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# The left tail of renewal measure

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ABSTRACT

In the paper, we find exact asymptotics of the left tail of renewal measure for a broad class of two-sided random walks. We only require that an exponential moment of the left tail is finite. Through a simple change of measure approach, our result turns out to be almost equivalent to Blackwell's Theorem.



## 1. Introduction

Let $(X_k)_{k\geq 1}$ be a sequence of independent copies of a random variable $X$ with $\mathbb{E}X > 0$ (we allow $\mathbb{E}X = \infty$). Further, define $S_n = X_1 + \cdots + X_n$, $n \geq 1$ and $S_0 = 0$. The measure defined by

$$H(B) := \sum_{n=0}^{\infty} \mathbb{P}(S_n \in B), \quad B \in \mathcal{B}(\mathbb{R})$$

is called the *renewal measure of* $(S_n)_{n\geq 1}$.

We say that the distribution of a random variable $X$ is *d-arithmetic* ($d > 0$) if it is concentrated on $d\mathbb{Z}$ and not concentrated on $d'\mathbb{Z}$ for any $d' > d$. A distribution is said to be *non-arithmetic* if it is not $d$-arithmetic for any $d > 0$.

A fundamental result of renewal theory is the Blackwell Theorem (Blackwell, 1953): if the distribution of $X$ is non-arithmetic, then for any $h > 0$,

$$H\left((x, x+h]\right) \longrightarrow \frac{h}{\mathbb{E}X} \quad \text{as } x \to \infty. \tag{1}$$

If the distribution of $X$ is $d$-arithmetic, then for any $h > 0$,

$$H\left((dn, dn+h]\right) \longrightarrow \frac{d\lfloor h/d \rfloor}{\mathbb{E}X} \quad \text{as } n \to \infty. \tag{2}$$

The above results remain true if $\mathbb{E}X = \infty$ with the usual convention that $c/\infty = 0$ for any finite $c$. In the infinite-mean case the exact asymptotics of $H((x, x+h])$ are also known. Assume that $X$ is a non-negative random variable with a non-arithmetic law such that $\mathbb{P}(X > x) = L(x)x^{-\alpha}$ with $\alpha \in (0, 1)$, where $L$ is a slowly varying function. Then $\mathbb{E}X = \infty$. If $\alpha \in (1/2, 1)$, then without additional assumptions the so called Strong Renewal Theorem holds, for $h > 0$,

$$m(x)H\left((x, x+h]\right) \longrightarrow \frac{h}{\Gamma(\alpha)\Gamma(2-\alpha)} \quad \text{as } x \to \infty, \tag{3}$$

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where $m(x) = \int_0^x \mathbb{P}(X > t)\mathrm{d}t \sim L(x)x^{1-\alpha}/(1-\alpha) \to \infty$. Here and later on $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$ as $x \to \infty$.

The case of $\alpha \in (0, 1/2]$ is much harder and was completely solved just recently by Caravenna and Doney (2016). It was shown that if $\alpha \in (0, 1/2]$ and $X$ is a non-negative random variable with regularly varying tail, then (3) holds if and only if (Caravenna and Doney, 2016 Proposition 1.11)

$$\lim_{\delta\to 0}\limsup_{x\to\infty}\int_1^{\delta x}\frac{F(x)-F(x-z)}{\overline{F}(z)z^2}\mathrm{d}z = 0, \tag{4}$$

where $F$ is the cumulative distribution function of $X$ and $\overline{F} = 1 - F$. It was already observed by Kevei (2016, Theorem 3.1) that this result generalizes to $X$ attaining negative values as well if additionally

$$\mathbb{P}(X \le -x) = o(e^{-rx}) \quad \text{as } x \to \infty \tag{5}$$

for some $r > 0$. This will be our setup. Full picture of SRT for random walks is also known (Caravenna and Doney, 2016 Theorem 1.12).

It is clear that $\lim_{x\to\infty} H((-\infty, -x)) = 0$. There are considerably fewer papers dedicated to analysis of exact asymptotics of such object than of $H((x, x+h])$ as in Blackwell's Theorem. Under some additional assumptions we know more about the asymptotic behaviour of the left tail. Stone (1965) proved that if for some $r > 0$ (5) holds, then for some $r_1 > 0$,

$$H((-\infty, -x)) = o(e^{-r_1 x}) \quad \text{as } x \to \infty. \tag{6}$$

Stone's result was strengthened by van der Genugten (1969), where exact asymptotics as well the speed of convergence of the remainder term are given for $d$-arithmetic and spread-out laws (i.e. laws, whose $n$th convolution has a nontrivial absolutely continuous part for some $n \in \mathbb{N}$). An important contribution regarding the asymptotics of the left tail of renewal measure was made by Carlsson (1983), who concerned with the case when $\mathbb{E}|X|^m < \infty$ for some $m \ge 2$, but this does not fit well into our setup. We allow $\mathbb{E}X_+ = \infty$, but on the other hand we require that some exponential moments of $X_-$ exist. The results mentioned above were obtained using some analytical methods, whereas we will use a simple probabilistic argument, which boils down the asymptotics of $H((-\infty, -x))$ to the asymptotics of $\widetilde{H}((x, x+h])$, where $\widetilde{H}$ is some new (possibly defective, see below) renewal measure.

### *1.1. Defective renewal measure*

For $\rho \in (0, 1)$ consider

$$H_\rho(B) := \sum_{n=0}^{\infty} \rho^n \mathbb{P}(S_n \in B), \quad B \in \mathcal{B}(\mathbb{R}),$$

where $(S_n)_{n\ge 1}$ is, as in the previous section, a random walk starting from 0. $H_\rho$ is called a *defective renewal measure of* $(S_n)_{n\ge1}$. In contrast to the renewal measure, $H_\rho$ is a finite measure. Let $\tau$ be independent of $(S_n)_{n\ge1}$ and $\mathbb{P}(\tau = n) = (1-\rho)\rho^n$, $n = 0, 1, \ldots$. Then $H_\rho(B) = \mathbb{P}(S_\tau \in B)/(1-\rho)$. It is well known that if the distribution of $S_1$ is subexponential, then $\mathbb{P}(S_\tau > x) \sim \mathbb{E}\tau\mathbb{P}(S_1 > x)$. Here, we are interested in exact asymptotics of $H_\rho(B)$ when $B = (x, x+T]$ for any $T > 0$. In this context, local subexponentiality is the key concept (Asmussen et al., 2003).

Let $\mu$ be a probability measure on $\mathbb{R}$. For $T > 0$ we write $\Delta = (0, T]$ and $x + \Delta = (x, x+T]$. We say that $\mu$ belongs to the class $\mathcal{L}_\Delta$ if $\mu(x+\Delta) > 0$ for sufficiently large $x$ and

$$\frac{\mu(x+s+\Delta)}{\mu(x+\Delta)} \to 1 \quad \text{as } x \to \infty, \tag{7}$$

uniformly in $s \in [0, 1]$.

We say that $\mu$ is $\Delta$-subexponential if $F \in \mathcal{L}_\Delta$ and

$$\mu^{*2}(x+\Delta) \sim 2\mu(x+\Delta).$$

Then we write $\mu \in \mathcal{S}_\Delta$. Finally, $\mu$ is called *locally subexponential* if $\mu \in \mathcal{S}_\Delta$ for any $T > 0$. We denote this class by $\mathcal{S}_{loc}$.

The following theorem is an obvious conclusion from Watanabe and Yamamuro (2009, Theorem 1.1).

**Theorem 1.1.** *Assume that* $\mu$ *is a probability measure on* $\mathbb{R}$ *such that*

$$\int_{\mathbb{R}} e^{-\varepsilon x}\mu(\mathrm{d}x) < \infty \quad \text{for some } \varepsilon > 0.$$

*For* $0 < \rho < 1$ *define*

$$\eta = \sum_{n=0}^{\infty}\rho^n\mu^{*n}.$$

*Then* $\mu \in \mathcal{S}_\Delta$ *if and only if* $\eta/(1-\rho) \in \mathcal{S}_\Delta$ *if and only if*

$$\eta(x+\Delta) \sim \frac{\rho}{(1-\rho)^2}\mu(x+\Delta).$$

Some examples of measures from $\mathcal{S}_{loc}$ may be found in Asmussen et al. (2003, Section 4).

## 2. Main result

Assume that $X$ is a random variable with $\mathbb{E}X \in (0,\infty]$. We define the Laplace transform of the distribution of $X$ by

$$g(\theta) := \mathbb{E}e^{-\theta X}.$$

Function $g$ is convex and lower-semicontinuous. We are interested in a situation of an exponentially decaying left tail, that is,

$$\mathbb{P}(X<0)>0 \quad \text{and} \quad g(\theta)<\infty \quad \text{for some } \theta>0. \tag{8}$$

Under (8) we define

$$\kappa := \sup\{\theta>0 : g(\theta)<1\} \quad \text{and} \quad \rho := g(\kappa). \tag{9}$$

Since $g'(0) = -\mathbb{E}X < 0$, $\kappa$ is strictly positive. Moreover, we have $g(\theta)\to\infty$ as $\theta\to\infty$ and thus $\kappa$ is finite. In general we have $0<\rho\le 1$ and a sufficient condition for $\rho=1$ is that $g(\theta)<\infty$ for all $\theta>0$.

**Theorem 2.1.** *Assume* $X$ *is a random variable with a positive (possibly infinite) expectation such that* (8) *holds. Let* $H$ *be the renewal measure of* $(S_n)_{n\ge 0}$, *where* $S_n = \sum_{k=1}^n X_k$ *for* $n\in\mathbb{N}$, $S_0=0$ *and* $X_k$ *are independent copies of* $X$. *Define* $\kappa$ *and* $\rho$ *as in* (9).

(a) *Assume that* $\rho = 1$.

(a-i) *Assume that* $X$ *has a non-arithmetic distribution. Then*

$$\lim_{x\to\infty} e^{\kappa x}H((-\infty,-x)) = \frac{1}{\kappa g'(\kappa)} \in [0,\infty).$$

*Moreover, if*

$$\mathbb{E}e^{-\kappa X}\mathbf{1}_{\{-X>t\}} \sim \frac{L(t)}{t^\alpha} \tag{10}$$

*for some* $\alpha\in(0,1)$ *and a slowly varying function* $L$, *then* $g'(\kappa)=\infty$. *For* $\alpha\in(0,1/2]$, *assume additionally that* $F(t) = \mathbb{E}e^{-\kappa X}\mathbf{1}_{\{-X\le t\}}$ *satisfies* (4). *In such case,*

$$e^{\kappa x}H((-\infty,-x)) \sim \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\frac{1}{\kappa m(x)},$$

*where* $m(x)\sim L(x)x^{1-\alpha}/(1-\alpha)$.

(a-ii) *Assume that* $X$ *has a d-arithmetic distribution. Then*

$$\lim_{n\to\infty} e^{\kappa dn}H((-\infty,-nd)) = \frac{d}{(e^{\kappa d}-1)g'(\kappa)}.$$

(b) *If* $\rho<1$, *then*

$$\lim_{x\to\infty} e^{\kappa x}H((-\infty,-x)) = 0. \tag{11}$$

*Moreover, if* $\rho^{-1}\mathbb{E}e^{-\kappa X}\mathbf{1}_{\{-X\in\cdot\}} \in \mathcal{S}_{loc}$, *then*

$$e^{\kappa x}H((-\infty,-x)) \sim \frac{\mathbb{E}e^{-\kappa X}\mathbf{1}_{x<-X\le x+1}}{\kappa(1-\rho)^2}.$$

**Remark 2.2.** Condition (10) is implied by

$$\mathbb{P}(-X>t) = \frac{\alpha}{\kappa}\frac{L(t)}{t^{\alpha+1}}e^{-\kappa t}, \quad t>0.$$

Indeed, for any slowly varying function $L$ and $\beta<-1$, Bingham et al. (1989, Proposition 1.5.10) asserts that

$$\int_x^\infty t^\beta L(t)\mathrm{d}t \sim x^{\beta+1}L(x)/(-\beta-1).$$

**Remark 2.3.** Under the same assumptions, a stronger result concerning (a-ii) is proved in van der Genugten (1969, Theorem 2) (the remainder term is also exponential).

**Remark 2.4.** If $X$ has a non-arithmetic distribution, for any $\delta > 0$, we obtain "more local" behaviour:

$$\lim_{x\to\infty} e^{\kappa x} H((-x-\delta,-x)) = \frac{1-e^{-\delta\kappa}}{\kappa g'(\kappa)}.$$

**Proof of Theorem 2.1.** Note that $g'(\kappa) = -\mathbb{E}Xe^{-\theta X}$ is positive ($1 = g(0) = g(\kappa)$ and $g$ is convex), but may be infinite.

Define $\mathcal{F}_n = \sigma(X_1,\dots,X_n)$ and let $\mathcal{F}_\infty$ be the smallest $\sigma$-field containing all $\mathcal{F}_n$. On $(\Omega,\mathcal{F}_\infty)$ we define a new measure $\mathbb{Q}$ via projections

$$\mathbb{Q}((X_1,\dots,X_n)\in B) = \rho^{-n}\mathbb{E}e^{-\kappa S_n}\mathbf{1}_{\{-(X_1,\dots,X_n)\in B\}}, \quad B\in\mathcal{B}(\mathbb{R}^n),$$

where $S_n = X_1+\cdots+X_n$, $n\in\mathbb{N}$ and $S_0 = 0$. By the definition of $\rho$, $\mathbb{Q}$ is a probability measure. Moreover, $(X_n)_{n\ge 1}$ is an iid sequence under $\mathbb{Q}$ as well. Let $\mathbb{E}_{\mathbb{Q}}$ denote the corresponding expectation. For any Borel function $f:\mathbb{R}\to\mathbb{R}_+$ one has

$$\mathbb{E}f(S_n) = \rho^n\mathbb{E}_{\mathbb{Q}}e^{-\kappa S_n}f(-S_n).$$

Thus,

$$\mathbb{P}(S_n < -x) = \rho^n\mathbb{E}_{\mathbb{Q}}e^{-\kappa S_n}\mathbf{1}_{S_n>x} = \rho^n\int_{(x,\infty)} e^{-\kappa t}\mathbb{Q}_X^{*n}(\mathrm{d}t).$$

Moreover, observe that $\mathbb{E}_{\mathbb{Q}}X = -\mathbb{E}Xe^{-\kappa X} = g'(\kappa)\in(0,\infty]$, thus $(S_n)_n$ has a positive drift under $\mathbb{Q}$ as well. Hence, for $x>0$,

$$H((-\infty,-x)) = H_{\mathbb{P}}((-\infty,-x)) = \sum_{n=1}^{\infty}\mathbb{P}(S_n < -x) = \int_{(x,\infty)} e^{-\kappa t}H_{\mathbb{Q}}(\mathrm{d}t),$$

where $H_{\mathbb{Q}} = \sum_{n=0}^{\infty}\rho^n\mathbb{Q}_X^{*n}$ is the (defective if $\rho<1$) renewal measure of $(S_n)_{n\ge0}$ under $\mathbb{Q}$.

Writing $e^{-\kappa t} = \kappa\int_t^\infty e^{-\kappa s}\mathrm{d}s$, through Tonelli's Theorem, we arrive at key identity:

$$H_{\mathbb{P}}((-\infty,-x)) = \kappa\int_x^\infty e^{-\kappa s}H_{\mathbb{Q}}((x,s])\mathrm{d}s = \kappa e^{-\kappa x}\int_0^\infty e^{-\kappa h}H_{\mathbb{Q}}((x,x+h])\mathrm{d}h. \tag{12}$$

Consider first the case of $\rho = 1$. For any renewal measure $H$ we have $H((x,x+h])\le\alpha h+\beta$ for some $\alpha,\beta>0$ and all $x$, thus by Lebesgue's Dominated Convergence Theorem and (1),

$$\lim_{x\to\infty} e^{\kappa x}H_{\mathbb{P}}((-\infty,-x)) = \kappa\int_0^\infty e^{-\kappa h}\lim_{x\to\infty}H_{\mathbb{Q}}((x,x+h])\mathrm{d}h = \frac{1}{\kappa\mathbb{E}_{\mathbb{Q}}X},$$

which gives the first part of (a-i). For (a-ii) use (2), instead of (1).

For the second part of (a-i), observe that

$$\mathbb{Q}(X>t) = \mathbb{E}e^{-\kappa X}\mathbf{1}_{\{-X>t\}} = \frac{L(t)}{t^\alpha},$$

thus the result follows by the Strong Renewal Theorem.

If $\rho<1$, then $H_{\mathbb{Q}}$ is a finite measure and (11) follows again by Lebesgue's Dominated Convergence Theorem.

Consider now the case, when $\mathbb{Q}_X\in\mathcal{S}_{loc}$. Since $\mathbb{E}_{\mathbb{Q}}e^{-\kappa X} = 1 < \infty$, by Theorem 1.1, we have $(1-\rho)H_{\mathbb{Q}}\in\mathcal{S}_{loc}$ and

$$H_{\mathbb{Q}}((x,x+1]) \sim \frac{\rho}{(1-\rho)^2}\mathbb{Q}_X((x,x+1]). \tag{13}$$

Define $L(y) := H_{\mathbb{Q}}((\log y,\log y+1])$. By (7), $L$ is a slowly varying function. Moreover, for any $h>0$

$$\frac{H_{\mathbb{Q}}((x,x+h])}{H_{\mathbb{Q}}((x,x+1])} \le \sum_{n=1}^{\lceil h\rceil}\frac{H_{\mathbb{Q}}((x+n-1,x+n])}{H_{\mathbb{Q}}((x,x+1])} = \sum_{n=1}^{\lceil h\rceil}\frac{L(e^{x+n-1})}{L(e^x)}.$$

By Potter bounds (Bingham et al., 1989 Theorem 1.5.6) for any $\varepsilon>0$ and $C>1$ there exists $x_0$ such that for $x>x_0$,

$$\frac{H_{\mathbb{Q}}((x,x+h])}{H_{\mathbb{Q}}((x,x+1])} \le \sum_{n=1}^{\lceil h\rceil} Ce^{\varepsilon(n-1)} \le C\lceil h\rceil e^{\varepsilon\lceil h\rceil}.$$

Observe that (7) implies for $h>0$,

$$H_{\mathbb{Q}}((x,x+h]) \sim h\,H_{\mathbb{Q}}((x,x+1]).$$

Indeed, for $h = k/n \in \mathbb{Q}_+$ one gets

$$H_{\mathbb{Q}}\left(\left(x, x+\tfrac{k}{n}\right]\right) = \sum_{i=1}^{k} H_{\mathbb{Q}}\left(\left(x+\tfrac{i-1}{n}, x+\tfrac{i}{n}\right]\right) \sim k\, H_{\mathbb{Q}}\left(\left(x, x+\tfrac{1}{n}\right]\right)$$

and

$$H_{\mathbb{Q}}\left(\left(x, x+\tfrac{1}{n}\right]\right) \sim \frac{1}{n}\sum_{i=1}^{n} H_{\mathbb{Q}}\left(\left(x+\tfrac{i-1}{n}, x+\tfrac{i}{n}\right]\right) = \frac{1}{n} H_{\mathbb{Q}}((x, x+1]).$$

By monotonicity, $f(h) := \lim_{x\to\infty} H_{\mathbb{Q}}((x, x+h])/H_{\mathbb{Q}}((x, x+1])$ exists for all $h > 0$ and $f(h) = h$. Thus, by (12) and Lebesgue's Dominated Convergence Theorem we conclude that

$$\lim_{x\to\infty} e^{\kappa x} \frac{H_{\mathbb{P}}((-\infty, -x))}{H_{\mathbb{Q}}((x, x+1])} = \kappa \int_0^\infty e^{-\kappa h} \lim_{x\to\infty} \frac{H_{\mathbb{Q}}((x, x+h])}{H_{\mathbb{Q}}((x, x+1])} \mathrm{d}h = 1/\kappa.$$

The use of (13) completes the proof. □

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