BIFURCATIONS IN STOCHASTIC DYNAMICAL SYSTEMS WITH SIMPLE SINGULARITIES

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The generalized Langevin stochastic dynamical system is introduced and the stationary probability density for its solution is investigated. The stochastic field is assumed to be singular with a simple singularity, and noise in the control parameters is modelled as dychotomous Markov noises. A classification of bifurcation diagrams for the stationary density probability is obtained. Two examples encountered from physics, the dye laser model and the Verhulst model, are investigated.

stochastic process * dynamical system * singularity * bifurcation diagram

1. Introduction

The main interest of recent bifurcation theory (cf. [2, 6]) has been to determine bifurcation diagrams and the corresponding structural changes in the system under consideration. For the generic, finitely-determined models of singularity theory (cf. [4, 14]) these bifurcation sets (also called catastrophe sets [15]) are well known and their relevance for the understanding of various physical, chemical and biological systems has been exhaustively proved in a number of recent publications (see e.g. [15, 6, 14, 17]). It appears, however, that in realistic laser systems (cf. [5]) critical region phenomena (cf. [8]) or various open or semiopen chemical or thermodynamical systems, random internal fluctuations play an important role. It is therefore necessary to include in the model the stochastic component of the noise. Stochasticity of some control parameters of the standard dynamical models tends to completely change bifurcation diagrams. Their use is an important physical branch of the theory of stochastic processes (cf. [7, 16]). Abstracting from the concrete models we see that the structure of these stochastic bifurcation sets is interesting even for the theory itself. In this paper we investigate the bifurcation sets for a wide class of stochastic dynamical systems, coming from standard singularity theory with Markovian dychotomous noise. The paper deals only with 1-dimensional stochastic dynamical systems. We do not pretend to give a complete description of such systems but merely to investigate some interesting examples and the universal problems suggested by them.

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In the first section we give some necessary preliminaries concerning stochastic dynamical systems and propose a geometrical method to investigate their probability densities. Section 3 is devoted to a presentation of various physical systems with stochastic noise, as for example the dye laser system with pre-Gaussian pump fluctuations. This problem can be seen as the main motivation for our study. In Section 4 we develop a theory for stable dynamical systems with a stochastic control space and prove some generic properties. We show that the equations for stationary probability densities reduce to equations with so-called regular singular points. The analysis of bifurcation diagrams for stable dynamical systems is provided in Section 5, where the appropriate "topological type function" is introduced. The fold model and the cusp model with dychotomous one dimensional noise are investigated and the corresponding bifurcation diagrams are obtained.

2. Stochastic dynamical systems driven by the dychotomous Markov noise

Many systems of practical interest can be described by a single relevant variable. This variable as a function of time is governed by a first-order differential equation of motion which additionally depends on external control parameters (see e.g. [14]). If it is assumed that there exist a fluctuating control parameter (external noise [16]) the deterministic equation of motion (usually) transforms into a stochastic one of the form

$$\dot{x} = f(x) + g(x)u(t) \tag{1}$$

where $u(t)$ is the external noise, $f(x)$ is the deterministic vector field and $g(x)$ represents a linear coupling of the fluctuating parameter $u(t)$ with the coordinate x. Equation (1) is the most common case in a large number of real situations; it is called a Langevin equation (cf. [12, 16]) Assuming that $u(t)$ is Gaussian white noise (cf. [7]), i.e.

$$\langle u(t) \rangle = 0, \qquad \langle u(t)u(t') \rangle = 2\Gamma\delta(t - t'), \tag{2}$$

we see that the stochastic process $x(t)$ is Markovian and the probability density $P(x, t)$ satisfies the differential Fokker–Planck equation (see [16])

$$\frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} f(x) P(x, t) - \frac{\partial}{\partial x} g(x) \frac{\partial}{\partial x} g(x) P(x, t). \tag{3}$$

If $u(t)$ is not white noise the process $x(t)$ is not a Markov process.

In this paper we consider (1) when the stochastic process $u(t)$, which is possibly multidimensional, is a dychotomous Markov noise. In this case $u(t)$ takes two possible values, a and $-a$, $a > 0$ with transition rates, from a to $-a$ and vice versa, equal to $\gamma/2$. Let $P_{\pm}(t) \stackrel{\text{def}}{=}$ Probability $\{u(t) = \pm a\}$. The Master equation for $P \pm (t)$

(see [1]) then reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} P_+(t) \\ P_-(t) \end{pmatrix} = -\gamma/2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} P_+(t) \\ P_-(t) \end{pmatrix}. \tag{4}$$

If $P_\pm(0) = \frac{1}{2}$ then $u(t)$ is a stationary process with the correlation function

$$\langle u(t)u(t') \rangle = a^2 \, \mathrm{e}^{-\gamma|t-t'|}. \tag{5}$$

For the detailed information on this type of processes we refer the reader to [16, 12].

Instead of the Langevin equation (1) for $x(t)$ we can write the Liouville equation satisfied by the corresponding density ϕ,

$$\phi(y, t) = \delta(y - x(t)), \tag{6}$$

namely

$$\frac{\partial}{\partial t}\phi(y, t) = -\frac{\partial}{\partial y}f(y)\phi(y, t) - u(t)\frac{\partial}{\partial y}g(y)\phi(y, t), \tag{7}$$

where $x(t)$ is the solution to equation (1) with initial condition x_0 and the definite realisation of the stochastic trajectory $u(t)$. In fact we are interested in the probability density $P(y, t)$, which is defined as

$$P(y, t) = \langle \phi(y, t) \rangle, \tag{8}$$

where $\langle \cdot \rangle$ defines an average over all realisations of $u(t)$ according to equation (4).

Combining the equations (4) and (7) we obtain two Master equations for the probability densities $P_\pm(x, t)$, namely

$$\frac{\partial}{\partial t}P_+(x, t) = -\frac{\partial}{\partial x}f(x)P_+(x, t) - a\frac{\partial}{\partial x}g(x)P_+(x, t) - \frac{\gamma}{2}P_+(x, t) + \frac{\gamma}{2}P_-(x, t), \tag{9}$$

$$\frac{\partial}{\partial t}P_-(x, t) = -\frac{\partial}{\partial x}f(x)P_-(x, t) + a\frac{\partial}{\partial x}g(x)P_-(x, t) + \frac{\gamma}{2}P_+(x, t) - \frac{\gamma}{2}P_-(x, t). \tag{10}$$

Here $P_\pm(x, t)\,\mathrm{d}x$ denotes the joint infinitesimal probability for x at time t and $u(t) = \pm a$. Using the new functions

$$P(x, t) = P_+(x, t) + P_-(x, t), \qquad Q(x, t) = P_+(x, t) - P_-(x, t), \tag{11}$$

where $P(x, t)$ is probability density of the variable x, independent of external noise $u(t)$, we can rewrite equations (9) and (10) in the form

$$\frac{\partial}{\partial t}P(x, t) = -\frac{\partial}{\partial x}f(x)P(x, t) - a\frac{\partial}{\partial x}g(x)Q(x, t), \tag{12}$$

$$\frac{\partial}{\partial t}Q(x, t) = -\frac{\partial}{\partial x}f(x)Q(x, t) - a\frac{\partial}{\partial x}g(x)P(x, t) - \gamma Q(x, t). \tag{13}$$

Now we are interested in the stationary solutions; $P_{st}(x)$, $Q_{st}(x)$, of the above equations. By (12) we have

$$f(x)P_{st}(x) + ag(x)Q_{st}(x) = j = \mathrm{const}(x). \tag{14}$$

Let us put j equal to zero. This assumption does not restrict the generality of our approach. We insert Q_{st}, derived from (14), to (13) and obtain the following differential equation for the stationary P_{st}:

$$\frac{\partial}{\partial x}\left(\frac{f^2(x) - a^2 g^2(x)}{g(x)} P_{st}(x)\right) = -\frac{\gamma f(x)}{g(x)} P_{st}(x). \tag{15}$$

This equation has the general solution (cf. [11])

$$P_{st}(x) = N\frac{g(x)}{a^2 g^2(x) - f^2(x)} \exp\left(\gamma \int^x dx' \frac{f(x')}{a^2 g^2(x') - f^2(x')}\right). \tag{16}$$

Additionally, we have

$$Q_{st}(x) = -\frac{f(x)}{ag(x)} P_{st}(x), \tag{17}$$

$$P_{st}^+(x) = -\frac{1}{2ag(x)} F_-(x) P_{st}(x), \tag{18}$$

$$P_{st}^-(x) = -\frac{1}{2ag(x)} F_-(x) P_{st}(x), \tag{19}$$

where $F_\pm(x) = f(x) \pm ag(x)$.

Remark 2.1. From physical arguments it is concluded that P_{st}^+ and P_{st}^- are not negative functions. From equations (18), (19) we see that we can satisfy this requirement if and only if F_+ and F_- have opposite signs. This observation is also very helpful in the geometrical interpretation of stable physical domains defining the stationary probability density involved in equation (15). This interpretation was introduced in [5] and describes the stable physical domains as defined by the stochastic vector fields F_+, F_- in competition.

3. Applications to the open systems in a fluctuating environment

To discuss the ideas for various applications we begin by noting that the random force which occurs in a Langevin equation (1) can have quite different origins. In an ordinary microscopic derivation of a Langevin equation (see e.g. [16]) the random term $u(t)g(t)$ is associated with the "thermal" or "internal" noise – thermal fluctuations, especially near a critical point (cf. [8, 7]).

A different and more common interpretation of the random term of a Langevin equation is necessary, however when this is thought to model what we call an "external noise" situation. This type of fluctuation can be due to a varying environment or can be the result of an externally applied random force. The mathematical modelling of these fluctuations is made by formulating the appropriate deterministic equation in the absence of external fluctuations. One then lets the external parameter,

which undergoes fluctuations, be a stochastic variable. The noise term of the stochastic differential equation obtained in this way, usually has a multiplicative character, i.e. it depends on the instantaneous value of the variables of the system. We can take the external noise as an external stochastic field which drives the system (cf. [16]).

Dynamical equations of motion with random noise have been first used in the theory of Brownian motion. The best known examples of such Langevin equations are

$$\frac{\mathrm{d}}{\mathrm{d}t} x = u(t) \tag{20}$$

where $f(x) = 0$, $g(x) = 1$ (as in equation (1)) and x denotes the position of the particle. Another example is

$$\frac{\mathrm{d}}{\mathrm{d}t} v = -\lambda v + u(t) \tag{21}$$

where $f(v) = -\lambda v$, $g(v) = 1$, v is the velocity of the particle and in both these examples $u(t)$ is a random noise due to to the external random collisions. If $u(t)$ is a Gaussian white noise (see Section 2), then equation (20) leads through equation (3) to the well known Wiener-Lévy process described by the diffusion equation for the probability density $P(x, t)$:

$$\frac{\partial}{\partial t} P(x, t) = \Gamma \frac{\partial^2}{\partial x^2} P(x, t).$$

Analogously, inserting f and g from example (21) into equation (3) (see Section 2) we obtain the following equation for the density probability $P(v, t)$:

$$\frac{\partial}{\partial t} P(v, t) = \lambda \frac{\partial}{\partial v} (vP(v, t)) + \Gamma \frac{\partial^2}{\partial v^2} P(v, t)$$

which describes the evolution of the Ornstein–Uhlenbeck process.

Another use of stochastic dynamical systems has been made in the modelling of a more realistic laser with pump fluctuations (see [5, 9, 10]). Again the random process composed of two random telegraph processes appears to be useful in approximation of the Ornstein–Uhlenbeck process representing the chaotic fluctuations in the system. Let us consider the dye-laser equation (cf. [5, 9])

$$\frac{\mathrm{d}}{\mathrm{d}t} x = (\lambda - x)x + xz(t), \tag{22}$$

where $z(t) = u_1(t) + u_2(t)$ is the two telegraph additive noise. The corresponding equation for the stationary probability density,

$$\left(2g - \frac{f^2}{2a^2 g}\right) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{2\gamma g}{f} - \frac{3\gamma f}{2a^2 g} - \frac{1}{2a^2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f^2}{g}\right) + 2 \frac{\mathrm{d}g}{\mathrm{d}x}\right) \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$+ \left(-\frac{\gamma^2}{a^2 g} - \frac{\gamma}{2a^2} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f}{g}\right)\right) y = 0, \tag{23}$$

where

$$g(x) = x, \qquad f(x) = \lambda x - x^2, \qquad y(x) = \int^x \frac{fP_{st}}{g},$$

reduces to the standard Fuchs type equation of second order. The local behaviour of P_{st} at singular points was investigated in [5]. The bifurcating family of supports for this model is illustrated in Fig. 1.

As an example of a Langevin equation used in quantum optics let us take the propagation of a pulse in a nonlinear random medium. The propagation equation in the z direction of a wave evolution has the form ([16])

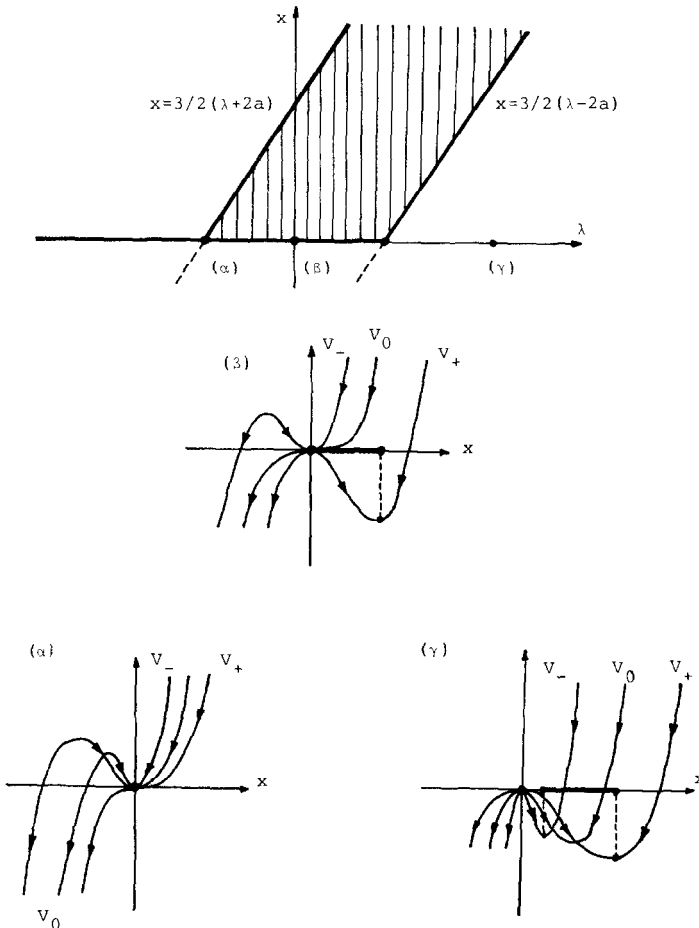$$\frac{\mathrm{d}}{\mathrm{d}z} I(z) = \frac{L_0 + u(z)}{1 + L_0 I(z)} I(z). \tag{24}$$

Fig. 1

In this equation, $u(z)$ describes the random properties of the medium. Another example of the Langevin equation with external noise in quantum optics can be given by the rate equations for multiphoton processes. The changes of atomic population due to one or two photon ionisation in the presence of a fluctuating laser light are given by the equations (cf. [7])

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -u^2(t)\rho(t), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\rho = -u^4(t)\rho(t), \tag{25}$$

where $u(t)$ is the external field amplitude which fluctuates according to external conditions.

In all these examples $u(t)$ may be an arbitrary noise process. The most trivial case is of white noise and such systems are described by the Fokker–Planck equations, except for (25) where the noise enters nonlinearly. In the rest of the paper we are interested in a more realistic situation with nonwhite external noise represented by random telegraphs, possibly entering nonlinearly.

4. Stable dynamical systems in the stochastic control space

Let us consider a general stochastic dynamical system with one internal variable (cf. [12]),

$$\dot{x} = -\mathrm{grad}_x\, V(x, u_1(t), \ldots, u_n(t)) = F(x, u_1(t), \ldots, u_n(t)), \tag{26}$$

where $u_1(t), \ldots, u_n(t)$ are independent stochastic processes and F is a smooth function. We allow (26) to be a general nonlinear system. Because of its simplicity and extensive use in applications (see Section 2 and 3) we restrict our considerations to the Markovian dychotomous noises $u_1(t), \ldots, u_n(t)$. In this case we have the following result.

Proposition 4.1. *Let*

$$\dot{x} = F(x; \bar{u}, u_1(t), \ldots, u_n(t)), \quad \bar{u} = (\bar{u}_1, \ldots, \bar{u}_k) \in \mathbb{R}^k \tag{27}$$

be a dynamical stochastic system parametrized by deterministic control variables $(\bar{u}_i)_1^k$ and Markovian dychotomous noises $(u_j)_1^n$. Then there exist smooth functions

$$(g_i(x, \bar{u}))_1^n, \ (g_{i_1 i_2}(x, \bar{u}))_{i_1 > i_2}^n, \ldots, (g_{i_1, \ldots, i_n}(x, \bar{u}))_{i_1 > \cdots > i_n}^n,$$

such that (27) has the form

$$\dot{x} = f(x, u) + \sum_{i=1}^n g_i(x, \bar{u})u_i(t) + \sum_{i_1 > i_2}^n g_{i_1 i_2}(x, \bar{u})u_{i_1}(t)u_{i_2}(t) + \cdots$$

$$+ \sum_{i_1 > \cdots > i_n}^n g_{i_1 \cdots i_n}(x, \bar{u})u_{i_1}(t) \cdots u_{i_n}(t). \tag{28}$$

In other words the general stochastic system (27) *is equal to one with stochastic variables entering linearly, i.e.*

$$\dot{x} = f(x, \bar{u}) + \sum_{i=1}^{N} g_i(x, \bar{u}) w_i(t),$$ (29)

where $w_i(t)$ are, not necessarily independent, stochastic variables representing also dychotomous processes as multiplicatively composed initial telegraphs (i.e. $w_i(t)$ is a product of some number of independent telegraph noises so is also dychotomous).

Proof. For the one telegraph noise, say $u(t)$ (realized by an integer valued stochastic process $n(t, t_0)$, i.e. $u(t) = a(-1)^{n(t, t_0)}$, $a > 0$, cf. [1]). For its two values $+a$, $-a$ we have

$$F(x, \bar{u}, +a) = \tfrac{1}{2}(F(x, \bar{u}, a) + F(x, \bar{u}, -a)) + \frac{(+a)}{2a}(F(x, \bar{u}, a) - F(x, \bar{u}, -a)),$$

$$F(x, \bar{u}, -a) = \tfrac{1}{2}(F(x, \bar{u}, a) + F(x, \bar{u}, -a)) + \frac{(-a)}{2a}(F(x, \bar{u}, a) - F(x, \bar{u}, -a)).$$

Thus the stochastic dependence of $F(x, \bar{u}, u(t))$ can be written in the form

$$F(x, \bar{u}, u(t)) = \tfrac{1}{2}(F(x, \bar{u}, a) + F(x, \bar{u}, -a))$$

$$+ \frac{u(t)}{2a}(F(x, \bar{u}, a) - F(x, \bar{u}, -a)).$$ (*)

Now we can treat (*) as a first step in the induction process with respect to the number of telegraphs in F. Repeating decomposition (*) with respect to second, third, ..., nth telegraphs successively we directly obtain the desired result (28).

Remark 4.2. For the deterministic dynamical system of type (27) its stationary surface (the set of zeros of the corresponding field F) gives basic information about the slow dynamics appearing in various dynamical models of physical systems (see [15, 17]). The general nonlinear system of type (27) can be very complicated with regard to its stationary properties. However, when we remove some very small class of "pathological" (unstable) systems, the theory of singularities (see [4, 14]) provides complete information about the local stationary properties of typical nonlinear systems.

Now we recall some necessary basic facts from standard singularity theory. Let $\dot{x} = -\text{grad}_x V(x, u)$ be a dynamical system with smooth potential $V: \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. We say that this system is stable if at each point $(x_0, u_0) \in \mathbb{R} \times R^k$ the following holds (cf. [14, 17]): the quotient ring $m_{(x,u)}/J(V)$ is generated by $\{\partial V(x, t)/\partial u_i\}_{i=1}^{k}$ as an $\xi_{(u)}$ module (cf. [4]). Here by $m_{(x,u)} \subset \xi_{(x,u)}$, $(\xi_{(u)})$ we denote the maximal ideal of the ring $\xi_{(x,u)}$ of germs at (x_0, u_0), $((u_0) - \text{resp.})$ of smooth functions and by $J(V)$ we denote the corresponding ideal generated by $\partial V(x, u)/\partial x$. It appears that for

structurally stable systems (by the Malgrange Preparation Theorem [4]), in a neigh-bourhood of each point (x_0, u_0) the system is equivalent, in its stationary dynamics, to one given in the form (cf. Proposition 4.1)

$$\dot{x} = -\text{grad}_x \, V_\mu(x, u) = f(x) + \sum_{i=1}^{\mu} u_i g_i(x), \quad \mu \leq k, \tag{30}$$

where we assume $(x_0, u_0) = (0, 0)$ and the $g_i(x)$ are smooth functions generating the space $\xi_{(x)}/J(f)$ (cf. [4, 15]).

Comparing the form of the system given by the above potentials to the stochastic one (28) with telegraphically fluctuating control parameters we are also encouraged to restrict ourselves to dynamical systems with stable potentials (30). It also suggests that we should consider additional stochastic noises entering additively in the deterministic parameters (u_i).

Thus, in what follows, we assume our stochastic dynamical system to be recon-structed from a stable (30) deterministic one by switching on the respective telegraphs associated with each control parameter, the values of which are treated as the fluctuating centres of external forces. Thus we have

$$\dot{x} = -\text{grad}_x \left(V_\mu(x, \bar{u}) + \sum_{i=1}^{\mu} u_i(t) h_i(x) \right), \tag{31}$$

where the functions $h_i(x)$ are determined by the stable deterministic potential V_μ at the neighbourhood of each point (x, u).

Let $\dot{x} = F(x, \bar{u}, u(t))$ be a stochastic system where $u(t) = (u_1(t), \ldots, u_n(t))$ is a direct product of independent dychotomous noises. The corresponding Master equations for the respective probabilities are

$$\dot{P}_+^{(i)}(t) = \frac{1}{2\tau_i} (P_-^{(i)} - P_+^{(i)}), \tag{32}$$

$$\dot{P}_-^{(i)}(t) = \frac{1}{2\tau_i} (P_+^{(i)} - P_-^{(i)}), \quad \tau_i > 0, \tag{33}$$

and the amplitudes $a_i > 0$, i.e. $\langle u_i(t) u_i(t') \rangle = a_i^2 \, e^{-|t-t'|/\tau_i}$.

The stochastic solution of (26) is characterized by its joint probability density, say $P_I(x, t)$, $I = (i_1, \ldots, i_n)$, $i_j = \pm$, which is governed by the joint probability Master equation (cf. [16, 11])

$$\frac{\partial}{\partial t} P_I(x, t) = -\frac{\partial}{\partial x} F(x, A_I) P_I(x, t) - \frac{1}{2} \sum_{j=1}^{n} \frac{1}{\tau_j} (P_I - P_{I_j})(x, t), \tag{34}$$

where, $I_j = (i_1, \ldots, -i_j, \ldots, i_n)$ and A_I is a value of $u(t)$ corresponding to I, i.e. $A_I = (i_1 a_1, \ldots, i_n a_n)$.

Let us order the set of indices I (say: $(+, +, \ldots), (-, +, \ldots), (+, -, \ldots), (-, -, \ldots)$ etc.). Then we can write the system $F(x, A_I) P_I(x, t)$ as a column vector, say $\tilde{P}$. Thus

for the modified stationary density probability $\bar{P}(x)$, on the basis of (34), we have the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\,\bar{P}(x) = B(x)\bar{P}(x), \tag{35}$$

where $B(x) = TF^{-1}(x)$, $F(x)$ is a matrix function with $F(x, A_I)$-entries on the diagonal, $(F^{-1}(x))_{JK} = \delta_{JK}/F(x, A_J)$ and $T = (T_{IJ})$ is a constant matrix defined by

$$T_{IJ} = \tfrac{1}{2}\sum_{j=1}^{n}\frac{1}{\tau_j}\,\delta_{JI_j} - \tfrac{1}{2}\left(\sum_{i=1}^{n}\frac{1}{\tau_i}\right)\delta_{IJ}. \tag{36}$$

Taking into account equation (28) and the form of matrix $B(x)$ with rational entries we have the following result.

Proposition 4.3. *For the generic stochastic system* (27) *the corresponding system of equations for the joint stationary probability, considered in the complex domain, is a system of equations with regular singularities* (*cf.* [13]).

Remark 4.4. On the basis of Proposition 4.3 we can use the standard theory of differential equations (see [13, 3]) and analyse the behaviour of a physically acceptable solution $\bar{P}(x)$ in the neighbourhood of singular points, for the generic stochastic systems without parameters. However in stochastically controlled systems, even generic ones, the confluencies of singularities appear and the complete analysis of any solution is much more complicated. In order to analyse the controlled systems we restrict ourselves to ones with one dychotomous noise, i.e. $n = 1$ in formula (27). We recall that μ is a number of control parameters nontrivially entering into F, while k denotes the dimension of control space – this is a standard setting of singularity theory [14]. More general analysis and the connection between the solutions of (34) and monodromy theorems in singularity theory (see [3]) we leave to a forthcoming paper.

In the case of one dychotomous noise the stochastic dynamics is governed by two potentials (see equation (26)), $V_{\pm}(x, \bar{u}) = V(x, \bar{u}, \pm a)$. Thus, on the basis of Remark 2.1, we can completely characterize the topological structure of the support of the stationary probability density (cf. [5, 9]).

Let x_0 be a finite critical point of one of the potentials $V_{\pm}$. Let $2i$ denote the order of this point if it is a minimum, and by $(2i+1)_{\pm}$ we distinguish the two kinds of injection point of order $(2i+1)$ (i.e. nondecreasing "+" or nonincreasing "−").

Proposition 4.5. *The following configurations of potentials* V_+, V_- (*with at most one critical point* x_0, *as illustrated in Fig. 2*) *form, at* x_0, *the support boundary point, s.b.p. for short*:

(a) *Right s.b.p. If* V_+ *and* V_- (*or in opposite order*) *are of order* $2i$ *or* $(2i+1)_-$ *and* $(2 \cdot 0 + 1)_+$ *respectively. We denote these configurations by* R^{2i}, R^{2i+1}.

(b) *Left s.b.p. If* V_+ *and* V_- (*or in opposite order*) *are of order* $2i$ *or* $(2i+1)_+$ *and* $(2 \cdot 0 + 1)_-$ *respectively. We denote these configurations by* L^{2i}, L^{2i+1}.

These four configurations are illustrated in Fig. 2.

Proof. We can integrate (34) and obtain

$$P_{st}(x) = N \left(\frac{1}{F_+(x, \bar{u})} - \frac{1}{F_-(x, \bar{u})} \right)$$

$$\times \exp \left(-\frac{\gamma}{2} \int^x \left(\frac{1}{F_+(x', \bar{u})} + \frac{1}{F_-(x', \bar{u})} \right) dx' \right). \tag{37}$$

Assuming that x_0 is a finite critical point of one of the potentials, say V_+, and $(d/dx) V_-(x_0, \bar{u}) = -F_-(x_0, \bar{u}) = 0$, we obtain by analysing the formula (37) that, in the neighbourhood of x_0,

$$P_{st}(x) \sim \frac{1}{F_+^{(k)}(x_0, \bar{u})(x - x_0)^k} \exp \left(\frac{\gamma k!}{2(k-1)F_+^{(k)}(x_0, \bar{u})} (x - x_0)^{1-k} \right) \tag{38}$$

for $k > 1$ and

$$P_{st} \sim |x - x_0|^{\gamma/2 F_+'(x_0, u) - 1} \tag{39}$$

for $k = 1$. Verifying the integrability (i.e. existence of $\int_{x_0 \pm \varepsilon}^{x_0} P_{st}(x) \, dx$) of these formulas we obtain the conclusion of Proposition 4.5.
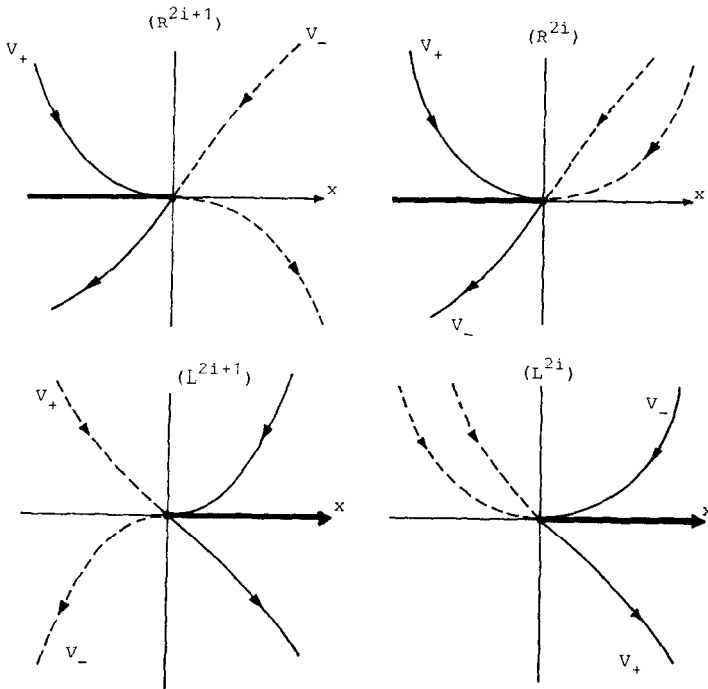

Fig. 2

Corollary 4.6. *For the dychotomous Markov noise, with potentials* V_+, V_- *in general position the only support boundary points which appear have the type* R^2 *or* L^2 *(see Fig. 2), with the corresponding index of divergence*

$$\frac{\gamma}{2F'_+(x, \bar{u})} - 1 \quad \text{or} \quad \frac{\gamma}{2F'_-(x_0, \bar{u})} - 1. \tag{40}$$

5. Stochastic bifurcation sets for elementary catastrophes

In the framework of standard singularity theory one classifies the potential functions $g : \mathbb{R}^n \to \mathbb{R}$ with respect to the properties and configurations of their degenerate critical points. From results on structural stability (cf. [15]) these investigations are usually conducted, locally in the neighbourhood of a critical point and using the language of germs (i.e. the classes of smooth functions identified in some neighbourhood of the source point of the germ [4]). However to avoid inessential rigour we speak about functions instead of their germs.

Let $g : \mathbb{R}^n, 0 \to \mathbb{R}$ be a singularity, $g \in \xi_{(x)}$, i.e. g has an isolated critical point at $0 \in \mathbb{R}^n$. An unfolding of a singularity is a "parametrized family of perturbations". These parameters are treated usually as control parameters [14]. The notion is useful mainly because, for finite codimension singularities (cf. [17]), there exists a "universal unfolding" which in a sense captures all possible unfoldings. More rigorously, let $g \in \xi_{(x)}$. Then an 1-parameter unfolding of g is a germ $V \in \xi_{(x,u)}$, $V : \mathbb{R}^{n+l}, 0 \to \mathbb{R}$, such that $V(x, 0) = g(x)$. An unfolding $\tilde{V} \in \xi_{(x,v)}$ of g is induced from $V_{(x,u)}$ if

$$\tilde{V}(x, u) = V(\rho_v(x), \psi(v)) + \gamma(v)$$

where $v = (v_1, \ldots, v_m) \in \mathbb{R}^m$, $\rho_v : \mathbb{R}^n \to \mathbb{R}^n$, $\psi : \mathbb{R}^m \to \mathbb{R}^l$, $\gamma : \mathbb{R}^l \to \mathbb{R}$, and ρ_v smoothly depends on v. The mapping $(v, x) \to (\rho_n(x), \psi(v))$ is called a morphism of unfoldings. Two unfoldings are equivalent if each can be induced from the other. An l-parameter unfolding is versal if all other unfoldings can be induced from it; universal if in addition l is as small as possible. Suppose that g has finite codimension μ, i.e. the codimension in the orbit structure of $m_{(x)}$. Let $h_1, \ldots, h_\mu$ be a basis for $m_{(x)}/J(g)$ (see Section 4). Then a universal unfolding of g is given by the germ

$$V(x, u) = g(x) + \sum_{i=1}^{\mu} h_i(x) u_i.$$

This is stable for $\mu \leq 5$, see [17]. While different choices of the u_i can be made, a universal unfolding is unique up to an equivalence.

One of the most spectacular results of elementary catastrophe theory is a complete classification of stable-universal unfoldings for $\mu \leq 5$ [14]. These form a generic subset of all parametrized potentials. The celebrated elementary catastrophes of Thom [15] are the universal unfoldings of the singularities on this list for $\mu \leq 4$.

Theorem [14, 17]. *The elementary catastrophes of codimention* ≤ 5 *are the universal unfoldings*

A_2: $x^3 + u_1 x$,

A_3: $\pm x^4 + u_1 x^2 + u_2 x$,

A_4: $x^5 + u_1 x^3 + u_2 x^2 + u_3 x$,

A_5: $\pm x^6 + u_1 x^4 + u_2 x^3 + u_3 x^2 + u_4 x$,

A_6: $x^7 + u_1 x^5 + u_2 x^4 + u_3 x^3 + u_4 x^2 + u_5 x$,

D_4: $x_1^3 \pm x_1 x_2^2 + u_1 x_1^2 + u_2 x_1 + u_3 x_2$,

D_5: $x_1^2 x_2 + x_2^4 + u_1 x_1^2 + u_2 x_2^2 + u_3 x_1 + u_4 x_2$,

D_6: $x_1^5 \pm x_1 x_2^2 + u_1 x_1^3 + u_2 x_1^2 + u_3 x_2^2 + u_4 x_1 + u_5 x_2$,

E_6: $x_1^3 + x_2^4 + u_1 x_1 x_2^2 + u_2 x_2^2 + u_3 x_1 x_2 + u_4 x_1 + u_5 x_2$.

We have the following immediate corollary.

Corollary. *For the smooth potentials* $V : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ *we have only the following normal forms of stable-universal unfoldings*:

$$A_{\mu-1}: \quad V_\mu(x, t) = x^{\mu+2} + \sum_{1=1}^{\mu} u_i x^i, \quad \mu \leq k.$$

Now we define the bifurcation set – the set of qualitative changes in the slow dynamics of gradient dynamical systems [17]. It is the set, also called a catastrophe set [14],

$$\Sigma = \{u \in \mathbb{R}^k; \ d_x V(x, u) = 0, \ \det(d_{xx}^2 V(x, u)) = 0\}.$$

The catastrophe sets for A_3-singularity and A_4-singularity are called "cusp" and "swallowtail" respectively (see Fig. 3). The higher A_k-singular catastrophe sets are called generalized swallowtails. The catastrophe set for A_2-singularity forms the smooth hypersurface ("fold").

Now we have the stochastic dynamical system

$$\dot{x} = F(x, \bar{u}, v, u(t)) = -\text{grad}_x V(x, \bar{u}, v, u(t))$$

$$= -\text{grad}_x(g(x) + \sum_{i=1}^{\mu} (\bar{u}_i + v_i u(t)) h_i(x)) = f(x) + \sum_{i=1}^{\mu} (\bar{u}_i + v_i u(t)) g_i(x),$$

$$(41)$$

where V is a universal unfolding of a finite g with the μ-dimensional unfolding-control space (cf. [14]) and $v \in \mathbb{R}^\mu$, $|v| = 1$ is a direction of the dychotomous fluctuation $u(t)$.

The bifurcation set of stationary points of (41) in the deterministic case, corresponds to the discriminant set (catastrophe set-generalized swallowtail) of F. Investigation of the catastrophe sets for controlled dynamical systems is one of the aims of bifurcation theory or catastrophe theory [2]. Applying the stochastic noise to control parameters the nature of a dynamical system is drastically changed but the bifurcation set for the stationary probability is still very important in the stochastic analysis of the system (cf. [9]). In this section we investigate the bifurcations for
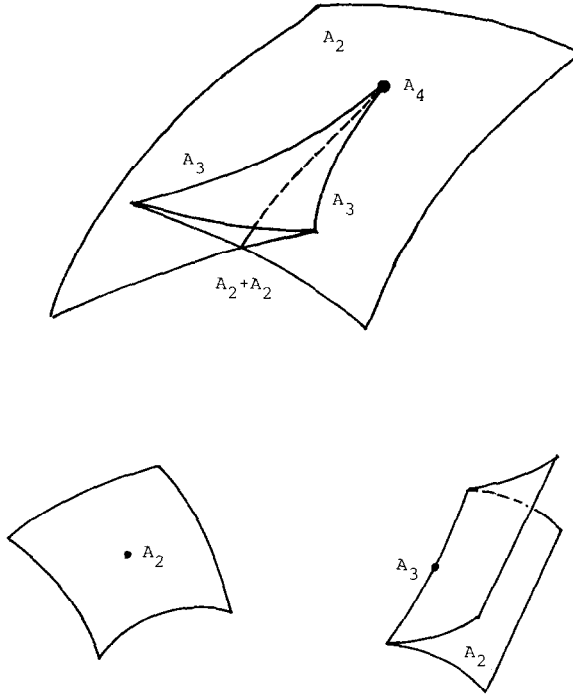
Fig. 3

(41) with dychotomous Markov noise and connect them to the standard catastrophe set for the corresponding deterministic system.

Using the formula (cf. Section 2)

$$P_{st} = N \frac{\sum_{i=1}^{\mu} v_i g_i(x)}{w(v_i, \bar{u}_i, x)} \exp\left\{ -\gamma \int^x dx' \frac{f(x') + \sum_{i=1}^{\mu} \bar{u}_i g_i(x')}{w(v_i, \bar{u}_i, x')} \right\}, \tag{42}$$

where

$$w(v_i, \bar{u}_i, x) = \left(f(x) + \sum_{i=1}^{\mu} (\bar{u}_i + a v_i) g_i(x) \right) \left(f(x) + \sum_{i=1}^{\mu} (\bar{u}_i - a v_i) g_i(x) \right)$$

one can completely analyse the bifurcations of topological structure of the domain of $P_{st}(x)$ as well as its divergence exponents on the boundaries.

Let us consider $F_{\pm}(x, u) = F(x, \bar{u}; v, \pm a)$, with fixed parameters $v \in \mathbb{R}^{\mu}$, $a \in \mathbb{R}_+$. We define

$$\Sigma_{\pm}^i = \{\bar{u} \in \bar{U}; \text{ number of zeros of } F_{\pm}(\cdot, \bar{u}),$$

$$\text{i.e. } \#(\Phi_{\bar{u}}^{\pm} = \{x; F_{\pm}(x, u) = 0\}) \text{ is equal to } i\}, \tag{43}$$

where by $\bar{U}$ we denote the space of control parameters. Thus we have the canonical stratifications (see [14, 2])

$$\bar{U} = \bigcup_{i=0}^{\mu+1} \Sigma_{+}^i = \bigcup_{j=0}^{\mu+1} \Sigma_{-}^j. \tag{44}$$

To each $x_k^\pm \in \Phi_{\bar{u}}^\pm$ we can associate $+1$ (-1) if it is a local minimum or inflection point of the potential $V_\pm$ (if it is a local maximum of $V_\pm$). We denote this function by sgn $x_k^\pm$. Now we can define the function

$$\chi : \bar{U} \to N \cup \{0\}, \tag{45}$$

$$\chi(\bar{u}) = \min\left\{\tfrac{1}{2} \sum_{k=1}^{i} (1+\text{sgn } x_k^+), \tfrac{1}{2} \sum_{k=1}^{j} (1+\text{sgn } x_k^-)\right\} \tag{46}$$

where $\bar{u} \in \Sigma_+^i \cap \Sigma_-^j$.

By straightforward calculations for our A_k-singularities of (41) (cf. [14]) we have the following result.

Proposition 5.1. *For a generic stochastic system* (41) *and sufficiently small $a > 0$, the integer-valued function χ defines the topological type of the support of P_{st}. The value of χ measures the number of connected components of the support. χ is equal to zero if and only if P_{st} is not defined at all. Discontinuities of χ define the points of the bifurcation diagram for the corresponding stochastic dynamical system.*

We now illustrate the bifurcation diagram for the concrete perturbations of the fold catastrophe and cusp catastrophe (cf. [14, 4]).

Let us consider the system (the fold catastrophe)

$$\dot{x} = x^2 - (\bar{u} + u(t)) = -\text{grad}_x V. \tag{47}$$

The two equilibrium surfaces for V_+ and V_- are shown in Fig. 4. Geometrical analysis of the pairs of potentials V_+, V_- corresponding to the regions α, β, γ, δ, ρ as illustrated in Fig. 5 (cf. Remark 2.1) immediately gives the supports for integrable
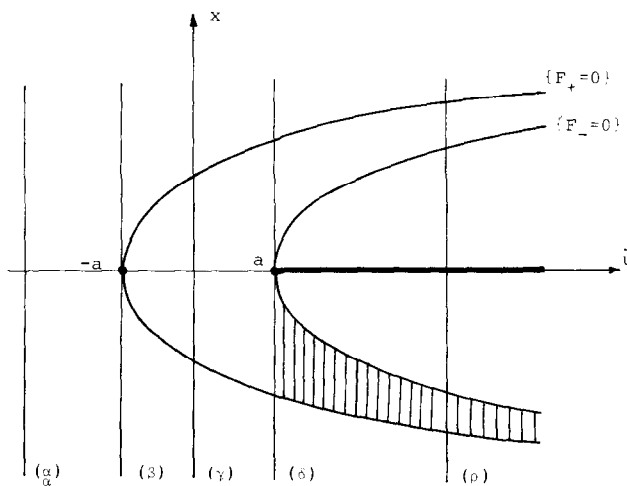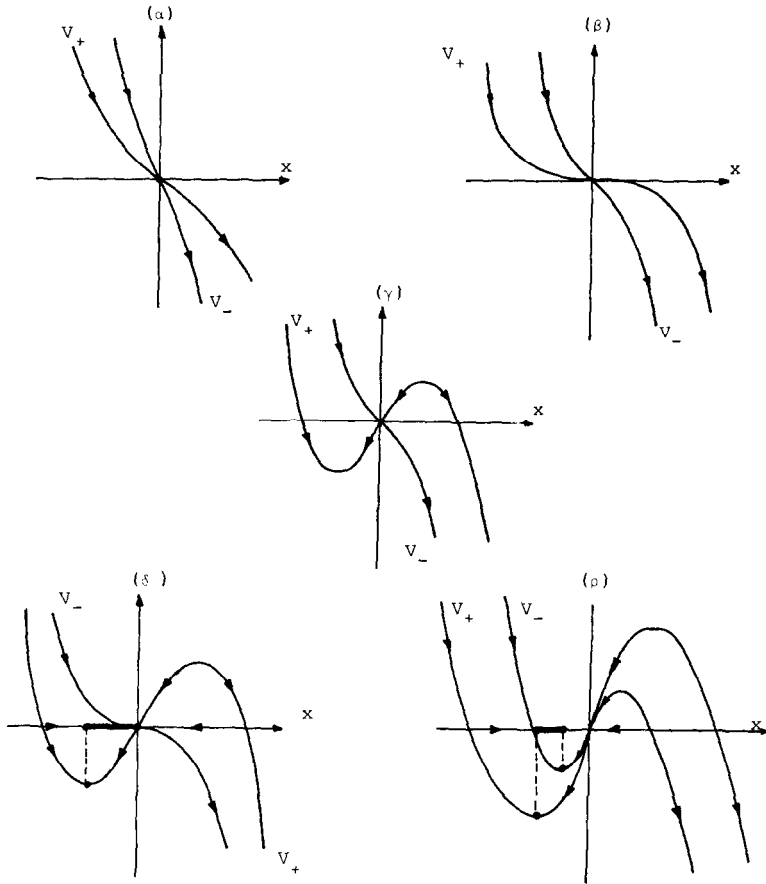

Fig. 4

Fig. 5

and thus physically accepted stationary probabilities (the dotted region in Fig. 4). We easily check the topological type function χ for this system

$$\chi(u) = \begin{cases} 0, & \bar{u} < a, \\ 1, & \bar{u} \geq a. \end{cases} \tag{48}$$

Thus, $\bar{u} = a$ is a bifurcation point; the shifted one corresponding to the deterministic fold catastrophe.

Remark 5.2. In the case of unstable systems, which are always induced from stable ones by appropriate morphisms, say $F \circ \Phi(x, u)$ (cf. [4]), the function χ gives only an upper bound for the topological type of the support. As an example of such a system we can take the Verhulst chemical reaction equation (cf. [9])

$$\dot{x} = \bar{F}(x, \bar{u}, u(t)) = -x^2 + x(\bar{u} + u(t)), \tag{49}$$

where the deterministic $\bar{F}$ is induced from the stable fold singularity $F(x, \bar{u}) = -x^2 + \bar{u}$ by the morphism $\Phi(x, \bar{u}) = (x - \frac{1}{2}\bar{u}, \frac{1}{4}\bar{u}^4)$, i.e. $\bar{F} = F \circ \Phi$. We can easily check that $\chi(\bar{u}) \equiv 1$ but the existence of an integrable P_{st} only occurs for $\bar{u} \geqslant 0$.

Let us consider the system (41) on the plane, $\mu = 2$, with direction of fluctuations $v = (0, 1)$, i.e. $\dot{x} = -x^3 - \bar{u}_1 x - (\bar{u}_2 + u(t))$. The corresponding stratifications of $\bar{U}$ are

$$\Sigma_{\pm}^1: \quad \{\Delta_{\pm} > 0\}, \tag{50}$$

$$\Sigma_{\pm}^2: \quad \{\Delta_{\pm} = 0\}, \tag{51}$$

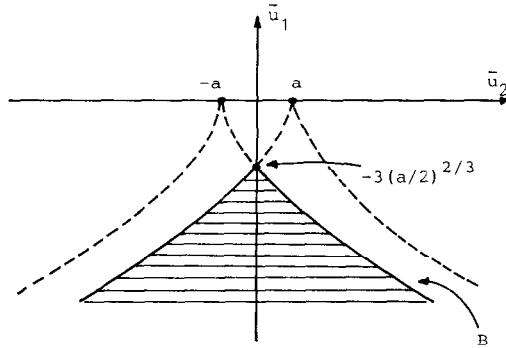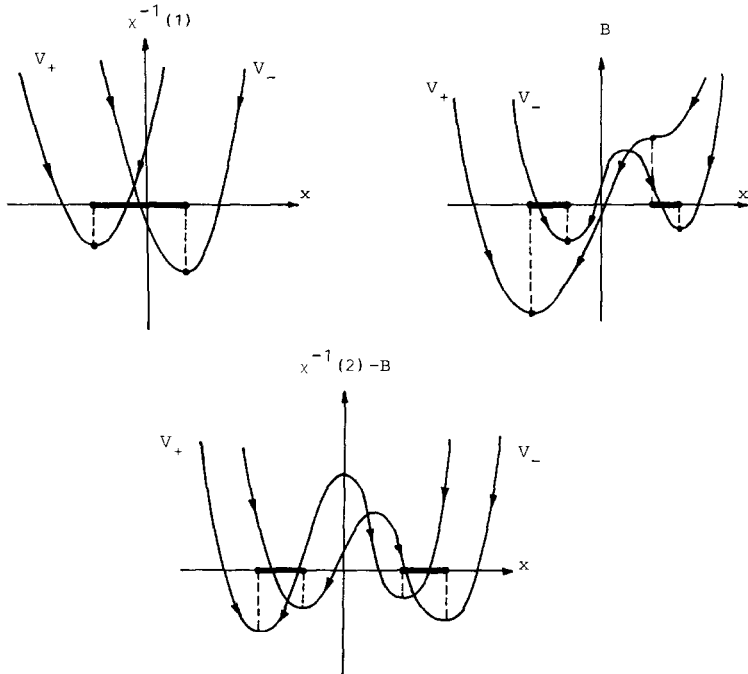$$\Sigma_{\pm}^3: \quad \{\Delta_{\pm} < 0\}, \tag{52}$$

Fig. 6

Fig. 7

where $\Delta_{\pm} = \frac{1}{4}(\bar{u}_2 \pm a)^2 + \frac{1}{27}\bar{u}_1^3$, and the function χ is given by

$$\chi(\bar{u}) = \begin{cases} 1, & \bar{u} \in \Sigma_+^1 \cup \Sigma_-^1, \\ 2, & \bar{u} \in \Sigma_+^3 \cap \Sigma_-^3. \end{cases}$$

By straightforward calculations we obtain the corresponding bifurcation set B (see Fig. 6)

$$(\bar{u}_1, \bar{u}_2) = (-3s^2, \mp 2s^3 \pm a) \quad \text{for } s \geqslant (a/2)^{1/3}. \tag{53}$$

The generalized analysis of potentials at distinguished points of the respective components of the stratification defined by the function χ, is illustrated in Fig. 7.

Acknowledgements

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