= \left(\frac{\partial F}{\partial \lambda}\middle|\varphi_i\right), \qquad 1 \leqslant i \leqslant b, \\ \tilde{E}_j \circ \mu &= \left(\frac{\partial F}{\partial x}\middle|\psi_j\right), \qquad 1 \leqslant j \leqslant a, \end{aligned} \qquad (5.5)$$

where $\tilde{H}_i, \tilde{E}_j \in \mathfrak{E}(k+r)$.

PROPOSITION 5.3. *Let $(L^G, 0) \subset (T^*\boldsymbol{R}^n, \omega)$ be the stable $\nu$-L-germ. Then the necessary infinitesimal $G$-L-stability condition for the corresponding $G$-Mf-germ $F$ can be written in the following form:*

$$\mathfrak{E}(k) = \langle H_1, \ldots, H_b\rangle_{\mathfrak{E}(k)} + \langle E_1, \ldots, E_a, 1\rangle_{\boldsymbol{R}} + M_\mu(k+r|k), \qquad (5.6)$$

*where $H_i = \tilde{H}_i|_{\boldsymbol{R}^k\times\{0\}}$, $(i = 1, \ldots, b)$, $E_j = \tilde{E}_j|_{\boldsymbol{R}^k\times\{0\}}$, $(j = 1, \ldots, a)$ and $M_\mu(k+r|k)$ is the restriction of $M_\mu(k+r)$ to $\boldsymbol{R}^k\times\{0\}$.*

*Proof*: Inserting the expressions (5.5) to the condition (5.4) and taking the surjective homomorphism $\mu^*$: $\mathfrak{E}(k+r) \to \mathfrak{E}_{\nu\oplus\sigma}(n+l)$, we obtain the equivalent condition

$$\mu^* \mathfrak{E}(k+r) = \langle \mu^* \tilde{H}_1, \ldots, \mu^* \tilde{H}_b \rangle_{\mu^*\mathfrak{E}(k+r)} + \langle \mu^* \tilde{E}_1, \ldots, \mu^* \tilde{E}_a, 1 \rangle_{\mu^*\pi_r^*\mathfrak{E}(r)},$$

where $\pi_r \colon \boldsymbol{R}^{k+r} \to \boldsymbol{R}^r$, $(z, y) \to (y)$ is the canonical projection. Thus we can take (5.4) in the following equivalent form:

$$\mathfrak{E}(k+r) = \langle \tilde{H}_1, \ldots, \tilde{H}_b \rangle_{\mathfrak{E}(k+r)} + \langle \tilde{E}_1, \ldots, \tilde{E}_a, 1 \rangle_{\pi_r^*\mathfrak{E}(r)} + M_\mu(k+r), \tag{5.7}$$

where $M_\mu(k+r)$ is defined in § 3. Let $A$ be the finite generated $\mathfrak{E}(k+r)$-module,

$$A = \mathfrak{E}(k+r)/\langle \tilde{H}_1, \ldots, \tilde{H}_b \rangle \mathfrak{E}(k+r) + M_\mu(k+r).$$

From (5.7) we have

$$A/\pi_r^*(\mathfrak{M}(r))A = \langle \tilde{E}_1, \ldots, \tilde{E}_a, 1 \rangle_{\boldsymbol{R}}.$$

Thus by applying the Malgrange preparation theorem we see that condition (5.7) is equivalent to (5.6). This completes the proof of Proposition 5.3.

Let us notice that the functions $H_i$, $E_j$ depend linearly on $F$, which shows some advantage of the Morse family approach comparing to the generating functions method presented in the preceding sections. These two approaches are equivalent, however the direct method of description of Lagrangian singularities by generating functions is convenient from the point of view of physical applications where the generating functions, usually, have a physical meaning of the equilibrum potentials. Similarly as in Section 3, the condition

$$\mathfrak{E}(k) = \langle H_1, \ldots, H_b)_{\mathfrak{E}(k)} + \langle E_1, \ldots, E_a, 1 \rangle_{\boldsymbol{R}} + M_\mu^*(k+r|k) \tag{5.8}$$

will be called a linear condition of infinitesimal $G$-L-stability. If we assume that $M_\mu(k+r|k) - M_\mu^*(k+r|k) \subset \mathfrak{M}^\infty(k)$, then by Nakayama's Lemma [20] we obtain equivalence of the two conditions (5.8) and (5.6).

EXAMPLE 5.4. Assume that $\nu \colon G \to O(n)$ is trivial. Let $(\xi_I, x_J) \to S(\xi_I, x_J)$ be a $IJ$-germ for $(L, 0) \subset T^* \boldsymbol{R}^n$ and the corresponding Morse family $F \in \mathfrak{E}(n+k)$ be given by (2.4), where $k = \# I$. In this case we can put $\mu = id_{\boldsymbol{R}^{n+k}}$. We also find easily that (5.7) takes the form

$$\mathfrak{E}(n+k) = \left\langle \left(\frac{\partial S}{\partial \lambda_1}(\lambda_I, x_J) - x_1\right), \ldots, \left(\frac{\partial S}{\partial \lambda_k}(\lambda_I, x_J) - x_K\right) \right\rangle_{\mathfrak{E}(n+k)} +$$

$$+ \left\langle \frac{\partial S}{\partial x_{J_1}}(\lambda_I, x_J), \ldots, \frac{\partial S}{\partial x_{J_{n-k}}}(\lambda_I, x_J), \lambda_1, \ldots, \lambda_k, 1 \right\rangle_{\mathfrak{E}(n)}.$$

And equivalently, we can write (5.6) in the following form:

$$\mathfrak{E}(\xi_I, x_J) = \left\langle \frac{\partial S}{\partial \xi_I}(\xi_I, x_J), x_J \right\rangle_{\mathfrak{E}(\xi_I, x_J)} + \left\langle \frac{\partial S}{\partial x_J}(\xi_I, x_J), \xi_I, 1 \right\rangle_{\boldsymbol{R}}.$$

We can write (5.6) even in a more compact form

$$\mathfrak{E}(\xi_I) = \left\langle \frac{\partial S}{\partial \xi_I}(\xi_I, 0) \right\rangle_{\mathfrak{E}(\xi_I)} + \left\langle \frac{\partial S}{\partial x_J}(\xi_I, 0), \xi_I, 1 \right\rangle_{\boldsymbol{R}},$$

which is exactly the standard condition for versality (infinitesimal stability) of unfoldings of singularity $\eta = S|_{\boldsymbol{R}^l \times \{0\}}$ (cf. [2], [20]).

Example 5.5. Let us take $G = \boldsymbol{Z}_2$, its action on $\boldsymbol{R}^n$ being defined as follows:

$$\nu_\varepsilon(x_1, x_2, \ldots, x_n) = (\varepsilon x_1, x_2, \ldots, x_n), \quad \varepsilon \in \boldsymbol{Z}_2 = \{\pm 1\}, \; x \in \boldsymbol{R}^n.$$

Let a $\nu$-L-germ $(L^G, 0)$ have the following $\nu$-$IJ$-germ of generating function:

$$S(\xi_1, x_2, \ldots, x_n) = \tilde{S} \circ \varrho(\xi_1, x_2, \ldots, x_n),$$

where $\varrho\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$, $(\xi_1, x_2, \ldots, x_n) \to (\xi_1^2, x_2, \ldots, x_n)$. The corresponding Morse family:

$$F(x, \lambda) = S(\lambda, x_2, \ldots, x_n) - \lambda x_1 \tag{5.9}$$

and the corresponding representation $\sigma$ has the form

$$\sigma_\varepsilon(\lambda) = \varepsilon\lambda.$$

Define a Hilbert map $\bar{\mu}\colon \boldsymbol{R}^{n+1} \to \boldsymbol{R}^{n+2}$ for $\nu \oplus \sigma$ as

$$\bar{\mu}(x, \lambda) = (\lambda^2, \lambda x_1, x_1^2, x_2, \ldots, x_n).$$

We find that $M_{\bar{\mu}}(n+2)$ is the set of smooth function-germs vanishing on the set

$$\bar{\mu}(\boldsymbol{R}^{n+1}) = \{(y_1, \ldots, y_{n+2});\; y_2^2 - y_1 y_3 = 0\} \tag{5.10}$$

and also we have

$$M_{\bar{\mu}}^*(n+2|2) = y_2^2\, \mathfrak{E}(y_1, y_2).$$

After straightforward calculations we obtain

$$\tilde{H}_1(y) = 2y_1 \tilde{S}_{,1}(y_1, y_4, \ldots, y_{n+2}) - y_2,$$
$$\tilde{H}_2(y) = 2y_2 \tilde{S}_{,1}(y_1, y_4, \ldots, y_{n+2}) - y_3,$$
$$\tilde{E}_1(y) = -y_2, \quad \tilde{E}_i(y) = \tilde{S}_{,i}(y'), \quad i = 2, \ldots, n.$$

Substituting them to (5.7), we get the condition

$$\mathfrak{E}(y) = \langle 2y_1 \tilde{S}_{,1}(y') - y_2, 2y_2 \tilde{S}_{,1}(y') - y_3 \rangle_{\mathfrak{E}(y)} + \\ + \langle y_2, \tilde{S}_{,2}(y'), \ldots, \tilde{S}_{,n}(y') \rangle_{\mathfrak{E}(y'')} + M_\mu(n+2),$$

where $y' = (y_1, y_4, \ldots, y_{n+2})$, $y'' = (y_3, \ldots, y_{n+2})$ and for $k$, $r$ in Proposition 5.3 we put $k = 2$, $r = n$. Thus the infinitesimal $\nu$-L-stability condition for the $\nu$-L-germ $(L^G, 0)$ has the form

$$\mathfrak{E}(y_1, y_2) = \langle 2y_1 \bar{S}_{,1}(y_1) - y_2,\ y_2 \bar{S}_{,1}(y_1),\ y_2^2 \rangle_{\mathfrak{E}(y_1, y_2)} + \\ + \langle y_2, \bar{S}_{,2}(y_1), \ldots, \bar{S}_{,n}(y_1), 1 \rangle_{\boldsymbol{R}}, \qquad (5.11)$$

where $\bar{S}_{,\alpha}(y_1) := \tilde{S}_{,\alpha}(y_1, 0, \ldots, 0)$. From the decomposition

$$\mathfrak{E}(y_1, y_2) = \mathfrak{E}(y_1) + y_2\, \mathfrak{E}(y_1) + y_2^2\, \mathfrak{E}(y_1, y_2),$$

on the basis of (5.11) we obtain

$$\mathfrak{E}(y_1) + y_2\, \mathfrak{E}(y_1) = (2y_1 \bar{S}_{,1}(y_1) - y_2)\, \mathfrak{E}(y_1) + y_2 \bar{S}_{,1}(y_1)\, \mathfrak{E}(y_1) + \\ + \langle y_2, \bar{S}_{,2}(y_1), \ldots, \bar{S}_{,n}(y_1), 1 \rangle_{\boldsymbol{R}}.$$

In other words, for every $a(y_1), b(y_1) \in \mathfrak{E}(y_1)$ there exist $h_1, h_2 \in \mathfrak{E}(y_1)$ and constants $c_0, \ldots, c_n \in \boldsymbol{R}$ such that

$$\begin{aligned} a(y_1) &= 2y_1 \bar{S}_{,1}(y_1) h_1(y_1) + \bar{S}_{,2}(y_1) c_2 + \ldots + \bar{S}_{,n}(y_1) c_n + c_0, \\ b(y_1) &= -h_1(y_1) + \bar{S}_{,1}(y_1) h_2(y_1) + c_1. \end{aligned} \qquad (5.12)$$

Eliminating $h_1$ from these equations, we get an equivalent condition

$$a(y_1) + 2y_1 \bar{S}_{,1}(y_1) b(y_1) = y_1 \bar{S}_{,1}^2(y_1) h_2(y_1) + y_1 \bar{S}_{,1}(y_1) c_1 +, \ldots, + \\ + \bar{S}_{,n}(y_1) c_n + c_0. \qquad (5.13)$$

We easily see that (5.13) can be written in the form

$$\mathfrak{E}(y_1) = \langle y_1 \bar{S}_{,1}^2(y_1) \rangle_{\mathfrak{E}(y_1)} + \langle y_1 \bar{S}_{,1}(y_1), \bar{S}_{,2}(y_1), \ldots, \bar{S}_{,n}(y_1), 1 \rangle_{\boldsymbol{R}}, \qquad (5.14)$$

which gives another form for infinitesimal $\nu$-L-stability of the $\nu$-L-germ $(L^G, 0) \subset T^* \boldsymbol{R}^n$.

*Remark* 5.6. We derived condition (5.14) in Section 3 (see formula (3.16)), in quite a different way. In Example 5.5 we showed the equivalence of these two approaches to the classification problem of stable $\nu$-L-germs of Lagrangian submanifolds. It seems that the Morse family approach is very useful in explicit calculations because of the linearity of the corresponding infinitesimal stability conditions.

## 6. Versality and stability of $\nu$-L-germs

In the preceding sections we characterized the infinitesimal stability of $\nu$-L-germs through the corresponding infinitesimal stability conditions for their $G$-Morse family germs. To have an adequate approach to local stability of $\nu$-L-germs by the corresponding locally stable generating families we have to introduce the modified notion of $G$-unfolding (cf. [16]) and adapt this notion to use it in the standard Morse family approach (cf. [9]).

Let $\eta \in \mathfrak{E}_\sigma(k)$ for some orhtogonal representation $\sigma$ of $G$. The pair $(v, f)$, where $v$: $G \to O_n(\boldsymbol{R})$ is a representation of $G$, and $f \in \mathfrak{E}_{v \oplus \sigma}(n+k)$ such that $f|_{\{0\} \times \boldsymbol{R}^k} = \eta$ is called an $n$-parametric $G$-unfolding of $\eta$ with respect to the representation $v$. Let $\sigma$ be fixed for all $G$-unfoldings of the germ $\eta$.

Let $\gamma$ be an orthogonal representation of $G$ in $\boldsymbol{R}^s$. A morphism of $G$-unfoldings $(\Phi, \alpha)$: $(\gamma, h) \to (v, f)$ of the germ $\eta$ is defined by the following maps:

(i) $\Phi = (\varphi, \psi) \in \mathfrak{E}(s+k, \gamma \oplus \sigma; k, \sigma) \oplus \mathfrak{E}(s, \gamma; n, v)$,

(ii) $\alpha \in \mathfrak{E}_\gamma(s)$

and the following condition:

$$h = f \circ \Phi + \alpha \circ \pi_s,$$

where $\pi_s$: $\boldsymbol{R}^s \times \boldsymbol{R}^k \to \boldsymbol{R}^s$ is the canonical projection. If $\psi$ is a diffeomorphism, then $(\Phi, \alpha)$ is called an isomorphism of $G$-unfoldings. We say that a $G$-unfolding $(v, f)$ of the germ $\eta$ is $G$-versal if for any other $G$-unfolding $(\gamma, h)$ of $\eta$ there exists a morphism $(\Phi, \alpha)$: $(\gamma, h) \to (v, f)$. The $G$-versal unfolding of $\eta$ is called $G$-miniversal if the dimension of the basis $n$ of the unfolding is the smallest possible number (cf. [6]). We see that the above introduced isomorphism of $G$-unfoldings $(\Phi, \alpha)$ defines the Lagrangian equivalence of $\gamma$-L-germ $(L_1^G, 0)$ generated by $h$ and the $v$-L-germ $(L_2^G, 0)$ generated by $f$, i.e. there is the $G$-equivariant symplectomorphism $R^G$: $T^* \boldsymbol{R}^n \to T^* \boldsymbol{R}^n$ preserving the fibre structure $\pi_{\boldsymbol{R}^n}$: $T^* \boldsymbol{R}^n \to \boldsymbol{R}^n$ and such that $R^G(L_1^G) = L_2^G$, $R^G(b) = 0$. We know (see [9], [6], p. 269) that $R^G$ can be locally written as follows:

$$(x, \xi) \to \left(\psi(x), {}^t D\psi(x)^{-1}(\xi + d\alpha(x))\right): \ T^* \boldsymbol{R}^n \to T^* \boldsymbol{R}^n \tag{6.1}$$

with $\psi \in \mathrm{Diff}(n, \gamma; n, v)$, $\alpha \in \mathfrak{E}_\gamma(n)$. The converse statement is also true, i.e. if the $\gamma$-L-germ $(L_1^G, (x_0, \xi_0)) \subset T^* \boldsymbol{R}^n$ is mapped into the $v$-L-germ $(L_2^G, (\bar{x}_0, \bar{\xi}_0)) \subset T^* \boldsymbol{R}^n$ by a germ of $G$-equivariant symplectomorphism $(T^* \boldsymbol{R}^n, (x_0, \xi_0)) \to (T^* \boldsymbol{R}^n, (\bar{x}_0, \bar{\xi}_0))$ of the form (6.1), then the corresponding $G$-unfoldings, say $h$ and $f$, generating $(L_1^G, (x_0, \xi_0))$ and $(L_2^G, (\bar{x}_0, \bar{\xi}_0))$ respectively, are isomorphic as $G$-unfoldings (cf. [22]).

Let $S$: $(\boldsymbol{R}^n \times \boldsymbol{R}^k, 0) \to \boldsymbol{R}$, $S \in \mathfrak{E}_{v \oplus \sigma}(n+k)$ be a germ of a generating family for the $v$-L-germ $(L^G, 0) \subset T^* \boldsymbol{R}^n$.

Definition 6.1. A $G$-invariant Lagrangian submanifold $L^G \subset T^* \boldsymbol{R}^n$ is called *G-versal* at $0 \in L^G$ if a germ $S$ of a generating family of $(L^G, 0)$ is a $G$-versal unfolding if the germ $\eta = S|_{\{0\} \times \boldsymbol{R}^k} \in \mathfrak{E}_\sigma(k)$.

Let us endow the space of $G$-unfoldings $C^\infty_{v \oplus \sigma}(n+k)$ and the space of $G$-

equivariant Lagrangian immersions $I(n, \varrho; n+n, v\oplus v)$ with the induced $C^\infty$-Whitney topology, then the $G$-versal $v$-L-germ $(L^G, 0)$ is locally stable, i.e. for every $G$-invariant neighbourhood $V$ of 0 in $T^* \boldsymbol{R}^n$ there exists an open neighbourhood $U$ of the $G$-equivariant Lagrangian immersion $i_{L^G}$: $(\boldsymbol{R}^n, 0) \to (T^* \boldsymbol{R}^n, 0)$ in $I(n, \varrho; n+n, v\oplus v)$ (where $\varrho$ is the linearised representation $v\oplus v|_L G$) such that for every $i \in U$ there exists $p \in \operatorname{Image} i \subset V$ with the property that the $v$-L-germs $(L^G, 0)$ and $(\operatorname{Image} i, p)$ are $v$-L-equivalent (or $(i_{L^G}, 0)$, $(i, i^{-1}(p))$ are $G$-equivalent as immersions [7]). Thus the local $v$-L-stability of $v$-L-germs has an adjoint formulation in terms of the stable $G$-unfoldings of invariant singularities (cf. [24], [2]).

Let $\eta \in \mathfrak{M}_\sigma^2(k)$, by $J(\eta)$ we denote the Jacobi ideal of $\eta$ generated by the partial derivatives $\partial\eta/\partial\lambda_1, \ldots, \partial\eta/\partial\lambda_k$. $J(\eta)$ is a $G$-submodule of the $G$-module $\mathfrak{E}(k)$. Following [9] (see also [16]) we obtain the main result on the $G$-versal $v$-L-germs.

PROPOSITION 6.2. *Let $\sigma$: $G \to O_k(\boldsymbol{R})$ be a fixed representation of $G$ in $\boldsymbol{R}^k$, let $(v, S)$ be a $G$-unfolding of a germ $\eta = S|_{\{0\}\times \boldsymbol{R}^k}$ which generates the $v$-L-germ $(L^G, 0) \subset T^* \boldsymbol{R}^n$, we set $n = \dim_{\boldsymbol{R}} \mathfrak{M}(k)/J(\eta) < \infty$. Let $\gamma$ be the representation of $G$ in the vector space $\mathfrak{M}(k)/J(\eta) \cong \boldsymbol{R}^n$ and $r$: $\mathfrak{M}(k)/J(\eta) \to \mathfrak{M}(k)$ an equivariant splitting of the exact sequence of $G$-modules $0 \to J(\eta) \to \mathfrak{M}(k) \overset{r}{\leftrightarrows} \mathfrak{M}(k)/J(\eta) \to 0$ such that the function $f$: $\mathfrak{M}(k)/J(\eta)\oplus \boldsymbol{R}^k \to \boldsymbol{R}$, $f(x, \lambda) = \eta(\lambda)+r(x)(\lambda)$ is a Morse family. Then*

(i) *$f$ is a generating family for the $G$-versal $\gamma$-L-germ $(L_1^G, 0)$.*

(ii) *The $v$-L-germ $(L^G, 0)$ with the generating family $(v, S)$ is $G$-versal if and only if a morphism of $G$-unfoldings $(\Phi, \alpha)$: $(v, S) \to (\gamma, f)$ is an isomorphism.*

The proof of this proposition can be found in [9] (p. 187).

The main tool in proving Proposition 6.2 as well as in classifying the corresponding normal forms for $G$-versal $v$-L-germs is the infinitesimal versality notion (cf. [24], [16]).

Let $\eta \in \mathfrak{E}_\sigma(k)$ and $f \in \mathfrak{E}_{v\oplus\sigma}(n+k)$ be a $G$-unfolding of $\eta$. Thus $df \in \mathfrak{E}(n+k)\otimes(X\oplus\Lambda)^*$ (where we denote $\Lambda \cong \boldsymbol{R}^k$, $X \cong \boldsymbol{R}^n$) has the two components $d_1 f \in \mathfrak{E}(n+k)\otimes\Lambda^*$, and $d_2 f \in \mathfrak{E}(n+k)\otimes X^*$. Let us consider the second component and the sequence of homomorphisms (cf. [16])

$$\begin{gathered}\mathfrak{E}(n+k) \to \mathfrak{E}(n+k)\otimes X^* \to \mathfrak{E}(k)\otimes X^* \to \mathfrak{E}(k)/J(\eta)\otimes X^*,\\ f \to d_2 f \to d_2 f|_\Lambda \to \overline{d_2 f}|_\Lambda = \delta f.\end{gathered} \tag{6.2}$$

We see that $\delta f$ is $G$-invariant, i.e. $\delta f \in (\mathfrak{E}(k)/J(\eta)\otimes X^*)^G$, $\delta f$ is identified also with a $G$-equivariant homomorphism $X \to \mathfrak{E}(k)/J(\eta)$. If the homomorphism $\delta f$ is surjective we say that the $G$-unfolding $(v, f)$ is infinitesimally versal. It is proved in [16] that the two notions: infinitesimal versality and versality, are equivalent.

We can adapt the above notions to the symplectic objects and write down, for

$G$-Mf-germs, the corresponding sequence

$$\mathfrak{M}(k)\,\mathfrak{E}(n+k)+\mathfrak{M}^2(n) \to (\mathfrak{M}(k)\,\mathfrak{E}(n+k)+\mathfrak{M}(n))\otimes(\mathbf{R}^n)^* \to \mathfrak{M}(k)\otimes(\mathbf{R}^n)^* \to \mathfrak{M}(k)/J(\eta)\otimes(\mathbf{R}^n)^*$$

$$f \to \delta f \in \mathrm{Hom}_G(\mathbf{R}^n, \mathfrak{M}(k)/J(\eta)).$$

DEFINITION 6.3. Let $(v, f)$ be $G$-Mf-germ for the $v$-L-germ $(L^G, 0) \subset T^*\mathbf{R}^n$. We say that $(L^G, 0)$ is an *infinitesimally G-versal* if the corresponding $G$-homomorphism $\delta f$ is surjective.

PROPOSITION 6.4. *The* $v$-*L-germ* $(L^G, 0)$ *is G-versal if and only if* $(L^G, 0)$ *is infinitesimally G-versal.*

*Proof*: Using Proposition 6.2 and Corollary 3.7 in [16] (cf. [9]).

Following the standard lines of Lagrangian singularity theory (see [2], [9], [7], [18]) we can summarize the stability theory of invariant Lagrangian submanifolds in the following

PROPOSITION 6.5. *Let* $i_{L^G}\colon (L^G, 0) \to (T^*\mathbf{R}^n, 0)$ *be a germ of G-equivariant Lagrangian immersion. Let* $S\colon (\mathbf{R}^n \times \mathbf{R}^k, 0) \to \mathbf{R}$ *be a corresponding generating family for* $(L^G, 0)$. *Then the following conditions are equivalent:*

(i) $(i_{L^G}, 0)$ *is locally stable,*

(ii) $(i_{L^G}, 0)$ *is infinitesimally stable,*

(iii) $(S, 0)$ *is a versal G-unfolding of the germ* $\eta = S|_{\{0\}\times\mathbf{R}^k}$,

(iv) $(S, 0)$ *is an infinitesimally versal G-unfolding of the germ* $\eta = S|_{\{0\}\times\mathbf{R}^k}$.

*Proof*: The equivalence of (i) and (ii) results immediately by the equivariant local version of the Theorem 5.1.3 in [6]. By Theorem 4 [22] and the previous results we obtain equivalence of conditions (i), (iii). The equivalence of the notion of infinitesimal stability for Lagrangian $G$-immersions and infinitesimal versality for generating $G$-invariant Morse families follow from the corresponding equivariant reformulation of standard arguments in [24] (see also [7]).

**Acknowledgments**

One of the authors (S. J.) is very grateful to H. Knörrer, R. Kulkarni and B. Moroz, for their help, stimulating advice and interest in our work.

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