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ON PERPETUITIES WITH LIGHT TAILS

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Abstract

In this paper we consider the asymptotics of logarithmic tails of a perpetuity $R \stackrel{\mathrm{D}}{=} \sum_{j=1}^{\infty} Q_j \prod_{k=1}^{j-1} M_k$, where $(M_n, Q_n)_{n=1}^{\infty}$ are independent and identically distributed copies of (M, Q), for the case when $\mathbb{P}(M \in [0, 1)) = 1$ and Q has all exponential moments. If M and Q are independent, under regular variation assumptions, we find the precise asymptotics of $-\log \mathbb{P}(R > x)$ as $x \to \infty$. Moreover, we deal with the case of dependent M and Q, and give asymptotic bounds for $-\log \mathbb{P}(R > x)$. It turns out that the dependence structure between M and Q has a significant impact on the asymptotic rate of logarithmic tails of R. Such a phenomenon is not observed in the case of heavy-tailed perpetuities.

Keywords: Perpetuity; dependence structure; regular variation; Tauberian theorem; convex conjugate

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1. Introduction

In the present paper, we consider a random variable R defined as a solution of the affine stochastic equation

$$R \stackrel{\mathrm{D}}{=} MR + Q, \qquad R \text{ and } (M, Q) \text{ independent.} \tag{1.1}$$

Under suitable assumptions (see (1.4) below) on (M, Q), we can think of R as a limit in distribution of the iterative scheme

$$R_n = M_n R_{n-1} + Q_n, \qquad n \geq 1, \tag{1.2}$$

where $(M_n, Q_n)_{n \geq 1}$ are independent and identically distributed (i.i.d.) copies of (M, Q), and R_0 is arbitrary and independent of $(M_n, Q_n)_{n \geq 1}$. Writing out the above recurrence and renumbering the random variables (M_n, Q_n), we see that R may also be defined by

$$R \stackrel{\mathrm{D}}{=} \sum_{j=1}^{\infty} Q_j \prod_{k=1}^{j-1} M_k, \tag{1.3}$$

provided that the series above converges in distribution. For a detailed discussion of sufficient and necessary conditions in the one-dimensional case, we refer the reader to [12] and [31]; here we only note that the conditions

$$\mathbb{E} \log^{+} |Q| < \infty \quad \text{and} \quad \mathbb{E} \log |M| < 0 \tag{1.4}$$

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suffice for the almost-sure convergence of the series in (1.3) and for uniqueness of a solution to (1.1). For a systematic approach to the probabilistic properties of the fixed point equation (1.1) and much more, we recommend two recent books, [3] and [17].

When R is the solution of (1.1), then following a custom from insurance mathematics, we call R a *perpetuity*. In this scheme, let Q represent a random payment and let M be a random discount factor. Then R is the present value of a commitment to pay the value of Q every year in the future; see (1.3). Such a stochastic equation appears in many areas of applied mathematics; for a broad list of references, consult, for example, [8] and [31]. If (R, M, Q) satisfy (1.1), we will say that perpetuity R is *generated* by (M, Q) and that the random vector (M, Q) is the *generator* of R.

For the sake of simplicity, we consider only the case when

$$\mathbb{P}(M \geq 0,\ Q \geq 0) = 1, \tag{1.5}$$

which implies that $\mathbb{P}(R \geq 0) = 1$.

The main focus of research on perpetuities is their tail behaviour. Assume for a moment that $Q = 1$ almost surely (a.s.). Then, for $x \geq 1$, on the set

$$\left\{M_1 > 1 - \frac{1}{x}, \dots, M_{\lfloor x \rfloor} > 1 - \frac{1}{x}\right\},$$

we have

$$R \geq \sum_{k=1}^{\lfloor x \rfloor + 1} M_1 \cdots M_{k-1} \geq \sum_{k=1}^{\lfloor x \rfloor + 1} \left(1 - \frac{1}{x}\right)^{k-1} > (1 - \mathrm{e}^{-1})x,$$

which gives a lower bound for the tails $\mathbb{P}(R > (1 - \mathrm{e}^{-1})x)$ of the form

$$\mathbb{P}\left(M_1 > 1 - \frac{1}{x}, \dots, M_{\lfloor x \rfloor} > 1 - \frac{1}{x}\right) = \mathbb{P}\left(M > 1 - \frac{1}{x}\right)^{\lfloor x \rfloor}.$$

It turns out that such an approach, proposed in [11], gives the appropriate logarithmic asymptotics for constant Q; in [21] (with an earlier contribution in [16]), under some weak assumptions on the distribution of M near $1-$, it was proved that

$$\log \mathbb{P}(R > x) \sim cx \log \mathbb{P}\left(M > 1 - \frac{1}{x}\right) \tag{1.6}$$

for an explicitly given positive constant c. As usual, we write $f(x) \sim g(x)$ if $f(x)/g(x)$ converges to 1 as $x \to \infty$.

The next step in [11] was to consider nonconstant Q. If Q and M are independent, and M has a distribution equivalent at 1 to uniform distribution, that is,

$$-\log \mathbb{P}\left(M > 1 - \frac{1}{x}\right) \sim \log x,$$

then (see [11, Theorem 3.1])

$$\lim_{x \to \infty} \frac{\log \mathbb{P}(R > x)}{x \log \mathbb{P}(M > 1 - 1/x)} = \frac{1}{q_+},$$

where $q_+ = \operatorname{ess\,sup} Q \in (0, \infty]$.

Two natural questions then arise.

1. What is the precise asymptotic if $q_+ = \infty$?
2. What is the asymptotic if M and Q are not independent?

This paper is devoted to answering both these questions in a unified manner. We will be particularly interested in the asymptotic behaviour of $\log \mathbb{P}(R > x)$ as $x \to \infty$, which is closely related to the asymptotic behaviour of $\log M_R(t)$, where M_R is the moment generating function of R. It is known that if $\mathbb{P}(M > 1) > 0$ then R is necessarily heavy tailed. In the present paper we are interested in the case when $\mathbb{P}(M \in [0, 1]) = 1$ and when

$$M_Q(t) = \mathbb{E}e^{tQ} < \infty \quad \text{for all } t \in \mathbb{R}. \tag{1.7}$$

In this case, R is always light tailed; by [1] and [4],

$$M_R(t) = \mathbb{E}e^{tR} \quad \text{is finite on the set } (-\infty, t_0),$$

where $t_0 := \sup\{t \colon \mathbb{E}e^{tQ}\mathbf{1}_{\{M=1\}} < 1\}$, which is positive since $\mathbb{P}(M = 1) < 1$. If t_0 is finite then, by [6, Lemma 5],

$$\liminf_{x\to\infty} \frac{-\log \mathbb{P}(R > x)}{x} = \sup\{t > 0 \colon M_R(t) < \infty\} = t_0, \tag{1.8}$$

which means that this case is completely solved. We have $t_0 = \infty$ if and only if either $\mathbb{P}(M = 1) = 0$ or $\mathbb{P}_{Q \mid M=1} = \delta_0$, but the second case can be reduced to the first case. To see this, assume that $\mathbb{P}(M = 1) > 0$ and $\mathbb{P}(Q = 0 \mid M = 1) = 1$, and define $N = \inf\{n \colon M_n < 1\}$. It is easy to see that N is a stopping time with respect to $\mathcal{F}_n := \sigma((M_k, Q_k) \colon k \le n)$ and $\mathbb{P}(N < \infty) = 1$. Then the distribution of

$$\left(M_1 \cdots M_{N-1}, \sum_{k=1}^{N} M_1 \cdots M_{k-1} Q_k \right)$$

is the same as the conditional distribution of (M, Q) given $\{M < 1\}$. Thus, if $(M', Q') \stackrel{\mathrm{D}}{=} (M, Q) \mid M < 1$, by [31, Lemma 1.2], we have

$$R \stackrel{\mathrm{D}}{=} M'R + Q', \qquad R \text{ and } (M', Q') \text{ independent},$$

and $\mathbb{P}(M' = 1) = 0$. Therefore, to exclude the case of finite t_0, we assume that

$$\mathbb{P}(M \in [0, 1)) = 1. \tag{1.9}$$

Observe that the case when $M \le m_+ < 1$ and $Q \le q_+ < \infty$ a.s. is uninteresting, since then R has no tail (actually, $R \le q_+/(1 - m_+)$ a.s.). We will always exclude this case by assuming that

$$\frac{Q}{1 - M} \quad \text{is not bounded.} \tag{1.10}$$

We note here that the structure of dependence between M and Q does not have a significant impact on the tails of heavy-tailed perpetuities. If

$$\mathbb{P}(r = Mr + Q) < 1, \qquad r \in \mathbb{R}, \tag{1.11}$$

then, in the cases considered in [7], [9], [10], [13], and [19], the rate of asymptotics of $\mathbb{P}(R > x)$ is not influenced by the dependence structure of (M, Q) (with possible exception in the very special unsolved case of [7] if $\mathbb{E}M^\alpha Q^{\alpha-\eta} = \infty$ for all $\eta \in (0, \alpha)$). The problem becomes more complicated if (M, Q) have lighter tails, that is, if the moment generating function of R exists in a neighbourhood of 0 (but not in $\mathbb{R}$), but there is still a relatively high insensitivity to the dependence structure of the tail of R for given marginals (this is because in such cases Q dominates M); see, e.g. [4, Theorem 1.3] and (1.8). If the moment generating function of R is finite over all $\mathbb{R}$, we will see that the dependence structure may have significant impact on the rate of convergence even for logarithmic tails; this can be observed in the following example (see also Example 5.2).

Example 1.1. Consider $(M, Q) = (U, U)$ and $(M', Q') = (U, 1 - U)$, where U is uniformly distributed on [0, 1] (note that (1.10) and (1.11) are not satisfied here). Let R and R' be the perpetuities generated by (M, Q) and (M', Q'), respectively. We have

$$-\log \mathbb{P}(R > x) \sim x \log x,$$

while $\mathbb{P}(R' = 1) = 1$. To see the first result, observe that $\tilde{R} = R + 1$ satisfies

$$\tilde{R} \stackrel{\text{D}}{=} U\tilde{R} + 1, \qquad \tilde{R} \text{ and } U \text{ are independent;}$$

thus, the results of [11] and [21] apply. For this example, the asymptotics of $\mathbb{P}(R > x)$ as $x \to \infty$ are also known [30].

Finally, we would like to mention here [28], where the authors considered generators fulfilling a certain dependence structure which somehow resembles the notion of asymptotic independence from [24]. A similar and significantly weaker, but still restrictive, condition was considered in [4, Equation (5)]. Here we will be able to give bounds for the logarithmic tails even if large values of M exclude large values of Q (and vice versa), which is in opposition to the asymptotic independence.

The paper is organized as follows. In Section 2 we give a short introduction to the theories that will be extensively exploited, that is, regular variation, convex analysis, Tauberian theorems, and concepts of dependence. In Section 3 we find precise asymptotics of the logarithmic tail of R when M and Q are independent, and Q is unbounded (Theorem 3.1) and bounded (Theorem 3.2). Particularly, we assume that

$$x \mapsto -\log \mathbb{P}\left(\frac{1}{1-M} > x\right) \in \mathcal{R}_{r-1}, \qquad r > 1,$$

and

$$x \mapsto -\log \mathbb{P}(Q > x) \in \mathcal{R}_\alpha, \quad \alpha > 1, \quad \text{or} \quad \mathbb{P}(Q \le q_+) = 1,$$

where $\mathcal{R}_\gamma$ denotes the class of regularly varying functions with index γ. Under these assumptions, (1.7), (1.9), and (1.10) are satisfied. We show that

$$-\log \mathbb{P}(R > x) \sim c\, h(x),$$

where the constant $c > 0$ is given explicitly and

$$h(x) := \inf_{t \ge 1}\left\{-t \log \mathbb{P}\left(\frac{1}{1-M} > t,\ Q > \frac{x}{t}\right)\right\}. \tag{1.12}$$

Observe that if $Q = 1$ a.s. then $h(x) = -x \log \mathbb{P}(M > 1 - 1/x)$, so we recover (1.6). Thus, we generalize the results of [11] and [21], but with new proofs, which are very different from those in [11] and [21]. Our proofs are based on a new formulation of the classical Tauberian theorems; see Section 2.4. The appearance of the function h is probably the most interesting phenomenon here. It should be noted that the function h (in the simple form when Q is degenerate) in the two-sided bounds for $\log \mathbb{P}(R > x)$ appeared for the first time in [15].

Section 4 is devoted to explaining some informal heuristics, which show that the function h is a natural candidate for describing the asymptotic of $-\log \mathbb{P}(R > x)$ when M and Q are not independent. In Theorem 4.1 we give basic properties of the function h. In Section 5 we find a lower bound for $\log \mathbb{P}(R > x)$ as $x \to \infty$ also in the case when we allow M and Q to be dependent. In Theorem 5.1, under some regularity assumptions on h, we are able to show that

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h(x)} \geq -c,$$

where the constant c is explicit and depends on properties of the function h. The constant c agrees with the results of Section 3, where independent M and Q are considered. In Section 6 we show that if R is generated by (M, Q) with an arbitrary dependence structure then

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h_{\mathrm{co}}(x)} \leq \lim_{x\to\infty} \frac{\log \mathbb{P}(R_{\mathrm{co}} > x)}{h_{\mathrm{co}}(x)} = -c,$$

where R_{co} is a perpetuity generated by the so-called comonotonic (M, Q) (see Section 2.3) and h_{co} is the corresponding function h. The constant c is given explicitly (see Theorem 6.1). We also give stronger results under additional assumptions that the vector (M, Q) is positively or negatively quadrant dependent (Theorem 6.2). Finally, Section 7 contains proofs of some results from preceding sections.

2. Preliminaries

2.1. Regular variation

In this section we give a brief introduction to the theory of regular variation. For further details, we refer the reader to [2].

A positive measurable function L defined in a neighbourhood of $+\infty$ is said to be *slowly varying* if

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \quad \text{for all } t > 0. \tag{2.1}$$

A positive measurable function f defined in a neighbourhood of $+\infty$ is said to be *regularly varying* with *index* $\rho \in \mathbb{R}$ if $f(x) = x^{\rho} L(x)$ with L slowly varying. We denote the class of regularly varying functions with index ρ by $\mathcal{R}_{\rho}$, so that $\mathcal{R}_0$ is the class of slowly varying functions.

We say that a positive function f *varies smoothly with index* ρ ($f \in \mathcal{SR}_{\rho}$) if $f \in C^{\infty}$ and, for all $n \in \mathbb{N}$,

$$\lim_{x\to\infty} \frac{x^n f^{(n)}(x)}{f(x)} = \rho(\rho - 1)\cdots(\rho - n + 1). \tag{2.2}$$

It is clear that $\mathcal{SR}_{\rho} \subset \mathcal{R}_{\rho}$. Moreover, if $f \in \mathcal{SR}_{\rho}$ then $x^2 f''(x)/f(x) \to \rho(\rho - 1)$; hence, f is ultimately strictly convex if $\rho > 1$; *ultimately* here and later means 'in the vicinity of infinity'. Furthermore, if $f \in \mathcal{SR}_{\rho}$ with $\rho > 0$ then, in the neighbourhood of infinity, f has an inverse in $\mathcal{SR}_{1/\rho}$ (see [2, Theorem 1.8.5]). For any $f \in \mathcal{R}_\rho$, there exist $\underline{f}, \overline{f} \in \mathcal{SR}_\rho$ with $\underline{f}(x) \sim \overline{f}(x)$ and $\underline{f} \leq f \leq \overline{f}$ in the neighbourhood of infinity (the smooth variation theorem [2, Theorem 1.8.2]).

If $f \in \mathcal{SR}_\gamma$ with $\gamma > 0$ then

$$\lim_{x\to\infty} \frac{f(x + uf(x)/f'(x))}{f(x)} = \left(1 + \frac{u}{\gamma}\right)^\gamma. \tag{2.3}$$

This follows by the fact that convergence in (2.1) and (2.2) is locally uniform; see, e.g. [2, Theorem 1.2.1]. In [21, Lemma 2.1] it was shown that, if $f \in \mathcal{R}_\rho$ with $\rho > 0$ and $g(x) \to \infty$ as $x \to \infty$, then the condition

$$\lim_{x\to\infty} \frac{f(x)}{f(g(x))} = L \quad \text{implies that} \quad \lim_{x\to\infty} \frac{x}{g(x)} = L^{1/\rho}.$$

This fact will be used several times.

We say that a measurable function f is *rapidly varying* ($f \in \mathcal{R}_\infty$) if

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = \infty \quad \text{for all } t > 1.$$

It is the subclass of $\mathcal{R}_\infty$ that we are interested in. The class Γ consists of nondecreasing and right-continuous functions f for which there exists a measurable function $g\colon \mathbb{R} \to (0, \infty)$ such that (see [2, Section 3.10])

$$\lim_{x\to\infty} \frac{f(x + ug(x))}{f(x)} = \mathrm{e}^u \quad \text{for all } u \in \mathbb{R}. \tag{2.4}$$

The function g in (2.4) is called an *auxiliary function* and if f has nondecreasing positive derivative, then we may take $g = f/f'$ (compare with (2.3)). It can be shown that if $f \in \Gamma$ and $t > 1$ then $\lim_{x\to\infty} f(tx)/f(x) = \infty$; thus, $\Gamma \subset \mathcal{R}_\infty$.

The class Γ is very rich. If $f_1 \in \mathcal{R}_\rho$, $\rho > 0$, and $f_2 \in \Gamma$, then $f_1 \circ f_2 \in \Gamma$ (see [2, Proposition 3.10.12]). The same holds if $f_1 \in \Gamma$ and $f_2' \in \mathcal{R}_\rho$ with $\rho > -1$ or if $f_1, f_2' \in \Gamma$ (see [2, p. 191]).

Finally, we note that the convergence in (2.3) is uniform on compact subsets of $(-\gamma, \infty)$ and that the convergence in (2.4) is uniform on compact subsets of $\mathbb{R}$ (see [2, Proposition 3.10.2]).

2.2. Convex conjugate

For a function $f\colon (0, \infty) \to \mathbb{R}$, we define its *convex conjugate* (or the Fenchel–Legendre transform) by

$$f^*(x) = \sup\{xz - f(z) \colon z > 0\}. \tag{2.5}$$

It is standard that f^* is convex, nondecreasing, and lower semicontinuous. Moreover, if f is convex and lower semicontinuous then $(f^*)^* = f$ (see [26]). Convex conjugacy is order reversing, that is, if $f \leq g$ then $f^* \geq g^*$.

If f is differentiable and strictly convex then supremum (2.5) is attained at $z = (f')^{-1}(x)$ and, thus, $f^*(x) = x(f')^{-1}(x) - f((f')^{-1}(x))$. Moreover, $f' \circ (f^*)' = (f^*)' \circ f' = \mathrm{Id}$ and so

$$f^*(x) = x(f^*)'(x) - f((f^*)'(x)). \tag{2.6}$$

We will be interested in the relation between f and f^* when f is regularly varying. We say that α and β are *conjugate numbers* if $\alpha, \beta > 1$ and $\alpha^{-1} + \beta^{-1} = 1$. Let L be a slowly varying function. Then (see [2, Theorem 1.8.10, Corollary 1.8.11])

$$f(x) \sim \frac{1}{\alpha} x^\alpha L(x^\alpha)^{1/\beta} \in \mathcal{R}_\alpha$$

if and only if

$$f^*(z) \sim \frac{1}{\beta} z^\beta L^\#(z^\beta)^{1/\alpha} \in \mathcal{R}_\beta,$$

where $L^\#$ is a dual, unique up to asymptotic equivalence, slowly varying function with

$$L(x)L^\#(xL(x)) \to 1, \qquad L^\#(x)L(xL^\#(x)) \to 1, \qquad \text{as } x \to \infty.$$

By the very definition of f^* we obtain Young's inequality:

$$f(s) + f^*(t) \geq st \quad \text{for all } s, t > 0.$$

If f and f^* are invertible then, taking $s = f^{-1}(x)$ and $t = (f^*)^{-1}(x)$ for $x > 0$, we have

$$\frac{(f^*)^{-1}(x) f^{-1}(x)}{x} \leq 2.$$

We will show that the left-hand side above has a limit as $x \to \infty$. If $f \in \mathcal{R}_\rho$ with $\rho > 0$ then there exists a function g such that $f(g(x)) \sim g(f(x)) \sim x$. Such a g is unique up to asymptotic equivalence (see [2, Theorem 1.5.12]) and is called the *asymptotic inverse* of f. If f is locally bounded on $(0, \infty)$ then we can take $g = f^\leftarrow$, where

$$f^\leftarrow(x) = \inf\{y \in (0, \infty) \colon f(y) > x\}.$$

Lemma 2.1. *Let $f \in \mathcal{R}_\alpha$ with $\alpha > 1$, and let β be a conjugate number to α. Then*

$$\frac{f^\leftarrow(x)(f^*)^\leftarrow(x)}{x} \to \alpha(\beta - 1)^{1/\beta} \quad \textit{as } x \to \infty.$$

The proof is postponed to Section 7.

The following theorem will be important. For a formulation in $\mathbb{R}^n$, see [14, Theorem 2.5.1].

Theorem 2.1. *Assume that functions a and b are lower semicontinuous and convex on $(0, \infty)$. If a is additionally nondecreasing then, for $x > 0$, we have*

$$(a \circ b)^*(x) = \inf_{z>0} \left\{ a^*(z) + z b^* \left(\frac{x}{z} \right) \right\}.$$

2.3. Dependence structure of random vectors

A function $f \colon \mathbb{R}^2 \to \mathbb{R}$ is said to be *supermodular* if

$$f(\min\{u, v\}) + f(\max\{u, v\}) \geq f(u) + f(v) \quad \text{for all } u, v \in \mathbb{R}^2,$$

where the minimum and maximum are calculated componentwise. If f has continuous second-order partial derivatives then f is supermodular if and only if $\partial^2 f / \partial x \partial y \geq 0$. An important example of supermodular functions is $f(x_1, x_2) = g(x_1 + x_2)$, when g is convex. We will use this fact in the proof of Lemma 2.2 below.

A random vector (X, Y) is said to be smaller than a random vector (X', Y') in the *supermodular order* if $\mathbb{E}f(X, Y) \le \mathbb{E}f(X', Y')$ for all supermodular functions f for which the expectations exist. The following theorem has many formulations with different assumptions (see, e.g. [23] and [29]), but we will use that given in [5].

Theorem 2.2. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous supermodular function. Let (X, Y) and (X', Y') be random vectors with the same marginal distributions. Assume that*

$$\mathbb{P}(X \le x,\ Y \le y) \le \mathbb{P}(X' \le x,\ Y' \le y) \quad \text{for all } x, y \in \mathbb{R}.$$

If the expectations $\mathbb{E}f(X, y_0)$ and $\mathbb{E}f(x_0, Y)$ are finite for some x_0 and y_0, then

$$\mathbb{E}f(X, Y) \le \mathbb{E}f(X', Y'),$$

provided that the above expectations exist (even if infinite valued).

Assume that X and Y are random variables defined on the same probability space. Let F_X and F_Y denote the cumulative distribution functions (CDFs) of X and Y, respectively. Define $\underline{F}(x, y) = (F_X(x) + F_Y(y) - 1)_+$ and $\overline{F}(x, y) = \min\{F_X(x), F_Y(y)\}$. It is clear that $\underline{F}$ and $\overline{F}$ are two-dimensional CDFs. Moreover, $\underline{F}$ and $\overline{F}$ have the same marginal distributions and, for any F with the same marginals, we have (Fréchet—Hoeffding bounds)

$$\underline{F} \le F \le \overline{F}.$$

If a random variable or vector X has a CDF F, we write $X \overset{\mathrm{D}}{\sim} F$. We say that a vector $(X, Y) \overset{\mathrm{D}}{\sim} F$ is *comonotonic* if $F = \overline{F}$ and that it is *countermonotonic* if $F = \underline{F}$. Thus, Theorem 2.2 implies that comonotonic (countermonotonic) random vectors are maximal (minimal) with respect to the supermodular order. For a CDF F, define

$$F^{-1}(x) = \inf\{y \in \mathbb{R}\colon F(y) \ge x\} \quad \text{for } x \in [0, 1].$$

It is known that if U is uniformly distributed on $[0, 1]$ then

$$(F_X^{-1}(U), F_Y^{-1}(U)) \overset{\mathrm{D}}{\sim} \overline{F}$$

and

$$(F_X^{-1}(U), F_Y^{-1}(1 - U)) \overset{\mathrm{D}}{\sim} \underline{F}.$$

We say that the pair (X, Y) is *positively quadrant dependent* (see [20] and [22]) if

$$\mathbb{P}(X \le x,\ Y \le y) \ge \mathbb{P}(X \le x)\mathbb{P}(Y \le y) \quad \text{for all } x, y \in \mathbb{R}.$$

Similarly, (X, Y) is *negatively quadrant dependent* if the above holds with the inequality sign reversed. We say that a function f is weakly monotonic if it is nondecreasing or nonincreasing.

Lemma 2.2. *Assume that (1.5) holds, and let (M', Q') be a random vector such that*

$$\mathbb{P}(M \le x,\ Q \le y) \le \mathbb{P}(M' \le x,\ Q' \le y) \quad \text{for all } x, y \in \mathbb{R},$$

with $M' \overset{\mathrm{D}}{=} M$ and $Q' \overset{\mathrm{D}}{=} Q$. Let R and R' denote the perpetuities generated by (M, Q) and (M', Q'), respectively. Then

$$\mathbb{E}f(R) \le \mathbb{E}f(R') \quad \text{for all convex and weakly monotonic functions } f \text{ on } \mathbb{R}, \tag{2.7}$$

provided that the above expectations exist (even if infinite valued).

The proof of Lemma 2.2 is postponed to Section 7.

Remark 2.1. Assume additionally that

$$\mathbb{E}M < 1 \quad \text{and} \quad \mathbb{E}Q < \infty.$$

In this case $\mathbb{E}R$ and $\mathbb{E}R'$ are finite, and

$$\mathbb{E}R = \frac{\mathbb{E}Q}{1 - \mathbb{E}M} = \frac{\mathbb{E}Q'}{1 - \mathbb{E}M'} = \mathbb{E}R'. \tag{2.8}$$

For convex and nondecreasing $f_x(r) = (r - x)_+$ with $x > 0$, we have

$$\mathbb{E}f_x(R) = \int_x^\infty \mathbb{P}(R > t)\,\mathrm{d}t,$$

and, thus, (2.7) gives

$$\int_x^\infty (\mathbb{P}(R' > t) - \mathbb{P}(R > t))\,\mathrm{d}t \geq 0 \quad \text{for all } x.$$

But, by (2.8) we obtain

$$\int_0^\infty (\mathbb{P}(R' > t) - \mathbb{P}(R > t))\,\mathrm{d}t = \mathbb{E}(R' - R) = 0,$$

which implies that

$$\int_{-\infty}^x (F_{R'}(t) - F_R(t))\,\mathrm{d}t \geq 0 \quad \text{for all } x,$$

which is equivalent to saying that R is second-order stochastically dominant over R'; see [27].

2.4. Useful Tauberian theorems

The Tauberian theorems presented below are classical, but here we formulate them in a new way. To see that these formulations are equivalent to the classical formulations, see Section 7.

Theorem 2.3. (Kasahara's Tauberian theorem.) *Let X be an a.s. nonnegative random variable such that the moment generating function*

$$M(z) = \mathbb{E}\mathrm{e}^{zX}$$

is finite for all $z > 0$. Let $k \in \mathcal{R}_\rho$ with $\rho > 1$. Then

$$-\log \mathbb{P}(X > x) \sim k(x)$$

if and only if

$$\log M(z) \sim k^*(z).$$

Moreover, the limits of oscillation satisfy

$$B_1 \leq \liminf_{x\to\infty} \frac{-\log \mathbb{P}(X > x)}{k(x)} \leq \limsup_{x\to\infty} \frac{-\log \mathbb{P}(X > x)}{k(x)} \leq B_2$$

for some constants $0 < B_1 < B_2 < \infty$ if and only if

$$\tilde{B}_1 \leq \liminf_{z\to\infty} \frac{\log M(z)}{f^*(z)} \leq \limsup_{z\to\infty} \frac{\log M(z)}{f^*(z)} \leq \tilde{B}_2$$

for some constants $0 < \tilde{B}_1 < \tilde{B}_2 < \infty$ (the above result can be strengthened by specifying the relation between B_i and $\tilde{B}_i$; see [2, Corollary 4.12.8]).

Theorem 2.4. (de Bruijn's Tauberian theorem.) *Let Y be a nonnegative random variable. Let $f \in \mathcal{R}_\rho$ with $\rho > 1$. Then*

$$-x \log \mathbb{P}\left(Y < \frac{1}{x}\right) \sim f(x) \quad \text{as } x \to \infty$$

if and only if

$$-\log \mathbb{E}\mathrm{e}^{-\lambda Y} \sim (f^*)^{\leftarrow}(\lambda) \quad \text{as } \lambda \to \infty.$$

3. Independent generators

In this section we consider M and Q independent under two regimes:

- both $1/(1-M)$ and Q are unbounded (Theorem 3.1),
- $1/(1-M)$ is unbounded, while Q is bounded (Theorem 3.2).

Both of the proofs use the two Tauberian theorems introduced in the previous section.

Theorem 3.1. *Let M and Q be independent, and assume that (1.5) holds. Let*

$$k(x) := -\log \mathbb{P}(Q > x) \quad \text{and} \quad f(x) := -x \log \mathbb{P}\left(M > 1 - \frac{1}{x}\right),$$

and assume that $f \in \mathcal{R}_r$ and $k \in \mathcal{R}_\alpha$ with $r, \alpha > 1$. Let r^* and β denote the conjugate numbers to r and α, respectively. Then $(f^* \circ k^*)^* \in \mathcal{R}_\gamma$ and

$$-\log \mathbb{P}(R > x) \sim \left(\frac{\gamma}{\gamma - 1}\right)^{\gamma - 1} (f^* \circ k^*)^*(x) \tag{3.1}$$

with $\gamma = \beta r^*/(\beta r^* - 1)$.

As will be seen in Remark 4.1 and Theorem 4.1 below, the function $(f^* \circ k^*)^*$ coincides with the function h introduced in (1.12).

Similarly, we can handle the case of bounded Q.

Theorem 3.2. *Let M and Q be independent, and assume that (1.5) holds. Let*

$$q_+ := \operatorname{ess\,sup} Q < \infty \quad \text{and} \quad f(x) := -x \log \mathbb{P}\left(M > 1 - \frac{1}{x}\right),$$

and assume that $f \in \mathcal{R}_r$ with $r > 1$. Then

$$-\log \mathbb{P}(R > x) \sim \left(\frac{r}{r-1}\right)^{r-1} f\left(\frac{x}{q_+}\right).$$

Proof of Theorem 3.1. Since M, Q, and R are independent on the right-hand side of $R \stackrel{\mathrm{D}}{=} MR + Q$, for

$$\psi(z) := \log \mathbb{E}\mathrm{e}^{zR},$$

we have

$$\mathrm{e}^{\psi(z)} = \mathbb{E}\mathrm{e}^{zMR}\mathbb{E}\mathrm{e}^{zQ} = \mathbb{E}\mathrm{e}^{\psi(zM)}\mathbb{E}\mathrm{e}^{zQ} \tag{3.2}$$

upon conditioning on M.

In view of Kasahara's Tauberian theorem, Theorem 2.3, it is enough to show that

$$\psi(z) \sim (\beta r^*)^{-1}(f^* \circ k^*)(z). \tag{3.3}$$

Indeed, observe that in this case

$$-\log \mathbb{P}(R > x) \sim \psi^*(x) \sim \sup_{z>0}\{zx - (\beta r^*)^{-1}(f^* \circ k^*)(z)\} = (\beta r^*)^{-1}(f^* \circ k^*)^*(\beta r^* x). \tag{3.4}$$

Since $f^* \circ k^* \in \mathcal{R}_{\beta r^*}$, (3.1) then follows by the regular variation of $(f^* \circ k^*)^* \in \mathcal{R}_\gamma$.

Moreover, by the Abelian (direct) parts of the Kasahara's and de Bruijn's Tauberian theorems (put $X = Q$ and $Y = 1 - M$) we have

$$\log \mathbb{E}e^{zQ} \sim k^*(z) \in \mathcal{R}_\beta$$

and

$$-\log \mathbb{E}e^{-(1-M)z} \sim (f^*)^{\leftarrow}(z) \in \mathcal{R}_{1/r^*}.$$

Assume for the moment that

$$\log \mathbb{E}e^{zQ} \sim -\log \mathbb{E}e^{-z\psi'(z)(1-M)}. \tag{3.5}$$

Then by the above considerations we obtain

$$k^*(z) \sim (f^*)^{\leftarrow}(z\psi'(z)),$$

or, equivalently (recall the definition of the asymptotic inverse in Section 2.2),

$$(f^* \circ k^*)(z) \sim z\psi'(z).$$

This implies that $\psi' \in \mathcal{R}_{\beta r^*-1}$ and so $z\psi'(z) \sim \beta r^* \psi(z)$, which, together with the above equation, gives (3.3) after applying Kasahara's Tauberian theorem (see (3.4)).

It remains to show that (3.5) holds. By the convexity of ψ we have

$$\mathbb{E}e^{\psi(zM)-\psi(z)} \geq \mathbb{E}e^{-z\psi'(z)(1-M)}. \tag{3.6}$$

Moreover, since R is a.s. nonnegative, ψ is nondecreasing. Thus, for any $m \in (0, 1)$, by the monotonicity and again by the convexity of ψ, we obtain

$$\mathbb{E}e^{\psi(zM)-\psi(z)} \leq \mathbb{E}e^{-z\psi'(zM)(1-M)}\,\mathbf{1}_{\{M>m\}} + e^{\psi(zm)-\psi(z)}\mathbb{P}(M \leq m) =: I_1 + I_2.$$

Since ψ is strictly convex, we have

$$I_1 \leq \mathbb{E}e^{-z\psi'(zm)(1-M)}\,\mathbf{1}_{\{M>m\}} \leq \mathbb{E}e^{-z\psi'(zm)(1-M)} \quad \text{and} \quad I_2 \leq e^{-z\psi'(zm)(1-m)}.$$

But

$$\frac{\mathbb{E}e^{-z\psi'(zm)(1-M)}}{e^{-z\psi'(zm)(1-m)}} = \mathbb{E}e^{-z\psi'(zm)(m-M)} \to \infty \quad \text{as } z \to \infty,$$

since $\mathbb{P}(M > m) > 0$; hence,

$$\mathbb{E}e^{\psi(zM)-\psi(z)} \leq I_1 + I_2 \leq \mathbb{E}e^{-z\psi'(zm)(1-M)}(1 + o(1)) \leq \mathbb{E}e^{-mz\psi'(zm)(1-M)}(1 + o(1)),$$

because $m < 1$. Thus, by (3.2) we obtain

$$\log \mathbb{E}e^{zQ/m} = -\log \mathbb{E}e^{\psi(zM/m)-\psi(z/m)} \geq -\log \mathbb{E}e^{-z\psi'(z)(1-M)} - \log(1+o(1)).$$

Hence, by (3.6) and the above inequality, for any $m \in (0,1)$, we have

$$\log \mathbb{E}e^{zQ} \leq -\log \mathbb{E}e^{-z\psi'(z)(1-M)} \leq \log \mathbb{E}e^{z/mQ} + o(1).$$

By the regular variation of $z \mapsto \log \mathbb{E}e^{zQ}$, we finally conclude that

$$1 \leq \liminf_{z\to\infty} \frac{-\log \mathbb{E}e^{-z\psi'(z)(1-M)}}{\log \mathbb{E}e^{zQ}} \leq \limsup_{z\to\infty} \frac{-\log \mathbb{E}e^{-z\psi'(z)(1-M)}}{\log \mathbb{E}e^{zQ}} \leq m^{-\beta}$$

for any $m \in (0,1)$, which is (3.5). □

Proof of Theorem 3.2. The proof proceeds in the same way as previously, but here we have $z \mapsto \log \mathbb{E}e^{zQ} \in \mathcal{R}_1$ so that $\beta = 1$. Indeed, for any $q \in (0, q_+)$, we have

$$zq_+ \geq \log \mathbb{E}e^{zQ} \geq \log \mathbb{E}e^{zQ}\,\mathbf{1}_{\{Q>q\}} \geq zq + \log \mathbb{P}(Q > q),$$

which means that $\log \mathbb{E}e^{zQ} \sim zq_+$. Let r^* be the conjugate number to r. Similarly as before, we show that

$$zq_+ \sim \log \mathbb{E}e^{zQ} = -\log \mathbb{E}e^{\psi(zM)-\psi(z)} \sim -\log \mathbb{E}e^{-z\psi'(z)(1-M)} \sim (f^*)^{\leftarrow}(z\psi'(z)),$$

so that

$$z\psi'(z) \sim f^*(zq_+) \sim r^*\psi(z)$$

since $f^* \in \mathcal{R}_{r^*}$. Then by Kasahara's Tauberian theorem we conclude that

$$-\log \mathbb{P}(R > x) \sim \psi^*(x) \sim \sup_{z>0}\left\{zx - \frac{1}{r^*} f^*(q_+ z)\right\} = \frac{1}{r^*} f\left(r^* \frac{x}{q_+}\right).$$ □

4. Heuristics and the function *h*

In this section we present some informal heuristics, which show that the function h defined in (1.12) is a natural candidate for explaining the asymptotic of $-\log \mathbb{P}(R > x)$ even if M and Q are not independent. By Kasahara's theorem we know that $x \mapsto -\log \mathbb{P}(R > x)$ is regularly varying with index $\gamma > 1$ if and only if $z \mapsto \psi(z) := \log \mathbb{E}e^{zR}$ is regularly varying with index $\gamma/(\gamma-1)$, where ψ is uniquely determined by the equation

$$\mathbb{E}e^{zQ+\psi(zM)-\psi(z)} = 1.$$

In this case, we expect that in some sense as $z \to \infty$ we have

$$\mathbb{E}e^{zQ-\psi(z)(1-M^{\gamma/(\gamma-1)})} \approx 1,$$

and, from this point, it is not far to considering a function λ defined by the equation

$$\mathbb{E}e^{zQ-\lambda(z)(1-M)} = 1 \quad \text{for } z > 0.$$

It seems reasonable to expect that, for large z and some constants B_i, $i = 1, 2$, we have (this is true if $m_- = \operatorname{ess\,inf} M > 0$)

$$0 < B_1 \le \frac{\psi(z)}{\lambda(z)} \le B_2 < \infty.$$

Assume now that λ is regularly varying. By Kasahara's Tauberian theorem, this would imply that (recall that $-\log \mathbb{P}(R > x) \sim \psi^*(x)$)

$$0 < \tilde{B}_1 \le \liminf_{x\to\infty} \frac{-\log \mathbb{P}(R > x)}{\lambda^*(x)} \le \limsup_{x\to\infty} \frac{-\log \mathbb{P}(R > x)}{\lambda^*(x)} \le \tilde{B}_2 < \infty$$

for some constants $\tilde{B}_i$, $i = 1, 2$. However, the definition of λ does not seem much more appealing than that of ψ, but it is the function λ^* that is of interest. By the definition of λ we have

$$1 = \mathbb{E}\mathrm{e}^{zQ-\lambda(z)(1-M)} \ge \mathbb{E}\mathrm{e}^{zQ-\lambda(z)(1-M)} \mathbf{1}_{\{1-M<1/t\}} \ge \mathbb{E}\mathrm{e}^{zQ} \mathbf{1}_{\{1-M<1/t\}} \mathrm{e}^{-\lambda(z)/t},$$

which gives, for any $t > 0$,

$$\lambda(z) \ge t \log \mathbb{E}\mathrm{e}^{zQ} \mathbf{1}_{\{1-M<1/t\}}. \tag{4.1}$$

Furthermore, by the exponential Markov inequality we have, for $z > 0$,

$$\mathbb{P}\bigg(1 - M < \frac{1}{t},\ Q > \frac{x}{t}\bigg) \le \frac{\mathbb{E}\mathrm{e}^{zQ} \mathbf{1}_{\{1-M<1/t\}}}{\mathrm{e}^{zx/t}},$$

which gives, together with (4.1),

$$-t \log \mathbb{P}\bigg(\frac{1}{1-M} > t,\ Q > \frac{x}{t}\bigg) \ge zx - t \log \mathbb{E}\mathrm{e}^{zQ} \mathbf{1}_{\{M>1-1/t\}} \ge zx - \lambda(z)$$

for any positive x, t, and z. Taking $\inf_{t\ge1}$ and $\sup_{z>0}$ of both sides, we obtain (recall the definition of h in (1.12))

$$h(x) \ge \lambda^*(x) \quad \text{for all } x > 0.$$

In general, we are not able to prove that $h(x) \sim \lambda^*(x)$ (or $\limsup_{x\to\infty} h(x)/\lambda^*(x) < \infty$), but there is strong evidence that such a claim holds for a wide class of distributions of (M, Q). This would eventually imply that $-\log \mathbb{P}(R > x)$ is comparable, up to a constant, with $h(x)$ as $x \to \infty$. Moreover, if M and Q are independent, then Theorems 3.1 and 3.2 give asymptotics for $-\log \mathbb{P}(R > x)$ in terms of h; see below.

Remark 4.1. Every convex conjugate is convex, nondecreasing, and lower semicontinuous. Thus, under the assumptions of Theorem 3.1, by Theorem 2.1 we have

$$(f^* \circ k^*)^*(x) = \inf_{t>0} \bigg\{ f(t) + tk\bigg(\frac{x}{t}\bigg) \bigg\} \sim \inf_{t\ge1} \bigg\{ -t \log \mathbb{P}\bigg(\bigg(M > 1 - \frac{1}{t}\bigg) \mathbb{P}\bigg(Q > \frac{x}{t}\bigg)\bigg) \bigg\},$$

since $f(t) = 0$ for $t \in (0, 1)$. In particular, if $f(x) = cx^r$ and $k(x) = dx^\alpha$ for some $c, d > 0$ and $r, \alpha > 1$, then direct calculation gives

$$(f^* \circ k^*)^*(x) = d \frac{\alpha + r - 1}{r} \bigg(\frac{c}{d} \frac{r}{\alpha - 1} \bigg)^{(\alpha-1)/(\alpha+r-1)} x^{\alpha r/(\alpha+r-1)}.$$

We gather the properties of the function h in the following theorem. Its proof is postponed to Section 7.

Theorem 4.1. *Assume that (1.5) holds, and define*

$$f(x) := -x \log \mathbb{P}\Big(M > 1 - \frac{1}{x}\Big), \qquad k(x) := -\log \mathbb{P}(Q > x).$$

(a) *There exists a function t such that*

$$h(x) = -t(x) \log \mathbb{P}\Big(\frac{1}{1-M} > t(x),\ Q > \frac{x}{t(x)}\Big) + o(1). \tag{4.2}$$

Moreover, if (1.10) holds then

$$t(x) \le \frac{h(x) + o(1)}{-\log \mathbb{P}(Q/(1-M) > x)}. \tag{4.3}$$

(b) *We have*

$$h_{\mathrm{co}} \le h \le h_{\mathrm{counter}},$$

where

$$h_{\mathrm{co}}(x) := \inf_{t \ge 1}\Big\{-t \log \min\Big\{\mathbb{P}\Big(\frac{1}{1-M} > t\Big), \mathbb{P}\Big(Q > \frac{x}{t}\Big)\Big\}\Big\}$$

and

$$h_{\mathrm{counter}}(x) := \inf_{t \ge 1}\Big\{-t \log\Big[\mathbb{P}\Big(\frac{1}{1-M} > t\Big) + \mathbb{P}\Big(Q > \frac{x}{t}\Big) - 1\Big]\Big\}$$

are functions corresponding to comonotonic and countermonotonic vectors (M, Q).

(c) *Let*

$$h_{\mathrm{ind}}(x) := \inf_{t \ge 1}\Big\{f(t) + tk\Big(\frac{x}{t}\Big)\Big\}$$

be the h function corresponding to independent M and Q. Then

$$h_{\mathrm{ind}}(x) \sim (f^* \circ k^*)^*(x).$$

If $f \in \mathcal{R}_r$ and $k \in \mathcal{R}_\alpha$ with $r, \alpha > 1$, then

$$h_{\mathrm{ind}} \in \mathcal{R}_\gamma,$$

where $\gamma = \alpha r/(\alpha + r - 1)$ and $x \mapsto t(x) \in \mathcal{R}_{\alpha/(\alpha+r-1)}$. If $f \in \mathcal{R}_r$ with $r > 0$ and $q_+ = \operatorname{ess\,sup} Q < \infty$, then $k^*(z) \sim zq_+$ and

$$h_{\mathrm{ind}}(x) \sim f\Big(\frac{x}{q_+}\Big).$$

(d) *We have*

$$h_{\mathrm{co}}(x) = \inf_{t \ge 1}\Big\{\max\Big\{f(t), tk\Big(\frac{x}{t}\Big)\Big\}\Big\}.$$

If $f \in \mathcal{R}_r$ and $k \in \mathcal{R}_\alpha$ with $r, \alpha > 1$, then

$$h_{\mathrm{co}}(x) \sim \frac{\alpha - 1}{\alpha + r - 1}\Big(\frac{r}{\alpha - 1}\Big)^{r/(\alpha+r-1)} h_{\mathrm{ind}}(x)$$

and $x \mapsto t(x) \in \mathcal{R}_{\alpha/(\alpha+r-1)}$.

(e) *If $f \in \mathcal{R}_r$ and $k \in \mathcal{R}_\alpha$ with $r, \alpha > 1$ and $q_- = \operatorname{ess\,inf} Q > 0$, then*

$$h_{\text{counter}}(x) \sim \min\left\{ f\left(\frac{x}{q_-}\right), \frac{k((1-m_-)x)}{1-m_-} \right\} \in \mathcal{R}_{\min\{r,\alpha\}}, \tag{4.4}$$

where $m_- = \operatorname{ess\,inf} M$.

Remark 4.2. The function t satisfying (4.2) is not unique, it is not necessarily monotone, nor may have a limit. An easy example may be constructed using Theorem 4.1(e), where $t(x) \in \{t_1(x), t_2(x)\}$ and $t_1(x) \sim x/q_-$ for $t_2(x) \sim (1-m_-)^{-1}$.

Another important example can be constructed as follows. Let $\gamma > 1$. Assume that (M, Q) has an atom $\mathbb{P}(M = 0,\ Q = 1) = 1 - \mathrm{e}^{-1}$ and an absolutely continuous part on $(0, 1) \times (1, \infty)$ given by

$$\mathbb{P}(M > x,\ Q > y) = \exp\left(-\frac{y^\gamma}{(1-x)^{\gamma-1}}\right), \qquad (x, y) \in [0, 1) \times [1, \infty),$$

so that $\mathbb{P}(M > 0,\ Q > 1) = \mathrm{e}^{-1}$. For $x > 1$, we have

$$f(x) = -x \log \mathbb{P}\left(M > 1 - \frac{1}{x}\right) = x^\gamma \quad \text{and} \quad k(x) = -x \log \mathbb{P}(Q > x) = x^\gamma.$$

If M and Q were independent then we would have $h_{\text{ind}} \in \mathcal{R}_{\gamma^2/(2\gamma-1)}$. However, in our case they are not independent and it is easy to see that, for any $x, t \geq 1$,

$$-t \log \mathbb{P}\left(\frac{1}{1-M} > t,\ Q > \frac{x}{t}\right) = \max\{x, t\}^\gamma,$$

so that $h(x) = x^\gamma$ for $x > 1$ and

$$h(x) = -t \log \mathbb{P}\left(\frac{1}{1-M} > t,\ Q > \frac{x}{t}\right)$$

for any $t = t(x) \in [1, x]$.

Remark 4.3. If

$$R = \sum_{k=1}^{\infty} M_1 \cdots M_{k-1} Q_k$$

then

$$R \geq R^{(1)} := \sum_{k=1}^{\infty} m_-^{k-1} Q_k$$

and (assume that $q_- > 0$)

$$R \geq R^{(2)} := \sum_{k=1}^{\infty} M_1 \cdots M_{k-1} q_-.$$

Let f and k be defined as in Theorem 4.1, and assume that $f \in \mathcal{R}_r$ and $k \in \mathcal{R}_\alpha$ with $r, \alpha > 1$. We have

$$\frac{\log \mathbb{E}\mathrm{e}^{zR^{(1)}}}{\log \mathbb{E}\mathrm{e}^{zQ}} = \sum_{k=1}^{\infty} \frac{\log \mathbb{E}\mathrm{e}^{zm_-^{k-1}Q}}{\log \mathbb{E}\mathrm{e}^{zQ}}.$$

Using the regular variation of $\log \mathbb{E}e^{zQ} \sim k^*(z)$ and Potter bounds (see [2, Theorem 1.5.6]), we may take the limit under the sum to obtain

$$\lim_{z\to\infty} \frac{\log \mathbb{E}e^{zR^{(1)}}}{k^*(z)} = \sum_{k=1}^{\infty} m_-^{(k-1)\beta} = \frac{1}{1-m_-^{\beta}}.$$

Thus, by Kasahara's theorem,

$$\log \mathbb{P}(R > x) \geq \log \mathbb{P}(R^{(1)} > x) \sim -\sup_{z>0}\left\{zx - \frac{1}{1-m_-^{\beta}} k^*(z)\right\} = -\frac{k((1-m_-^{\beta})x)}{1-m_-^{\beta}}.$$

On the other hand, by [21] we have

$$\log \mathbb{P}(R > x) \geq \log \mathbb{P}(R^{(2)} > x) \sim -\left(\frac{r}{r-1}\right)^{r-1} f\left(\frac{x}{q_-}\right),$$

which gives, by Theorem 4.1(e),

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h_{\text{counter}}(x)} \geq -C$$

for some $C > 0$. In the next section we give a more accurate lower bound.

5. Lower bound

By Theorem 4.1(a) we know that there exists a function t such that

$$h(x) = -t(x) \log \mathbb{P}\left(\frac{1}{1-M} > t(x),\ Q > \frac{x}{t(x)}\right) + o(1); \tag{5.1}$$

however, the function t is not unique. An eye-opener example was introduced in Remark 4.2, where we had

$$h(x) = -t \log \mathbb{P}\left(\frac{1}{1-M} > t,\ Q > \frac{x}{t}\right) = x^{\gamma} \quad \text{for all } t \in [1, x].$$

Below we present a lower bound for the logarithmic asymptotics of the tail of R. The rate of convergence is described by the regularly varying function h, while the constant depends on the index of h and the limit of a function t. If there is no uniqueness of the function t then the following result holds for any such function provided that it converges to a limit at infinity.

Theorem 5.1. *Assume that (1.5) holds. Assume that the function h defined in (1.12) belongs to $\mathcal{R}_{\gamma}$ with $\gamma \in [1, \infty]$. If $\gamma = \infty$, assume additionally that $h \in \Gamma \subset \mathcal{R}_{\infty}$. Finally, assume that h is such that (5.1) holds for a function t with* $\lim_{x\to\infty} t(x) = t_{\infty} \in (1, \infty]$. *Then*

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h(x)} \geq -c_{t_{\infty},\gamma},$$

where $c_{t,\gamma}$ is a finite positive constant given below; if $t \in (1, \infty)$ and $\gamma \in (1, \infty)$ then

$$c_{t,1} = c_{\infty,1} = 1, \tag{5.2}$$

$$c_{t,\gamma} = \left[t\left\{1 - \left(1 - \frac{1}{t}\right)^{\gamma/(\gamma-1)}\right\}\right]^{\gamma-1}; \tag{5.3}$$

otherwise,

$$c_{\infty,\gamma} = \left(\frac{\gamma}{\gamma-1}\right)^{\gamma-1}, \tag{5.4}$$

$$c_{\infty,\infty} = \mathrm{e}, \qquad c_{t,\infty} = \left(1+\frac{1}{t}\right)^{1+t}.$$

Example 5.1. Let us consider a perpetuity R generated by (M, Q) such that $\mathbb{P}(M = m) = 1$ with $m \in (0, 1)$ and $x \mapsto -\log \mathbb{P}(Q > x) =: k(x) \in \mathcal{R}_\alpha$ with $\alpha > 1$. Then we have $t(x) = t_\infty = 1/(1-m)$ and

$$h(x) = -t_\infty \log \mathbb{P}\left(Q > \frac{x}{t_\infty}\right) \sim t_\infty^{1-\alpha} k(x).$$

On the other hand (by calculations from Remark 4.3),

$$\log \mathbb{P}(R > x) \sim -(1-m^\beta)^{\alpha-1} k(x) \sim -(1-m^\beta)^{\alpha-1} t_\infty^{\alpha-1} h(x)$$

with $\beta = \alpha/(\alpha-1)$. Finally, we see that

$$(1-m^\beta)^{\alpha-1} t_\infty^{\alpha-1} = c_{t_\infty,\gamma},$$

where $\gamma = \alpha$. This means that the constant obtained in (5.3) is optimal.

Proof of Theorem 5.1. Without loss of generality, we may assume that h is differentiable and, if $\gamma > 1$, ultimately convex. For $\gamma \in [1, \infty)$, use the smooth variation theorem; for $\gamma = \infty$, use the arguments given in [21, p. 5].

Case 1: $t_\infty < \infty$ and $\gamma = 1$. Observe that, on the set

$$\bigcap_{k=1}^{n} \{M_k > 1-\delta,\ Q_k > q\},$$

we have

$$R \geq \sum_{k=1}^{n} M_1 \cdots M_{k-1} Q_k > q\,\frac{1-(1-\delta)^n}{\delta},$$

which means that, for any $\delta \in (0, 1)$, $q > 0$, and $n \in \mathbb{N}$, we have

$$\begin{aligned}\log \mathbb{P}\left(R > q\,\frac{1-(1-\delta)^n}{\delta}\right) &\geq \log \mathbb{P}\left(\bigcap_{k=1}^{n} \{M_k > 1-\delta,\ Q_k > q\}\right)\\ &= n \log \mathbb{P}(M > 1-\delta,\ Q > q).\end{aligned} \tag{5.5}$$

For given $x > 0$, set

$$\delta = \delta(x) = \frac{1}{t(x)}, \qquad q = q(x) = \frac{x}{t(x)}, \qquad \text{and} \qquad n = 1,$$

so that

$$\log \mathbb{P}(M > 1-\delta(x),\ Q > q(x)) \sim -\frac{h(x)}{t_\infty}$$

and

$$q\,\frac{1-(1-\delta)^n}{\delta} = \frac{x}{t(x)}.$$

Then (5.5) gives

$$\frac{\log \mathbb{P}(R > x/t(x))}{h(x/t(x))} \geq \frac{\log \mathbb{P}(M > 1-\delta(x),\ Q > q(x))}{h(x)} \frac{h(x)}{h(x/t(x))} \sim -\frac{1}{t_\infty}\frac{1}{1/t_\infty} = -1.$$

We will show that this implies that $\liminf_{x\to\infty} \log \mathbb{P}(R > x)/h(x) \geq -1$. Let x_0 be such that $t(x)/t_\infty \in (1-\varepsilon, 1+\varepsilon)$ for $\varepsilon \in (0,1)$ and all $x > x_0$. Then $x/(t_\infty(1+\varepsilon)) \leq x/t(x) \leq x/(t_\infty(1-\varepsilon))$ and

$$\frac{\log \mathbb{P}(R > x/t(x))}{h(x/t(x))} \leq \frac{\log \mathbb{P}(R > x/t_\infty(1+\varepsilon))}{h(x/t_\infty(1-\varepsilon))} \tag{5.6}$$

for $x > x_0$; thus,

$$\begin{aligned}\liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h(x)} &= \liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x/t_\infty(1+\varepsilon))}{h(x/t_\infty(1+\varepsilon))} \\ &\geq \liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x/t(x))}{h(x/t(x))} \frac{h(x/t_\infty(1-\varepsilon))}{h(x/t_\infty(1+\varepsilon))} \\ &\geq -1\frac{1+\varepsilon}{1-\varepsilon}\end{aligned}$$

by (5.6) and the regular variation of h. Letting $\varepsilon \to 0$ we obtain the first part of (5.2).

Case 2: $t_\infty = \infty$ and $\gamma = 1$. We proceed similarly as in case 1. For arbitrary $\alpha > 0$, set

$$\delta = \frac{1}{t(x)}, \qquad q = \frac{x}{t(x)}, \quad \text{and} \quad n = \lfloor \alpha t(x) \rfloor$$

in (5.5) to obtain, for any $x > 0$,

$$\begin{aligned}&\frac{\log \mathbb{P}(R > x(1-(1-1/t(x))^{\lfloor \alpha t(x)\rfloor}))}{h(x(1-(1-1/t(x))^{\lfloor \alpha t(x)\rfloor}))} \\ &\quad\geq \frac{n \log \mathbb{P}(M > 1-\delta,\ Q > q)}{h(x)} \frac{h(x)}{h(x(1-(1-1/t(x))^{\lfloor \alpha t(x)\rfloor}))}.\end{aligned}$$

Since $t(x) \to \infty$ as $x \to \infty$, by the regular variation of h, we see that the right-hand side converges to

$$-\frac{\alpha}{1-\mathrm{e}^{-\alpha}}.$$

Using a similar approach as in the $t_\infty < \infty$ case, we show that

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h(x)} \geq -\frac{\alpha}{1-\mathrm{e}^{-\alpha}}.$$

Considering the limit as $\alpha \to 0$, we obtain the second part of (5.2).

Case 3: $t_\infty < \infty$ and $\gamma \in (1,\infty)$. For given $n \in \mathbb{N}$, consider sequences $(\delta_k)_{k=1}^n$ and $(q_k)_{k=1}^n$ satisfying

$$x \leq \sum_{k=1}^{n} (1-\delta_1)\cdots(1-\delta_{k-1})q_k. \tag{5.7}$$

Then we have

$$\log \mathbb{P}(R > x) \geq \sum_{k=1}^{n} \log \mathbb{P}(M > 1-\delta_k,\ Q > q_k). \tag{5.8}$$

For $k = 1, \dots, n$, set

$$y_k = u_k x, \qquad \delta_k = \frac{1}{t(y_k)}, \quad \text{and} \quad q_k = \frac{y_k}{t(y_k)},$$

where $u_1, \dots, u_n$ are some positive constants such that (compare with (5.7))

$$1 \le \sum_{k=1}^{n} \pi_k(x) u_k, \tag{5.9}$$

where

$$\pi_k(x) = (1 - \delta_1) \cdots (1 - \delta_{k-1}) \frac{1}{t(y_k)} \to \left(1 - \frac{1}{t_\infty}\right)^{k-1} \frac{1}{t_\infty} \quad \text{as } x \to \infty,$$

since $y_i \to \infty$ for $i = 1, \dots, n$. Considering the limit as $x \to \infty$ in the right-hand side of (5.9) we obtain

$$\frac{1}{t_\infty} \sum_{k=1}^{n} \left(1 - \frac{1}{t_\infty}\right)^{k-1} u_k. \tag{5.10}$$

We will choose $(u_k)_k$ in such a way that the above expression is strictly greater than 1 and this will ensure that (5.9) holds for large x. Let us consider

$$u_k = t_\infty (1 - t_\infty^{-1})^{1-k} A B^{k-1}, \qquad k = 1, \dots, n, \tag{5.11}$$

for positive A and $B \in (0, 1)$. Substituting into (5.10) we get

$$A \frac{1 - B^n}{1 - B}.$$

If, additionally, $A > 1 - B$ then there exists N such that, for all $n \ge N$, the above expression is strictly larger than 1. Thus, (5.9) is established for large x. Moreover, by the definitions of h and the function t, we have, for any $\varepsilon > 0$ and $x > 0$,

$$h(x) \le -t(x) \log \mathbb{P}\left(M > 1 - \frac{1}{t(x)},\ Q > \frac{x}{t(x)}\right) \le (1 + \varepsilon) h(x),$$

and so, by (5.8),

$$\frac{\log \mathbb{P}(R > x)}{h(x)} \ge -(1 + \varepsilon) \sum_{k=1}^{n} \frac{h(u_k x)}{t(y_k) h(x)}. \tag{5.12}$$

Taking $\liminf_x$ of both sides of (5.12), we obtain, for any $n \ge N$,

$$\liminf_{x \to \infty} \frac{\log \mathbb{P}(R > x)}{h(x)} \ge -\frac{1 + \varepsilon}{t_\infty} \sum_{k=1}^{n} u_k^\gamma,$$

and taking the limit as $n \to \infty$ along with the substitution of (5.11) we obtain

$$\liminf_{x \to \infty} \frac{\log \mathbb{P}(R > x)}{h(x)} \ge -(1 + \varepsilon) t_\infty^{\gamma - 1} \frac{A^\gamma}{1 - (B t_\infty / t_\infty - 1)^\gamma}.$$

The above inequality holds for any $A > 1 - B \in (0, 1)$. Let us set $A = 1 - B + \varepsilon$. Then the expression on the right-hand side above attains its supremum for

$$B_\varepsilon = (1 + \varepsilon)^{1/(1-\gamma)} \left(1 - \frac{1}{t_\infty}\right)^{\gamma/(\gamma-1)},$$

and, for such B, this supremum equals

$$-t_\infty^{\gamma-1}(1-B_\varepsilon+\varepsilon)^{\gamma-1}.$$

Letting $\varepsilon\to 0$ we obtain (5.3).

Case 4: $t_\infty=\infty$ and $\gamma\in(1,\infty]$. Let $x_0=0$ and $R_0=Q_0$, and define a random sequence $(R_n)_{n\geq 1}$ and a sequence of scalars $(x_n)_{n\geq 1}$ through

$$R_n=M_nR_{n-1}+Q_n \quad \text{and} \quad x_n=(1-\delta_n)x_{n-1}+q_n, \qquad n\geq 1,$$

where $(M_n,Q_n)_{n\geq 0}$ is an i.i.d. sequence of the generic element (M,Q), and $(\delta_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ are scalar sequences yet to be determined.

Since M and Q are assumed to be a.s. nonnegative and

$$R=\sum_{k=1}^{\infty}Q_k\prod_{j=1}^{k-1}M_j\geq\sum_{k=1}^{n+1}Q_k\prod_{j=1}^{k-1}M_j\stackrel{\mathrm{D}}{=}R_n,$$

we have

$$\mathbb{P}(R>x)\geq\mathbb{P}(R_n>x).$$

Moreover, since (M_n,Q_n) and R_{n-1} are independent, we have

$$\begin{aligned}\mathbb{P}(R_n>x_n)&\geq\mathbb{P}(M_nR_{n-1}+Q_n>(1-\delta_n)x_{n-1}+q_n,\ M_n>1-\delta_n,\ Q_n>q_n)\\&\geq\mathbb{P}(M_n>1-\delta_n,\ Q_n>q_n)\mathbb{P}(R_{n-1}>x_{n-1})\\&\geq\prod_{k=1}^{n}\mathbb{P}(M>1-\delta_k,\ Q>q_k)\mathbb{P}(Q>0)\end{aligned}\tag{5.13}$$

and $\mathbb{P}(Q>0)>0$. If $(x_n)_n$ is strictly increasing and $x_{n-1}<x\leq x_n$, then

$$\frac{\log\mathbb{P}(R>x)}{h(x)}\geq\frac{\log\mathbb{P}(R>x_n)}{h(x_{n-1})},$$

and, therefore, if additionally $(x_n)_n$ is divergent and $h(x_n)/h(x_{n-1})$ has a limit as $n\to\infty$, we have

$$\begin{aligned}\liminf_{x\to\infty}\frac{\log\mathbb{P}(R>x)}{h(x)}&\geq\liminf_{n\to\infty}\frac{\log\mathbb{P}(R>x_n)}{h(x_{n-1})}\\&\stackrel{(5.13)}{\geq}\liminf_{n\to\infty}\frac{h(x_n)}{h(x_{n-1})}\frac{\sum_{k=1}^{n}\log\mathbb{P}(M>1-\delta_k,\ Q>q_k)}{h(x_n)}\\&\geq\lim_{n\to\infty}\frac{h(x_n)}{h(x_{n-1})}\liminf_{n\to\infty}\frac{\sum_{k=1}^{n}\log\mathbb{P}(M>1-\delta_k,\ Q>q_k)}{h(x_n)}\\&\geq\lim_{n\to\infty}\frac{h(x_n)}{h(x_{n-1})}\liminf_{n\to\infty}\frac{\log\mathbb{P}(M>1-\delta_n,\ Q>q_n)}{h(x_n)-h(x_{n-1})},\end{aligned}$$

where the last inequality follows by the Stoltz–Cesàro theorem (recall that $h(x)\to\infty$ as $x\to\infty$).

We now choose the sequences $(\delta_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ in such a way that $x_n\to\infty$ and the above limit is finite and negative. Set

$$I_n:=\frac{\log\mathbb{P}(M>1-\delta_n,\ Q>q_n)}{h(x_n)-h(x_{n-1})},$$

and let

$$\delta_n = \frac{1}{t(y_n)} \quad \text{and} \quad q_n = \frac{y_n}{t(y_n)},$$

where

$$y_n = x_{n-1} + c\frac{h(x_{n-1})}{h'(x_{n-1})}$$

for some positive constant c. Inserting the above into the definition of $(x_n)_{n\geq 1}$ we obtain

$$x_n - x_{n-1} = q_n - \delta_n x_{n-1} = \frac{c}{t(y_n)}\frac{h(x_{n-1})}{h'(x_{n-1})}. \tag{5.14}$$

Since the right-hand side of (5.14) is positive, x_n is strictly increasing. This means that x_n has a limit, possibly infinite. Assume that $p := \lim_n x_n < \infty$. Then $y_n \to p + ch(p)/h'(p) < \infty$ and, by (5.14), we see that

$$0 = \lim_{n\to\infty} \frac{c}{t(y_n)}\frac{h(x_{n-1})}{h'(x_{n-1})} = \frac{ch(p)}{h'(p)} \lim_{n\to\infty} \frac{1}{t(y_n)}.$$

But this is impossible because, for any finite $x > 0, t(x)$ is finite (Theorem 4.1(a)). Thus, $x_n \to \infty$.

Furthermore,

$$I_n = -C_n \frac{h(y_n)}{t(y_n)(h(x_n) - h(x_{n-1}))},$$

where

$$C_n := \frac{-t(y_n)\log \mathbb{P}(M > 1 - 1/t(y_n),\ Q > y_n/t(y_n))}{h(y_n)} \to 1 \quad \text{as } n \to \infty.$$

Using the convexity of h, we obtain

$$I_n \geq -C_n \frac{h(y_n)}{t(y_n)(x_n - x_{n-1})h'(x_{n-1})} = -C_n \frac{h(x_{n-1} + ch(x_{n-1})/h'(x_{n-1}))}{ch(x_{n-1})}.$$

Letting $n \to \infty$, we have (see (2.3) and (2.4))

$$\liminf_{n\to\infty} I_n \geq \begin{cases} -\dfrac{1}{c}\left(\dfrac{c+\gamma}{\gamma}\right)^{\gamma} & \text{if } \gamma \in [1, \infty), \\ -\dfrac{\mathrm{e}^c}{c} & \text{if } \gamma = \infty. \end{cases}$$

If $\gamma \in (1, \infty)$ then the supremum on the right-hand side above is attained at $c = \gamma/(\gamma - 1)$ and this supremum equals $-(\gamma/(\gamma - 1))^{\gamma-1}$. For $\gamma = \infty$, the supremum is attained at $c = 1$ and then equals $-e$. It remains to show that $\lim_{n\to\infty} h(x_n)/h(x_{n-1}) = 1$. We have

$$\frac{h(x_n)}{h(x_{n-1})} = \frac{h(x_{n-1} + ch(x_{n-1})/t(y_n)h'(x_{n-1}))}{h(x_{n-1})} \to 1,$$

since $\lim_{n\to\infty} t(y_n) = \infty$ (the convergence in (2.4) is uniform; see [2, Proposition 3.10.2]).

Case 5: $t_\infty < \infty$ and $\gamma = \infty$. Proceeding in the same way as in case 4, we obtain

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h(x)} \geq \lim_{n\to\infty} \frac{h(x_n)}{h(x_{n-1})} \liminf_{n\to\infty} I_n,$$

where $x_n \to \infty$ and

$$I_n = -\frac{C_n}{t(y_n)} \frac{h(x_{n-1} + ch(x_{n-1})/h'(x_{n-1}))}{h(x_{n-1} + ch(x_{n-1})/t(y_n)h'(x_{n-1})) - h(x_{n-1})},$$

where $C_n \to 1$ as $n \to \infty$. Thus, using (2.4), we obtain

$$\lim_{n\to\infty} I_n = -\frac{\mathrm{e}^c}{t_\infty(\mathrm{e}^{c/t_\infty} - 1)}$$

and

$$\lim_{n\to\infty} \frac{h(x_n)}{h(x_{n-1})} = \mathrm{e}^{c/t_\infty}.$$

Thus,

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h(x)} \geq -\inf_{c>0}\{\mathrm{e}^{c/t_\infty} \frac{\mathrm{e}^c}{t_\infty(\mathrm{e}^{c/t_\infty} - 1)}\} = -\left(1 + \frac{1}{t_\infty}\right)^{t_\infty+1}. \qquad \square$$

Remark 5.1. In the example introduced in Remark 4.2, we have $h \in \mathcal{R}_\gamma$ with $\gamma \in (1, \infty)$ and

$$h(x) \sim -t \log \mathbb{P}\left(\frac{1}{1-M} > t,\ Q > \frac{x}{t}\right)$$

for any $t \in (1, \infty)$, so that Theorem 5.1 gives, for any $t > 1$,

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h(x)} \geq -\left[t\left\{1 - \left(1 - \frac{1}{t}\right)^{\gamma/(\gamma-1)}\right\}\right]^{\gamma-1}.$$

We have

$$\inf_{t>1}\left[t\left\{1 - \left(1 - \frac{1}{t}\right)^{\gamma/(\gamma-1)}\right\}\right]^{\gamma-1} = 1,$$

so that

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h(x)} \geq -1.$$

Below we give an example of two perpetuities with logarithmic tails of different asymptotic order whose generators have the same marginals.

Example 5.2. Let $X = (X(t))_{t\geq 0}$ be a drift-free nonkilled subordinator with Laplace exponent $\Phi(s) = -\log \mathbb{E}\mathrm{e}^{-sX(1)}$, $s \geq 0$, and let T be an exponentially distributed random variable of parameter 1 which is independent of X. The random variable $R := \int_0^\infty \mathrm{e}^{-X(t)}\,\mathrm{d}t$ is a perpetuity generated by

$$(M, Q) := \left(\mathrm{e}^{-X(T)}, \int_0^T \mathrm{e}^{-X(t)}\,\mathrm{d}t\right).$$

A semi-explicit formula for the joint moments of M and Q is given in [18, Equation (2.6)].

Assume now that $\Phi \in \mathcal{R}_\alpha$ with $\alpha \in (0, 1)$. Then it was proved in [25] that

$$-\log \mathbb{P}(R > x) \sim (1 - \alpha)\Psi(x) \quad \text{as } t \to \infty, \tag{5.15}$$

with $\Psi(x) := \inf\{s > 0 \colon s/\Phi(s) > x\}$.

If the Lévy measure of X is of the form

$$\nu(\mathrm{d}t) = \frac{\mathrm{e}^{-t/\alpha}}{(1-\mathrm{e}^{-t/\alpha})^{\alpha+1}} \mathbf{1}_{(0,\infty)}(t)\,\mathrm{d}t$$

then

$$\Phi(s) = \int_{[0,\infty)} (1-\mathrm{e}^{-st})\nu(\mathrm{d}t) = \frac{\Gamma(1-\alpha)\Gamma(1+\alpha s)}{\Gamma(1+\alpha(s-1))} - 1 \sim \alpha^\alpha \Gamma(1-\alpha) s^\alpha,$$

and we can find marginal distributions of (M, Q) (see Example 2.1.2 of [17]). In this special case, Q has a Mittag-Leffler distribution with parameter α and $M^{1/\alpha}$ has a beta distribution with parameters $1-\alpha$ and α. This implies that

$$f(x) := -x \log \mathbb{P}\bigg(M > 1 - \frac{1}{x}\bigg) \sim \alpha x \log x,$$

and (see, e.g. [2, Theorem 8.1.12])

$$k(x) := -\log \mathbb{P}(Q > x) \sim (1-\alpha)\alpha^{\alpha/(1-\alpha)} x^{1/(1-\alpha)},$$

that is, $k \in \mathcal{R}_{1/(1-\alpha)}$ and $f \in \mathcal{R}_1$.

Let us now consider a perpetuity R_{ind} generated by independent M and Q with the same distributions. We can show that the corresponding function $h_{\text{ind}} = (f^* \circ k^*)^*$ belongs to $\mathcal{R}_1$. Thus, by Theorem 5.1 we have

$$-\log \mathbb{P}(R_{\text{ind}} > x) \lesssim C h_{\text{ind}}(x)$$

for some $C > 0$. On the other hand, (5.15) implies that $x \mapsto -\log \mathbb{P}(R > x)$ is regularly varying at ∞ with index $(1-\alpha)^{-1} > 1$ ($\Psi = \rho^{\leftarrow}$, where $\rho(s) = s/\Phi(s) \in \mathcal{R}_{1-\alpha}$). Therefore, we obtain

$$\lim_{x\to\infty} \frac{\log \mathbb{P}(R_{\text{ind}} > x)}{\log \mathbb{P}(R > x)} = 0.$$

6. Upper bound

In this section we give asymptotic upper bounds for $\log \mathbb{P}(R > x)$ when (M, Q) is negatively quadrant dependent (Theorem 6.2) and when (M, Q) is dependent in an arbitrary way (Theorem 6.1, which is the most important result of this section).

Let us assume that

$$k(x) = -\log \mathbb{P}(Q > x) \in \mathcal{R}_\alpha, \qquad f(x) = -x \log \mathbb{P}\bigg(M \geq 1 - \frac{1}{x}\bigg) \in \mathcal{R}_r, \qquad \alpha, r > 1. \tag{6.1}$$

Let r^* and β denote the conjugate numbers to r and α, respectively and denote

$$\gamma = \frac{\beta r^*}{\beta r^* - 1}.$$

Let R_{co} and R_{ind} denote perpetuities generated by comonotonic and independent (M, Q), respectively, and let h_{co} and h_{ind} denote h functions corresponding to these two cases. Recall that in Theorem 4.1 we showed that

$$(f^* \circ k^*)^*(x) \sim h_{\text{ind}}(x) \sim \frac{\alpha + r - 1}{\alpha - 1} \bigg(\frac{\alpha - 1}{r}\bigg)^{r/(\alpha+r-1)} h_{\text{co}}(x). \tag{6.2}$$

Theorem 6.1. *Assume that (1.5) and (6.1) hold. Then*

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h_{\mathrm{co}}(x)} \le -\left(\frac{\gamma}{\gamma-1}\right)^{\gamma-1} \tag{6.3}$$

and

$$\lim_{x\to\infty} \frac{\log \mathbb{P}(R_{\mathrm{co}} > x)}{h_{\mathrm{co}}(x)} = -\left(\frac{\gamma}{\gamma-1}\right)^{\gamma-1}. \tag{6.4}$$

If we additionally assume that (M, Q) is negatively or positively quadrant dependent, then we can prove slightly stronger results.

Theorem 6.2. *Assume that (1.5) and (6.1) hold.*

(i) *If (M, Q) is negatively quadrant dependent then*

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h_{\mathrm{ind}}(x)} \le -\left(\frac{\gamma}{\gamma-1}\right)^{\gamma-1}.$$

(ii) *If (M, Q) is positively quadrant dependent then*

$$\begin{aligned} -\left(\frac{\gamma}{\gamma-1}\right)^{\gamma-1} &\le \liminf_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h_{\mathrm{ind}}(x)} \\ &\le \limsup_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{h_{\mathrm{ind}}(x)} \\ &\le -\frac{\alpha-1}{\alpha+r-1}\left(\frac{r}{\alpha-1}\right)^{r/(\alpha+r-1)}\left(\frac{\gamma}{\gamma-1}\right)^{\gamma-1}. \end{aligned} \tag{6.5}$$

To prove (6.3), we will need the following lemma, whose proof is postponed to Section 7.

Lemma 6.1. *Assume that there exists a function ϕ such that*

$$\mathbf{1}_\phi(z) := \mathbb{E}\mathrm{e}^{zQ+\phi(zM)-\phi(z)} \le 1$$

for large values of z. Then there exists a constant $C > 0$ such that

$$\mathbb{E}\mathrm{e}^{zR} \le \mathrm{e}^{\phi(z)+C}$$

for large enough z.

Proof of Theorem 6.1. Observe that (6.4) follows by (6.3). Indeed, by Theorem 5.1 (see (5.4)) we have

$$\liminf_{x\to\infty} \frac{\log \mathbb{P}(R_{\mathrm{co}} > x)}{h_{\mathrm{co}}(x)} \ge -\left(\frac{\gamma}{\gamma-1}\right)^{\gamma-1}.$$

Since $r \mapsto \exp(zr)$ is convex and monotonic, by Lemma 2.2 we see that

$$\mathbb{E}\exp(zR) \le \mathbb{E}\exp(zR_{\mathrm{co}}), \qquad z \ge 0,$$

where

$$R_{\mathrm{co}} \stackrel{\mathrm{D}}{=} \overline{M} R_{\mathrm{co}} + \overline{Q}, \quad (\overline{M}, \overline{Q}) \text{ and } R_{\mathrm{co}} \text{ are independent,}$$

and $(\overline{M}, \overline{Q}) \overset{\mathrm{D}}{=} (F_M^{-1}(U), F_Q^{-1}(U))$, $U \overset{\mathrm{D}}{\sim} \mathrm{U}([0, 1])$, is a comonotonic vector with given marginals. Let $\overline{\psi}(z) := \log \mathbb{E} \exp(zR_{co})$. By the exponential Markov inequality,

$$\mathbb{P}(R > x) \leq \frac{\mathbb{E}\mathrm{e}^{zR}}{\mathrm{e}^{zx}} \leq \frac{\mathbb{E}\mathrm{e}^{zR_{\mathrm{co}}}}{\mathrm{e}^{zx}}.$$

After taking log and $\inf_{z>0}$ of both sides we arrive at

$$\log \mathbb{P}(R > x) \leq -\overline{\psi}^*(x). \tag{6.6}$$

By the smooth variation theorem, there exist $\underline{f}, \overline{f} \in \mathcal{SR}_r$ and $\underline{k}, \overline{k} \in \mathcal{SR}_\alpha$ with

$$\underline{f}(x) \sim \overline{f}(x) \quad \text{and} \quad \underline{k}(x) \sim \overline{k}(x),$$

and

$$\underline{f} \leq f \leq \overline{f} \quad \text{and} \quad \underline{k} \leq k \leq \overline{k}$$

in a neighbourhood of infinity. Define

$$\phi = \underline{f}^* \circ \underline{k}^* \quad \text{and} \quad \phi_B(x) = B\phi\left(\frac{x}{B}\right) \quad \text{for } B > 0,$$

and

$$B_{co} = \frac{\alpha - 1}{\alpha + r - 1}\left(\frac{r}{\alpha - 1}\right)^{r/(\alpha+r-1)}\left(\frac{\gamma}{\gamma - 1}\right)^{\gamma-1}.$$

Assume for the moment that, for any $B \in (0, B_{co})$,

$$I_B(z) := \mathbb{E}\mathrm{e}^{z\overline{Q}+\phi_B(z\overline{M})-\phi_B(z)} \to 0 \quad \text{as } z \to \infty. \tag{6.7}$$

By Lemma 6.1, this implies that, for any $B < B_{co}$,

$$\overline{\psi}(z) \leq \phi_B(z) + C_B$$

for large z and some constant C_B. Since convex conjugation is order reversing, we have

$$\overline{\psi}^*(x) \geq (\phi_B + C_B)^*(x) = \phi_B^*(x) - C_B.$$

Moreover,

$$\phi_B^*(x) = B\phi^*(x) = B(\underline{f}^* \circ \underline{k}^*)^*(x).$$

The above, together with (6.6), imply that, for any $B < B_{co}$, we have

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{\phi^*(x)} \leq -B.$$

Taking the limit as $B \uparrow B_{co}$, we obtain

$$\limsup_{x\to\infty} \frac{\log \mathbb{P}(R > x)}{(f^* \circ k^*)^*(x)} \leq -\frac{\alpha - 1}{\alpha + r - 1}\left(\frac{r}{\alpha - 1}\right)^{r/(\alpha+r-1)}\left(\frac{\gamma}{\gamma - 1}\right)^{\gamma-1},$$

and, by (6.2), this is equivalent to (6.3).

It remains to show that (6.7) holds. For any $\varepsilon \in (0,1)$, we have

$$I_B(z) = \mathbb{E}e^{zQ+\phi_B(zM)-\phi_B(z)}\,\mathbf{1}_{\{M\le 1-\varepsilon\}} + \mathbb{E}e^{z\overline{Q}+\phi_B(z\overline{M})-\phi_B(z)}\,\mathbf{1}_{\{M>1-\varepsilon\}} =: K_1(z)+K_2(z).$$

Since $\psi_B \in \mathcal{R}_{\beta r^*}$ and, by Kasahara's Tauberian theorem, $z \mapsto \log \mathbb{E}e^{zQ} \sim k^*(z) \in \mathcal{R}_\beta$, we have

$$K_1(z) \le e^{\log \mathbb{E}\exp(zQ)+\phi_B(z(1-\varepsilon))-\phi_B(z)} = o(1).$$

By the definition of the generalized inverse we have

$$U \le F_M(F_M^{-1}(U)) \quad \text{and} \quad F_M^{-1}(U) \overset{\mathrm{D}}{=} M.$$

Thus,

$$\begin{aligned} K_2 &= \mathbb{E}e^{zF_Q^{-1}(U)+\phi_B(zF_M^{-1}(U))-\phi_B(z)}\,\mathbf{1}_{\{F_M^{-1}(U)>1-\varepsilon\}} \\ &\le \mathbb{E}e^{zF_Q^{-1}(F_M(M))+\phi_B(zM)-\phi_B(z)}\,\mathbf{1}_{\{M>1-\varepsilon\}}. \end{aligned}$$

Let us define $s(x) := F_Q^{-1}(F_M(x))$. By the definitions of f and k we have, for $x \in (0,1)$,

$$(1-x) f\bigg(\frac{1}{1-x}\bigg) = -\log(1-F_M(x)) \quad \text{and} \quad F_Q^{-1}(x) \le \underline{k}^{-1}(-\log(1-x)).$$

Hence, it is easy to see that in a left neighbourhood of 1 we have $s \le \overline{s}$, where

$$\overline{s}(x) := \underline{k}^{-1}\bigg((1-x)\overline{f}\bigg(\frac{1}{1-x}\bigg)\bigg). \tag{6.8}$$

Since ϕ is ultimately convex, we have, for $x \in (1-\varepsilon, 1]$,

$$\phi_B(zx) - \phi_B(z) \le -z\phi_B'(zx)(1-x) \le -z\phi_B'(z(1-\varepsilon))(1-x) = -z\phi_{B_\varepsilon}'(z)(1-x),$$

where $B_\varepsilon := B/(1-\varepsilon)$. Thus,

$$\begin{aligned} K_2(z) &\le \int_{(1-\varepsilon,1]} \exp(z\overline{s}(x) - z\phi_{B_\varepsilon}'(z)(1-x))\,\mathrm{d}F_M(x) \\ &= \int_{(1-\varepsilon,1]} \exp\bigg(z\overline{s}(x) - z\phi_{B_\varepsilon}'(z)(1-x) - \eta(1-x) f\bigg(\frac{1}{1-x}\bigg)\bigg) \\ &\qquad \times \frac{1}{(1-F_M(x))^\eta}\,\mathrm{d}F_M(x) \end{aligned}$$

since $\log \mathbb{P}(M > x) = -(1-x) f(1/(1-x))$. Furthermore,

$$K_2(z) \le \exp\bigg(\sup_{t\in[0,\varepsilon)} \bigg\{ z\overline{s}(1-t) - zt\phi_{B_\varepsilon}'(z) - \eta t \underline{f}\bigg(\frac{1}{t}\bigg)\bigg\}\bigg) \int_{(1-\varepsilon,1]} \frac{\mathrm{d}F_M(x)}{(1-F_M(x))^\eta},$$

and the integral is finite for any $\eta \in (0,1)$.

Since all functions involved are smooth, we can show that, for small enough ε, the expression under sup as a function of $t \in (0,\varepsilon)$ is concave (calculate the second derivative and use the fact that $x \mapsto \overline{s}(1-1/x) \in \mathcal{R}_{(r-1)/\alpha}$ and $x \mapsto \underline{f}(x)/x \in \mathcal{R}_{r-1}$). Hence, the supremum above is attained at $t_0 = t_0(z)$ such that

$$z\overline{s}'(1-t_0) + z\phi'_{B_\varepsilon}(z) = \eta \frac{1}{t_0} \underline{f}'\left(\frac{1}{t_0}\right) - \eta \underline{f}\left(\frac{1}{t_0}\right). \tag{6.9}$$

Put $t_0 = 1/(\underline{f}^*)'(\underline{k}^*(x))$. Then, by (2.6),

$$z\overline{s}'\left(1 - \frac{1}{(\underline{f}^*)'(\underline{k}^*(x))}\right) + z\phi'_{B_\varepsilon}(z) = \eta \underline{f}^*(\underline{k}^*(x)) = \eta\phi(x).$$

It is clear that if $z \to \infty$ then $x = x(z) \to \infty$ and $t_0 \to 0$. Moreover, since $\phi = \underline{f}^* \circ \underline{k}^* \in \mathcal{R}_{\beta r^*}$, we have

$$z\phi'_{B_\varepsilon}(z) \sim \beta r^* \phi_{B_\varepsilon}(z) \sim \beta r^* B_\varepsilon^{1-\beta r^*} \phi(z)$$

and (see Lemma 6.2 below)

$$\overline{s}'\left(1 - \frac{1}{(\underline{f}^*)'(\underline{k}^*(x))}\right) \sim r^{1/\beta}(\beta-1)^{1/\beta} \frac{\phi(x)}{x}.$$

Thus,

$$r^{1/\beta}(\beta-1)^{1/\beta} \frac{z}{x}(1+o(1)) + \beta r^* B_\varepsilon^{1-\beta r^*} \frac{\phi(z)}{\phi(x)}(1+o(1)) = \eta.$$

Take an arbitrary sequence $z_n \to \infty$, set $x_n = x(z_n)$, and define $y_n = z_n/x_n$. We have

$$\underline{C}_1 \frac{z_n}{x_n} + \underline{C}_2 \frac{\phi(z_n)}{\phi(x_n)} \le \eta \le \overline{C}_1 \frac{z_n}{x_n} + \overline{C}_2 \frac{\phi(z_n)}{\phi(x_n)} \tag{6.10}$$

for some positive constants $\underline{C}_i, \overline{C}_i$, $i = 1, 2$. Thus, by the first inequality above we quickly infer that $y_n = z_n/x_n \le \eta/\underline{C}_1$. By the Potter bounds (see [2, Theorem 1.5.6]) we have, for any $A > 1$ and $\delta > 0$,

$$\frac{\phi(z)}{\phi(x)} \le A \max\left\{\left(\frac{z}{x}\right)^{\beta r^*+\delta}, \left(\frac{z}{x}\right)^{\beta r^*-\delta}\right\}$$

for sufficiently large z and x. Hence, the second inequality in (6.10) gives

$$0 < \eta \le \max\{\overline{C}_1, A\overline{C_2}\} \max\{y_n, y_n^{\beta r^* \pm \delta}\},$$

and so $\lambda_1 \le y_n \le \lambda_2$ for some positive constants λ_1 and λ_2. Thus, there exists a convergent subsequence y_{n_k} to D, say, for which we also have ($x_n = z_n/y_n$)

$$r^{1/\beta}(\beta-1)^{1/\beta} y_{n_k}(1+o(1)) + \beta r^* B_\varepsilon^{1-\beta r^*} \frac{\phi(z_{n_k})}{\phi(z_{n_k}/y_{n_k})}(1+o(1)) = \eta,$$

where $o(1)$ is with respect to $z_{n_k} \to \infty$. Thanks to the uniform convergence in (2.1), we see that

$$\frac{\phi(z_{n_k})}{\phi(z_{n_k}/y_{n_k})} \to D^{\beta r^*}$$

and $D = D(B, \eta, \varepsilon)$ satisfies

$$r^{1/\beta}(\beta-1)^{1/\beta} D + \beta r^* B_\varepsilon^{1-\beta r^*} D^{\beta r^*} = \eta. \tag{6.11}$$

Since such a D is unique (the left-hand side of (6.11) is strictly increasing for $D > 0$), we conclude that $z \sim Dx$.

Recall that we have

$$K_2(z) \le C_\eta \exp\left(z\overline{s}(1-t_0) - zt_0\phi'_{B_\varepsilon}(z) - \eta t_0 \underline{f}\left(\frac{1}{t_0}\right)\right)$$

for some finite constant C_η. By (6.9),

$$z\overline{s}(1-t_0) - zt_0\phi'_{B_\varepsilon}(z) - \eta t_0 \underline{f}\left(\frac{1}{t_0}\right) = z\overline{s}(1-t_0) + zt_0\overline{s}'(1-t_0) - \eta \underline{f}'\left(\frac{1}{t_0}\right).$$

By Lemma 6.2 below we have

$$z\overline{s}(1-t_0 \sim \alpha(\beta-1)^{1/\beta} r^{-1/\alpha} D\underline{k}^*(x)$$

and

$$\begin{aligned} zt_0\overline{s}'(1-t_0) &\sim Dx\frac{1}{(\underline{f}^*)'(\underline{k}^*(x))}\frac{r^{1/\beta}(\beta-1)^{1/\beta}\underline{f}^*(\underline{k}^*(x))}{x} \\ &\sim r^{1/\beta}(\beta-1)^{1/\beta}(r^*)^{-1}Dk^*(x). \end{aligned}$$

Thus,

$$\limsup_{z\to\infty} \frac{\log K_2(z)}{k^*(x)} \le \alpha(\beta-1)^{1/\beta} r^{-1/\alpha} D + r^{1/\beta}(\beta-1)^{1/\beta}(r^*)^{-1} D - \eta. \tag{6.12}$$

If the right-hand side above is negative then, for some $\zeta > 0$ and large z, we have $K_2(z) \le \exp(-\zeta k^*(x(z))) \to 0$ as $z \to \infty$, and the same holds for I_B. We will show that if $B < B_{co}$ then the right-hand side of (6.12) is negative for some $\eta, \varepsilon \in (0, 1)$. The right-hand side of (6.12) is negative if

$$D < \eta \frac{r^{1/\alpha}}{(\alpha + r - 1)(\beta-1)^{1/\beta}} =: \overline{D},$$

where we have used the fact that

$$\alpha r^{-1/\alpha} + r^{1/\beta}(r^*)^{-1} = r^{-1/\alpha}(\alpha + r - 1).$$

We will show that, for fixed η and ε, the function $B \mapsto D(B, \eta, \varepsilon)$ is strictly increasing. Let $0 < B_1 < B_2$, and put $D_i = D(B_i, \eta, \varepsilon)$, $i = 1, 2$. Then by (6.11) we obtain (recall that $1 - \beta r^* < 0$)

$$\begin{aligned} 0 &= r^{1/\beta}(\beta-1)^{1/\beta}(D_1 - D_2) + \frac{\beta r^*}{(1-\varepsilon)^{1-\beta r^*}}(B_1^{1-\beta r^*} D_1^{\beta r^*} - B_2^{1-\beta r^*} D_2^{\beta r^*}) \\ &> r^{1/\beta}(\beta-1)^{1/\beta}(D_1 - D_2) + \frac{\beta r^*}{(1-\varepsilon)^{1-\beta r^*}} B_2^{1-\beta r^*}(D_1^{\beta r^*} - D_2^{\beta r^*}), \end{aligned}$$

which implies that $D_2 > D_1$. Moreover, after tedious but straightforward calculations we can show that, for

$$\overline{B} := \eta(1-\varepsilon)\frac{\alpha-1}{\alpha+r-1}\left(\frac{r}{\alpha-1}\right)^{r/(\alpha+r-1)}\left(\frac{\gamma}{\gamma-1}\right)^{\gamma-1} = \eta(1-\varepsilon)B_{co},$$

we have $D(\overline{B}, \eta, \varepsilon) = \overline{D}$. To see this, insert the definition of $\overline{D}$ into (6.11) and calculate the corresponding B. It is equal to $\overline{B}$. Thus, for any $B < B_{co}$, there exists $\eta, \varepsilon \in (0, 1)$ such that $B < \overline{B}$ and, thus, $D(B, \eta, \varepsilon) < \overline{D}$. □

Lemma 6.2. *Under the assumptions of Theorem 6.1, assume that z and t_0 are related by (6.9). Let $t_0 = 1/(\underline{f}^*)'(\underline{k}^*(x))$, $\phi = \underline{f}^* \circ \underline{k}^*$, and function $\overline{s}$ be defined as in (6.8). Then, as $z \to \infty$, we have*

(a) $\overline{s}(1-t_0) \sim \alpha(\beta-1)^{1/\beta} r^{-1/\alpha} \underline{k}^*(x)/x$,

(b) $\overline{s}'(1-t_0) \sim r^{1/\beta}(\beta-1)^{1/\beta}\phi(x)/x$.

Proof. (a) Since f is regularly varying and $\underline{f} \sim \overline{f}$, we have $\underline{f}^* \sim \overline{f}^*$. Thus,

$$t_0\overline{f}(t_0^{-1}) \sim \frac{\overline{f}'(t_0^{-1})}{r} \sim \frac{\underline{k}^*(x)}{r}.$$

Hence,

$$x\overline{s}(1-t_0) = x\underline{k}^{-1}\left(t_0\overline{f}\left(\frac{1}{t_0}\right)\right) \sim x\underline{k}^{-1}\left(\frac{\underline{k}^*(x)}{r}\right) \sim xr^{-1/\alpha}\underline{k}^{-1}(\underline{k}^*(x)).$$

Moreover, by Lemma 2.1, with the substitution $x \mapsto k^*(x)$, we have

$$\underline{k}^{-1}(\underline{k}^*(x)) \sim \alpha(\beta-1)^{1/\beta}\frac{\underline{k}^*(x)}{x}.$$

(b) We have

$$\overline{s}'(1-t_0) = \frac{\overline{f}'(1/t_0)/t_0 - \overline{f}(1/t_0)}{\underline{k}'(\overline{s}(1-t_0))}.$$

By (2.6), the numerator equals

$$\overline{f}^*\left(\overline{f}'\left(\frac{1}{t_0}\right)\right) = \overline{f}^*(\overline{f}'(\underline{f}'(\underline{k}^*(x)))) \sim \phi(x).$$

By (a),

$$\overline{s}(1-t_0) \sim \alpha(\beta-1)^{1/\beta} r^{-1/\alpha}\beta^{-1}(\underline{k}^*)'(x),$$

and, thus,

$$\underline{k}'(\overline{s}(1-t_0)) \sim \underline{k}'(\alpha(\beta-1)^{1/\beta}r^{-1/\alpha}\beta(\underline{k}^*)'(x)) \sim (\alpha(\beta-1)^{1/\beta}r^{-1/\alpha}\beta^{-1})^{\alpha-1}x,$$

where the latter asymptotic equivalence follows from the fact that $\underline{k}' \in \mathcal{R}_{\alpha-1}$ and $\underline{k}' \circ (\underline{k}^*)' = \mathrm{Id}$. Finally, observe that, since $\alpha^{-1} + \beta^{-1} = 1$, we have

$$r^{1/\beta}(\beta-1)^{1/\beta} = (\alpha(\beta-1)^{1/\beta}r^{-1/\alpha}\beta^{-1})^{1-\alpha}.$$ □

Proof of Theorem 6.2. (i) Since $r \mapsto \mathrm{e}^{zr}$ is convex, by Lemma 2.2 we see that

$$\mathbb{E}\mathrm{e}^{zR} \le \mathbb{E}\mathrm{e}^{zR_{\mathrm{ind}}}, \qquad z \ge 0.$$

Let $\psi_{\mathrm{ind}}(z) := \log \mathbb{E}\mathrm{e}^{zR_{\mathrm{ind}}}$. By the exponential Markov inequality,

$$\mathbb{P}(R > x) \le \frac{\mathbb{E}\mathrm{e}^{zR}}{\mathrm{e}^{zx}} \le \frac{\mathbb{E}\mathrm{e}^{zR_{\mathrm{ind}}}}{\mathrm{e}^{zx}}.$$

After taking log and $\inf_{z>0}$ of both sides we arrive at

$$\log \mathbb{P}(R > x) \le -\psi_{\mathrm{ind}}^*(x).$$

By Kasahara's Tauberian theorem we conclude that $-\psi^*_{\mathrm{ind}}(x) \sim -\log \mathbb{P}(R_{\mathrm{ind}} > x)$. The assertion follows from Theorem 3.1.

(ii) The upper bound follows by Theorem 6.1. The lower bound in (6.5) is immediate if we examine the proof of Theorem 5.1. By positive quadrant dependence we have

$$\log \mathbb{P}(M > \delta_k,\ Q > q_k) \geq \log[\mathbb{P}(M > \delta_k)\mathbb{P}(Q > q_k)].$$

Thus, using the above inequality and repeating the steps in the proof of Theorem 5.1 (case $t_\infty = \infty$) with $h(x) = (f^* \circ k^*)^*(x) \in \mathcal{R}_\gamma$, we arrive at the lower bound in (6.5). □

7. Proofs of the auxiliary results

Proof of Lemma 2.1. Suppose first that $f \in \mathcal{SR}_\alpha$. Then $f' \in \mathcal{SR}_{\alpha-1}$ has an inverse in some neighbourhood of infinity. Since $(f^*)' = (f')^{-1}$, we see that $(f^*)' \in \mathcal{SR}_{1/(\alpha-1)}$, and so $f^* \in \mathcal{SR}_\beta$. By (2.2) and (2.6) we have

$$\frac{f((f^*)'(x))}{f^*(x)} = \frac{x(f^*)'(x) - f^*(x)}{f^*(x)} \to \beta - 1 \quad \text{as } x \to \infty.$$

Since $(f')^{-1} = (f^*)'$, setting above $x(z) = f^(f^{-1}(z)) \to \infty$ we obtain

$$\frac{z}{f^*(x(z))} \to \beta - 1.$$

Thus, e.g. [21, Lemma 2.1] gives

$$(f^*)^{-1}(z) \sim (\beta - 1)^{1/\beta} x(z) = (\beta - 1)^{1/\beta} f'(f^{-1}(z))$$

and

$$\frac{f^{-1}(x)(f^*)^{-1}(x)}{x} \sim (\beta - 1)^{1/\beta} \frac{f^{-1}(x) f'(f^{-1}(x))}{x} \to (\beta - 1)^{1/\beta}\alpha,$$

by the definition of $\mathcal{SR}_\alpha$.

In the general case, the smooth variation theorem yields the existence of $\underline{f}, \overline{f} \in \mathcal{SR}_\alpha$ with $\underline{f} \leq f \leq \overline{f}$ on some neighbourhood of infinity. Since conjugacy is order reversing, we have $\overline{f}^* \leq f^* \leq \underline{f}^*$. Moreover, $\overline{f}^{-1} \leq f^{\leftarrow} \leq \underline{f}^{-1}$ in a vicinity of infinity, and similar inequalities hold for $(f^*)^{\leftarrow}$. The conclusion follows from the fact that $\underline{f}(x) \sim \overline{f}(x)$. □

Proof of Lemma 2.2. We will use the fact that a stochastic recursion (1.2) converges in distribution to the solution of an affine equation. Take $R_0 = R'_0 = 0$ a.s. We proceed by induction. Assume that, for some $n \in \mathbb{N}$, we have

$$\mathbb{E}f(R_n) \leq \mathbb{E}f(R'_n) \quad \text{for all convex functions } f \text{ on } \mathbb{R}. \tag{7.1}$$

Let f be a convex function. By the fact that $r \mapsto \mathbb{E}f(Mr + Q)$ is convex and by the inductive assumption, we first infer that

$$\mathbb{E}f(M_{n+1}R_n + Q_{n+1}) \leq \mathbb{E}f(M_{n+1}R'_n + Q_{n+1}).$$

Furthermore, for any $r \geq 0$, the function $h_r(m, q) := f(mr + q)$ is supermodular. Note that, since $R'_0 = 0$ and M and Q are a.s. nonnegative, R'_n is a.s. nonnegative as well. Then

$$\begin{aligned}\mathbb{E}f(R_{n+1}) &\leq \mathbb{E}f(M_{n+1}R'_n + Q_{n+1}) \\ &= \mathbb{E}h_{R'_n}(M_{n+1}, Q_{n+1}) \\ &\leq \mathbb{E}h_{R'_n}(M'_{n+1}, Q'_{n+1}) \\ &= \mathbb{E}f(R'_{n+1}).\end{aligned}$$

Thus, we have established (7.1) for any $n \in \mathbb{N}$. Observe that $(R_n)_n$ is stochastically nondecreasing, that is,

$$R_{n+1} \overset{\mathrm{D}}{=} \sum_{k=1}^{n+1} M_1 \cdots M_{k-1} Q_k \geq \sum_{k=1}^{n} M_1 \cdots M_{k-1} Q_k \overset{\mathrm{D}}{=} R_n.$$

Thus, for any weakly monotonic function f, $(f(R_n))_n$ is stochastically monotonic as well, and the same holds for $(f(R'_n))_n$. The assertion follows from the fact that $\mathbb{E}f(R_n)$ and $\mathbb{E}f(R'_n)$ are monotonic and so have a limit (possibly infinite) as $n \to \infty$. □

Proof of Theorem 2.3. In [2, Theorem 4.12.7] a different formulation of the same result was proposed. Namely, for $\alpha \in (0, 1)$ and $\phi \in \mathcal{R}_\alpha$, define $\psi(z) = z/\phi(z) \in \mathcal{R}_{1-\alpha}$. Then

$$-\log \mathbb{P}(X > x) \sim \phi^{\leftarrow}(x)$$

if and only if

$$\log M(z) \sim (1-\alpha)\alpha^{\alpha/(1-\alpha)} \psi^{\leftarrow}(z).$$

We have to show that

$$k(x) \sim \phi^{\leftarrow}(x) \quad \text{if and only if} \quad k^*(z) \sim (1-\alpha)\alpha^{\alpha/(1-\alpha)} \psi^{\leftarrow}(z).$$

Let $\rho = 1/\alpha$, and put $f = \phi^{\leftarrow} \in \mathcal{R}_\rho$. By Lemma 2.1 we have

$$\frac{f^{\leftarrow}(x)(f^*)^{\leftarrow}(x)}{x} \to \rho(\rho-1)^{-(\rho-1)/\rho} = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} \quad \text{as } x \to \infty. \tag{7.2}$$

But

$$\frac{f^{\leftarrow}(x)(f^*)^{\leftarrow}(x)}{x} \sim \frac{(f^*)^{\leftarrow}(x)}{\psi(x)},$$

and so (7.2) is equivalent to (use the definition of the asymptotic inverse and Lemma 2.1 of [21])

$$f^*(z) \sim (1-\alpha)\alpha^{\alpha/(1-\alpha)} \psi^{\leftarrow}(z).$$ □

Proof of Theorem 2.4. Recall that $h \in \mathcal{R}_\rho(0+)$ if $x \mapsto h(1/x) \in \mathcal{R}_{-\rho}$. Moreover, if $h \in \mathcal{R}_\rho(0+)$ then $h^{\leftarrow} \in \mathcal{R}_{1/\rho}$. Indeed, we have $h(1/x) = x^{-\rho}L(x)$ for some slowly varying function L. The (asymptotic) inverse g of $x \mapsto x^{-\rho}L(x)$ is regularly varying with index $-1/\rho$. But then we have $h^{\leftarrow}(x) \sim 1/g(x)$.

In [2, Theorem 4.12.9] the following result was proved. For $\alpha < 0$ and $\phi \in \mathcal{R}_\alpha(0+)$, define $\psi(z) = \phi(z)/z \in \mathcal{R}_{\alpha-1}(0+)$. Then

$$-\log \mathbb{P}(Y \leq x) \sim \frac{1}{\phi^{\leftarrow}}\left(\frac{1}{x}\right) \quad \text{as } x \to 0+$$

if and only if

$$-\log \mathbb{E}\mathrm{e}^{-\lambda Y} \sim \frac{(1-\alpha)(-\alpha)^{\alpha/(1-\alpha)}}{\psi^{\leftarrow}(\lambda)} \quad \text{as } \lambda \to \infty.$$

First observe that, under regular variation, asymptotics of $-\log \mathbb{P}(Y \leq 1/x)$ and $-\log \mathbb{P}(Y < 1/x)$ are the same. Indeed, for any $\varepsilon > 0$, we have $-\log \mathbb{P}(Y \leq 1/(x+\varepsilon)) \sim -\log \mathbb{P}(Y \leq 1/(x-\varepsilon))$. Furthermore, it is easy to see that if f is regularly varying then

$$f(x) \sim \frac{x}{\phi^{\leftarrow}(x)} \quad \text{if and only if} \quad f^{\leftarrow}(x) \sim x\psi^{\leftarrow}(x).$$

It remains to show that

$$f(x) \sim \frac{x}{\phi^{\leftarrow}(x)} \quad \text{if and only if} \quad (f^*)^{\leftarrow}(\lambda) \sim \frac{(1-\alpha)(-\alpha)^{\alpha/(1-\alpha)}}{\psi^{\leftarrow}(\lambda)}.$$

Since $x \mapsto \phi(1/x) \in \mathcal{R}_{-\alpha}$, we see that $\phi^{\leftarrow} \in \mathcal{R}_{1/\alpha}$, and so $f \in \mathcal{R}_{\rho}$ with $\rho = 1 - \alpha^{-1}$. By Lemma 2.1 we have

$$\frac{f^{\leftarrow}(x)(f^*)^{\leftarrow}(x)}{x} \to (1-\alpha)(-\alpha)^{\alpha/(1-\alpha)} \quad \text{as } x \to \infty,$$

which completes the proof. □

Proof of Theorem 4.1. (a) By the definition of h, for any x and any positive number $g(x)$, there exists a number $t(x)$ such that

$$h(x) \le -t(x) \log \mathbb{P}\biggl(\frac{1}{1-M} > t(x),\ Q > \frac{x}{t(x)}\biggr) \le h(x) + g(x).$$

If $g(x) = o(1)$, we obtain the first part of the assertion.

Using the fact that

$$\mathbb{P}\biggl(\frac{1}{1-M} > t(x),\ Q > \frac{x}{t(x)}\biggr) \le \mathbb{P}\biggl(\frac{Q}{1-M} > x\biggr),$$

we obtain (4.3).

(b) The assertion follows from the Fréchet–Hoeffding bounds:

$$\begin{aligned}\mathbb{P}\biggl(\frac{1}{1-M} > t\biggr) + \mathbb{P}\biggl(Q > \frac{x}{t}\biggr) - 1 &\le \mathbb{P}\biggl(\frac{1}{1-M} > t,\ Q > \frac{x}{t}\biggr)\\ &\le \min\biggl\{\mathbb{P}\biggl(\frac{1}{1-M} > t\biggr), \mathbb{P}\biggl(Q > \frac{x}{t}\biggr)\biggr\}.\end{aligned}$$

(c) The first part follows quickly by Theorem 2.1; see Remark 4.1. Moreover, we already know that, if $f \in \mathcal{R}_r$ and $k \in \mathcal{R}_\alpha$ with $r, \alpha > 1$, then $(f^* \circ k^*)^* \in \mathcal{R}_\gamma$. Thanks to the smooth variation theorem, we may only consider the case when $f \in \mathcal{SR}_r$ and $k \in \mathcal{SR}_\alpha$. The infimum in the definition of $h_{\mathrm{ind}}(x)$ is attained at a point $t_1 = t_1(x)$ such that

$$f'(t_1) = \frac{x}{t_1} k'\biggl(\frac{x}{t_1}\biggr) - k\biggl(\frac{x}{t_1}\biggr) = k^*\biggl(k'\biggl(\frac{x}{t_1}\biggr)\biggr), \tag{7.3}$$

where the last equality is (2.6). Thus, by the regular variation of f and k, we obtain

$$x = t_1[(k^*)' \circ (k^*)^{-1} \circ f'](t_1) = t_1^{(r+\alpha-1)/\alpha} L(t_1)$$

for some slowly varying function L. This means that $t_1 \to \infty$ and $x/t_1 \to \infty$ as $x \to \infty$ and that $x \mapsto t_1(x) \in \mathcal{R}_{\alpha/(r+\alpha-1)}$.

Consider now the case when $\operatorname{ess\,sup} Q = q_+ < \infty$. In this case $k(x) = \infty$ if $x \ge q_+$, and so

$$k^*(z) = \sup_{x>0}\{zx - k(x)\} = \sup_{x<q_+}\{zx - k(x)\} \le \sup_{x<q_+}\{zx\} = zq_+.$$

On the other hand, for any $x < q_+$,

$$\frac{k^*(z)}{z} \ge x - \frac{k(x)}{z} \to x \quad \text{as } z \to \infty.$$

Then we have

$$
\begin{aligned}
(f^* \circ k^*)^*(x) &= \sup_{z>0}\{zx - f^*(k^*(z))\} \\
&\sim \sup_{z>0}\{zx - f^*(q_+ z)\} \\
&= \sup_{y>0}\left\{\frac{x}{q_+} y - f^*(y)\right\} \\
&= f\left(\frac{x}{q_+}\right).
\end{aligned}
$$

(d) As previously, we work with $f \in \mathcal{SR}_r$ and $k \in \mathcal{SR}_\alpha$. Let us first make a simple observation that if functions a and b are continuous, $a(x_0) < b(x_0)$ and $a(x_1) > b(x_1)$, a is increasing and b is decreasing, then there exists a unique t_0 such that $a(t_0) = b(t_0)$ and, moreover, $\inf_{t\in[x_0,x_1]} \max\{a(t), b(t)\} = a(t_0)$. Our first step here will be to show that the infimum in the definition of

$$
h_{\mathrm{co}} = \inf_{t\geq 1}\left\{\max\left\{f(t), tk\left(\frac{x}{t}\right)\right\}\right\}
$$

is (for large enough x) attained at a point $t_2 = t_2(x)$ such that $f(t_2) = t_2 k(x/t_2)$. In our case, the function $[1,\infty) \ni t \mapsto tk(x/t)$ may not be decreasing and so we cannot use our observation directly. However, note that, since $k \in \mathcal{SR}_\alpha$, the limit

$$
\lim_{z\to\infty} \frac{zk'(z)}{k(z)} = \alpha
$$

is strictly larger than 1 and so $k(z) < zk'(z)$ for large enough z, say $z \geq 1/T$ for some $T > 0$. Calculating the derivative of $t \mapsto tk(x/t)$ we obtain

$$
k\left(\frac{x}{t}\right) - \frac{x}{t} k'\left(\frac{x}{t}\right),
$$

which is strictly negative if $x/t \geq 1/T$, that is, $t \leq Tx$. Hence, for $t \in [1, Tx)$, $f(t)$ is increasing and $tk(x/t)$ is decreasing. Moreover, for large x, we have $f(1) < k(x)$ and $f(x) > xk(1)$, and so

$$
\inf_{t\in[1,Tx]}\left\{\max\left\{f(t), tk\left(\frac{x}{t}\right)\right\}\right\} = f(t_2).
$$

It is enough to show that, for large enough x, the infimum in the definition of h_{co} is not attained on the set (Tx, ∞). We have

$$
\inf_{t>Tx}\left\{\max\left\{f(t), tk\left(\frac{x}{t}\right)\right\}\right\} \geq \max\left\{f(Tx), Tx \sup_{z\in(0,1/T)} k(z)\right\} \sim T^r f(x) \in \mathcal{R}_r.
$$

Let us assume for the moment that

$$
x \mapsto f(t_2(x)) \in \mathcal{R}_{\alpha r/(\alpha+r-1)}. \tag{7.4}
$$

Since $r > \alpha r/(\alpha + r - 1)$, we see that, under (7.4), our claim holds and we have

$$
h_{\mathrm{co}}(x) = \inf_{t\in[1,Tx]}\left\{\max\left\{f(t), tk\left(\frac{x}{t}\right)\right\}\right\} = f(t_2(x)).
$$

The equality $f(t_2) = t_2 k(x/t_2)$ is equivalent to $x = m(t_2)$, where $m(t) := t\,k^{-1}(f(t)/t) \in \mathcal{R}_{(r+\alpha-1)/\alpha}$. This implies that $x \mapsto t_2(x) \in \mathcal{R}_{\alpha/(r+\alpha-1)}$, so it is easy to see that, as before, t_2 and x/t_2 go to infinity as $x \to \infty$. Thus,

$$k\left(\frac{x}{t_2}\right) = \frac{f(t_2)}{t_2} \sim \frac{1}{r} f'(t_2).$$

Hence,

$$x \sim \frac{t_2 g(t_2)}{r^{1/\alpha}},$$

where $g = k^{-1} \circ f' \in \mathcal{R}_{(r-1)/\alpha}$. In this way we have established (7.4).

On the other hand, in the case of independent M and Q, (7.3) implies that

$$f'(t_1) \sim (\alpha - 1)k\left(\frac{x}{t_1}\right),$$

and so

$$x \sim \frac{t_1 g(t_1)}{(\alpha - 1)^{1/\alpha}}.$$

Thus, $r^{-1/\alpha} t_2 g(t_2) \sim (\alpha - 1)^{-1/\alpha} t_1 g(t_1)$, $t \mapsto t g(t) \in \mathcal{R}_{(r+\alpha-1)/\alpha}$, and so by Lemma 2.1 of [21] we obtain

$$t_2 \sim t_1 \left(\frac{r}{\alpha - 1}\right)^{1/(\alpha+r-1)}.$$

Finally,

$$h_{\mathrm{co}}(x) = f(t_2) \sim \left(\frac{r}{\alpha - 1}\right)^{r/(\alpha+r-1)} f(t_1) \sim \frac{\alpha - 1}{\alpha + r - 1}\left(\frac{r}{\alpha - 1}\right)^{r/(\alpha+r-1)} h_{\mathrm{ind}}(x),$$

since

$$h_{\mathrm{ind}}(x) = f(t_1) + t_1 k\left(\frac{x}{t_1}\right) \sim f(t_1) + (\alpha - 1)^{-1} t_1 f'(t_1) \sim \left(1 + \frac{r}{\alpha - 1}\right) f(t_1).$$

(e) The infimum in the definition of $h_{\mathrm{counter}}(x)$ is calculated for $t > 0$ such that

$$\mathbb{P}\left(M > 1 - \frac{1}{t}\right) + \mathbb{P}\left(Q > \frac{x}{t}\right) > 1. \tag{7.5}$$

We will show that, as $x \to \infty$, the infimum is actually calculated for

$$t \in I_x := [1, (1 - m_-)^{-1}] \cup \left[\frac{x}{q_-}, \infty\right).$$

Take $t \in [1, \infty) \setminus I_x$. Then there exist $m \in (m_-, m_+)$ and $q > q_-$ with $t \in (1/(1 - m), x/q)$. We have $\mathbb{P}(M > m) < 1$ and $\mathbb{P}(Q > q) < 1$. Consider first the case when $t \in (1/(1-m), \sqrt{x}]$. Then

$$\mathbb{P}\left(M > 1 - \frac{1}{t}\right) + \mathbb{P}\left(Q > \frac{x}{t}\right) \le \mathbb{P}(M > m) + \mathbb{P}(Q > \sqrt{x}),$$

and we obtain a contradiction with (7.5) as $x \to \infty$. Similarly, if $t \in [\sqrt{x}, x/q)$ then we obtain

$$\mathbb{P}\left(M > 1 - \frac{1}{\sqrt{x}}\right) + \mathbb{P}(Q > q) > 1,$$

and this yields a contradiction as well if $x \to \infty$.

So far, we have shown that

$$h_{\mathrm{counter}}(x) \sim \inf_{t\in I_x}\left\{-t\log\left[\mathbb{P}\left(M>1-\frac{1}{t}\right)+\mathbb{P}\left(Q>\frac{x}{t}\right)-1\right]\right\}$$
$$=\min\left\{\inf_{t\in[1,1/(1-m_-)]}\{\cdots\},\ \inf_{t\geq x/q_-}\{\cdots\}\right\}.$$

If $\mathbb{P}(M=m_-)=0=\mathbb{P}(Q=q_-)$, this is exactly (4.4) since (f and k are right continuous and nondecreasing)

$$\inf_{t\geq x/q_-}\{\cdots\}=\inf_{t\geq x/q_-} f(t)=f\left(\frac{x}{q_-}\right),$$

and, similarly,

$$\inf_{t\in[1,1/(1-m_-)]}\{\cdots\}=\inf_{t\leq 1/(1-m_-)} tk\left(\frac{x}{t}\right)=\frac{k((1-m_-)x)}{1-m_-}.$$

On the other hand, if $\mathbb{P}(Q=q_-)>0$ then by (7.5) we see that $t=x/q_-$ is impossible as $x\to\infty$ and, thus, the infimum is calculated for $t\in(x/q_-,\infty)$. However, this introduces virtually no changes to the proof since $\inf_{1\leq t<x/q_-} f(t)\sim f(x/q_-)$. If $\mathbb{P}(M=m_-)>0$ then $t=1/(1-m_-)$ is impossible and we eventually obtain (4.4). □

Proof of Lemma 6.1. Assume that $\mathbf{1}_\phi(z)\leq 1$ for $z>N$, and define

$$\overline{\phi}(x)=\begin{cases} ax, & x\leq N,\\ \phi(x)+C, & x>N.\end{cases}$$

We will show that, for sufficiently large a and C,

$$\mathbf{1}_{\overline{\phi}}(z)\leq 1 \quad \text{for all } z\geq 0. \tag{7.6}$$

Observe that $\mathbf{1}_{\overline{\phi}}(0)=1$. If $a>\mathbb{E}Q(1-\mathbb{E}M)^{-1}$ then $\mathbf{1}'_{\overline{\phi}}(0)=\mathbb{E}Q-a+a\mathbb{E}M<0$; thus, there exists $\varepsilon>0$ such that $\mathbf{1}_{\overline{\phi}}(z)\leq 1$ for $z\in[0,\varepsilon)$. For $z\in[\varepsilon,N]$, we have

$$\mathbf{1}_{\overline{\phi}}(z)\leq \mathbb{E}\mathrm{e}^{NQ-a\varepsilon(1-M)},$$

and the right-hand side tends to 0 as $a\to\infty$. Thus, for sufficiently large a, we also have $\mathbf{1}_{\overline{\phi}}(z)\leq 1$ for $z\in[\varepsilon,N]$. Furthermore, for $z>N$, we have

$$\mathbf{1}_{\overline{\phi}}(z)=\mathbf{1}_\phi(z)+\mathbb{E}\mathrm{e}^{zQ}(e^{azM-\phi(z)-C}-\mathrm{e}^{\phi(zM)-\phi(z)})\,\mathbf{1}_{\{zM\leq N\}},$$

and we may find C such that $ax-C\leq\phi(x)$ for any $x\in[0,N]$, so the second term above is nonpositive.

Proceeding by induction, assume that $\mathbb{E}\mathrm{e}^{zR_n}\leq\mathrm{e}^{\overline{\phi}(z)}$ for all $z\geq 0$ and some $n\in\mathbb{N}$. Then, for $z\geq 0$,

$$\mathbb{E}\mathrm{e}^{zR_{n+1}})=\mathbb{E}\mathrm{e}^{zM_{n+1}R_n+zQ_{n+1}}\leq\mathbb{E}\mathrm{e}^{\overline{\phi}(zM)+zQ}\leq\mathrm{e}^{\overline{\phi}(z)}$$

by (7.6). Moreover, we can start the induction since R_0 can be chosen arbitrarily and, thus, passing to the limit as $n\to\infty$, we obtain the assertion. □

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References

[1] Alsmeyer, G., Iksanov, A. and Rösler, U. (2009). On distributional properties of perpetuities. *J. Theoret. Prob.* **22,** 666–682.
[2] Bingham, N. H., Goldie, C. M. and Teugels, J. L. (1989). *Regular Variation* (Encyclopedia Math. Appl. **27**). Cambridge University Press.
[3] Buraczewski, D., Damek, E. and Mikosch, T. (2016). *Stochastic Models with Power-Law Tails. The Equation $X = AX + B$*. Springer, Cham.
[4] Buraczewski, D., Dyszewski, P., Iksanov, A. and Marynych, A. (2018). On perpetuities with gamma-like tails. *J. Appl. Prob.* **55,** 368–389.
[5] Cambanis, S., Simons, G. and Stout, W. (1976). Inequalities for $Ek(X, Y)$ when the marginals are fixed. *Z. Wahrscheinlichkeitsth.* **36,** 285–294.
[6] Daley, D. J. and Goldie, C. M. (2006). The moment index of minima. II. *Statist. Prob. Lett.* **76,** 831–837.
[7] Damek, E. and Kołodziejek, B. (2017). Stochastic recursions: between Kesten's and Grey's assumptions. Preprint. Available at https://arxiv.org/abs/1701.02625v1.
[8] Diaconis, P. and Freedman, D. (1999). Iterated random functions. *SIAM Rev.* **41,** 45–76.
[9] Dyszewski, P. (2016). Iterated random functions and slowly varying tails. *Stoch. Process. Appl.* **126,** 392–413.
[10] Goldie, C. M. (1991). Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Prob.* **1,** 126–166.
[11] Goldie, C. M. and Grübel, R. (1996). Perpetuities with thin tails. *Adv. Appl. Prob.* **28,** 463–480.
[12] Goldie, C. M. and Maller, R. A. (2000). Stability of perpetuities. *Ann. Prob.* **28,** 1195–1218.
[13] Grey, D. R. (1994). Regular variation in the tail behaviour of solutions of random difference equations. *Ann. Appl. Prob.* **4,** 169–183.
[14] Hiriart-Urruty, J.-B. and Lemaréchal, C. (1993). *Convex Analysis and Minimization Algorithms. II* (Fundamental Principles Math. Sci. **306**). Springer, Berlin.
[15] Hitczenko, P. (2010). On tails of perpetuities. *J. Appl. Prob.* **47,** 1191–1194.
[16] Hitczenko, P. and Wesołowski, J. (2009). Perpetuities with thin tails revisited. *Ann. Appl. Prob.* **19,** 2080–2101.
[17] Iksanov, A. (2016). *Renewal Theory for Perturbed Random Walks and Similar Processes*. Birkhäuser/Springer, Cham.
[18] Iksanov, A. and Möhle, M. (2008). On the number of jumps of random walks with a barrier. *Adv. Appl. Prob.* **40,** 206–228.
[19] Kevei, P. (2016). A note on the Kesten-Grincevičius-Goldie theorem. *Electron. Commun. Prob.* **21,** 12pp.
[20] Kimeldorf, G. and Sampson, A. R. (1989). A framework for positive dependence. *Ann. Inst. Statist. Math.* **41,** 31–45.
[21] Kołodziejek, B. (2017). Logarithmic tails of sums of products of positive random variables bounded by one. *Ann. Appl. Prob.* **27,** 1171–1189.
[22] Lehmann, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37,** 1137–1153.
[23] Lorentz, G. G. (1953). An inequality for rearrangements. *Amer. Math. Monthly* **60,** 176–179.
[24] Maulik, K., Resnick, S. and Rootzén, H. (2002). Asymptotic independence and a network traffic model. *J. Appl. Prob.* **39,** 671–699.
[25] Rivero, V. (2003). A law of iterated logarithm for increasing self-similar Markov processes. *Stoch. Stoch. Rep.* **75,** 443–472.
[26] Rockafellar, R. T. (1970). *Convex Analysis* (Princeton Math. Ser. **28**). Princeton University Press.
[27] Rothschild, M. and Stiglitz, J. E. (1970). Increasing risk. I. A definition. *J. Econom. Theory* **2,** 225–243.
[28] Tang, Q. and Yuan, Z. (2016). Random difference equations with subexponential innovations. *Sci. China Math.* **59,** 2411–2426.
[29] Tchen, A. H. (1980). Inequalities for distributions with given marginals. *Ann. Prob.* **8,** 814–827.
[30] Vervaat, W. (1972). *Success Epochs in Bernoulli Trials (with Applications in Number Theory)* (Math. Centre Tracts **42**). Mathematisch Centrum, Amsterdam.
[31] Vervaat, W. (1979). On a stochastic difference equation and a representation of nonnegative infinitely divisible random variables. *Adv. Appl. Prob.* **11,** 750–783.