

The Lukacs–Olkin–Rubin Theorem on Symmetric Cones Without Invariance of the “Quotient”

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Abstract We prove the Lukacs–Olkin–Rubin theorem without invariance of the distribution of the “quotient,” which was the key assumption in the original proof of (Olkin–Rubin in *Ann Math Stat* 33:1272–1280, 1962). Instead, we assume existence of strictly positive continuous densities of respective random variables. We consider the (cone variate) “quotient” for any division algorithm satisfying some natural conditions. For that purpose, a new proof of the Olkin–Baker functional equation on symmetric cones is given.

Keywords Lukacs characterization · Division algorithm · Wishart distribution · Riesz distribution · Symmetric cones · Functional equations

Mathematics Subject Classification 62H05

1 Introduction

The Lukacs [17] theorem is one of the most celebrated characterizations of probability distributions. It states that *if X and Y are independent, positive, non-degenerate random variables such that their sum and quotient are also independent, then X and Y have gamma distributions with the same scale parameter.*

This theorem has many generalizations. The most important in the multivariate setting was given by Olkin and Rubin [21] and Casalis and Letac [7], where the authors extended characterization to matrix and symmetric cones variate distributions, respectively. There is no unique way of defining the quotient of elements of the cone

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of positive definite symmetric matrices Ω_+ , and in these papers, the authors have considered very general form $U = g(X + Y) \cdot X \cdot g^T(X + Y)$, where g is the so-called division algorithm, that is, $g(\mathbf{a}) \cdot \mathbf{a} \cdot g^T(\mathbf{a}) = I$ for any $\mathbf{a} \in \Omega_+$, where I is the identity matrix and $g(\mathbf{a})$ is invertible for any $\mathbf{a} \in \Omega_+$ (later on, abusing notation, we will write $g(\mathbf{x})\mathbf{y} = g(\mathbf{x}) \cdot \mathbf{y} \cdot g^T(\mathbf{x})$, that is, in this case, $g(\mathbf{x})$ denotes the linear operator acting on Ω_+). The drawback of their extension was the additional strong assumption of invariance of the distribution of U under a group of automorphisms. This result was generalized to homogeneous cones in Boutouria et al. [6].

There were successful attempts in replacing the invariance of the “quotient” assumption with the existence of regular densities of random variables X and Y . Bobecka and Wesołowski [2] assuming existence of strictly positive, twice differentiable densities proved a characterization of Wishart distribution on the cone Ω_+ for division algorithm $g_1(\mathbf{a}) = \mathbf{a}^{-1/2}$, where $\mathbf{a}^{1/2}$ denotes the unique positive definite symmetric root of $\mathbf{a} \in \Omega_+$. These results were generalized to all non-octonion symmetric cones of rank >2 and to the Lorentz cone for strictly positive and continuous densities by Kołodziejek [12, 13].

Exploiting the same approach, with the same technical assumptions on densities as in Bobecka and Wesołowski [2], it was proven by Hassairi et al. [11] that the independence of $X + Y$ and the quotient defined through the Cholesky decomposition, i.e., $g_2(\mathbf{a}) = T_{\mathbf{a}}^{-1}$, where $T_{\mathbf{a}}$ is a lower triangular matrix such that $\mathbf{a} = T_{\mathbf{a}} \cdot T_{\mathbf{a}}^T \in \Omega_+$, characterizes a wider family of distributions called Riesz (or sometimes called Riesz-Wishart). This fact shows that the invariance property assumed in Olkin and Rubin [21] and Casalis and Letac [7] is not of technical nature only. Analogous results for homogeneous cones were obtained by Boutouria [4, 5].

In this paper, we deal with the density version of Lukacs–Olkin–Rubin theorem on symmetric cones for division algorithm satisfying some natural properties. We assume that the densities of X and Y are strictly positive and continuous. We consider quotient U for an arbitrary, fixed division algorithm g as in the original paper of Olkin and Rubin [21], additionally satisfying some natural conditions. In the known cases ($g = g_1$ and $g = g_2$), this improves the results obtained in Bobecka and Wesołowski, Hassairi et al. and Kołodziejek [2, 11, 13]. In general case, the densities of X and Y are given in terms of, so-called, w -multiplicative Cauchy functions, that is functions satisfying

$$f(\mathbf{x})f(w(I)\mathbf{y}) = f(w(\mathbf{x})\mathbf{y}), \quad (\mathbf{x}, \mathbf{y}) \in \Omega_+^2,$$

where $w(\mathbf{x})\mathbf{y} = w(\mathbf{x}) \cdot \mathbf{y} \cdot w^T(\mathbf{x})$ (i.e., $g(\mathbf{x}) = w(\mathbf{x})^{-1}$ is a division algorithm). Consistently, we will call w a multiplication algorithm. Such functions were recently considered in Kołodziejek [14].

Unfortunately, we cannot answer the question whether there exists division (or equivalently multiplication) algorithm resulting in characterizing other distribution than Riesz or Wishart. Moreover, the simultaneous removal of the assumptions of the invariance of the “quotient” and the existence of densities remains a challenge.

This paper is organized as follows. We start in the next section with basic definitions and theorems regarding analysis on symmetric cones. The statement and proof of the main result are given in Sect. 4. Section 3 is devoted to consideration of w -logarithmic

Cauchy functions and the Olkin–Baker functional equation. In that section, we offer much shorter, simpler and covering more general cones proof of the Olkin–Baker functional equation than given in Bobecka and Wośowski, Hassairi et al. and Kołodziejek [2, 11, 13].

2 Preliminaries

In this section, we give a short introduction to the theory of symmetric cones. For further details, we refer to Faraut and Korányi [8].

A *Euclidean Jordan algebra* is a Euclidean space \mathbb{E} (endowed with scalar product denoted $\langle \mathbf{x}, \mathbf{y} \rangle$) equipped with a bilinear mapping (product)

$$\mathbb{E} \times \mathbb{E} \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{xy} \in \mathbb{E}$$

and a neutral element \mathbf{e} in \mathbb{E} such that for all $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in \mathbb{E} :

- (i) $\mathbf{xy} = \mathbf{yx}$,
- (ii) $\mathbf{x}(\mathbf{x}^2\mathbf{y}) = \mathbf{x}^2(\mathbf{xy})$,
- (iii) $\mathbf{x}\mathbf{e} = \mathbf{x}$,
- (iv) $\langle \mathbf{x}, \mathbf{yz} \rangle = \langle \mathbf{xy}, \mathbf{z} \rangle$.

For $\mathbf{x} \in \mathbb{E}$ let $\mathbb{L}(\mathbf{x}) : \mathbb{E} \rightarrow \mathbb{E}$ be linear map defined by

$$\mathbb{L}(\mathbf{x})\mathbf{y} = \mathbf{xy},$$

and define

$$\mathbb{P}(\mathbf{x}) = 2\mathbb{L}^2(\mathbf{x}) - \mathbb{L}(\mathbf{x}^2).$$

The map $\mathbb{P} : \mathbb{E} \mapsto \text{End}(\mathbb{E})$ is called the *quadratic representation* of \mathbb{E} .

An element \mathbf{x} is said to be *invertible* if there exists an element \mathbf{y} in \mathbb{E} such that $\mathbb{L}(\mathbf{x})\mathbf{y} = \mathbf{e}$. Then, \mathbf{y} is called the *inverse of \mathbf{x}* and is denoted by $\mathbf{y} = \mathbf{x}^{-1}$. Note that the inverse of \mathbf{x} is unique. It can be shown that \mathbf{x} is invertible if and only if $\mathbb{P}(\mathbf{x})$ is invertible, and in this case, $(\mathbb{P}(\mathbf{x}))^{-1} = \mathbb{P}(\mathbf{x}^{-1})$.

Euclidean Jordan algebra \mathbb{E} is said to be *simple* if it is not a Cartesian product of two Euclidean Jordan algebras of positive dimensions. Up to linear isomorphism, there are only five kinds of Euclidean simple Jordan algebras. Let \mathbb{K} denotes either the real numbers \mathbb{R} , the complex ones \mathbb{C} , quaternions \mathbb{H} or the octonions \mathbb{O} and write $S_r(\mathbb{K})$ for the space of $r \times r$ Hermitian matrices with entries valued in \mathbb{K} , endowed with the Euclidean structure $\langle \mathbf{x}, \mathbf{y} \rangle = \text{Trace}(\mathbf{x} \cdot \bar{\mathbf{y}})$ and with the Jordan product

$$\mathbf{xy} = \frac{1}{2}(\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x}), \tag{1}$$

where $\mathbf{x} \cdot \mathbf{y}$ denotes the ordinary product of matrices and $\bar{\mathbf{y}}$ is the conjugate of \mathbf{y} . Then, $S_r(\mathbb{R}), r \geq 1, S_r(\mathbb{C}), r \geq 2, S_r(\mathbb{H}), r \geq 2$, and the exceptional $S_3(\mathbb{O})$ are the first four kinds of Euclidean simple Jordan algebras. Note that in this case

$$\mathbb{P}(\mathbf{y})\mathbf{x} = \mathbf{y} \cdot \mathbf{x} \cdot \mathbf{y}. \tag{2}$$

The fifth kind is the Euclidean space \mathbb{R}^{n+1} , $n \geq 2$, with Jordan product

$$(x_0, x_1, \dots, x_n)(y_0, y_1, \dots, y_n) = \left(\sum_{i=0}^n x_i y_i, x_0 y_1 + y_0 x_1, \dots, x_0 y_n + y_0 x_n \right). \tag{3}$$

To each Euclidean simple Jordan algebra, one can attach the set of Jordan squares

$$\bar{\Omega} = \{ \mathbf{x}^2 : \mathbf{x} \in \mathbb{E} \}.$$

The interior Ω is a symmetric cone. Moreover, Ω is *irreducible*, i.e., it is not the Cartesian product of two convex cones. One can prove that an open convex cone is symmetric and irreducible if and only if it is the cone Ω of some Euclidean simple Jordan algebra. Each simple Jordan algebra corresponds to a symmetric cone; hence, there exist up to linear isomorphism also only five kinds of symmetric cones. The cone corresponding to the Euclidean Jordan algebra \mathbb{R}^{n+1} equipped with Jordan product (3) is called the Lorentz cone.

We denote by $G(\mathbb{E})$ the subgroup of the linear group $GL(\mathbb{E})$ of linear automorphisms which preserves Ω , and we denote by G the connected component of $G(\mathbb{E})$ containing the identity. Recall that if $\mathbb{E} = S_r(\mathbb{R})$ and $GL(r, \mathbb{R})$ is the group of invertible $r \times r$ matrices, elements of $G(\mathbb{E})$ are the maps $g : \mathbb{E} \rightarrow \mathbb{E}$ such that there exists $\mathbf{a} \in GL(r, \mathbb{R})$ with

$$g(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x} \cdot \mathbf{a}^T.$$

We define $K = G \cap O(\mathbb{E})$, where $O(\mathbb{E})$ is the orthogonal group of \mathbb{E} . It can be shown that

$$K = \{ k \in G : k\mathbf{e} = \mathbf{e} \}.$$

A *multiplication algorithm* is a map $\Omega \rightarrow G : \mathbf{x} \mapsto w(\mathbf{x})$ such that $w(\mathbf{x})\mathbf{e} = \mathbf{x}$ for all $\mathbf{x} \in \Omega$. This concept is consistent with, so-called, division algorithm g , which was introduced by Olkin and Rubin [21] and Casalis and Letac [7], that is a mapping $\Omega \ni \mathbf{x} \mapsto g(\mathbf{x}) \in G$ such that $g(\mathbf{x})\mathbf{x} = \mathbf{e}$ for any $\mathbf{x} \in \Omega$. If w is a multiplication algorithm, then $g = w^{-1}$ (that is, $g(\mathbf{x})w(\mathbf{x}) = w(\mathbf{x})g(\mathbf{x}) = Id_\Omega$ for any $\mathbf{x} \in \Omega$) is a division algorithm and vice versa; if g is a division algorithm, then $w = g^{-1}$ is a multiplication algorithm. One of two important examples of multiplication algorithms is the map $w_1(\mathbf{x}) = \mathbb{P}(\mathbf{x}^{1/2})$.

We will now introduce a very useful decomposition in \mathbb{E} , called *spectral decomposition*. An element $\mathbf{c} \in \mathbb{E}$ is said to be a *idempotent* if $\mathbf{c}\mathbf{c} = \mathbf{c} \neq 0$. Idempotents \mathbf{a} and \mathbf{b} are *orthogonal* if $\mathbf{a}\mathbf{b} = 0$. Idempotent \mathbf{c} is *primitive* if \mathbf{c} is not a sum of two non-null idempotents. A *complete system of primitive orthogonal idempotents* is a set $(\mathbf{c}_1, \dots, \mathbf{c}_r)$ such that

$$\sum_{i=1}^r \mathbf{c}_i = \mathbf{e} \text{ and } \mathbf{c}_i \mathbf{c}_j = \delta_{ij} \mathbf{c}_i \text{ for } 1 \leq i \leq j \leq r.$$

The size r of such system is a constant called the *rank* of \mathbb{E} . Any element \mathbf{x} of a Euclidean simple Jordan algebra can be written as $\mathbf{x} = \sum_{i=1}^r \lambda_i \mathbf{c}_i$ for some complete system of primitive orthogonal idempotents $(\mathbf{c}_1, \dots, \mathbf{c}_r)$. The real numbers $\lambda_i, i = 1, \dots, r$ are the *eigenvalues* of \mathbf{x} . One can then define *trace* and *determinant* of \mathbf{x} by, respectively, $\text{tr } \mathbf{x} = \sum_{i=1}^r \lambda_i$ and $\det \mathbf{x} = \prod_{i=1}^r \lambda_i$. An element $\mathbf{x} \in \mathbb{E}$ belongs to Ω if and only if all its eigenvalues are strictly positive.

The rank r and $\dim \Omega$ of irreducible symmetric cone are connected through relation

$$\dim \Omega = r + \frac{dr(r - 1)}{2},$$

where d is an integer called the *Peirce constant*.

If \mathbf{c} is a primitive idempotent of \mathbb{E} , the only possible eigenvalues of $\mathbb{L}(\mathbf{c})$ are $0, \frac{1}{2}$ and 1 . We denote by $\mathbb{E}(\mathbf{c}, 0), \mathbb{E}(\mathbf{c}, \frac{1}{2})$ and $\mathbb{E}(\mathbf{c}, 1)$ the corresponding eigenspaces. The decomposition

$$\mathbb{E} = \mathbb{E}(\mathbf{c}, 0) \oplus \mathbb{E}(\mathbf{c}, \frac{1}{2}) \oplus \mathbb{E}(\mathbf{c}, 1)$$

is called the *Peirce decomposition of \mathbb{E} with respect to \mathbf{c}* . Note that $\mathbb{P}(\mathbf{c})$ is the orthogonal projection of \mathbb{E} onto $\mathbb{E}(\mathbf{c}, 1)$.

Fix a complete system of orthogonal idempotents $(\mathbf{c}_i)_{i=1}^r$. Then for any $i, j \in \{1, 2, \dots, r\}$, we write

$$\begin{aligned} \mathbb{E}_{ii} &= \mathbb{E}(\mathbf{c}_i, 1) = \mathbb{R}\mathbf{c}_i, \\ \mathbb{E}_{ij} &= \mathbb{E}\left(\mathbf{c}_i, \frac{1}{2}\right) \cap \mathbb{E}\left(\mathbf{c}_j, \frac{1}{2}\right) \text{ if } i \neq j. \end{aligned}$$

It can be proved (see Faraut and Korányi [8, Theorem IV.2.1]) that

$$\mathbb{E} = \bigoplus_{i \leq j} \mathbb{E}_{ij}$$

and

$$\begin{aligned} \mathbb{E}_{ij} \cdot \mathbb{E}_{ij} &\subset \mathbb{E}_{ii} + \mathbb{E}_{ij}, \\ \mathbb{E}_{ij} \cdot \mathbb{E}_{jk} &\subset \mathbb{E}_{ik}, \text{ if } i \neq k, \\ \mathbb{E}_{ij} \cdot \mathbb{E}_{kl} &= \{0\}, \text{ if } \{i, j\} \cap \{k, l\} = \emptyset. \end{aligned}$$

Moreover (Faraut and Korányi [8, Lemma IV.2.2]), if $\mathbf{x} \in \mathbb{E}_{ij}, \mathbf{y} \in \mathbb{E}_{jk}, i \neq k$, then

$$\begin{aligned} \mathbf{x}^2 &= \frac{1}{2} \|\mathbf{x}\|^2 (\mathbf{c}_i + \mathbf{c}_j), \\ \|\mathbf{xy}\|^2 &= \frac{1}{8} \|\mathbf{x}\|^2 \|\mathbf{y}\|^2. \end{aligned} \tag{4}$$

The dimension of \mathbb{E}_{ij} is the Peirce constant d for any $i \neq j$. When \mathbb{E} is $S_r(\mathbb{K})$, if (e_1, \dots, e_r) is an orthonormal basis of \mathbb{R}^r , then $\mathbb{E}_{ii} = \mathbb{R}e_i e_i^T$ and $\mathbb{E}_{ij} = \mathbb{K} \left(e_i e_j^T + e_j e_i^T \right)$ for $i < j$ and d is equal to $\dim_{\mathbb{K}} \mathbb{E}_{ij}$.

For $1 \leq k \leq r$, let P_k be the orthogonal projection onto $\mathbb{E}^{(k)} = \mathbb{E}(\mathbf{c}_1 + \dots + \mathbf{c}_k, 1)$, $\det^{(k)}$ the determinant in the subalgebra $\mathbb{E}^{(k)}$, and, for $\mathbf{x} \in \Omega$, $\Delta_k(\mathbf{x}) = \det^{(k)}(P_k(\mathbf{x}))$. Then, Δ_k is called the principal minor of order k with respect to the Jordan frame $(\mathbf{c}_k)_{k=1}^r$. Note that $\Delta_r(\mathbf{x}) = \det \mathbf{x}$. For $s = (s_1, \dots, s_r) \in \mathbb{R}^r$ and $\mathbf{x} \in \Omega$, we write

$$\Delta_s(\mathbf{x}) = \Delta_1(\mathbf{x})^{s_1 - s_2} \Delta_2(\mathbf{x})^{s_2 - s_3} \dots \Delta_r(\mathbf{x})^{s_r}.$$

Δ_s is called a generalized power function. If $\mathbf{x} = \sum_{i=1}^r \alpha_i \mathbf{c}_i$, then $\Delta_s(\mathbf{x}) = \alpha_1^{s_1} \alpha_2^{s_2} \dots \alpha_r^{s_r}$.

We will now introduce some basic facts about triangular group. For \mathbf{x} and \mathbf{y} in Ω , let $\mathbf{x} \square \mathbf{y}$ denote the endomorphism of \mathbb{E} defined by

$$\mathbf{x} \square \mathbf{y} = \mathbb{L}(\mathbf{x}\mathbf{y}) + \mathbb{L}(\mathbf{x})\mathbb{L}(\mathbf{y}) - \mathbb{L}(\mathbf{y})\mathbb{L}(\mathbf{x}).$$

If \mathbf{c} is an idempotent and $\mathbf{z} \in \mathbb{E}(\mathbf{c}, \frac{1}{2})$, we define the *Frobenius transformation* $\tau_{\mathbf{c}}(\mathbf{z})$ in G by

$$\tau_{\mathbf{c}}(\mathbf{z}) = \exp(2\mathbf{z} \square \mathbf{c}).$$

Since $2\mathbf{z} \square \mathbf{c}$ is nilpotent of degree 3 (see Faraut and Korányi [8, Lemma VI.3.1]), we get

$$\tau_{\mathbf{c}}(\mathbf{z}) = I + (2\mathbf{z} \square \mathbf{c}) + \frac{1}{2}(2\mathbf{z} \square \mathbf{c})^2. \tag{5}$$

Given a Jordan frame $(\mathbf{c}_i)_{i=1}^r$, the subgroup of G ,

$$\mathcal{T} = \left\{ \tau_{\mathbf{c}_1}(\mathbf{z}^{(1)}) \dots \tau_{\mathbf{c}_{r-1}}(\mathbf{z}^{(r-1)}) \mathbb{P} \left(\sum_{i=1}^r \alpha_i \mathbf{c}_i \right) : \alpha_i > 0, \mathbf{z}^{(j)} \in \bigoplus_{k=j+1}^r \mathbb{E}_{jk} \right\}$$

is called the *triangular group corresponding to the Jordan frame* $(\mathbf{c}_i)_{i=1}^r$. For any \mathbf{x} in Ω , there exists a unique $t_{\mathbf{x}}$ in \mathcal{T} such that $\mathbf{x} = t_{\mathbf{x}}$, that is, there exist (see Faraut and Korányi [8, Theorem IV.3.5]) elements $\mathbf{z}^{(j)} \in \bigoplus_{k=j+1}^r \mathbb{E}_{jk}$, $1 \leq j \leq r - 1$ and positive numbers $\alpha_1, \dots, \alpha_r$ such that

$$\mathbf{x} = \tau_{\mathbf{c}_1}(\mathbf{z}^{(1)}) \tau_{\mathbf{c}_2}(\mathbf{z}^{(2)}) \dots \tau_{\mathbf{c}_{r-1}}(\mathbf{z}^{(r-1)}) \left(\sum_{k=1}^r \alpha_k \mathbf{c}_k \right). \tag{6}$$

Mapping $w_2: \Omega \rightarrow \mathcal{T}$, $\mathbf{x} \mapsto w_2(\mathbf{x}) = t_{\mathbf{x}}$ realizes a multiplication algorithm.

For $\mathbb{E} = S_r(\mathbb{R})$, we have $\Omega = \Omega_+$. Let us define for $1 \leq i, j \leq r$ matrix $\mu_{ij} = (\gamma_{kl})_{1 \leq k, l \leq r}$ such that $\gamma_{ij} = 1$ and all other entries are equal 0. Then for Jordan

frame $(\mathbf{c}_i)_{i=1}^r$, where $\mathbf{c}_k = \mu_{kk}$, $k = 1, \dots, r$, we have $\mathbf{z}_{jk} = (\mu_{jk} + \mu_{kj}) \in \mathbb{E}_{jk}$ oraz $\|\mathbf{z}_{jk}\|^2 = 2$, $1 \leq j, k \leq r$, $j \neq k$. if $\mathbf{z}^{(i)} \in \bigoplus_{j=i+1}^r \mathbb{E}_{ij}$, $i = 1, \dots, r - 1$, then there exists $\alpha^{(i)} = (\alpha_{i+1}, \dots, \alpha_r) \in \mathbb{R}^{r-i}$ such that $\mathbf{z}^{(i)} = \sum_{j=i+1}^r \alpha_j \mathbf{z}_{ij}$. Then, the Frobenius transformation reads

$$\tau_{\mathbf{c}_i}(\mathbf{z}^{(i)})\mathbf{x} = \mathcal{F}_i(\alpha^{(i)}) \cdot \mathbf{x} \cdot \mathcal{F}_i(\alpha^{(i)})^T,$$

where $\mathcal{F}_i(\alpha^{(i)})$ is so-called Frobenius matrix:

$$\mathcal{F}_i(\alpha^{(i)}) = I + \sum_{j=i+1}^r \alpha_j \mu_{ji},$$

i.e., below i th one of identity matrix there is a vector $\alpha^{(i)}$, particularly

$$\mathcal{F}_2(\alpha^{(2)}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & \alpha_3 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_r & 0 & \dots & 1 \end{pmatrix}.$$

It can be shown (Faraut and Korányi [8, Proposition VI.3.10]) that for each $t \in \mathcal{T}$, $\mathbf{x} \in \Omega$ and $s \in \mathbb{R}^r$,

$$\Delta_s(t\mathbf{x}) = \Delta_s(t\mathbf{e})\Delta_s(\mathbf{x}) \tag{7}$$

and for any $\mathbf{z} \in \mathbb{E}(\mathbf{c}_i, \frac{1}{2})$, $i = 1, \dots, r$,

$$\Delta_s(\tau_{\mathbf{c}_i}(\mathbf{z})\mathbf{e}) = 1, \tag{8}$$

if only Δ_s and \mathcal{T} are associated with the same Jordan frame $(\mathbf{c}_i)_{i=1}^r$.

We will now introduce some necessary basics regarding certain probability distribution on symmetric cones. Absolutely continuous Riesz distribution $R_{s,\mathbf{a}}$ on Ω is defined for any $\mathbf{a} \in \Omega$ and $s = (s_1, \dots, s_r) \in \mathbb{R}^r$ such that $s_i > (i - 1)d/2$, $i = 1, \dots, r$, though its density

$$R_{s,\mathbf{a}}(d\mathbf{x}) = \frac{\Delta_s(\mathbf{a})}{\Gamma_\Omega(s)} \Delta_{s-\dim \Omega/r}(\mathbf{x}) e^{-(\mathbf{a},\mathbf{x})} I_\Omega(\mathbf{x}) d\mathbf{x}, \quad \mathbf{x} \in \Omega,$$

where Δ_s is the generalized power function with respect to a Jordan frame $(\mathbf{c}_i)_{i=1}^r$ and Γ_Ω is the Gamma function of the symmetric cone Ω . It can be shown that $\Gamma_\Omega(s) = (2\pi)^{(\dim \Omega - r)/2} \prod_{j=1}^r \Gamma(s_j - (j - 1)\frac{d}{2})$ (see Faraut and Korányi [8, VII.1.1.]). Riesz distribution was introduced in Hassairi and Lajmi [10].

Absolutely continuous Wishart distribution $\gamma_{p,\mathbf{a}}$ on Ω is a special case of Riesz distribution for $s_1 = \dots = s_r = p$. If $\mathbf{a} \in \Omega$ and $p > \dim \Omega/r - 1$ it has density

$$\gamma_{p,\mathbf{a}}(d\mathbf{x}) = \frac{(\det \mathbf{a})^p}{\Gamma_{\Omega}(p)} (\det \mathbf{x})^{p-\dim \Omega/r} e^{-\langle \mathbf{a}, \mathbf{x} \rangle} I_{\Omega}(\mathbf{x}) d\mathbf{x}, \quad \mathbf{x} \in \Omega,$$

where $\Gamma_{\Omega}(p) := \Gamma_{\Omega}(p, \dots, p)$. Wishart distribution is a generalization of gamma distribution (case $r = 1$).

In generality, Riesz and Wishart distributions does not always have densities, but due to the assumption of existence of densities in Theorem 4.2, we are not interested in other cases.

3 Functional Equations

3.1 Logarithmic Cauchy Functions

As will be seen, the densities of respective random variables will be given in terms of w -logarithmic Cauchy functions, i.e., functions $f : \Omega \rightarrow \mathbb{R}$ that satisfy the following functional equation

$$f(\mathbf{x}) + f(w(\mathbf{e})\mathbf{y}) = f(w(\mathbf{x})\mathbf{y}), \quad (\mathbf{x}, \mathbf{y}) \in \Omega^2, \tag{9}$$

where w is a multiplication algorithm. If f is w -logarithmic, then e^f is called w -multiplicative. In the following section, we will give the form of w -logarithmic Cauchy functions for two basic multiplication algorithms, one connected with the quadratic representation

$$w_1(\mathbf{x}) = \mathbb{P}(\mathbf{x}^{1/2}), \tag{10}$$

and the other related to a triangular group \mathcal{T} ,

$$w_2(\mathbf{x}) = t_{\mathbf{x}} \in \mathcal{T}. \tag{11}$$

Such functions were recently considered without any regularity assumptions in Kołodziejek [14].

It should be stressed that there exists infinite number of multiplication algorithms. If w is a multiplication algorithm, then trivial extensions are given by $w^{(k)}(\mathbf{x}) = w(\mathbf{x})k$, where $k \in K$ is fixed (Remark 4.3 explains why this extension is trivial when it comes to multiplicative functions). One may consider also multiplication algorithms of the form $\mathbb{P}(\mathbf{x}^{\alpha})t_{\mathbf{x}^{1-2\alpha}}$, which interpolate between the two main examples: w_1 (which is $\alpha = 1/2$) and w_2 (which is $\alpha = 0$). In general, any multiplication algorithm may be written in the form $w(x) = \mathbb{P}(\mathbf{x}^{1/2})k_x$, where $\mathbf{x} \mapsto k_x \in K$.

Functional equation (9) for w_1 was already considered by Bobecka and Wesolowski [3] for differentiable functions and by Molnár [19] for continuous functions of real

or complex Hermitian positive definite matrices of rank >2 . Without any regularity assumptions, it was solved on the Lorentz cone by Wesołowski [23].

Case of $w_2(\mathbf{x}) = t_{\mathbf{x}} \in \mathcal{T}$ for a triangular group \mathcal{T} , perhaps a bit surprisingly, leads to a different solution. It was indirectly solved for differentiable functions by Hassairi et al. [11, Proof of Theorem 3.3].

By Faraut and Korányi [8, Proposition III.4.3], for any g in the group G ,

$$\det(g\mathbf{x}) = (\text{Det } g)^{r/\dim \Omega} \det \mathbf{x},$$

where Det denotes the determinant in the space of endomorphisms on Ω . Inserting a multiplication algorithm $g = w(\mathbf{y})$, $\mathbf{y} \in \Omega$, and $\mathbf{x} = \mathbf{e}$, we obtain

$$\text{Det}(w(\mathbf{y})) = (\det \mathbf{y})^{\dim \Omega/r} \tag{12}$$

and hence

$$\det(w(\mathbf{y})\mathbf{x}) = \det \mathbf{y} \det \mathbf{x}$$

for any $\mathbf{x}, \mathbf{y} \in \Omega$. This means that $f(\mathbf{x}) = H(\det \mathbf{x})$, where H is generalized logarithmic function, i.e., $H(ab) = H(a) + H(b)$ for $a, b > 0$, is always a solution to (9), regardless of the choice of multiplication algorithm w . If a w -logarithmic functions f is additionally K -invariant ($f(\mathbf{x}) = f(k\mathbf{x})$ for any $k \in K$), then $H(\det \mathbf{x})$ is the only possible solution (Theorem 3.4).

In Kołodziejek [14], the following theorems have been proved. They will be useful in the proof of the main theorems in this paper.

Theorem 3.1 (w_1 -logarithmic Cauchy functional equation) *Let $f: \Omega \rightarrow \mathbb{R}$ be a function such that*

$$f(\mathbf{x}) + f(\mathbf{y}) = f\left(\mathbb{P}\left(\mathbf{x}^{1/2}\right)\mathbf{y}\right), \quad (\mathbf{x}, \mathbf{y}) \in \Omega^2.$$

Then, there exists a logarithmic function H such that for any $\mathbf{x} \in \Omega$,

$$f(\mathbf{x}) = H(\det \mathbf{x}).$$

Theorem 3.2 (w_2 -logarithmic Cauchy functional equation) *Let $f: \Omega \rightarrow \mathbb{R}$ be a function satisfying*

$$f(\mathbf{x}) + f(\mathbf{y}) = f(t_{\mathbf{y}}\mathbf{x})$$

for any \mathbf{x} and \mathbf{y} in the cone Ω of rank r , $t_{\mathbf{y}} \in \mathcal{T}$, where \mathcal{T} is the triangular group with respect to the Jordan frame $(\mathbf{c}_i)_{i=1}^r$. Then, there exist generalized logarithmic functions H_1, \dots, H_r such that for any $\mathbf{x} \in \Omega$,

$$f(\mathbf{x}) = \sum_{k=1}^r H_k(\Delta_k(\mathbf{x})),$$

where Δ_k is the principal minor of order k with respect to $(\mathbf{c}_i)_{i=1}^r$.

If we assume in Theorem 3.2 that f is additionally measurable, then functions H_k are measurable. This implies that there exist constants $s_k \in \mathbb{R}$ such that $H_k(\alpha) = s_k \log \alpha$ and

$$f(\mathbf{x}) = \sum_{k=1}^r s_k \log(\Delta_k(\mathbf{x})) = \log \prod_{k=1}^r \Delta_k^{s_k}(\mathbf{x}).$$

Thus, we obtain the following

Remark 3.3 If we impose on f in Theorem 3.2 some mild conditions (e.g., measurability), then there exists $s \in \mathbb{R}^r$ such that for any $\mathbf{x} \in \Omega$,

$$f(\mathbf{x}) = \log \Delta_s(\mathbf{x}).$$

Theorem 3.4 Let $f: \Omega \rightarrow \mathbb{R}$ be a function satisfying (9). Assume additionally that f is K -invariant, i.e., $f(k\mathbf{x}) = f(\mathbf{x})$ for any $k \in K$ and $\mathbf{x} \in \Omega$. Then, there exists a logarithmic function H such that for any $\mathbf{x} \in \Omega$,

$$f(\mathbf{x}) = H(\det \mathbf{x}).$$

Lemma 3.5 (*w*-logarithmic Pexider functional equation) Assume that a, b, c defined on the cone Ω satisfy following functional equation

$$a(\mathbf{x}) + b(\mathbf{y}) = c(w(\mathbf{x})\mathbf{y}), \quad (\mathbf{x}, \mathbf{y}) \in \Omega^2.$$

Then, there exist *w*-logarithmic function f and real constants a_0, b_0 such that for any $\mathbf{x} \in \Omega$,

$$\begin{aligned} a(\mathbf{x}) &= f(\mathbf{x}) + a_0, \\ b(\mathbf{x}) &= f(w(\mathbf{e})\mathbf{x}) + b_0, \\ c(\mathbf{x}) &= f(\mathbf{x}) + a_0 + b_0. \end{aligned}$$

3.2 The Olkin–Baker Functional Equation

In the following section, we deal with the Olkin–Baker functional equation on irreducible symmetric cones, which is related to the Lukacs independence condition (see proof of the Theorem 4.2).

Henceforth, we will assume that multiplication algorithm w additionally is homogeneous of degree 1, that is, $w(s\mathbf{x}) = s w(\mathbf{x})$ for any $s > 0$ and $\mathbf{x} \in \Omega$. It is easy to create a multiplication algorithm without this property, for example:

$$w(\mathbf{x}) = \begin{cases} w_1(\mathbf{x}), & \text{if } \det \mathbf{x} > 1, \\ w_2(\mathbf{x}), & \text{if } \det \mathbf{x} \leq 1. \end{cases}$$

The problem of solving

$$f(x)g(y) = p(x+y)q(x/y), \quad (x, y) \in (0, \infty)^2 \quad (13)$$

for unknown positive functions f , g , p and q was first posed in Olkin [20]. Note that in one-dimensional case, it does not matter whether one considers $q(x/y)$ or $q(x/(x+y))$ on the right-hand side of (13). Its general solution was given in Baker [1] and later analyzed in Lajkó [15] using a different approach. Recently, in Mészáros [18] and Lajkó and Mészáros [16], Eq. (13) was solved assuming that it is satisfied almost everywhere on $(0, \infty)^2$ for measurable functions which are nonnegative on its domain or positive on some sets of positive Lebesgue measure, respectively. Finally, a new derivation of solution to (13), when the equation holds almost everywhere on $(0, \infty)^2$ and no regularity assumptions on unknown positive functions are imposed, was given in Ger et al. [9]. The following theorem is concerned with an adaptation of (13), after taking logarithm, to the symmetric cone case.

Theorem 3.6 (Olkin–Baker functional equation on symmetric cones) *Let a , b , c and d be real continuous functions on an irreducible symmetric cone Ω of rank r . Assume*

$$a(\mathbf{x}) + b(\mathbf{y}) = c(\mathbf{x} + \mathbf{y}) + d(g(\mathbf{x} + \mathbf{y})\mathbf{x}), \quad (\mathbf{x}, \mathbf{y}) \in \Omega^2, \quad (14)$$

where $g^{-1} = w$ is a homogeneous of degree 1 multiplication algorithm. Then, there exist constants $C_i \in \mathbb{R}$, $i = 1, \dots, 4$, $\Lambda \in \mathbb{E}$ such that for any $\mathbf{x} \in \Omega$ and $\mathbf{u} \in \mathcal{D} = \{\mathbf{x} \in \Omega : \mathbf{e} - \mathbf{x} \in \Omega\}$,

$$\begin{aligned} a(\mathbf{x}) &= \langle \Lambda, \mathbf{x} \rangle + e(\mathbf{x}) + C_1, \\ b(\mathbf{x}) &= \langle \Lambda, \mathbf{x} \rangle + f(\mathbf{x}) + C_2, \\ c(\mathbf{x}) &= \langle \Lambda, \mathbf{x} \rangle + e(\mathbf{x}) + f(\mathbf{x}) + C_3, \\ d(\mathbf{u}) &= e(w(\mathbf{e})\mathbf{u}) + f(\mathbf{e} - w(\mathbf{e})\mathbf{u}) + C_4, \end{aligned}$$

where e and f are continuous w -logarithmic Cauchy functions and $C_1 + C_2 = C_3 + C_4$.

We will need following simple lemma. For the elementary proof, we refer to Kołodziejek [13, Lemma 3.2].

Lemma 3.7 (Additive Pexider functional equation on symmetric cones) *Let a , b and c be measurable functions on a symmetric cone Ω satisfying*

$$a(\mathbf{x}) + b(\mathbf{y}) = c(\mathbf{x} + \mathbf{y}), \quad (\mathbf{x}, \mathbf{y}) \in \Omega^2. \quad (15)$$

Then, there exist constants $\alpha, \beta \in \mathbb{R}$ and $\lambda \in \mathbb{E}$ such that for all $\mathbf{x} \in \Omega$,

$$\begin{aligned} a(\mathbf{x}) &= \langle \lambda, \mathbf{x} \rangle + \alpha, \\ b(\mathbf{x}) &= \langle \lambda, \mathbf{x} \rangle + \beta, \\ c(\mathbf{x}) &= \langle \lambda, \mathbf{x} \rangle + \alpha + \beta. \end{aligned} \quad (16)$$

Now, we can come back and give a new proof the Olkin–Baker functional equation.

Proof of Theorem 3.6 In the first part of the proof, we adapt the argument given in Ger et al. [9], where the analogous result on $(0, \infty)$ was analyzed, to the symmetric cone setting.

For any $s > 0$ and $(\mathbf{x}, \mathbf{y}) \in \Omega^2$, we get

$$a(s\mathbf{x}) + b(s\mathbf{y}) = c(s(\mathbf{x} + \mathbf{y})) + d(g(s\mathbf{x} + s\mathbf{y})s\mathbf{x}). \tag{17}$$

Since w is homogeneous of degree 1, we have $g(s\mathbf{x}) = \frac{1}{s}g(\mathbf{x})$ and so $g(s\mathbf{x} + s\mathbf{y})s\mathbf{x} = g(\mathbf{x} + \mathbf{y})\mathbf{x}$ for any $s > 0$. Subtracting now (14) from (17) for any $s > 0$, we arrive at the additive Pexider equation on symmetric cone Ω ,

$$a_s(\mathbf{x}) + b_s(\mathbf{y}) = c_s(\mathbf{x} + \mathbf{y}), \quad (\mathbf{x}, \mathbf{y}) \in \Omega^2,$$

where a_s, b_s and c_s are functions defined by $a_s(\mathbf{x}) := a(s\mathbf{x}) - a(\mathbf{x})$, $b_s(\mathbf{x}) := b(s\mathbf{x}) - b(\mathbf{x})$ and $c_s(\mathbf{x}) := c(s\mathbf{x}) - c(\mathbf{x})$.

Due to continuity of a, b and c and Lemma 3.7, it follows that for any $s > 0$, there exist constants $\lambda(s) \in \mathbb{E}$, $\alpha(s) \in \mathbb{R}$ and $\beta(s) \in \mathbb{R}$ such that for any $\mathbf{x} \in \Omega$,

$$\begin{aligned} a_s(\mathbf{x}) &= \langle \lambda(s), \mathbf{x} \rangle + \alpha(s), \\ b_s(\mathbf{x}) &= \langle \lambda(s), \mathbf{x} \rangle + \beta(s), \\ c_s(\mathbf{x}) &= \langle \lambda(s), \mathbf{x} \rangle + \alpha(s) + \beta(s). \end{aligned}$$

By the definition of a_s and the above observation, it follows that for any $(s, t) \in (0, \infty)^2$ and $\mathbf{z} \in \Omega$

$$a_{st}(\mathbf{z}) = a_t(s\mathbf{z}) + a_s(\mathbf{z}).$$

Hence,

$$\langle \lambda(st), \mathbf{z} \rangle + \alpha(st) = \langle \lambda(t), s\mathbf{z} \rangle + \alpha(t) + \langle \lambda(s), \mathbf{z} \rangle + \alpha(s). \tag{18}$$

Since (18) holds for any $\mathbf{z} \in \Omega$, we see that $\alpha(st) = \alpha(s) + \alpha(t)$ for all $(s, t) \in (0, \infty)^2$. That is $\alpha(s) = k_1 \log s$ for $s \in (0, \infty)$, where k_1 is a real constant.

On the other hand

$$\langle \lambda(st), \mathbf{z} \rangle = \langle \lambda(s), \mathbf{z} \rangle + \langle \lambda(t), s\mathbf{z} \rangle = \langle \lambda(t), \mathbf{z} \rangle + \langle \lambda(s), t\mathbf{z} \rangle \tag{19}$$

since one can interchange s and t on the left-hand side. Putting $s = 2$ and denoting $\Lambda = \lambda(2)$, we obtain

$$\langle \lambda(t), \mathbf{z} \rangle = \langle \Lambda, \mathbf{z} \rangle (t - 1)$$

for $t > 0$ and $\mathbf{z} \in \Omega$. It then follows that for all $s \in (0, \infty)$ and $\mathbf{z} \in \Omega$,

$$a_s(\mathbf{z}) = a(s\mathbf{z}) - a(\mathbf{z}) = \langle \Lambda, \mathbf{z} \rangle (s - 1) + k_1 \log s. \tag{20}$$

Let us define function \bar{a} by formula

$$\bar{a}(\mathbf{x}) = a(\mathbf{x}) - \langle \Lambda, \mathbf{x} \rangle.$$

From (20), we get

$$\bar{a}(s\mathbf{x}) = \bar{a}(\mathbf{x}) + k_1 \log s \tag{21}$$

for $s > 0$ and $\mathbf{x} \in \Omega$.

Analogous considerations for function b_s gives existence of constant k_2 such that $\bar{b}(s\mathbf{x}) = \bar{b}(\mathbf{x}) + k_2 \log s$, where

$$\bar{b}(\mathbf{x}) = b(\mathbf{x}) - \langle \Lambda, \mathbf{x} \rangle,$$

hence $\bar{c}(s\mathbf{x}) = \bar{c}(\mathbf{x}) + (k_1 + k_2) \log s$ and

$$\bar{c}(\mathbf{x}) = c(\mathbf{x}) - \langle \Lambda, \mathbf{x} \rangle$$

for any $s > 0$ and $\mathbf{x} \in \Omega$.

Functions $\bar{a}, \bar{b}, \bar{c}$ and d satisfy original Olkin–Baker functional equation:

$$\bar{a}(\mathbf{x}) + \bar{b}(\mathbf{y}) = \bar{c}(\mathbf{x} + \mathbf{y}) + d(g(\mathbf{x} + \mathbf{y})\mathbf{x}), \quad (\mathbf{x}, \mathbf{y}) \in \Omega^2. \tag{22}$$

Taking $\mathbf{x} = \mathbf{y} = \mathbf{v} \in \Omega$ in (22), we arrive at

$$\bar{a}(\mathbf{v}) + \bar{b}(\mathbf{v}) = \bar{c}(2\mathbf{v}) + d(\frac{1}{2}\mathbf{e}) = \bar{c}(\mathbf{v}) + (k_1 + k_2) \log 2 + d(\frac{1}{2}\mathbf{e}). \tag{23}$$

Insert $\mathbf{x} = \alpha w(\mathbf{v})\mathbf{u}$ and $\mathbf{y} = w(\mathbf{v})(\mathbf{e} - \alpha\mathbf{u})$ into (22) for $0 < \alpha < 1$ and $(\mathbf{u}, \mathbf{v}) \in (\mathcal{D}, \Omega)$. Using (21), we obtain

$$\bar{a}(w(\mathbf{v})\mathbf{u}) + \bar{b}(w(\mathbf{v})(\mathbf{e} - \alpha\mathbf{u})) = \bar{c}(\mathbf{v}) + d(\alpha\mathbf{u}) - k_1 \log \alpha, \quad (\mathbf{u}, \mathbf{v}) \in (\mathcal{D}, \Omega).$$

Let us observe, that due to continuity of \bar{b} on Ω and $\lim_{\alpha \rightarrow 0} \{w(\mathbf{v})(\mathbf{e} - \alpha\mathbf{u})\} = w(\mathbf{v})\mathbf{e} = \mathbf{v} \in \Omega$ (convergence in the norm generated by scalar product $\langle \cdot, \cdot \rangle$), limit as $\alpha \rightarrow 0$ of the left-hand side of the above equality exists. Hence, the limit of the right-hand side also exists and

$$\bar{a}(w(\mathbf{v})\mathbf{u}) + \bar{b}(\mathbf{v}) = \bar{c}(\mathbf{v}) + \lim_{\alpha \rightarrow 0} \{d(\alpha\mathbf{u}) - k_1 \log \alpha\}, \quad (\mathbf{u}, \mathbf{v}) \in (\mathcal{D}, \Omega). \tag{24}$$

Subtracting (24) from (23), we have

$$\bar{a}(w(\mathbf{v})\mathbf{u}) = \bar{a}(\mathbf{v}) + g(\mathbf{u}) \tag{25}$$

for $\mathbf{u} \in \mathcal{D}, \mathbf{v} \in \Omega$, where $g(\mathbf{u}) = \lim_{\alpha \rightarrow 0} \{d(\alpha\mathbf{u}) - k_1 \log \alpha\} - (k_1 + k_2) \log 2 - d(\frac{1}{2}\mathbf{e})$. Due to the property (21), equation (25) holds for any $\mathbf{u} \in \Omega$, so we arrive at the

w -logarithmic Pexider equation. Lemma 3.5 implies that there exists w -logarithmic function e such that

$$\bar{a}(\mathbf{x}) = e(\mathbf{x}) + C_1$$

for any $\mathbf{x} \in \Omega$ and a constant $C_1 \in \mathbb{R}$. Function e is continuous, because \bar{a} is continuous. Coming back to the definition of \bar{a} , we obtain

$$a(\mathbf{x}) = \langle \Lambda, \mathbf{x} \rangle + e(\mathbf{x}) + C_1, \quad \mathbf{x} \in \Omega.$$

Analogously for function b , considering equation (22) for $\mathbf{x} = w(\mathbf{v})(\mathbf{e} - \alpha\mathbf{u})$ and $\mathbf{y} = \alpha w(\mathbf{v})\mathbf{u}$ after passing to the limit as $\alpha \rightarrow 0$, we show that there exists continuous w -logarithmic function f such that

$$b(\mathbf{x}) = \langle \Lambda, \mathbf{x} \rangle + f(\mathbf{x}) + C_2, \quad \mathbf{x} \in \Omega$$

for a constant $C_2 \in \mathbb{R}$. The form of c follows from (23). Taking $\mathbf{x} = w(\mathbf{e})\mathbf{u}$ and $\mathbf{y} = \mathbf{e} - w(\mathbf{e})\mathbf{u}$ in (22) for $\mathbf{u} \in \mathcal{D}$, we obtain the form of d . □

4 The Lukacs–Olkin–Rubin Theorem Without Invariance of The Quotient

In the following section, we prove the density version of Lukacs–Olkin–Rubin theorem for any multiplication algorithm w satisfying

- (i) $w(s\mathbf{x}) = sw(\mathbf{x})$ for $s > 0$ and $\mathbf{x} \in \Omega$,
- (ii) differentiability of mapping $\Omega \ni \mathbf{x} \mapsto w(\mathbf{x}) \in G$.

We assume (ii) to ensure that Jacobian of the considered transformation exists. We start with the direct result, showing that the considered measures have desired property. The converse result is given in Theorem 4.2. For every generalized multiplication w , the family of these w -Wishart measures (as defined in (26)) contains the Wishart laws. For $w = w_1$, there are no other distributions, while the w_2 -Wishart measures consist of the Riesz distributions. It is an open question whether there is a generalized multiplication w that leads to other probability measures in this family.

Theorem 4.1 *Let w be a multiplication algorithm satisfying condition (ii) and define $g = w^{-1}$. Suppose that X and Y are independent random variables with densities given by*

$$\begin{aligned} f_X(\mathbf{x}) &= C_X e(\mathbf{x}) \exp \langle \Lambda, \mathbf{x} \rangle I_\Omega(\mathbf{x}), \\ f_Y(\mathbf{x}) &= C_Y f(\mathbf{x}) \exp \langle \Lambda, \mathbf{x} \rangle I_\Omega(\mathbf{x}), \end{aligned} \tag{26}$$

where e and f are w -multiplicative functions, $\Lambda \in \mathbb{E}$ and \mathbb{E} is the Euclidean Jordan algebra associated with the irreducible symmetric cone Ω .

Then, vector $(U, V) = (g(X + Y)X, X + Y)$ have independent components.

Note that if $w(\mathbf{x}) = w_1(\mathbf{x}) = \mathbb{P}(\mathbf{x}^{1/2})$, then there exist positive constants κ_X and κ_Y such that $e(\mathbf{x}) = (\det \mathbf{x})^{\kappa_X - \dim \Omega / r}$ and $f(\mathbf{x}) = (\det \mathbf{x})^{\kappa_Y - \dim \Omega / r}$. In this case $-\Lambda =: \mathbf{a} \in \Omega$ and $(X, Y) \sim \gamma_{\kappa_X, \mathbf{a}} \otimes \gamma_{\kappa_Y, \mathbf{a}}$. Similarly, if $w(\mathbf{x}) = w_2(\mathbf{x}) = t_{\mathbf{x}}$, X and Y follow Riesz distributions with the same scale parameter $-\Lambda \in \Omega$. In general, we do not know whether $\mathbf{a} = -\Lambda$ should always belong to Ω .

Proof Let $\psi : \Omega \times \Omega \rightarrow \mathcal{D} \times \Omega$ be a mapping defined through

$$\psi(\mathbf{x}, \mathbf{y}) = (g(\mathbf{x} + \mathbf{y})\mathbf{x}, \mathbf{x} + \mathbf{y}) = (\mathbf{u}, \mathbf{v}).$$

Then, $(U, V) = \psi(X, Y)$. The inverse mapping $\psi^{-1} : \mathcal{D} \times \Omega \rightarrow \Omega \times \Omega$ is given by

$$(\mathbf{x}, \mathbf{y}) = \psi^{-1}(\mathbf{u}, \mathbf{v}) = (w(\mathbf{v})\mathbf{u}, w(\mathbf{v})(\mathbf{e} - \mathbf{u})),$$

hence ψ is a bijection. We are looking for the Jacobian of the map ψ^{-1} , that is, the determinant of the linear map

$$\begin{pmatrix} d\mathbf{u} \\ d\mathbf{v} \end{pmatrix} \mapsto \begin{pmatrix} d\mathbf{x} \\ d\mathbf{y} \end{pmatrix} = \begin{pmatrix} d\mathbf{x}/d\mathbf{u} & d\mathbf{x}/d\mathbf{v} \\ d\mathbf{y}/d\mathbf{u} & d\mathbf{y}/d\mathbf{v} \end{pmatrix} \begin{pmatrix} d\mathbf{u} \\ d\mathbf{v} \end{pmatrix}.$$

We have

$$J = \begin{vmatrix} w(\mathbf{v}) & d\mathbf{x}/d\mathbf{v} \\ -w(\mathbf{v}) & Id_{\Omega} - d\mathbf{x}/d\mathbf{v} \end{vmatrix} = \begin{vmatrix} w(\mathbf{v}) & d\mathbf{x}/d\mathbf{v} \\ 0 & Id_{\Omega} \end{vmatrix} = \text{Det}(w(\mathbf{v})).$$

where Det denotes the determinant in the space of endomorphisms on Ω . By (12), we get

$$\text{Det}(w(\mathbf{v})) = (\det \mathbf{v})^{\dim \Omega / r}.$$

Now, we can find the joint density of (U, V) . Since (X, Y) have independent components, we obtain

$$f_{(U, V)}(\mathbf{u}, \mathbf{v}) = (\det \mathbf{v})^{\dim \Omega / r} f_X(w(\mathbf{v})\mathbf{u}) f_Y(w(\mathbf{v})(\mathbf{e} - \mathbf{u})) \tag{27}$$

We assumed (26), thus there exist $\Lambda \in \mathbb{E}$, $C_X, C_Y \in \mathbb{R}$ and w -multiplicative functions e, f such that

$$\begin{aligned} f_{(U, V)}(\mathbf{u}, \mathbf{v}) &= (\det \mathbf{v})^{\dim \Omega / r} f_X(w(\mathbf{v})\mathbf{u}) f_Y(w(\mathbf{v})(\mathbf{e} - \mathbf{u})) \\ &= C_1 C_2 (\det \mathbf{v})^{\dim \Omega / r} e(w(\mathbf{v})\mathbf{u}) f(w(\mathbf{v})(\mathbf{e} - \mathbf{u})) \\ &\quad e^{(\Lambda, \mathbf{v})} I_{\Omega}(w(\mathbf{v})\mathbf{u}) I_{\Omega}(w(\mathbf{v})(\mathbf{e} - \mathbf{u})) \\ &= C_1 C_2 (\det \mathbf{v})^{\dim \Omega / r} e(\mathbf{v}) f(\mathbf{v}) e^{(\Lambda, \mathbf{v})} I_{\Omega}(\mathbf{v}) \\ &\quad e(w(\mathbf{e})\mathbf{u}) f(w(\mathbf{e})(\mathbf{e} - \mathbf{u})) I_{\mathcal{D}}(\mathbf{u}), \\ &= f_U(\mathbf{u}) f_V(\mathbf{v}), \end{aligned}$$

what completes the proof. □

To prove the characterization of given measures, we need to show that the inverse implication is also valid. The following theorem generalizes results obtained in Bobecka and Wesolowski, Hassairi et al. and Kołodziejek [2, 11, 13]. We consider quotient U for any multiplication algorithm w satisfying conditions (i) and (ii) given at the beginning of this section (note that multiplication algorithms w_1 and w_2 defined in (10) and (11), respectively, satisfy both of these conditions). Respective densities are then expressed in terms of w -multiplicative Cauchy functions.

Theorem 4.2 (The Lukacs–Olkin–Rubin theorem with densities on symmetric cones) *Let X and Y be independent rv’s valued in irreducible symmetric cone Ω with strictly positive and continuous densities. Set $V = X + Y$ and $U = g(X + Y)X$ for any multiplication algorithm $w = g^{-1}$ satisfying conditions (i) and (ii). If U and V are independent, then there exist $\Lambda \in \mathbb{E}$ and w -multiplicative functions e, f such that (26) holds.*

In particular,

- (1) *if $g(\mathbf{x}) = g_1(\mathbf{x}) = \mathbb{P}(\mathbf{x}^{-1/2})$, then there exist constants $p_i > \dim \Omega / r - 1$, $i = 1, 2$, and $\mathbf{a} \in \Omega$ such that $X \sim \gamma_{p_1, \mathbf{a}}$ and $Y \sim \gamma_{p_2, \mathbf{a}}$,*
- (2) *if $g(\mathbf{x}) = g_2(\mathbf{x}) = t_{\mathbf{x}}^{-1}$, then there exist constants $s_i = (s_{i,j})_{j=1}^r$, $s_{i,j} > (j-1)d/2$, $i = 1, 2$, and $\mathbf{a} \in \Omega$ such that $X \sim R_{s_1, \mathbf{a}}$ and $Y \sim R_{s_2, \mathbf{a}}$.*

Proof We start from (27). Since (U, V) is assumed to have independent components, the following identity holds almost everywhere with respect to Lebesgue measure:

$$(\det(\mathbf{x} + \mathbf{y}))^{\dim \Omega / r} f_X(\mathbf{x}) f_Y(\mathbf{y}) = f_U(g(\mathbf{x} + \mathbf{y})\mathbf{x}) f_V(\mathbf{x} + \mathbf{y}), \tag{28}$$

where f_X, f_Y, f_U and f_V denote densities of X, Y, U and V , respectively.

Since the respective densities are assumed to be continuous, the above equation holds for every $\mathbf{x}, \mathbf{y} \in \Omega$. Taking logarithms of both sides of the above equation (it is permitted since $f_X, f_Y > 0$ on Ω), we get

$$a(\mathbf{x}) + b(\mathbf{y}) = c(\mathbf{x} + \mathbf{y}) + d(g(\mathbf{x} + \mathbf{y})\mathbf{x}), \tag{29}$$

where

$$\begin{aligned} a(\mathbf{x}) &= \log f_X(\mathbf{x}), \\ b(\mathbf{x}) &= \log f_Y(\mathbf{x}), \\ c(\mathbf{x}) &= \log f_V(\mathbf{x}) - \frac{\dim \Omega}{r} \log \det(\mathbf{x}), \\ d(\mathbf{u}) &= \log f_U(\mathbf{u}), \end{aligned}$$

for $\mathbf{x} \in \Omega$ and $\mathbf{u} \in \mathcal{D}$.

The first part of the conclusion follows now directly from Theorem 3.6. Thus, there exist constants $\Lambda \in \mathbb{E}$, $C_i \in \mathbb{R}$, $i \in \{1, 2\}$ and w -logarithmic functions e and f such that

$$\begin{aligned} f_X(\mathbf{x}) &= e^{a(\mathbf{x})} = e^{C_1} e(\mathbf{x}) e^{\langle \Lambda, \mathbf{x} \rangle}, \\ f_Y(\mathbf{x}) &= e^{b(\mathbf{x})} = e^{C_2} f(\mathbf{x}) e^{\langle \Lambda, \mathbf{x} \rangle}, \end{aligned}$$

for any $\mathbf{x} \in \Omega$.

Let us observe that if $w(\mathbf{x}) = w_1(\mathbf{x}) = \mathbb{P}(\mathbf{x}^{1/2})$, then for Theorem 3.1, there exist constants $\kappa_i, i = 1, 2$, such that $e(\mathbf{x}) = (\det \mathbf{x})^{\kappa_1}$ and $f(\mathbf{x}) = (\det \mathbf{x})^{\kappa_2}$. Since f_X and f_Y are densities, it follows that $\mathbf{a} = -\Lambda \in \Omega, k_i = p_i - (\dim \Omega)/r > -1$ and $e^{C_i} = (\det(\mathbf{a}))^{p_i} / \Gamma_{\Omega}(p_i), i = 1, 2$.

Analogously, if $w(\mathbf{x}) = w_2(\mathbf{x}) = t_{\mathbf{x}}$, then Theorem 3.2 and Remark 3.3 imply that there exist constants $s_i = (s_{i,j})_{j=1}^r, s_{i,j} > (j - 1)d/2, i = 1, 2$, and $\mathbf{a} = -\Lambda \in \Omega$ such that $X \sim R_{s_1, \mathbf{a}}$ i $Y \sim R_{s_2, \mathbf{a}}$. □

Remark 4.3 Fix $k \in K$ and consider $w^{(k)}(\mathbf{x}) = w(\mathbf{x})k$. The $w^{(k)}$ -multiplicative function f satisfies equation

$$f(\mathbf{x})f(w(\mathbf{e})k\mathbf{y}) = f(w(\mathbf{x})k\mathbf{y}).$$

Substituting $\mathbf{y} \mapsto k^{-1}\mathbf{y} \in \Omega$, we obtain

$$f(\mathbf{x})f(w(\mathbf{e})\mathbf{y}) = f(w(\mathbf{x})\mathbf{y}),$$

that is $w^{(k)}$ -multiplicative functions are the same as w -multiplicative functions. This leads to the rather unsurprising observation that if we consider Theorem 4.2 with $w(\mathbf{x}) = \mathbb{P}(\mathbf{x}^{1/2})k$ or $w(\mathbf{x}) = t_{\mathbf{x}}k$, regardless of $k \in K$, we will characterize the same distributions as in points (1) and (2) of Theorem 4.2.

With Theorem 4.2, one can easily reprove original Lukacs–Olkin–Rubin theorem (version of Olkin and Rubin [22] and Casalis and Letac [7]), when the distribution of U is invariant under a group of automorphisms:

Remark 4.4 Let us additionally assume in Theorem 4.2, that the quotient U has distribution which is invariant under a group of automorphisms, that is $kU \stackrel{d}{=} U$ for any $k \in K$. From the proof of Theorem 4.1, it follows that there exist continuous w -multiplicative functions e and f and constant C such that for $\mathbf{u} \in \mathcal{D}$,

$$f_U(\mathbf{u}) = Ce(w(\mathbf{e})\mathbf{u})f(\mathbf{e} - w(\mathbf{e})\mathbf{u}).$$

The distribution of U is invariant under K , thus density f_U is a K -invariant function, that is $f_U(\mathbf{u}) = f_U(k\mathbf{u})$ for any $k \in K$. Note that $w(\mathbf{e}) \in K$, thus

$$e(\mathbf{u})f(\mathbf{e} - \mathbf{u}) = e(k\mathbf{u})f(\mathbf{e} - k\mathbf{u}), \quad (k, \mathbf{u}) \in K \times \mathcal{D}. \tag{30}$$

We will show that both functions e and f are K -invariant. Recall that $e(\mathbf{x}) e(w(\mathbf{e})\mathbf{y}) = e(w(\mathbf{x})\mathbf{y})$, therefore after taking $\mathbf{y} = \alpha\mathbf{e}$, we obtain $e(\alpha\mathbf{x}) = e(\mathbf{x})e(\alpha\mathbf{e})$ for any $\alpha > 0$ and $\mathbf{x} \in \Omega$. Inserting $\mathbf{u} = \alpha\mathbf{v}$ into (30), we arrive at

$$e(\mathbf{v})e(\alpha\mathbf{e})f(\mathbf{e} - \alpha\mathbf{v}) = e(\alpha\mathbf{v})f(\mathbf{e} - \alpha\mathbf{v}) = e(\alpha k\mathbf{v})f(\mathbf{e} - \alpha k\mathbf{v}) = e(k\mathbf{v})e(\alpha\mathbf{e})f(\mathbf{e} - k\alpha\mathbf{v}).$$

Thus, $e(\mathbf{v})f(\mathbf{e} - \alpha\mathbf{v}) = e(k\mathbf{v})f(\mathbf{e} - k\alpha\mathbf{v})$ for any $\alpha \in (0, 1], \mathbf{v} \in \mathcal{D}$. Since $f(\mathbf{e}) = 1$ and f is continuous on Ω , by passing to the limit as $\alpha \rightarrow 0$, we get that e is K -invariant

and so is f . By Theorem 3.4 and continuity of e and f , we get that there exist constants κ_1, κ_2 such that $e(\mathbf{x}) = (\det \mathbf{x})^{\kappa_1}$ and $f(\mathbf{x}) = (\det \mathbf{x})^{\kappa_2}$; hence, X and Y have Wishart distributions.

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References

1. Baker, J.A.: On the functional equation $f(x)g(y) = p(x+y)q(x/y)$. *Aequ. Math.* **14**(3), 493–506 (1976)
2. Bobecka, K., Wesolowski, J.: The Lukacs-Olkin-Rubin theorem without invariance of the “quotient”. *Studia Math.* **152**(2), 147–160 (2002)
3. Bobecka, K., Wesolowski, J.: Multiplicative Cauchy functional equation in the cone of positive-definite symmetric matrices. *Ann. Pol. Math.* **82**(1), 1–7 (2003)
4. Boutouria, I.: Characterization of the Wishart distributions on homogeneous cones. *C. R. Math. Acad. Sci. Paris* **341**(1), 43–48 (2005)
5. Boutouria, I.: Characterization of the Wishart distribution on homogeneous cones in the Bobecka and Wesolowski way. *Commun. Stat. Theory Methods* **38**(13–15), 2552–2566 (2009)
6. Boutouria, I., Hassairi, A., Massam, H.: Extension of the Olkin and Rubin characterization to the Wishart distribution on homogeneous cones. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **14**(4), 591–611 (2011)
7. Casalis, M., Letac, G.: The Lukacs-Olkin-Rubin characterization of Wishart distributions on symmetric cones. *Ann. Stat.* **24**(2), 763–786 (1996)
8. Faraut, J., Korányi, A.: *Analysis on Symmetric Cones*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (1994)
9. Ger, R., Misiewicz, J., Wesolowski, J.: The Lukacs theorem and the Olkin-Baker equation. *J. Math. Anal. Appl.* **399**(2), 599–607 (2013)
10. Hassairi, A., Lajmi, S.: Riesz exponential families on symmetric cones. *J. Theor. Probab.* **14**(4), 927–948 (2001). ISSN 0894–9840
11. Hassairi, A., Lajmi, S., Zine, R.: A characterization of the Riesz probability distribution. *J. Theor. Probab.* **21**(4), 773–790 (2008)
12. Kołodziejek, B.: The Wishart distribution on the Lorentz cone. Master’s thesis, Fac. Math. Infor. Sci., Warsaw Univ. Tech., - in Polish (2010)
13. Kołodziejek, B.: The Lukacs-Olkin-Rubin theorem on symmetric cones through Gleason’s theorem. *Studia Math.* **217**, 1–17 (2013)
14. Kołodziejek, B.: Multiplicative Cauchy functional equation on symmetric cones. *Aequ. Math.* (2014). doi:10.1007/s00010-014-0320-4
15. Lajkó, K.: Remark to a paper: “On the functional equation $f(x)g(y) = p(x+y)q(x/y)$ ” by J. A. Baker. *Aequ. Math.* **19**(2–3), 227–231 (1979)
16. Lajkó, K., Mészáros, F.: Multiplicative type functional equations arising from characterization problems. *Aequ. Math.* **83**(3), 199–208 (2012)
17. Lukacs, E.: A characterization of the gamma distribution. *Ann. Math. Stat.* **26**, 319–324 (1955)
18. Mészáros, F.: A functional equation and its application to the characterization of gamma distributions. *Aequ. Math.* **79**(1–2), 53–59 (2010)
19. Molnár, L.: A remark on the Kochen-Specker theorem and some characterizations of the determinant on sets of Hermitian matrices. *Proc. Am. Math. Soc.* **134**(10), 2839–2848 (2006)
20. Olkin, I.: Problem (p128). *Aequ. Math.* **12**, 290–292 (1975)
21. Olkin, I., Rubin, H.: A characterization of the Wishart distribution. *Ann. Math. Stat.* **33**, 1272–1280 (1962)

22. Olkin, I., Rubin, H.: Multivariate beta distributions and independence properties of the Wishart distribution. *Ann. Math. Stat.* **35**, 261–269 (1964)
23. Wesołowski, J.: Multiplicative Cauchy functional equation and the equation of ratios on the Lorentz cone. *Studia Math.* **179**(3), 263–275 (2007)