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Diffeomorphisms Preserving Symplectic Data on Submanifolds

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Abstract—We characterize general symplectic manifolds and their structure groups through a family of isotropic or symplectic submanifolds and their diffeomorphic invariance. In this way we obtain a complete geometric characterization of symplectic diffeomorphisms and a reinterpretation of symplectomorphisms as diffeomorphisms acting purely on isotropic or symplectic submanifolds.

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1. INTRODUCTION

A symplectic manifold is a $2n$-manifold X together with a symplectic form ω on X, i.e., a differential 2-form that is closed and nondegenerate. Diffeomorphisms $\phi\colon X \to X$ that preserve the symplectic structure, i.e., $\phi^*\omega = \omega$, are called symplectic diffeomorphisms or symplectomorphisms. If N is a submanifold of X, say symplectic (isotropic, etc.), i.e., $\omega|_N$ is a symplectic structure on N, then we say that ϕ is symplectic on N; in other words, ϕ preserves the symplectic data $\omega|_N$ (see [5, 9, 16]).

The main question we would like to answer in this paper is whether we can collect information from the symplectic data on a family of submanifolds and construct a complete system of invariants for the group of symplectomorphisms.

In the case of linear X ($\dim X = 2n$), if we collect the data $\omega^k|_{L_i}$ from the finite (minimal) family $L_1, \dots, L_N$ of linear $2k$-dimensional subspaces that are not co-planar (i.e., they do not belong to any hyperplane in the appropriate Grassmannian), then for odd k these data form a complete system of symplectic invariants. If k is even, this is a complete system of invariants for ϵ_k-symplectomorphisms (i.e., $\phi^*\omega = \epsilon\omega$, $\epsilon^k = 1$).

Diffeomorphic invariance of symplectic data on proper smooth submanifolds in symplectic space provides a sufficient condition for the group of conformal symplectic diffeomorphisms. Let (X, ω_X) and (Y, ω_Y) be symplectic manifolds or compact symplectic manifolds of dimension $2n > 2$. Let us fix a number s with $0 < s < n$ and assume that a diffeomorphism $\Phi\colon X \to Y$ transforms all $2s$-dimensional symplectic submanifolds of X into symplectic submanifolds of Y or transforms all isotropic k-dimensional tori of X into isotropic tori of Y ($1 < k \le n$). We find that in both these cases (symplectic and isotropic ones) Φ is a conformal symplectomorphism, i.e., there is a constant $c \neq 0$ such that $\Phi^*\omega_Y = c\omega_X$ (see [19]).

The important properties of symplectic manifolds we have to apply to get this result are mainly the fact that they are k-point connected; i.e., if $a_1, \dots, a_m$ is a family of points in X and for every $i = 1, \dots, m$ we choose a linear k-dimensional isotropic subspace ($0 < k \le n$) $H_i \subset T_{a_i}X$, then there is a closed isotropic k-dimensional torus $Y \subset X$ such that $a_i \in Y$ and $T_{a_i}Y = H_i$.

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This is a survey article based on [9–11] and build of three components; the first is devoted purely to the linear symplectic geometry considerations and the second gives a characterization of symplectic manifolds and their structure groups through the isotropic or symplectic submanifolds and their basic invariants. The third part presented in Section 5 is a study of the group of polynomial symplectomorphisms and their k-transitivity.

2. LINEAR AUTOMORPHISMS IN A SYMPLECTIC SPACE

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Let (X,ω) be a symplectic vector space over $\mathbb{K}$, i.e., $X \cong \mathbb{K}^{2n}$ is a vector space and ω is a bilinear nondegenerate skew-symmetric form on X. The symplectic complement of a linear subspace $L \subset X$ is defined as the subspace $L^\omega = \{x \in X : \omega(x,y) = 0\ \forall y \in L\}$. A subspace $L \subset X$ is called isotropic if $L \subset L^\omega$, coisotropic if $L^\omega \subset L$, symplectic if $L \cap L^\omega = \{0\}$, and Lagrangian if $L^\omega = L$. A subspace L is symplectic if and only if $\omega|_L$ is a nondegenerate form. For any subspace L we have $\dim L + \dim L^\omega = \dim X$ and $(L^\omega)^\omega = L$. There exists a basis of X, called a symplectic basis, $u_1,\dots,u_n,v_1,\dots,v_n$, such that

$$\omega(u_i,u_j) = \omega(v_i,v_j) = 0, \qquad \omega(u_i,v_j) = \delta_{ij}.$$

If $L \subset X$ is a subspace, then there is a basis $u_1,\dots,u_k,v_1,\dots,v_k,w_1,\dots,w_l$ of L such that $\omega|_L(u_j,v_k) = \delta_{jk}$ and all other pairings $\omega|_L(\cdot,\cdot)$ vanish. This basis extends to a symplectic basis for (X,ω) and the integer $2k$ is the rank of $\omega|_L$.

We say that a linear automorphism $F\colon X \to X$ is a symplectomorphism (or is symplectic on X) if $F^*\omega = \omega$, i.e., $\omega(x,y) = \omega(F(x),F(y))$ for every $x,y \in X$. If $L \subset X$ is a linear subspace, then we say that F is symplectic on L if $\omega(x,y) = \omega(F(x),F(y))$ for every $x,y \in L$. The group of automorphisms of (X,ω) is called the symplectic group and is denoted by $\mathbf{Sp}(X,\omega)$. Via a symplectic basis, $\mathbf{Sp}(X,\omega)$ can be identified with the group $\mathrm{Sp}(2n,\mathbb{R})$ of real $2n\times 2n$ matrices A that satisfy $A^{\mathrm{T}}J_0A = J_0$, where J_0 is the $2n \times 2n$ matrix of ω (in a symplectic basis).

Let X be a vector space of dimension $2n$. Let $L \subset X$ be an l-dimensional linear subspace $(0 < l < 2n)$. If vectors $v_1,\dots,v_l$ form a basis of L, then the line $\mathbb{K}(v_1 \wedge \ldots \wedge v_l)$ is uniquely determined by L and it does not depend on the basis $v_1,\dots,v_l$. This line determines a unique point $\Psi(L)$ in the Grassmannian $G(l,2n) \subset \mathbb{P}^{N-1}$, where $N = \binom{2n}{l}$ and $\mathbb{P}^{N-1}$ denotes the $(N-1)$-dimensional projective space. We have the following notion of co-planar spaces:

Definition 2.1. Let $L_1,\dots,L_N$ be l-dimensional linear subspaces of X. We say that they are co-planar if the points $\Psi(L_1),\dots,\Psi(L_N) \in G(l,2n)$ are co-planar, i.e., if there is a hyperplane $\Lambda \subset \mathbb{P}^{N-1}$ containing all points $\Psi(L_1),\dots,\Psi(L_N)$.

Note that subspaces $L_1,\dots,L_N$ are not co-planar if the points $\Psi(L_1),\dots,\Psi(L_N) \in G(l,2n)$ span linearly the whole space $\mathbb{P}^{N-1}$. Let $\mathbf{e}_1,\dots,\mathbf{e}_{2n}$ be a basis of X. Since the Grassmannian $G(l,2n)$ contains the subset $\{\mathbf{e}_{i_1} \wedge \ldots \wedge \mathbf{e}_{i_l}\}_{0 \le i_1 < \ldots < i_l \le 2n}$, we see that the subspaces $L_1,\dots,L_N$ are not co-planar if the points $\Psi(L_1),\dots,\Psi(L_N)$ span linearly the Grassmannian $G(l,2n)$.

If $L_1,\dots,L_m$ are not co-planar, then we can always choose a subfamily $L_1,\dots,L_k$ of non-co-planar subspaces with $k = N$, and conversely, it is easy to construct a collection $L_1,\dots,L_N$ that is not co-planar. Hence we can always assume that $m = N$.

It is well known (see, e.g., [1]) that we can always choose a vector basis $\mathbf{e}_1,\dots,\mathbf{e}_{2n}$ in X (symplectic basis) such that $\omega\big((\sum v_i\mathbf{e}_i),(\sum w_i\mathbf{e}_i)\big) = \sum_{0<i\le n}(v_iw_{i+n} - v_{i+n}w_i)$. We have $\omega = \sum_{i=1}^n \mathbf{e}_i^* \wedge \mathbf{e}_{i+n}^*$ in the dual basis $\mathbf{e}_1^*,\dots,\mathbf{e}_{2n}^*$.

Denote by $\overline{G(2,2n)} \subset \bigwedge^2 X := Y$ the set of all vectors $v \wedge w$, where $v,w \in X$. Let (v_1^i, v_2^i) be a basis of the linear subspace L_i, $i = 1,\dots,N$. It is easy to see that the linear subspaces $L_1,\dots,L_N$ are co-planar if the vectors $\{v_1^i \wedge v_2^i\}_{i=1,\dots,N} \subset Y$ are co-planar in Y.

Set $u_i = v_1^i \wedge v_2^i$. Now, consider the mapping $R := \bigwedge^2 F\colon Y \to Y$. In Y we have the basis $\mathbf{e}_{ij} = \mathbf{e}_i \wedge \mathbf{e}_j$, $0 < i < j \le 2n$. For $y = \sum y_{ij}\mathbf{e}_{ij}$ let $\eta(y) = \sum_{i=1}^n y_{ii+n}$. Of course, η is a linear form on Y.

Observe that $\eta(v \wedge w) = \omega(v, w)$. Consequently, the mapping F is symplectic if and only if for all $v, w \in X$ we have $\eta(v \wedge w) = \eta(R(v \wedge w))$. However, the form $\eta(y) - \eta(R(y))$ is linear on Y and by the assumption it vanishes on the vectors u_i, $i = 1, \dots, N$. Since the latter set is not co-planar in Y, the form $\eta(y) - \eta(R(y))$ vanishes identically on Y. This means that $\omega(v, w) = \omega(F(v), F(w))$ for all $v, w \in X$, i.e., F is a symplectomorphism. Thus we proved

Proposition 2.1. *Let X be a symplectic vector space of dimension $2n$, and let $F\colon X \to X$ be a linear automorphism. Assume that F is symplectic on a collection $L_1, \dots, L_N$ of 2-dimensional subspaces that are not co-planar. Then F is a symplectomorphism.*

Definition 2.2. Let (X, ω) be a symplectic vector space, and let $F\colon X \to X$ be a linear automorphism. We say that F is an ϵ_k-symplectomorphism if $F^*\omega = \epsilon\omega$, where $\epsilon^k = 1$. Moreover, we say that F is an antisymplectomorphism if $F^*\omega = -\omega$.

We can treat an ϵ_k-symplectomorphism as a root of a symplectomorphism. Indeed, if F is an ϵ_k-symplectomorphism, then F^k is a symplectomorphism.

Lemma 2.1. *Let (X, ω) be a symplectic vector space. Let $W \subset X$ be a $2k$-dimensional symplectic subspace of X; i.e., $(W, \omega|_W)$ is a symplectic vector space. If*

$$\omega^k(v, w_1, \dots, w_{2k-1}) = 0$$

for any $\{w_1, \dots, w_{2k-1}\}$, $w_i \in W$, then v is complementary to W with respect to ω.

Using Proposition 2.1 and Lemma 2.1, we prove a general result.

Theorem 2.1 (cf. [9]). *Let (X, ω) be a symplectic vector space of dimension $2n$, and let $F\colon X \to X$ be a linear automorphism. Let $0 < k < n$. Assume that F preserves the form ω^k on a collection $L_1, \dots, L_N$ of $2k$-dimensional subspaces that are not co-planar. Then F is an ϵ_k-symplectomorphism. In particular, if $\mathbb{K} = \mathbb{R}$ and the number k is odd, then F is a symplectomorphism.*

Corollary 2.1. *Let (X, ω) be a symplectic vector space, and let $F\colon X \to X$ be a linear automorphism. Let $0 < l < k \le n$ be natural numbers such that $(k, l) = 1$. Assume that F preserves the forms ω^k and ω^l. Then F is a symplectomorphism.*

Let $\mathcal{A}_{l,2r} \subset G(l, 2n)$ denote the set of all l-dimensional linear subspaces of X on which the form ω has rank $\le 2r$. Of course, $\mathcal{A}_{l,2r} \subset \mathcal{A}_{l,2r+2}$ if $2r + 2 \le l$. We have the following:

Theorem 2.2 [9]. *Let (X, ω) be a symplectic vector space of dimension $2n$. Then the set $\mathcal{A}_{2k,2k-2} \subset G(2k, 2n) \subset \mathbb{P}^{N-1}$ is an irreducible algebraic subset of $G(2k, 2n)$ and it linearly spans a hyperplane in $\mathbb{P}^{N-1}$. More generally, for $r < k$ the set $\mathcal{A}_{2k,2r}$ is also irreducible and linear in $G(2k, 2n)$; i.e., there is a linear projective subspace $L \subset \mathbb{P}^{N-1}$ such that $\mathcal{A}_{2k,2r} = G(2k, 2n) \cap L$. Moreover, the set $\mathcal{A}_{2k,2r}$ can be computed effectively.*

Proof. First, assume that $\mathbb{K} = \mathbb{C}$. Let $\mathcal{A} = \mathcal{A}_{2k,2k-2}$ denote the set of all $2k$-dimensional subspaces on which the form ω has rank $< 2k$ (also called the set of subspaces of rank $< 2k$).

Now recall the notion of a projectively factorial variety. Let $X \subset \mathbb{P}^n$ be a complex algebraic subvariety of a complex projective space, and let $C(X)$ be an affine cone over X. We consider the projective coordinate ring $R(X)$ of X as the ring $\mathbb{C}[C(X)] = \mathbb{C}[x_0, \dots, x_n]/I(C(X))$, where $I(C(X)) = \{F \in \mathbb{C}[x_0, \dots, x_n]\colon F|_{C(X)} = 0\}$. We say that X is projectively factorial if the ring $R(X)$ is factorial.

If X is a smooth projective variety, we can consider the Picard group $\mathrm{Pic}(X)$ of all algebraic line bundles on X (for details see, e.g., [7, p. 133]). It is well known that if X is projectively factorial, then $\mathrm{Pic}(X) = \mathbb{Z}$ and $\mathrm{Pic}(X)$ is generated by the line bundle $\mathcal{O}(H)$, where $H \subset X$ is a hyperplane section.

By the Andreotti–Salmon theorem (see, e.g., [14]) the embedded Grassmannian $G(2k, 2n) \subset \mathbb{P}^{N-1}$ is projectively factorial. In particular, the Picard group of $G(2k, 2n)$ is generated by a hyperplane section. It can be easily deduced from this that for every hyperplane $H \subset \mathbb{P}^{N-1}$ the set $H \cap G(2k, 2n)$ is an irreducible variety that is not contained in a proper linear subspace of H (cf. [12, Lemma 3.18]). Let $H \subset \mathbb{P}^{N-1}$ be a hyperplane such that $\mathcal{A} = H \cap G(2k, 2n)$. Thus, by the above, $\mathcal{A}$ linearly spans the hyperplane H.

Now consider the set $\mathcal{A}_{2k,2r}$. It is irreducible since it contains an orbit of the symplectic group as a dense subset (in fact, the orbit of any subspace l of dimension $2k$ on which the form ω^k has rank exactly $2r$ is dense in $\mathcal{A}_{2k,2r}$). We show that there is a linear subspace L such that $\mathcal{A}_{2k,2r} = L \cap G(2k, 2n)$. Take a sequence of vectors $(x_1, \dots, x_{2k}) \in X^{2k}$. Let $\mathbb{A}$ be the matrix that has (the coordinates of) the vectors $x_{i_1}, \dots, x_{i_{2k-2r-2}}$ as rows. Let $\delta_{s_1,\dots,s_{2k-2r-2}}(x_{i_1}, \dots, x_{i_{2k-2r-2}})$ denote the principal minor of $\mathbb{A}$ determined by the columns indexed by $s_1, \dots, s_{2k-2r-2}$. Consider all possible skew-symmetric forms of the type

$$\omega^{r+1}(x_{j_1}, \dots, x_{j_{2r+2}})\delta_{s_1,\dots,s_{2k-2r-2}}(x_{i_1}, \dots, x_{i_{2k-2r-2}}),$$

where $\{j_1, \dots, j_{2r+2}\} \cup \{i_1, \dots, i_{2k-2r-2}\} = \{1, \dots, 2k\}$. It is not difficult to check that they simultaneously vanish only at the vectors $x_1 \wedge \ldots \wedge x_{2k}$ that belong to $\mathcal{A}_{2k,2r}$. On the other hand, these skew-symmetric functions can be treated as linear forms on $\mathbb{P}^{N-1}$ (cf. the proof of Theorem 2.1). Of course, we can find these linear forms effectively (as functions of the variables $y_{i_1,\dots,i_{2k}}$, cf. the proof of Theorem 2.1). Since we know the equations of the Grassmannian $G(2k, 2n)$ (see [7, p. 211]), we can compute the set $\mathcal{A}_{2k,2r}$ effectively. This finishes the proof in the case $\mathbb{K} = \mathbb{C}$.

Now we sketch the proof for the case $\mathbb{K} = \mathbb{R}$. As before, define $\mathcal{A} \subset G(2k, 2n)$ as the set of all $2k$-dimensional linear subspaces of X of rank $< 2k$. This set has a stratification into smooth subsets $\mathcal{A}_r = \{W \in \mathcal{A}\colon \operatorname{rank} \omega|_W = 2r\}$, where $r = 0, 1, \dots, k-1$ and $\mathcal{A}_i \subset \mathrm{closure}(\mathcal{A}_{i+1})$. Moreover, every such subset is homogeneous with respect to the induced action of the group $\mathrm{Sp}(2n, \mathbb{K})$. Take a real subspace $L \in \mathcal{A}_{k-1}$. Let $H \subset \mathrm{Sp}(2n, \mathbb{R})$ be the stabilizer of L in the group $\mathrm{Sp}(2n, \mathbb{R})$. Thus

$$\dim \mathcal{A} = \dim \mathcal{A}_{k-1} = \dim \mathrm{Sp}(2n, \mathbb{R}) - \dim H.$$

Now let us complexify X. Let $H' \subset \mathrm{Sp}(2n, \mathbb{C})$ be the stabilizer of $L \otimes \mathbb{C}$ in the group $\mathrm{Sp}(2n, \mathbb{C})$.

Let $\mathcal{A}' \subset G(2k, 2n, \mathbb{C})$ be the set of all complex $2k$-dimensional linear subspaces of $X \otimes \mathbb{C}$ of rank $< 2k$. Then $\mathcal{A}'$ has the same (complex) dimension as the orbit of $L \otimes \mathbb{C}$, and this dimension is equal to $\dim \mathrm{Sp}(2n, \mathbb{C}) - \dim H'$. But H' contains the complexification of the subgroup H; thus $\dim_{\mathbb{C}} H \geq \dim_{\mathbb{R}} H$ and consequently $\dim_{\mathbb{C}} \mathcal{A}' \leq \dim_{\mathbb{R}} \mathcal{A}$. However, in the complex case we have $\dim \mathcal{A}' = \dim G(2k, 2n, \mathbb{C}) - 1$. From this we see immediately that $\dim \mathcal{A} = \dim G(2k, 2n, \mathbb{R}) - 1$. This means that the complexification of $\mathcal{A}$ is $\mathcal{A}'$. Thus $\mathcal{A}$ spans linearly a (real) hyperplane if and only if $\mathcal{A}'$ spans a (complex) hyperplane. Now we can finish the proof as above. $\square$

From the proof we see that Theorem 2.2 can be partly generalized:

Corollary 2.2. *Let (X, ω) be a symplectic vector space of dimension $2n$. Let l and r be integers such that $l \leq 2n$ and $2r + 2 \leq l$. Then there is a proper linear subspace $L \subset \mathbb{P}^{N-1}$ such that $\mathcal{A}_{l,2r} = L \cap G(l, 2n)$. Moreover, we can compute the equations of $\mathcal{A}_{l,2r}$ effectively. In particular, this is true for the Lagrangian Grassmannian manifold $\Lambda_n = \mathcal{A}_{n,0}$.*

Definition 2.3. Let $L_1, \dots, L_{N-1} \in \mathcal{A}_{2k,2k-2}$ be $2k$-dimensional linear subspaces of X (of rank $< 2k$). We say that they are in general position if they linearly span a hyperplane in $\mathbb{P}^{N-1}$.

Remark 2.1. By Theorem 2.2 every sufficiently general subset $\{L_1, \dots, L_{N-1}\} \subset \mathcal{A}_{2k,2k-2}$ is in general position. Moreover, we can find such subspaces L_i with rank $L_i = 2k - 2$.

A slightly more general version of Theorem 2.1 (see also [9]) is as follows:

Theorem 2.3. *Let (X,ω) be a symplectic vector space of dimension $2n$, and let $F\colon X \to X$ be a linear automorphism. Let $0 < k < n$. Let $L_1, \dots, L_{N-1}$ be $2k$-dimensional linear subspaces of X of rank $<2k$ that are in general position. Assume that F transforms $L_1, \dots, L_{N-1}$ into subspaces of rank $<2k$. Then there is a nonzero constant c such that $F^*\omega = c\omega$.*

Proof. We can assume that $\mathbb{K} = \mathbb{C}$. Let L be a $2k$-dimensional subspace of rank $2k$. There is a constant a such that $F^*\omega^k = a\omega^k$ on L. As in the proof of Theorem 2.1 (see [9]), denote by $\overline{G(2k,2n)} \subset \bigwedge^{2k} X := Y$ the set of all vectors $v_1 \wedge \dots \wedge v_{2k}$, where $v_1, \dots, v_{2k} \in X$. Let $\mathcal{A} = \{L_1, \dots, L_{N-1}\}$.

Let $H \subset \mathbb{P}^N$ be a hyperplane such that $\mathcal{A}_{2k,2k-2} = H \cap G(2k,2n)$. Thus, by assumption, $\mathcal{A}$ linearly spans the hyperplane H. Since $L \notin H$, we can easily deduce that the set $\mathcal{B} := \{L\} \cup \mathcal{A}$ is not co-planar in Y. By assumption, for $W \in \mathcal{B}$ we have

$$a\eta(u_W) = \eta(R(u_W)),$$

where $u_W \in \overline{G(2k,2n)} \subset \bigwedge^{2k} X := Y$ is a vector determined by W.

This implies that the linear form $a\eta(y) - \eta(R(y))$ vanishes on the vectors u_W, $W \in \mathcal{B}$. Since the set $\mathcal{B}$ is not co-planar in Y, the form $a\eta(y) - \eta(R(y))$ vanishes identically on Y. This means that $a\omega^k(v_1, \dots, v_{2k}) = \omega^k(F(v_1), \dots, F(v_{2k}))$ for every $v_1, \dots, v_{2k} \in X$, i.e., $F^*\omega^k = a\omega^k$.

It is easy to see (see [9]) that there is a constant c such that $F^*(\omega) = c\omega$ and $c^k = a$. Since F is a linear automorphism, we have $c \neq 0$. □

Corollary 2.3. *Let (X,ω) be a symplectic vector space of dimension $2n$, and let $F\colon X \to X$ be a linear automorphism. Let $0 < k < n$. Assume that F transforms $2k$-dimensional subspaces of rank $2k-2$ into subspaces that have rank $<2k$. Then there is a nonzero constant c such that $F^*\omega = c\omega$.*

Corollary 2.4. *Let (X,ω) be a symplectic vector space of dimension $2n$, and let $F\colon X \to X$ be a linear automorphism. Let $0 < l < 2n$ and $2r+2 \le l$. Assume that $\mathcal{A}_{l,2r} \neq \varnothing$ and F transforms the set $\mathcal{A}_{l,2r}$ into the same set. Then there is a nonzero constant c such that $F^*\omega = c\omega$.*

Proof. Let $\mathcal{B}_{2r+2,2r}$ denote the set of $(2r+2)$-dimensional subspaces of rank $2r$. Since every subspace from $\mathcal{B}_{2r+2,2r}$ is contained in some subspaces from $\mathcal{A}_{l,2r}$, it is easy to see that F transforms the set $\mathcal{B}_{2r+2,2r}$ into the same set. Hence we are done by Corollary 2.3. □

Corollary 2.5. *Let (X,ω) be a symplectic vector space of dimension $2n$, and let $F\colon X \to X$ be a linear automorphism. Let $2 \le l \le n$ and assume that F transforms l-dimensional isotropic (e.g., Lagrangian) subspaces into subspaces of the same type. Then there is a nonzero constant c such that $F^*\omega = c\omega$.*

Theorem 2.4. *Let (X,ω) be a symplectic vector space of dimension $2n$, and let $F\colon X \to X$ be a linear automorphism. Let $0 < 2r < 2n$. Assume F transforms $\mathcal{A}_{2r,2r-2}$ into $\mathcal{A}_{2r,2r-2}$. Then there is a nonzero constant c such that $F^*\omega = c\omega$.*

From Theorem 2.4 we can deduce the following interesting facts:

Proposition 2.2 [10]. *Let (X,ω_X) and (Y,ω_Y) be symplectic vector spaces of dimension $2n$, and let $F\colon X \to Y$ be a linear isomorphism. Fix a number s: $0 < s < n$ and assume that F transforms all $2s$-dimensional symplectic subspaces of X into symplectic subspaces of Y. Then there is a nonzero constant c such that $F^*\omega_Y = c\omega_X$.*

Proof. Via a symplectic basis we can assume that $(X,\omega_X) \cong (\mathbb{R}^{2n},\omega_0) \cong (Y,\omega_Y)$. By assumption the mapping F^* induced by F transforms the set $A = \mathcal{A}_{2s,2s} \setminus \mathcal{A}_{2s,2s-2}$ into the same set A. Of course, $F^*\colon A \to A$ is an injection. Since A is a smooth algebraic variety and F^* is regular, the Borel theorem (see [4]) implies that F^* is a bijection. This means that F transforms $\mathcal{A}_{2s,2s-2}$ into the same set, and we conclude the proof by applying Theorem 2.4. □

Proposition 2.3 [9]. *Let (X,ω_X) and (Y,ω_Y) be symplectic vector spaces of dimension $2n$, and let $F\colon X\to Y$ be a linear isomorphism. Fix a number k: $1<k\le n$ and assume that F transforms all k-dimensional isotropic subspaces of X into isotropic subspaces of Y. Then there is a nonzero constant c such that $F^*\omega_Y=c\omega_X$.*

Proof. For $k=2$ it follows immediately from Theorem 2.4. Assume that $k>2$. Take a plane H belonging to $\mathcal{A}_{2,0}$. Since H is isotropic, we can extend H to a k-dimensional isotropic subspace L. By assumption L is transformed into an isotropic subspace $F(L)$. Observe that $F(H)$ is contained in $F(L)$. Then $F(H)$ is also isotropic. In particular, $F(H)\in\mathcal{A}_{2,0}$. Then on the basis of Theorem 2.4 we have the thesis. □

We end this section by

Proposition 2.4 [10]. *Let X be a vector space of dimension $2n$, and let ω_1 and ω_2 be two symplectic forms on X. If $\mathbf{Sp}(X,\omega_1)\subset\mathbf{Sp}(X,\omega_2)$, then there exists a nonzero constant c such that $\omega_2=c\omega_1$.*

Proof. If $n=1$, then the theorem is obvious. Assume that $n>1$. Let $\mathcal{A}_1$ ($\mathcal{A}_2$) be the set of all 2-dimensional subspaces of X that are symplectic with respect to ω_1 (ω_2). These sets are open and dense in the Grassmannian $G(2,2n)$. Hence $\mathcal{A}_1\cap\mathcal{A}_2\neq\varnothing$. Take $H\in\mathcal{A}_1\cap\mathcal{A}_2$. We have $\mathcal{A}_1=\mathbf{Sp}(X,\omega_1)H\subset\mathbf{Sp}(X,\omega_2)H=\mathcal{A}_2$. Now apply Proposition 2.2 to $X=(X,\omega_1)$, $Y=(X,\omega_2)$ and $F=$ identity. □

3. SYMPLECTOMORPHISMS GENERATED BY HAMILTONIANS

Here we recall some basic facts about the linear symplectic group. The group of automorphisms of (X,ω) is called the symplectic group and is denoted by $\mathbf{Sp}(X,\omega)$. Via a symplectic basis, X can be identified with the standard symplectic space $(\mathbb{R}^{2n},\omega_0)$ and $\mathbf{Sp}(X,\omega)$ can be identified with the group of $2n\times 2n$ real matrices A that satisfy $A^{\mathrm{T}}J_0A=J_0$, where J_0 is the $2n\times 2n$ matrix of ω_0 (in the standard basis), i.e.,

$$J_0=\begin{pmatrix}0 & -I_n\\ I_n & 0\end{pmatrix}.$$

We can define the following "elementary" symplectomorphisms:

(1) $L_i(c_i)(x_1,\dots,x_n,y_1,\dots,y_n)=(x_1,\dots,x_n,y_1,\dots,y_{i-1},y_i+c_ix_i,y_{i+1},\dots,y_n)$;

(2) $L_{ij}(c_{ij})(x_1,\dots,x_n,y_1,\dots,y_n)=(x_1,\dots,x_n,y_1,\dots,y_{i-1},y_i+c_{ij}x_j,y_{i+1},\dots,y_{j-1},y_j+c_{ij}x_i,$ $y_{j+1},\dots,y_n)$;

(3) $R_i(d_i)(x_1,\dots,x_n,y_1,\dots,y_n)=(x_1,\dots,x_{i-1},x_i+d_iy_i,x_{i+1},\dots,x_n,y_1,\dots,y_n)$;

(4) $R_{ij}(d_{ij})(x_1,\dots,x_n,y_1,\dots,y_n)=(x_1,\dots,x_{i-1},x_i+d_{ij}y_j,x_{i+1},\dots,x_{j-1},x_j+d_{ij}y_i,x_{j+1},\dots,$ $x_n,y_1,\dots,y_n)$,

where c_i, c_{ij}, d_i, d_{ij} are real numbers and $0\le i<j\le n$.

Now we have the following basic result:

Theorem 3.1 [10]. *Let $X=(\mathbb{R}^{2n},\omega_0)$ be the standard symplectic vector space. Then the group $\mathbf{Sp}(X)$ is generated by the following family of elementary symplectomorphisms*:

$$\bigl\{L_i(c_i),L_{ij}(c_{ij}),R_i(d_i),R_{ij}(d_{ij})\colon\ 0<i<j\le n\ \text{ and }\ c_i,c_{ij},d_i,d_{ij}\in\mathbb{R}\bigr\};$$

i.e., if $g\in\mathbf{Sp}(X)$, then $g=\prod_{i=1}^m e_i$, where e_i is one of the elementary symplectomorphisms and $m\in\mathbb{N}$.

Let $X=(\mathbb{R}^{2n},\omega_0)$ be the standard symplectic vector space. In X we consider the norm $\|(a_1,\dots,a_{2n})\|=\max_{i=1}^{2n}|a_i|$. Take a smooth function $H\colon X\times\mathbb{R}\ni(z,t)\to\mathbb{R}$ and consider a system of differential equations (cf. [6])

$$\phi'(t,z) = J_0(\nabla_z H)(\phi(t,z),t), \qquad \phi(0,z) = z.$$

Assume that this system has a solution $\phi(t,z)$ for every z and every t (this is satisfied, e.g., if the supports of all functions H_t, $t \in \mathbb{R}$, are contained in a compact set). Then we can define the diffeomorphism

$$\Phi(z) = \phi(1,z). \tag{3.1}$$

It is not difficult to check that Φ is a symplectomorphism.

Definition 3.1. Let $\Phi\colon X \to X$ be a symplectomorphism. We say that Φ is a *Hamiltonian symplectomorphism* if it is given by formula (3.1) for some smooth function H. We also say that H is a Hamiltonian of Φ.

Lemma 3.1. *All elementary linear symplectomorphisms are Hamiltonian symplectomorphisms.*

Proof. Indeed, we have the following:

(1) $L_i(c)$ is given by the Hamiltonian $H(x,y) = (c/2)x_i^2$;

(2) $L_{ij}(c)$ is given by the Hamiltonian $H(x,y) = cx_ix_j$;

(3) $R_i(c)$ is given by the Hamiltonian $H(x,y) = -(c/2)y_i^2$;

(4) $R_{ij}(c)$ is given by the Hamiltonian $H(x,y) = -cy_iy_j$. $\square$

Now we show how to compute a Hamiltonian of a linear symplectomorphism:

Theorem 3.2. *Let $L\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear symplectomorphism. Then L has a polynomial Hamiltonian*

$$H_L(z,t) = \sum_{i,j=1}^{2n} a_{i,j}(t) z_i z_j, \tag{3.2}$$

where $a_{i,j}(t) \in \mathbb{R}[t]$ are polynomials in one variable t. Moreover, we can compute H_L effectively.

Proof. Let $L = L_m \circ \ldots \circ L_1$, where L_i are elementary symplectomorphisms. We proceed by induction with respect to m. If $m = 1$, then we can use Lemma 3.1. In this case the flow $L_1(t)$ depends linearly on t.

Now consider $L' = L_{m-1} \circ \ldots \circ L_1$. By the induction hypothesis $L'(t) = L_{m-1}(t) \circ \ldots \circ L_1(t)$ is given by a Hamiltonian H' of the form (3.2). Let H'' be the Hamiltonian of L_m (as in Lemma 3.1). Now the flow $L(t) = L_m(t) \circ L'(t)$ is given by the Hamiltonian

$$H(z,t) = H''(z) + H'(L_m(t)^{-1}(z), t).$$

Of course, it also has the form (3.2). Since we can decompose L into the product $L = L_m \circ \ldots \circ L_1$ effectively (see the proof of Theorem 3.1), we can also compute H in an effective way. $\square$

Proposition 3.1. *Let $L\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a Hamiltonian symplectomorphism given by the flow $z \to \phi(t,z)$, $t \in \mathbb{R}$. Assume that $\phi(t,0) = 0$ for $t \in [0,1]$. For every $\eta > 0$ there is an $\epsilon > 0$ and a Hamiltonian symplectomorphism $\Phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that*

(1) *$\Phi(z) = L(z)$ for all z with $\|z\| \le \epsilon$;*

(2) *$\Phi(z) = z$ for all z with $\|z\| \ge \eta$.*

Proof. We know that $L(z) = \phi(1,z)$, where $\phi(t,z)$ is the solution of some differential equation

$$\phi'(t,z) = J_0(\nabla_z H)(\phi(t,z),t), \qquad \phi(0,z) = z.$$

Since $\phi(t,0) = 0$ for every $t \in [0,1]$, we can find $\epsilon > 0$ so small that all trajectories $\{\phi(t,z),\ 0 \le t \le 1\}$ that start from the ball $B(0,\epsilon)$ are contained in the ball $B(0,\eta/2)$. Let $\sigma\colon \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function such that

$$\sigma(z) = \begin{cases} 1 & \text{if } \|z\| \le \eta/2, \\ 0 & \text{if } \|z\| \ge \eta. \end{cases}$$

Take $S = \sigma H$. The Hamiltonian symplectomorphism Φ given by the differential equation

$$\phi'(t,z) = J_0(\nabla_z S)(\phi(t,z),t), \qquad \phi(0,z) = z,$$

is well defined on the whole of $\mathbb{R}^{2n}$ and

$$\Phi(z) = \begin{cases} L(z) & \text{if } \|z\| \le \epsilon, \\ z & \text{if } \|z\| \ge \eta. \end{cases} \qquad \square$$

Now Theorem 3.2 easily yields the following important

Corollary 3.1. *Let $L\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a linear symplectomorphism. For every $\eta > 0$ there is an $\epsilon > 0$ and a Hamiltonian symplectomorphism $\Phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that*

(1) *$\Phi(z) = L(z)$ for all z with $\|z\| \le \epsilon$;*

(2) *$\Phi(z) = z$ for all z with $\|z\| \ge \eta$.*

Before we formulate our next result concerning characterization of symplectomorphisms by finite data, we need the following:

Lemma 3.2. *Let $X = (\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space. Fix $\eta > 0$ and let $a, b \in B(0,\eta)$. Then there exists a symplectomorphism $\Phi\colon X \to X$ such that*

$$\Phi(a) = b \qquad \text{and} \qquad \Phi(z) = z \quad \text{for } \|z\| \ge 2\eta.$$

Proof. Let $c = (c_1, \dots, c_{2n}) = b - a$. Define a sequence of points as follows:

(1) $a_0 = a$,

(2) $a_i = a_{i-1} + (0, \dots, 0, c_i, 0, \dots, 0)$.

Of course, $a_i \in B(0,\eta)$ and $a_{2n} = b$. Now consider the translation

$$T_i\colon\ \mathbb{R}^{2n} \ni (x,y) \mapsto (x,y) + (0, \dots, 0, c_i, 0, \dots, 0) \in \mathbb{R}^{2n}.$$

We have $T_i(a_{i-1}) = a_i$ for $i = 1, \dots, 2n$.

The translation T_i is a Hamiltonian symplectomorphism given by the Hamiltonian

$$H_i(x,y) = \begin{cases} -c_i y_i & \text{if } i \le n, \\ c_i x_{i-n} & \text{if } i > n. \end{cases}$$

Let V_i be the symplectic vector field determined by the Hamiltonian H_i. Since the ball $B(0,r)$ is a convex set, all trajectories $\phi(t)$, $0 \le t \le 1$, of the symplectic vector fields V_i that begin at a_i lie in the ball $B(0,\eta)$. Let $\sigma\colon \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function such that

$$\sigma(z) = \begin{cases} 1 & \text{if } \|z\| \le \eta, \\ 0 & \text{if } \|z\| \ge 2\eta. \end{cases}$$

Let $F_i\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be the Hamiltonian symplectomorphism given by the Hamiltonian $G_i = \sigma H_i$. Then

$$G_i(a_{i-1}) = a_i \qquad \text{and} \qquad G_i(z) = z \quad \text{if } \|z\| \ge 2\eta.$$

Now it is enough to take $\Phi = G_{2n} \circ G_{2n-1} \circ \dots \circ G_1$. $\square$

We apply Proposition 3.1 to the general case:

Theorem 3.3 [10]. *Let (X,ω) be a symplectic manifold. Let $a_1,\dots,a_m$ and $b_1,\dots,b_m$ be two families of points of X. For every $i=1,\dots,n$ choose a linear symplectomorphism $L_i\colon T_{a_i}X \to T_{b_i}X$. Then there is a symplectomorphism $\Phi\colon X\to X$ such that*

(1) $\Phi(a_i)=b_i$,

(2) $d_{a_i}\Phi = L_i$.

Proof. By the Darboux theorem every point $z\in X$ has an open neighborhood V_z that is symplectically isomorphic to the ball $B(0,r_z)$ in the standard vector space $(\mathbb{R}^{2n},\omega_0)$. Denote by $U_z\subset V_z$ the open set that corresponds to the ball $B(0,r_z/3)$.

Since $\dim X\ge 2$, the manifold $X\setminus\{a_2,\dots,a_m\}$ is also connected. Hence there exists a smooth path $\gamma\colon I\to X$ such that $\gamma(0)=a_1$, $\gamma(1)=b_1$ and $\{a_2,\dots,a_m\}\cap\gamma(I)=\varnothing$. Additionally we can assume that the sets V_z that cover $\gamma(I)$ are also disjoint from $\{a_2,\dots,a_m\}$.

Let ϵ be a Lebesgue number for the function $\gamma\colon I\to X$ with respect to the cover $\{U_z\}_{z\in X}$. Choose an integer N with $1/N<\epsilon$. If $I_k:=[k/N,(k+1)/N]$, then $\gamma(I_k)$ is contained in some $\{U_z\}$; denote it by U_k, the set V_z by V_k, and r_z by r_k. Let $A_k:=\gamma(k/N)$, in particular, $A_0=a_1$ and $A_N=b_1$.

Since $V_k\cong B(0,r_k)$ and $A_k,A_{k+1}\in B(0,r_k/3)$, we can apply Lemma 3.2 to obtain a symplectomorphism $\Phi\colon B(0,r_k)\to B(0,r_k)$ such that

$$\Phi(A_k)=A_{k+1} \qquad \text{and} \qquad \Phi(z)=z \quad \text{for} \quad \|z\|\ge (2/3)r_k.$$

We can extend Φ to the whole of X (we glue it with the identity); denote this extension by Φ_k. Put

$$\Psi=\Phi_N\circ\Phi_{N-1}\circ\ldots\circ\Phi_0.$$

Then $\Psi(a_1)=b_1$ and $\Psi(a_i)=a_i$ for $i>1$. Repeating this process, we finally arrive at a symplectomorphism $\Sigma\colon X\to X$ such that $\Sigma(a_i)=b_i$ for $i=1,\dots,m$. In a similar way, using Proposition 3.1, we can construct a symplectomorphism $\Pi\colon X\to X$ such that

(1) $\Pi(b_i)=b_i$,

(2) $d_{b_i}\Pi = L_i\circ(d_{a_i}\Sigma)^{-1}$.

Now it is enough to take $\Phi=\Pi\circ\Sigma$. $\square$

Remark 3.1. Statement (1) of Theorem 3.3 is well known; however, a new ingredient is given by statement (2) of this theorem.

Since for a compact symplectic manifold (X,ω) of dimension $2n$ it is well known (cf. [15]) that for a fixed number $0<s\le n$ there exists a closed $2s$-dimensional symplectic submanifold $Z\subset X$, we can use Theorem 3.3 to obtain

Corollary 3.2 [10]. *Let (X,ω) be a compact symplectic manifold of dimension $2n$. Let $a_1,\dots,a_m$ be a family of points of X. Take $0<s\le n$. For every $i=1,\dots,m$ choose a linear $2s$-dimensional symplectic subspace $H_i\subset T_{a_i}X$. Then there is a closed symplectic $2s$-dimensional submanifold $Y\subset X$ such that*

(1) $a_i\in Y$,

(2) $T_{a_i}Y=H_i$.

In a similar way we get

Corollary 3.3 [10]. *Let (X,ω) be a symplectic manifold of dimension $2n$. Let $a_1,\dots,a_m$ be a family of points of X. Take $0<k\le n$. For every $i=1,\dots,m$ choose a linear k-dimensional isotropic subspace $H_i\subset T_{a_i}X$. Then there is a closed isotropic k-dimensional torus $Y\subset X$ such that*

(1) $a_i\in Y$,

(2) $T_{a_i}Y=H_i$.

4. DIFFEOMORPHISMS THAT ARE SYMPLECTOMORPHISMS

Finally we show that a symplectomorphism can be described as a diffeomorphism that preserves isotropic or symplectic submanifolds of a given fixed dimension (cf. [10]).

Theorem 4.1. *Let (X,ω_X) and (Y,ω_Y) be compact symplectic manifolds of dimension $2n>2$. Fix a number $0<s<n$. Assume that $\Phi\colon X\to Y$ is a diffeomorphism that transforms all $2s$-dimensional symplectic submanifolds of X into symplectic submanifolds of Y. Then Φ is a conformal symplectomorphism*; *i.e., there exists a nonzero number $c\in\mathbb{R}$ such that*

$$\Phi^*\omega_Y=c\omega_X.$$

Proof. Fix $z\in X$ and let $H\subset T_zX$ be a $2s$-dimensional symplectic subspace of T_zX. By Proposition 3.2 (applied for $m=1$, $a_1=z$ and $H_1=H$) there exists a $2s$-dimensional symplectic submanifold M of X such that $z\in M$ and $T_zM=H$.

Let $\Phi(M)=M'$ and $z'=\Phi(z)$. By assumption the submanifold $M'\subset Y$ is symplectic. This means that the space $d_z\Phi(H)=T_{z'}M'$ is symplectic. Hence the mapping $d_z\Phi$ transforms all linear $2s$-dimensional symplectic subspaces of T_zX into subspaces of the same type. By Proposition 2.2 this implies that $d_z\Phi$ is a conformal symplectomorphism, i.e.,

$$(d_z\Phi)^*\omega_Y=\lambda(z)\omega_X,$$

where $\lambda(z)\neq 0$. This means that there is a smooth function $\lambda\colon X\to\mathbb{R}^*$ $(=\mathbb{R}\setminus\{0\})$ such that

$$\Phi^*\omega_Y=\lambda\omega_X.$$

Since the form ω_X is closed, so is $\Phi^*\omega_Y$. Since $n>1$, this implies that the derivative $d\lambda$ vanishes, i.e., the function λ is constant. □

Theorem 4.2. *Let (X,ω_X) and (Y,ω_Y) be symplectic manifolds of dimension $2n>2$. Fix a number $1<k\le n$. Assume that $\Phi\colon X\to Y$ is a diffeomorphism that transforms all k-dimensional isotropic tori of X into isotropic tori of Y. Then Φ is a conformal symplectomorphism*; *i.e., there exists a nonzero constant $c\in\mathbb{R}$ such that*

$$\Phi^*\omega_Y=c\omega_X.$$

Proof. Fix $z\in X$ and let $H\subset T_zX$ be a k-dimensional isotropic subspace of T_zX. By Theorem 3.3 (applied for $m=1$, $a_1=z$ and $H_1=H$) there exists a k-dimensional isotropic torus M of X such that $z\in M$ and $T_zM=H$.

Let $\Phi(M)=M'$ and $z'=\Phi(z)$. By assumption the torus $M'\subset Y$ is isotropic. This means that the space $d_z\Phi(H)=T_{z'}M'$ is isotropic. Hence the mapping $d_z\Phi$ transforms all linear k-dimensional isotropic subspaces of T_zX into subspaces of the same type. By Proposition 2.3 this implies that $d_z\Phi$ is a conformal symplectomorphism. The rest of the proof is the same as in the case of Theorem 4.1 above. □

Remark 4.1. Let us note that, in particular, if Φ maps Lagrangian tori onto tori of the same type, then Φ is a conformal symplectomorphism.

Corollary 4.1. *Let X be a manifold of dimension $2n>2$. Let ω_1 and ω_2 be two symplectic forms on X. Fix a number $1<k\le n$. Assume that the family of all k-dimensional ω_1-isotropic tori of X is contained in the family of all k-dimensional ω_2-isotropic tori of X. Then there exists a nonzero number $c\in\mathbb{R}$ such that*

$$\omega_1=c\omega_2.$$

Proof. It is enough to apply Theorem 4.2 to $X=(X,\omega_1)$, $Y=(X,\omega_2)$ and $\Phi=$ identity. □

Corollary 4.2. *Let (X,ω) be a compact symplectic manifold of dimension $2n > 2$. Fix a number $1 < k \le n$. Assume that $\Phi\colon X \to X$ is a diffeomorphism that transforms all k-dimensional isotropic tori of X into isotropic tori. Then Φ is a symplectomorphism or antisymplectomorphism; i.e., $\Phi^*\omega = \pm\omega$. If Φ preserves the orientation and n is odd, then Φ is a symplectomorphism. Moreover, if n is even, then Φ must preserve the orientation.*

Proof. Indeed, we have $\Phi^*\omega = c\omega$. And we can write

$$\mathrm{vol}(X) = \int\limits_X \omega^n = \pm\int\limits_X \Phi^*\omega^n = \pm c^n \int\limits_X \omega^n; \tag{4.1}$$

hence $c = \pm 1$. Moreover, if Φ preserves the orientation and n is odd, then we get $c = 1$. If n is even, then $(-\omega)^n = \omega^n$ and Φ must preserve the orientation. $\square$

Remark 4.2. Corollaries similar to Corollary 4.1 and Corollary 4.2 are true for a compact symplectic manifold X in the case of symplectic submanifolds. Also a similar concept of geometric characterization of symplectomorphisms was already used for capacity-preserving diffeomorphisms, which imply symplectic and antisymplectic diffeomorphisms (cf. [8, 15]).

Example 4.1. We show that in the general case Φ need not be a symplectomorphism. Let $Y = (S^2,\omega)$ (where ω is a standard volume form on the sphere), and let $(X_n,\omega_n) = \prod_{i=1}^n Y$ be a standard symplectic product. Further let $\sigma\colon S^2 \ni (x,y,z) \to (x,y,-z) \in S^2$ be a mirror symmetry. Of course, $\sigma^*\omega = -\omega$. More generally, if $\Sigma = \prod_{i=1}^n \sigma\colon X_n \to X_n$, then $\Sigma^*\omega_n = -\omega_n$. Hence it is possible that Φ from Corollary 4.2 is an antisymplectomorphism.

However, in any case either Φ or $\Phi\circ\Phi$ is a symplectomorphism.

Now let (X,ω) be a symplectic manifold, and let us denote by $\mathbf{Symp}(X,\omega)$ the group of symplectomorphisms of X. At the end of this note we show that this group also determines a conformal symplectic structure on X:

Theorem 4.3. *Let X be a smooth manifold of dimension $2n > 2$, and let ω_1 and ω_2 be two symplectic forms on X. If $\mathbf{Symp}(X,\omega_1) \subset \mathbf{Symp}(X,\omega_2)$, then there exists a nonzero constant c such that $\omega_2 = c\omega_1$.*

Proof. Take $z \in X$ and consider the symplectic vector spaces $V_1 = (T_zX,\omega_1)$ and $V_2 = (T_zX,\omega_2)$. By Theorem 3.3, for every linear symplectomorphism S of V_1, there is a symplectomorphism $\Phi_S \in \mathbf{Symp}(X,\omega_1)$ such that

(a) $\Phi_S(z) = z$,

(b) $d_z\Phi_S = S$.

Since $\mathbf{Symp}(X,\omega_1) \subset \mathbf{Symp}(X,\omega_2)$, we easily find that $\mathbf{Sp}(V_1) \subset \mathbf{Sp}(V_2)$. Consequently, by Proposition 2.4 there exists a nonzero number $\lambda(z)$ such that $\omega_2(z) = \lambda(z)\omega_1(z)$. Now we finish the proof as in the proof of Theorem 4.1. $\square$

5. POLYNOMIAL SYMPLECTOMORPHISMS

Throughout this section, (X,ω) will be a symplectic affine space over $\mathbb{K}$ (the field of real or complex numbers) of dimension $2n$; i.e., $X \cong \mathbb{K}^{2n}$ (unless mentioned otherwise) and $\omega = \sum_i dx_i \wedge dy_i$ is the standard nondegenerate skew-symmetric form on X. Linear symplectomorphisms of (X,ω) are characterized (cf., e.g., [13]) as linear automorphisms of X preserving some minimal complete data defined by ω on systems of linear subspaces. In this way the linear symplectic group $\mathbf{Sp}(X)$ may be characterized geometrically together with its natural conformal and antisymplectic extensions.

The purpose of this investigation is to put the linear considerations of symplectic invariants into the more general context of polynomial automorphisms. We say that a polynomial automorphism $F\colon X \to X$ is a *symplectomorphism* (or is *symplectic* on X) if $F^*\omega = \omega$, i.e.,

$\omega(u,v) = \omega(d_xF(u), d_xF(v))$ for every $x \in X$ and all $u,v \in T_xX$. The group $\mathbf{PlSp}(X)$ of polynomial symplectomorphisms is an important tool in affine algebraic geometry (see [2, 3, 18, 17]). In particular, the group of polynomial symplectomorphisms of $\mathbb{C}^{2n}$ is conjectured to be isomorphic to the group of automorphisms of the Weyl algebra $A_n(\mathbb{C})$ (see [3]).

The first property of $\mathbf{PlSp}(X)$ that we prove is its transitivity (k-transitivity) on finite collections of points of X. First, we show that the group $\mathbf{PlSp}(X)$ is quite large. We start with the following lemma:

Lemma 5.1. *Let $a_i = (\alpha_{i,1}, \dots, \alpha_{i,2n}) \in \mathbb{K}^{2n}$ and $\mathcal{A} = \{a_1, \dots, a_m\}$ be a finite family of points. Let $\pi_k \colon \mathbb{K}^{2n} \ni (\alpha_1, \dots, \alpha_{2n}) \to \alpha_k \in \mathbb{K}$ be the projection. There is a linear symplectomorphism L such that if $\mathcal{A}' = L(\mathcal{A})$, then all projections π_k, $k = 1, \dots, 2n$, restricted to $\mathcal{A}'$ are one-to-one; i.e., if*

$$L(a_i) = (\alpha'_{i,1}, \dots, \alpha'_{i,2n}),$$

then for every $\{i,j\} \subset \{1, \dots, m\}$, $\alpha'_{i,s} = \alpha'_{j,s}$ for some s implies $\alpha'_{i,s} = \alpha'_{j,s}$ for all s.

Let us recall the following

Definition 5.1. Let G be a group that acts on a set X. We say that G acts *k-transitively* on X if for any two k-element subsets $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ of X, there is a $g \in G$ such that $g(a_i) = b_i$ for $i = 1, \dots, k$.

We have the following basic result (cf. [11]):

Theorem 5.1. *Let (X, ω) be an affine symplectic space of dimension $2n$. For every $m \in \mathbb{N}$ the group $\mathbf{PlSp}(X)$ acts m-transitively on X.*

Proof. Let $a_i = (\alpha_i, \beta_i) \in \mathbb{K}^{2n}$ and $b_i = (\gamma_i, \delta_i) \in \mathbb{K}^{2n}$, $i = 1, \dots, m$, be finite families of distinct points. By Lemma 5.1 there are linear symplectomorphisms L and T such that

$$L(a_i) = (\alpha'_i, \beta'_i) \qquad \text{and} \qquad T(b_i) = (\gamma'_i, \delta'_i),$$

where for every i and every j we have $L(a_j) = L(a_i)$ if and only if $\alpha'_{is} = \alpha'_{js}$ for some $s \in \{1, \dots, n\}$, and $T(b_j) = T(b_i)$ if and only if $\delta'_{is} = \delta'_{js}$ for some $s \in \{1, \dots, n\}$.

Let $\phi_i(t)$ be a polynomial in one variable such that

$$\phi_i(\alpha'_{is}) = \beta'_{is} \qquad \text{for} \quad s = 1, \dots, n.$$

Consider the polynomial symplectomorphism

$$\Phi(x,y) = \bigl(x, y_1 - \phi_1(x_1), y_2 - \phi_2(x_2), \dots, y_n - \phi_n(x_n)\bigr).$$

By the construction we have

$$\Phi \circ L(a_i) = (\alpha'_i, 0) \qquad \text{for} \quad i = 1, \dots, m.$$

In a similar way we can construct a polynomial symplectomorphism

$$\Psi(x,y) = \bigl(x, y_1 + \psi_1(x_1), y_2 + \psi_2(x_2), \dots, y_n + \psi_n(x_n)\bigr)$$

such that

$$\Psi(\alpha'_i, 0) = (\alpha'_i, \delta'_i) \qquad \text{for} \quad i = 1, \dots, m.$$

Further, there exists a polynomial symplectomorphism

$$\Lambda(x,y) = \bigl(x_1 - \lambda_1(y_1), x_2 - \lambda_2(y_2), \dots, x_n - \lambda_n(y_n), y\bigr)$$

such that

$$\Lambda(\alpha_i', \delta_i') = (0, \delta_i') \qquad \text{for} \quad i = 1, \dots, m.$$

Finally, we can construct a polynomial symplectomorphism

$$\Sigma(x, y) = \bigl(x_1 + \sigma_1(y_1), x_2 + \sigma_2(y_2), \dots, x_n + \sigma_n(y_n), y\bigr)$$

such that

$$\Sigma(0, \delta_i') = (\gamma_i', \delta_i') \qquad \text{for} \quad i = 1, \dots, m.$$

Set

$$P = T^{-1} \circ \Sigma \circ \Lambda \circ \Psi \circ \Phi \circ L.$$

Then

$$P(a_i) = b_i \qquad \text{for} \quad i = 1, \dots, m. \quad \square$$

Example 5.1. Theorem 5.1 does not hold for an arbitrary symplectic algebraic variety. We construct a smooth rational algebraic symplectic manifold Y with trivial automorphism group; in particular, $\mathbf{PlSp}(Y) = \{\mathrm{id}\}$. Let $G \subset \mathbb{C}^{2n}$ be a sufficiently generic hypersurface of degree $d > 2n$. Set $Y = \mathbb{C}^{2n} \setminus G$ and equip Y with the symplectic structure induced by the inclusion $i\colon Y \to \mathbb{C}^{2n}$.

We show that $\mathrm{Aut}(Y) = \{\mathrm{id}\}$. Let $F\colon Y \to Y$ be a polynomial automorphism of Y. Since the hypersurface G is not uniruled (for details see, e.g., [12, 13]), F has a unique extension to a polynomial automorphism $\overline{F}\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}$. Moreover, by [12] we have $\mathrm{Aut}(G) = \{\mathrm{id}\}$ and we know that the hypersurface G is the identity set for automorphisms; i.e., if $\overline{F}|_G = \{\mathrm{id}\}$, then $\overline{F} = \mathrm{id}$. Altogether this implies that $\mathrm{Aut}(Y) = \{\mathrm{id}\}$.

Now we are ready to provide a geometric characterization of polynomial symplectomorphisms.

Definition 5.2. Let (X, ω) be an affine symplectic space, and let $Y \subset X$ be a smooth algebraic subvariety. We say that Y is a *Lagrangian variety* if for every $y \in Y$ the linear space T_yY is a Lagrangian subspace of T_yX. In an analogous way we define a symplectic, pseudo-symplectic and isotropic variety.

Of course, in X there are affine linear isotropic (or symplectic) subvarieties—these are varieties of the form $a + H$, where H is a linear isotropic (or symplectic) linear subspace of X. We show that there are also quite a lot of nonlinear ones. The measure of nonlinearity is the degree of a variety. Let us recall the definition:

Definition 5.3. Let $Y \subset \mathbb{C}^n$ be a complex variety of dimension k. By the degree of Y ($\deg Y$) we mean the number

$$\#(L^{n-k} \cap Y),$$

where L^{n-k} is an $(n-k)$-dimensional sufficiently general affine linear subspace of $\mathbb{C}^n$. If $Y \subset \mathbb{R}^n$ is a real variety, then by $\deg Y$ we mean $\deg Y_{\mathbb{C}}$, where $Y_{\mathbb{C}}$ denotes the Zariski closure of Y in $\mathbb{C}^n$.

It is not difficult to prove

Proposition 5.1. *Let $Y \subset \mathbb{K}^n$ be an algebraic variety. Assume that there is an affine line l that intersects Y in precisely D points. Then*

$$\deg Y \geq D.$$

Proof. We use induction on n. If $n = 1$ or $n = 2$, then Y is a set of points or a curve and the result is clear. Now let $n > 2$ and assume that our result holds for $n-1$. Take a general hyperplane H that contains l. Then by the Bézout theorem $\deg Y \cap H \leq \deg Y$, and by the induction hypothesis we have $\deg Y \cap H \geq D$. $\square$

Now we can prove

Proposition 5.2 [11]. *Let (X,ω) be a symplectic $2n$-dimensional affine space. For any positive integers $s \le n$ and D, there is an algebraic isotropic s-dimensional subvariety $Y \subset X$ such that*

$$\deg Y \ge D.$$

Proof. Fix a linear isotropic s-dimensional subvariety $H \subset X$. Choose D points $a_1, \dots, a_D$ on H and additionally a point $a_0 \notin H$.

Now take a line $l \subset X$ and choose distinct points $b_1, \dots, b_D, b_0$ on it. By Theorem 5.1 there is a polynomial symplectomorphism Φ of X such that

$$\Phi(a_j) = b_j \qquad \text{for} \quad j = 0, 1, \dots, D.$$

Now set $Y = \Phi(H)$. By the construction the line l intersects Y in at least D points and it is not contained in Y. This implies that $\deg Y \ge D$. $\square$

In the same way we can prove

Proposition 5.3. *For any even integer $0 < s < 2n$ and any positive integer D, there is an algebraic symplectic s-dimensional subvariety $Y \subset X$ such that*

$$\deg Y \ge D.$$

Similarly, for any odd integer $0 < s < 2n$ and any positive integer D, there is an algebraic pseudo-symplectic s-dimensional subvariety $Y \subset X$ such that

$$\deg Y \ge D.$$

Finally, we show that a polynomial symplectomorphism can be described as one that preserves symplectic, pseudo-symplectic or isotropic algebraic subvarieties of X.

Proposition 5.4 [11]. *Let (X,ω) be an affine symplectic space of dimension $2n > 2$. Fix an integer $2 \le s \le n$. Assume that $\Phi\colon X \to X$ is a polynomial automorphism that preserves the family of all s-dimensional isotropic subvarieties of X. Then Φ is a conformal symplectomorphism; i.e., there exists a nonzero number $c \in \mathbb{K}$ such that*

$$\Phi^*(\omega) = c\omega.$$

Proof. Fix $x \in H \subset X$, where H is an affine linear s-dimensional isotropic subvariety of X. Let $x' = \Phi(x)$ and $H' = \Phi(H)$. By assumption the variety H' is isotropic. This means that the space $d_x\Phi(T_xH) = T_{x'}H'$ is also isotropic. Hence $d_x\Phi$ transforms all linear l-dimensional isotropic subspaces of T_xX into subspaces of the same type. By Proposition 2.3 this implies that $d_x\Phi$ is a conformal symplectomorphism, i.e.,

$$(d_x\Phi)^*\omega = \lambda(x)\omega,$$

where $\lambda(x) \ne 0$. This means that there is a smooth (even polynomial) function $\lambda\colon X \to \mathbb{K}^*$ ($\mathbb{K}^* = \mathbb{K} \setminus \{0\}$) such that

$$\Phi^*(\omega) = \lambda\omega.$$

But since the form ω is closed, so is $\Phi^*(\omega)$. Since $n > 1$, this implies that the derivative $d\lambda$ vanishes; i.e., the function λ is constant. $\square$

In a similar way (we now use Proposition 2.2) we can prove

Proposition 5.5. *Let (X, ω) be an affine symplectic space of dimension $2n > 2$. Fix an integer $0 < s < n$. Assume that $\Phi\colon X \to X$ is a polynomial automorphism that preserves the family of all $2s$-dimensional symplectic subvarieties of X or (if $1 < s < n-1$) the family of all $(2s+1)$-dimensional pseudo-symplectic subvarieties of X. Then Φ is a conformal symplectomorphism; i.e., there exists a nonzero $c \in \mathbb{K}$ such that*

$$\Phi^*(\omega) = c\omega.$$

To end this section, we introduce the notion of symplectic type of an algebraic variety.

Proposition 5.6. *Let (X, ω) be an affine symplectic space of dimension $2n$. Let $Y \subset X$ be a smooth k-dimensional algebraic variety. Then there are even integers $r_1 > \dots > r_s$ (where $s \geq 1$) and disjoint algebraic locally closed subvarieties $Y_{r_1}, \dots, Y_{r_s}$ that cover Y such that the form ω has rank r_i on Y_i. Moreover, $Y_{r_{i+1}} \subset \mathrm{cl}(Y_{r_i})$. The sequence $\{r_1, \dots, r_s\}$ is a symplectic invariant, which we call the symplectic type of the variety Y.*

Proof. Consider the Gauss mapping

$$G\colon\ Y \ni y \to T_yY \in G(k, 2n).$$

This is a regular (locally polynomial) mapping. By [13] the linear spaces in the Grassmannian $G(k, 2n)$ on which the rank of ω is equal to r form a smooth locally closed (in the Zariski topology) subset S_r, and $S_{r-2} \subset \mathrm{cl}(S_r)$. Now it is enough to take $Y_r = G^{-1}(S_r)$ if this set is not empty. □

Example 5.2. (a) A $2k$-dimensional subvariety $Y \subset X$ is a symplectic subvariety if and only if the symplectic type of Y is $\{2k\}$.

(b) A $(2k+1)$-dimensional subvariety $Y \subset X$ is a pseudo-symplectic subvariety if and only if the symplectic type of Y is $\{2k\}$.

(c) A subvariety $Y \subset X$ is an isotropic subvariety if and only if the symplectic type of Y is $\{0\}$.

Now the following statement is obvious:

Theorem 5.2. *Let (X, ω) be an affine symplectic space of dimension $2n > 2$. Fix an integer $2 \leq k \leq 2n-2$. A polynomial automorphism $\Phi\colon X \to X$ is a conformal symplectomorphism if and only if it preserves the symplectic types of all algebraic k-dimensional subvarieties of X.*

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