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Classification of Lagrangian stars and their symplectic reductions

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Abstract. The Lagrangian star is a germ of the system $(\{L_1, \ldots, L_k\}, \tilde{p})$ of Lagrangian submanifolds in the symplectic manifold (M, ω). We investigate the symplectic group action on Lagrangian stars and construct the basic invariants of such action. The Kashiwara signature for 3-Lagrangian linear stars is generalized to the nonlinear case and the generalized contact classes for Lagrangian stars are constructed. Finally, we obtain the generic classification of simple normal forms of reduced Lagrangian stars with respect to a hypersurface.

1. Introduction

Let l_1, l_2, l_3 be three Lagrangian subspaces in the symplectic vector space (M, ω). The natural invariant of the group of symplectic transformations of M, acting on the triplets of Lagrangian subspaces, is a signature (Maslov index [6]) of the Kashiwara quadratic form $Q(x_1, x_2, x_3) = \omega(x_1, x_2)+\omega(x_2, x_3)+\omega(x_3, x_1)$ defined on the direct sum $l_1 \oplus l_2 \oplus l_3$. In this paper we generalize this notion to the case of germs of triplets of Lagrangian submanifolds in a symplectic manifold. The problem considered is related to the classification of Lagrangian germs with respect to the subgroups of the group of symplectomorphisms. The natural subgroups are induced by f-liftable (cf [1]) vector fields V on M such that $\mathrm{d}(V \rfloor \omega) = 0$ and f is a smooth mapping between two manifolds, $f : N^{2n} \to M^{2n}$. Using the action of these groups one investigates the geometry of the maximal isotropic submanifolds in the degenerated symplectic structures (cf [3, 7]) and show the direct way of generalizing the Lagrangian singularities. Using the symplectic invariants of contact (cf [4, 5]), in section 2 we find the algebraic invariants of the triplets of Lagrangian submanifolds containing two transversal submanifolds (basic Lagrangian star). We show that for the special class of tangential Lagrangian stars these invariants are determined by the equivalence class of right equivalence in the space of function-germs on R^n. Classification of reduced Lagrangian stars and basic Lagrangian stars, on a hypersurface H, under some genericity conditions is given in section 3. As an extension of this result the reduced local models, in the case of some non-transversal positions of Lagrangian stars with respect to H, are calculated.

2. Lagrangian stars

Let (M, ω) be a symplectic manifold. Let $\{L_1, \ldots, L_k\}$ be a system of Lagrangian submanifolds of (M, ω) intersecting at the common point $\tilde{p} \in L_1 \cap \cdots \cap L_k$.

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Definition 2.1. The germ of Lagrangian submanifolds $(\{L_1, \ldots, L_k\}, \tilde{p})$ is called a k-Lagrangian star at $\tilde{p}$. If $k = 2$ and L_1 is transversal to L_2 then the 2-Lagrangian star $(\{L_1, L_2\}, \tilde{p})$ is called the basic Lagrangian star. The 3-Lagrangian star we simply call the Lagrangian star.

Let $(\{L_1, \ldots, L_k\}, \tilde{p})$ and $(\{L'_1, \ldots, L'_k\}, \tilde{p})$ be two k-Lagrangian stars at $\tilde{p}$. Then we say that they are symplectically equivalent (or equivalent) if there is a germ of symplectomorphism $\Phi : ((M, \omega), \tilde{p}) \to ((M, \omega), \tilde{p})$ such that $\Phi(L_j) = L_{i_j}$ for some permutation i_j of $\{1, \ldots, k\}$ and $\Phi(\tilde{p}) = \tilde{p}$. The basic Lagrangian star forms a system of local symplectic coordinates of (M, ω). There are Darboux coordinates around $\tilde{p} \in M$ such that the basic Lagrangian star $(\{L_1, L_2\}, \tilde{p})$ is symplectically equivalent to the one defined by $L_1 = \{(p, q) \in R^{2n}, p = 0\}$ and $L_2 = \{(p, q) \in R^{2n}, q = 0\}$ with $(M, \omega) \cong (R^{2n}, \sum_{i=1}^n \mathrm{d}p_i \wedge \mathrm{d}q_i)$.

To classify the Lagrangian stars we have to introduce the notion of contact equivalence and subsequently the symplectic contact equivalence. Let X, L_1, L_2 be equi-dimensional submanifolds of M with $p \in X \cap L_1 \cap L_2$. Then we say that L_1 and L_2 have the same contact with X at p if there is a germ of diffeomorphism $\phi : (M, p) \to (M, p)$ such that $\phi(L_1) = L_2$ and $\phi(X) = X$. Orbits of the group of these defined contact equivalences are called the contact classes. Using this definition, if L_1, L_2 have the same contact with X at p then the local rings $\bar{R}(X, L_i) = \mathcal{E}_X/\rho_1(X, L_i)$, where $\mathcal{E}_X$ denotes the local ring of smooth function-germs on X at p and $\rho_1(X, L_i)$ denotes the ideal of germs of functions on M at p which vanish to first order on L_i restricted to X, are isomorphic. The corresponding isomorphism is induced by the pullback map $\phi^\star$, $\phi^\star f = f \circ \phi$, for $f \in \mathcal{E}_X$. The converse statement is true provided additionally $\dim \bar{R}(X, L_i) < \infty$.

The group of symplectomorphism-germs of $((M, \omega), \tilde{p})$ is a subgroup of the group of diffeomorphism-germs of $(M, \tilde{p})$ so that the contact data is a much more subtle invariant. If X, L_1, L_2 are Lagrangian submanifolds then the natural symplectic contact data is a pair

$$(\bar{R}_s = \mathcal{E}_X/\rho_2(X, L_i), \sigma_i)$$

where $\rho_2(X, L_i)$ denotes the ideal of germs of functions on M at $\tilde{p}$ which vanish to second order on L_i restricted to X, and the element $\sigma_i \in \bar{R}_s$ is naturally associated to L_i. In each case σ is defined using a special cotangent bundle structure on a neighbourhood $\tilde{M}$ of X in M, such that $\tilde{M} = T^\star X$ and $L_i = \text{graph}\ \mathrm{d}\psi_i$ for some smooth functions ψ_i on X, σ_i is the image of ψ_i in $\tilde{R}_s$. Obviously σ is defined up to the choice of the special cotangent bundle structure $T^\star X$ on M. The special symplectic structure is a quadruple (M, X, π, θ), where (M, X, π) is a differentiable fibre bundle, θ is a 1-form on M, $\mathrm{d}\theta = \omega$, such that there exists a diffeomorphism $\alpha : M \to T^\star X$ such that $\pi = \pi_X \circ \alpha$, $\theta = \alpha^\star \theta_X$. Let L be a Lagrangian submanifold in M and let θ_1 and θ_2 be 1-forms corresponding to two special symplectic structures on (M, ω) with the same base X. Then $\theta_1|_X = \theta_2|_X = 0$ and near X we have $\theta_1 - \theta_2 = \mathrm{d}H$, where H is a function on M which vanish to second order on X. The corresponding generating functions ψ_{θ_1} and ψ_{θ_2} of L in both special symplectic structures are right equivalent with a diffeomorphism $g : X \to X$ defined by the formula

$$g^\star \psi_{\theta_2} = \psi_{\theta_1} + \sum_{ij=1}^n h_{ij}(x,\ \mathrm{d}\psi_{\theta_1}) \frac{\partial \psi_{\theta_1}}{\partial x_i} \frac{\partial \psi_{\theta_1}}{\partial x_j}$$

where $H = \sum_{ij=1}^n h_{ij}(x, p) p_i p_j$. From this consideration we easily see the geometric sense of the local ring

$$\bar{R}_s = \mathcal{E}_X \Big/ \left\langle \frac{\partial \psi}{\partial x_1}, \ldots, \frac{\partial \psi}{\partial x_n} \right\rangle^2.$$

Now we assume that the Lagrangian star $S = (\{L_1, L_2, L_3\}, \tilde{p})$ contains the basic Lagrangian star, say $(\{L_1, L_2\}, \tilde{p})$. It is natural to define the pair of local rings $\boldsymbol{R} = \boldsymbol{R}_1 \oplus \boldsymbol{R}_2$ associated to S, being a local invariant of the group of germs of symplectomorphisms acting on the space of Lagrangian stars (cf [4]). By considering germs of functions on M near $\tilde{p}$ which vanish to second order on L_3 and taking the restrictions of these functions to L_1 (and respectively to L_2) we obtain an ideal $\Delta_1(L_1, L_3)$ (and respectively an ideal $\Delta_2(L_2, L_3)$).

Definition 2.2. By the basic invariant of the Lagrangian star S we denote the pair of local rings

$$\boldsymbol{R} = \boldsymbol{R}_1 \oplus \boldsymbol{R}_2 = \mathcal{E}_{L_1}/\Delta_1(L_1, L_3) \oplus \mathcal{E}_{L_2}/\Delta_2(L_2, L_3)$$

where $\mathcal{E}_{L_1}$ (respectively $\mathcal{E}_{L_2}$) denotes the local ring of smooth function-germs on L_1 (respectively on L_2) near $\tilde{p}$. We call S finite if $\dim_R \boldsymbol{R} < \infty$.

Now we have a natural realization of $\boldsymbol{R}_1 \oplus \boldsymbol{R}_2$.

Proposition 2.1. For the considered Lagrangian star S

$$\boldsymbol{R}_i = \mathcal{E}_{L_i} \Big/ \left\langle \frac{\partial \phi_i}{\partial x_1}, \ldots, \frac{\partial \phi_i}{\partial x_n} \right\rangle^2$$

where $i = 1, 2$ and ϕ_1 and ϕ_2 are the function germs associated with the realizations of L_3 in two different cotangent bundle structures over L_1 and L_2. The generating functions ϕ_1 and ϕ_2 are defined up to an automorphism of $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ induced by the corresponding diffeomorphism-germs $L_1 \to L_1$ and $L_2 \to L_2$.

Proof. At first we recall some basic properties of Lagrangian submanifolds. If X is a Lagrangian submanifold in (M, ω) then in some neighbourhood of X the symplectic manifold M is isomorphic to $T^\star X$. We say that $T^\star X$ is a special symplectic structure on M. Let L be another Lagrangian submanifold in (M, ω), then around a point $p \in L \cap X$, the submanifold L is generated by the generating function $F(p_I, q_J)$ (cf [1]), i.e. in local Darboux coordinates on $T^\star X$, L is described by the equations

$$p_J = \frac{\partial F}{\partial q_J}(p_I, q_J) \qquad q_I = -\frac{\partial F}{\partial p_I}(p_I, q_J) \tag{$*$}$$

for some $J, I \subset \{1, \ldots, n\}$, $I \cap J = \emptyset$, $I \cup J = \{1, \ldots, n\}$. If the second equation of $(*)$ cannot be solved according to p_I (around $\tilde{p}$) then obviously L is vertical in directions p_I, so it cannot be generated by a function only on q. We see that X is described by $\{p_i = 0, i = 1, \ldots, n\}$, so the ideal $\Delta(X, L)$ does not change if we perturb L (make it transversal to the fibration $T^\star X \to X$) by adding the linear terms in p to the second part of $(*)$ and making it solvable according to p_I. Thus we can represent the local ring $\boldsymbol{R} = \mathcal{E}_X/\Delta(X, L)$ by a generating function on X.

For the basic Lagrangian star $(\{L_1, L_2\}, \tilde{p})$ we consider the special symplectic structures around $\tilde{p}$, $T^\star L_1 \cong M$ and $T^\star L_2 \cong M$. In both these structures the manifold L_3 can be defined by generating functions using the corresponding Liouville forms

$$\theta_{L_i}|_{L_3} = \mathrm{d}\tilde{\phi}_i \qquad i = 1, 2.$$

We see that the ideals $\Delta_i(L_i, L_3)$ describe the order of contact of L_3 to L_i, (cf [4]) so by the small deformation of L_3 making it transversal to the fibrations $T^\star L_i \to L_i$ we get the generating functions ϕ_i of L_3 which may be defined on L_i keeping Δ_i unchanged. These deformations may be achieved by changing the canonical 1-forms associated to the two cotangent bundle structures of $T^\star L_i$. □

If the following two Lagrangian stars

$$S = (\{L_1, L_2, L_3\}, \tilde{p}), S' = (\{L'_1, L'_2, L'_3\}, \tilde{p})$$

are symplectically equivalent then their corresponding basic invariants $\boldsymbol{R}_1 \oplus \boldsymbol{R}_2$ and $\boldsymbol{R}'_1 \oplus \boldsymbol{R}'_2$ are isomorphic. Now we would like to show that under certain conditions the converse is true.

Let the basic Lagrangian star of S be in Darboux form, then L_3 is generated by a generating family $F_q(p_I, q) = S_3(p_I, q_J) + p_I q_I$ in $T^\star L_1$ and by $F_p(q_J, p) = S_3(p_I, q_J) - p_J q_J$ in $T^\star L_2$, (which is the Legendre transform of F_q), for some $J, I \subset \{1, \ldots, n\}$, $I \cup J = \{1, \ldots, n\}$ and $I \cap J = \emptyset$. We choose S_3 such that

$$\frac{\partial^2 S_3}{\partial p_I \partial p_I}(0) = 0.$$

This condition says that L_3 projects along p with the kernel parametrized by p_I. In usual Lagrange equivalency preserving the fibration $(p, q) \to q$ we reduce S_3 to the form such that $S_3(p_I, q_J) \in \boldsymbol{m}^3_{IJ}$. However, in this case we have to preserve the basic Lagrangian star $(\{L_1, L_2\}, 0)$, where $L_1 = \{(p, q) \in R^{2n}, p = 0\}$ and $L_2 = \{(p, q) \in R^{2n}, q = 0\}$, so that the quadratic terms in some q-variables cannot be reduced. Thus we can write S_3 in the following final form

$$S_3(p_I, q_J) = \tilde{S}(p_I, q_J) + Q(q_{J'})$$

where $J' \subset J$, $\tilde{S} \in \boldsymbol{m}^3_{IJ}$ ($\boldsymbol{m}_{IJ}$ is the maximal ideal of smooth function-germs depending on p_I, q_J-variables, $\tilde{p}$ we assume to be 0 in these local coordinates), and $Q(q_{J'})$ is a non-degenerated quadratic form of $q_{J'}$-variables, $\#J' = l$. Now we can deduce the following result.

Proposition 2.2. Let S and S' be two finite Lagrangian stars containing the basic Lagrangian star, then S and S' are symplectically equivalent iff

(1) the quadratic forms Q and Q' are equivalent, and

(2) the basic invariants $\boldsymbol{R}_1 \oplus \boldsymbol{R}_2$ and $\boldsymbol{R}'_1 \oplus \boldsymbol{R}'_2$ are isomorphic and the corresponding isomorphisms γ_1 and γ_2 send the images of ϕ_1 and ϕ_2 in $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ into the images of ϕ'_1 and ϕ'_2 in $\boldsymbol{R}'_1$ and $\boldsymbol{R}'_2$, respectively.

The basic invariant of the Lagrangian star S is a C^∞-invariant, i.e. an equivalence of Lagrangian stars is not necessarily symplectic. Now we see that the following data

$$(\boldsymbol{R}, Q, \phi_1, \phi_2)$$

form the complete symplectic invariant for Lagrangian stars under the symplectic group equivalence.

Remark 2.1. If L_3 is generated, in the basic Lagrangian star by $S_3(p_I, q_J)$, $I \cup J = \{1, \ldots, n\}$, $I \cap J = \emptyset$ and $\partial^2 S_3(0)/\partial p_I \partial p_I = 0$, then the class of S is preserved if we apply the right equivalence group to S_3 preserving $\{p_I\}$ and $\{q_J\}$ spaces separately. There is a natural question, what does the relation between the local ring

$$\mathcal{E}_{p_I, q_J} \Big/ \left\langle -\frac{\partial S_3}{\partial p_I}, \frac{\partial S_3}{\partial q_J} \right\rangle^2$$

and the basic symplectic invariant of the Lagrangian star S look like?

Now we consider the special case. We assume S contains the basic Lagrangian star. We call S the *tangential star* if there are two Lagrangian germs in S tangent at $\tilde{p}$. If S is tangential then there exists local Darboux coordinates at $\tilde{p}$ in which $L_1 = \{(p,q) \in R^{2n} : p = 0\}$, $L_2 = \{(p,q) \in R^{2n} : q = 0\}$ and L_3 is generated by a generating function $q \to F(q)$, $F'(0) = 0$ and $F''(0) = 0$. The basic invariant for the tangential Lagrangian stars is reduced to the local ring

$$\boldsymbol{R} = \mathcal{E}_q \Big/ \left\langle \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_n} \right\rangle^2.$$

Symplectic equivalence of the tangential Lagrangian stars say S and S', is equivalent to right equivalence of their generating functions F and F'. So the equivalence classes of contact are determined mainly by the A_k, D_k and E_k classification of singularities (cf [1]).

Remark 2.2. If Q has a maximal rank then the main symplectic invariant of the Lagrangian star is a signature of Q. It is a signature (Maslov index) of the Kashiwara quadratic form (cf [6])

$$\omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$$

defined on the tangent (at $\tilde{p}$) Lagrangian star $l_1 \oplus l_2 \oplus l_3$. We denote this signature by $\tau(l_1, l_2, l_3)$. This is a symplectic invariant for any Lagrangian star, not only if $l_1 \cap l_2 = l_2 \cap l_3 = l_3 \cap l_1 = \{0\}$. In general we can write

$$\tau(l_1, l_2, l_3) = n + \dim(l_1 \cap l_2) + \dim(l_2 \cap l_3) + \dim(l_3 \cap l_1) (\mathrm{mod}\, 2).$$

The basic symplectic invariant introduced here is a natural generalization of τ for the nonlinearizable Lagrangian stars. Generalization of this invariant for a Lagrangian star of four Lagrangian submanifolds goes through the composed 3-Lagrangian stars (cf [6]).

3. Reduction of Lagrangian stars

As far as the basic symplectic stars are all symplectically equivalent there is a natural question how they pass through the reduction on a hypersurface or a general co-isotropic submanifold? First, we consider the reduction of co-dimension 1, which is the very special reduction along the integral curves of the Hamiltonian system with the Hamiltonian function defining the hypersurface as its zero-level set.

Let H be a hypersurface in (M, ω). We consider the basic stars $(\{L_1, L_2\}, \tilde{p})$ transversal to H at $\tilde{p}$. Let $\pi_H : H \to (\tilde{M}, \tilde{\omega})$ be the projection along bicharacteristics into the reduced symplectic manifold $\tilde{M}$ and $\tilde{\omega}$ is the corresponding reduced symplectic form $\pi_H^\star \tilde{\omega} = \omega|_H$. We define the reduced star as follows

$$(\{L_1^r, L_2^r\}, \tilde{p}^r)$$

where $L_i^r = \pi_H(H \cap L_i)$ and $\tilde{p}^r = \pi_H(\tilde{p})$.

Now we pass to the classification of reduced basic stars according to the symplectomorphisms of (M, ω) and $(\tilde{M}, \tilde{\omega})$ preserving the projection π_H.

Proposition 3.1. Any simple, reduced basic Lagrangian star can be written in one form from the following normal forms

$$A_k : (\{L_1^r, L_2^r\}, 0)$$

where

$$L_1^r = \{(q, p) \in \tilde{M} : p_i = 0, i = 1, \ldots, n-1\}$$

$$L_2^r = \{(q, p) \in \tilde{M} : p_1 = \partial S/\partial q_1(q_1, p_2, \ldots, p_{n-1}),$$
$$q_i = -\partial S/\partial p_i(q_1, p_2, \ldots, p_{n-1}), i = 2, \ldots, n-1\}$$

and

$$S(q_1, p_2, \ldots, p_{n-1}) = \pm q_1^{k+1} + q_1^{k-1} p_{k-1} + \cdots + q_1^2 p_2$$

for $2 \leqslant k \leqslant n-1$ and A_1 with $L_2^r = \{(q, p) \in \tilde{M} : q_i = 0, i = 1, \ldots, n-1\}$.

Proof. Now we have to classify the triplets $(\{L_1, L_2, H\}, \tilde{p})$ in (R^{2n}, ω), where L_1, L_2 and H are mutually transversal at $\tilde{p} \in L_1 \cap L_2 \cap H$. We find Darboux coordinates $\{x_1, \ldots x_n, y_1, \ldots, y_n\}$ in which L_1 and H may be written in the following normal form in $R^{2n} = T^\star L_1$

$$L_1 = \{y_1 = 0, \ldots, y_n = 0\} \qquad H = \{x_1 = 0\}.$$

Then L_2 can be written by the generating function $y \to S(y)$, such that

$$\mathrm{d}\left(\frac{\partial S}{\partial y_1}\right)(0) \neq 0$$

because of transversality of L_1 to L_2 and L_2 to H. Now we need to use the symplectomorphisms of $T^\star L_1$ preserving $(L_1 \cup H, 0)$ and reduce L_2 to its simple normal form. So we need the group $G_{L_1 \cup H}$ of germs of symplectomorphisms which preserve the fibration $(x, y) \to y$ and the hypersurface $H = \{x_1 = 0\}$. Every element Φ of this group can be defined as a lifting of a diffeomorphism $\phi : R^n \ni y \to \phi(y) \in R^n$, which preserve the fibration over $(y_2, \ldots, y_n)$, i.e. $y \to \bar{y} = (y_2, \ldots, y_n)$ with adding the gradient of a function f which depends on $\bar{y}$

$$\Phi(x, y) = ((\phi^\star)^{-1}(y)x + \mathrm{d}f(\bar{y}), \phi(y)).$$

Using this group we can reduce the function S to the form

$$S(y) = y_1 \tilde{S}(y).$$

Then using the theorem on versal deformations (cf [8]) we reduce it successively to the form

$$L_2 : S(y) = \pm y_1^{k+1} + y_1^{k-1} y_k + \cdots + y_1 y_2 \qquad \text{for } 1 \leqslant k \leqslant n.$$

At first we consider the case when $k = 1$. In this case $S(y) = \pm y_1^2$ and the reduction equations $\{x_1 = 0, y_1 = 0\}$ give us the reduced Lagrangian germ in the form

$$\pi_{\{x_1=0\}}(L_2) = \{(q, p) \in \tilde{M} : q_i = 0, i = 1, \ldots, n-1\}$$

which corresponds to the case A_1 in the proposition.

Now we consider the case when $k \geqslant 2$. Taking the image $\pi_{\{x_1=0\}}(L_2)$ we obtain the following equations

$$-x_1 = \frac{\partial S}{\partial y_1}(y) = \pm(k+1)y_1^k + (k-1)y_1^{k-2} y_k + \cdots + 2y_1 y_3 + y_2 = 0$$

$$-x_2 = \frac{\partial S}{\partial y_2}(y) = y_1, \ldots, \ -x_k = \frac{\partial S}{\partial y_k}(y) = y_1^{k-1}$$

$$-x_{k+1} = \frac{\partial S}{\partial y_{k+1}}(y) = 0, \ldots, \ -x_n = \frac{\partial S}{\partial y_n}(y) = 0.$$

From the first equation we derive

$$-y_2 = \pm(k+1)y_1^k + (k-1)y_1^{k-2}y_k + \cdots + 2y_1y_3$$

and renumerating the corresponding Darboux coordinates

$$p_1 = -y_2, p_2 = y_3, \ldots, p_{n-1} = y_n$$

$$q_1 = -x_2, q_2 = x_3, \ldots, q_{n-1} = x_n$$

we rewrite the equations for $\pi_{\{x_1=0\}}(L_2)$ in the form

$$p_1 = \frac{\partial \bar{S}}{\partial q_1} = \pm(k+1)q_1^k + (k-1)q_1^{k-2}p_{k-1} + \cdots + 2q_1p_2$$

$$q_2 = -\frac{\partial \bar{S}}{\partial p_2} = -q_1^2, \ldots, q_{k-1} = -\frac{\partial \bar{S}}{\partial p_{k-1}} = -q_1^{k-1}$$

$$q_k = -\frac{\partial \bar{S}}{\partial p_k} = 0, \ldots, q_{n-1} = -\frac{\partial \bar{S}}{\partial p_{n-1}} = 0$$

with the generating function

$$\bar{S}(q_1, p_1, \ldots, p_{n-1}) = \pm q_1^{k+1} + q_1^{k-1}p_{k-1} + \cdots + q_1^2 p_2$$

for the reduced Lagrangian germ L_2^r. □

Remark 3.1. We see that the only stable case of the triplet $(\{L_1, L_2, H\}, \tilde{p})$ is equivalent to the local model of type A_1 for the submanifold L_2 and that it corresponds to the basic reduced star which is the basic star in the reduced symplectic space.

Let $S = (\{L_1, L_2, L_3\}, \tilde{p})$ be a Lagrangian star and let H be a hypersurface-germ at $\tilde{p}$.

Proposition 3.2. We assume that the Lagrangian star S contains the star, say $(\{L_1, L_2\}, \tilde{p})$ which is of type A_1 (stable) with respect to $(H, \tilde{p})$. Then in the transversal case i.e. L_3 is transversal to L_1, L_2, and H, the typical reduced stars $(\{L_1^r, L_2^r, L_3^r\}, \tilde{p}^r)$ are classified by the following normal forms: $(\{L_1^r, L_2^r\}, \tilde{p}^r)$ is a basic Lagrangian star in $\tilde{M} \equiv (R^{2(n-1)}, \omega = \sum_{i=1}^{n-1} \mathrm{d}y_i \wedge \mathrm{d}x_i)$ and L_3^r is generated by the following Morse family

$$F(\lambda, y) = \lambda^{k+1} + \sum_{i=1}^{k-1} \lambda^{k-i} y_i + \phi(y_1, \ldots, y_{k-1}) \pm y_k^2 \pm \cdots \pm y_{n-1}^2$$

where $\phi \in \boldsymbol{m}^2_{y_1,\ldots,y_{k-1}}$.

Proof. By proposition 3.1 we can reduce $(\{L_1, L_2, H\}, \tilde{p})$ to the following normal form in $\tilde{p}$

$$L_1 : y_1 = 0, \ldots, y_n = 0$$

$$L_2 : x_1 = \pm 2y_1, x_2 = 0, \ldots, x_n = 0$$

$$H : x_1 = 0.$$

By transversality assumptions L_3 can be generated by the generating function $y \to F(y)$ such that

$$\mathrm{d}\left(\frac{\partial F}{\partial y_1}\right)(0) \neq 0.$$

By the reduction projection π_H we get

$$L_1^r : y_2 = 0, \ldots, y_n = 0$$
$$L_2^r : x_2 = 0, \ldots, x_n = 0$$
$$L_3^r : x_2 = \frac{\partial F}{\partial y_2}(y_1, \bar{y}), \ldots, \ x_n = \frac{\partial F}{\partial y_n}(y_1, \bar{y}) \qquad 0 = \frac{\partial F}{\partial y_1}(y_1, \bar{y})$$

where $\bar{y} = (y_2, \ldots, y_n)$.

Any liftable (through π_H) equivalence of $(\{L_1^r, L_2^r, L_3^r\}, 0)$ is determined by an $\mathcal{R}$-equivalence of Morse families $F(y_1, \bar{y})$, where the diffeomorphism of $\bar{y}$ is preserving zero. By reordering $\bar{y}$, treating y_1 as a Morse parameter, λ and applying the group of equivalences we obtain the prenormal forms of proposition 3.2. □

Now we consider the situation when the basic Lagrangian star $(\{L_1, L_2\}, \tilde{p})$ is not transversal to the hypersurface $(H, \tilde{p})$. In this case at least one of the two Lagrangian germs L_1, L_2 have to be transversal to H. We assume that it is L_1. Then we have the following result.

Proposition 3.3. If the basic Lagrangian star $(\{L_1, L_2\}, \tilde{p})$ is not transversal to $(H, \tilde{p})$ then the generic reduced Lagrangian star $(\{L_1^r, L_2^r\}, \tilde{p}^r)$ can be written in one form from the following normal forms: $L_1^r : y_1 = 0, \ldots, y_{n-1} = 0$, L_2^r : is generated by the following Morse family

$$S(\lambda, \bar{y}) = \pm\lambda^k + \sum_{i=1}^{k-3} y_i \lambda^{k-i-1} + \left(g(y_1, \ldots, y_{k-3}) \pm \sum_{i=k-2}^{n} y_i^2 \right)\lambda$$

where $k = \dim_R \mathcal{E}_\lambda / \Delta(S(\lambda, 0)) + 1 \leqslant n + 2$, $g \in \boldsymbol{m}^2_{y_1,\ldots,y_{k-3}} - \boldsymbol{m}^3_{y_1,\ldots,y_{k-3}}$ and $\Delta(S(\lambda, 0))$ is an ideal in $\mathcal{E}_\lambda$ generated by $\partial S/\partial\lambda \ (\lambda, 0)$.

Proof. If L_1 is transversal to H then we obtain the Darboux coordinates such that

$$L_1 : y_1 = 0, \ldots, y_n = 0 \qquad H : x_1 = 0.$$

Because L_2 is transversal to L_1, then L_2 may be generated in the form

$$x_i = -\frac{\partial S}{\partial y_i}(y) \qquad i = 1, \ldots, n$$

where $\mathrm{d}(\partial S/\partial y_1)|_0 = 0$. By the equivalence group of symplectomorphisms preserving $(H \cup L_1, 0)$ we can reduce S to the form (cf [2])

$$S(y) = \pm y_1^k + \sum_{i=2}^{k-2} y_i y_1^{k-i} + \left(g(y_2, \ldots, y_{k-2}) \pm \sum_{i=k-1}^{n} y_i^2 \right) y_1$$

where g is a smooth function (functional invariant) and $g \in \boldsymbol{m}^2_{y_2,\ldots,y_{k-2}}$. By the reduction projection π_H and reordering the variables 'y' we get the corresponding Morse family $S(\lambda, \bar{y})$, $\bar{y} = (y_1, \ldots, y_{n-1})$, generating L_2^r. □

We see that the reduction of the basic Lagrangian star, which is not transversal to H is no more basic. Moreover it is not even smooth. The only simple model of the reduced Lagrangian star in the non-transversal case is the one generated by the following Morse family

$$L_2^r : S(\lambda, y_1, y_2) = \lambda^3 \pm y_1^2 \lambda$$

which is the singular Lagrangian set.

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References

[1] Arnold V I, Gusein-Zade S M and Varchenko A N 1985 *Singularities of Differentiable Maps* vol 1 (Boston: Birkhäuser)

[2] Domitrz W and Janeczko S 1997 Equivalence of Lagrangian germs in the presence of a surface *Symplectic Singularities and Geometry of Gauge Fields (Banach Center Publications 39)* (Warsaw: Polish Academy of Sciences) pp 31–37

[3] Janeczko S and Kowalczyk A 1990 On singularities in the degenerated symplectic geometry *Hokkaido Math. J.* **19** 103–23

[4] Golubitsky M and Guillemin V 1975 Contact equivalence for Lagrangian manifolds *Adv. Math.* **15** 375–87

[5] Golubitsky M and Guillemin V 1973 *Stable Mappings and Their Singularities* GTM 14 (New York: Springer)

[6] Lion G and Vergne M 1980 The Weil representation, Maslov index and theta series *Progress in Math.* vol 6 (Basel: Birkhäuser)

[7] Martinet J 1970 Sur les singularités des formes differentielles *Ann. Inst. Fourier* **20** 95–178

[8] Martinet J 1982 *Singularities of Smooth Functions and Maps* (Cambridge: Cambridge University Press)