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STOCHASTIC FIXED-POINT EQUATION AND LOCAL DEPENDENCE MEASURE

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We study solutions to the stochastic fixed-point equation $X \stackrel{d}{=} AX + B$ where the coefficients A and B are nonnegative random variables. We introduce the "local dependence measure" (LDM) and its Legendre-type transform to analyze the left tail behavior of the distribution of X. We discuss the relationship of LDM with earlier results on the stochastic fixed-point equation and we apply LDM to prove a theorem on a Fleming–Viot-type process.

1. Introduction. Our research on the stochastic fixed-point equation is motivated by a problem arising in the theory of the so-called Fleming–Viot processes considered in [9, 15]. This article contains new ideas that lead to the complete solution of a specific problem; see Section 7 for details. Needless to say, we hope that the new technique developed in this paper will have applications beyond the theory of Fleming–Viot processes.

Given a pair of random variables (A, B), an independent random variable X is said to satisfy the stochastic fixed-point equation if

$$X \stackrel{d}{=} AX + B. \tag{1.1}$$

The behavior of the solution, especially the left and right tails, has been extensively studied. A classical result ([12, 17]) says that under some assumptions on (A, B), for some $\alpha, C_-, C_+ > 0$,

$$\mathbb{P}(X > x) \sim C_+ x^{-\alpha} \quad \text{and} \quad \mathbb{P}(X < -x) \sim C_- x^{-\alpha}, \tag{1.2}$$

as $x \to \infty$ (see Theorem 7.12 for a fully rigorous version). An excellent review of the subject can be found in [6].

It can be shown that if A and B are nonnegative random variables then a nonconstant solution X to (1.1) must be also a nonnegative random variable (we do not present a proof because this claim is not needed for the main application of (1.1) in Section 7). If X is nonnegative then the first estimate in (1.2) is still meaningful and informative, but the second one is not because for $x > 0$ we have $\mathbb{P}(X < -x) = 0$. In this article, we will continue the analysis of the behavior of $\mathbb{P}(X < x)$ as $x \to 0^+$ initiated in [9].

We will introduce a new concept of "local dependence measure" (LDM) and its Legendre-type transform. We will relate LDM to concepts discussed in [9]: inverse exponential decay of the tail of B, and positive quadrant dependence of A and B. We will illustrate the power of LDM by a few examples, including the proof of a result on the Fleming–Viot model.

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1.1. *Organization of the paper.* Section 2 is devoted to the basic general properties of solutions to the stochastic fixed-point equation (1.1). We recall the conditions that guarantee the existence and uniqueness of the solution in Theorem 2.2 and Corollary 2.3.

In Section 3 we define the local dependence measure (LDM) for the random variables (A, B) in (1.1), and its Legendre-type transform. We study basic properties of these functions and present their first application to the stochastic fixed-point equation.

In Section 4 we show that if LDM for the random variables (A, B) exists then the solution to (1.1) is a random variable with an "inverse exponential decay" left tail.

Section 5 is devoted to calculating explicit formulas for LDM (Proposition 5.3) and its Legendre-type transform (Proposition 5.4) when A and B are positively quadrant dependent random variables.

In Section 6 we prove that if $X_n = A_n X_{n-1} + B_n$ for $n \geq 1$, $X_0 = 0$ and $(A_n, B_n)_{n\geq 1}$ is a sequence of independent copies of (A, B), then

$$\liminf_{n\to\infty} \frac{X_n}{H^{-1}(\log n)} = (\lambda^*)^{1/\rho},$$

where H is a regularly varying function introduced in the definition of LDM, λ^* is the fixed point for the Legendre-type transform, and ρ is a parameter in the definition of LDM.

In Section 7 we apply an LDM to prove a version of the law of iterated logarithm for a Fleming–Viot-type process.

2. General results on stochastic fixed-point equation. In this section we will introduce notation and conventions used in the rest of the paper, and present some known general results, with references but no proofs.

In this section, and this section only, we will allow the coefficients A and B of the stochastic fixed-point equation

$$X \stackrel{d}{=} AX + B, \tag{2.1}$$

to take arbitrary (positive and negative) values. Starting with Section 3, we will assume that $A, B \geq 0$, a.s.

We will say that the law of a random variable X with values in $\mathbb{R}$ is a solution to (2.1) if one can construct X, A and B on the same probability space in such a way that X is independent of (A, B) and (2.1) is satisfied.

We will always use (A_n, B_n) to denote a vector with the same distribution as (A, B) (the distribution of (A, B) can change from one context to another).

Let $(A_n, B_n)_{n\geq 1}$ be an i.i.d. sequence and define random affine maps from $\mathbb{R}$ to itself by

$$\Psi_n(t) = A_n t + B_n, \quad t \in \mathbb{R}.$$

Clearly, (Ψ_n) is an i.i.d. sequence. Suppose that X_0 is independent from $(A_n, B_n)_{n\geq 1}$ and let

$$X_n = \Psi_n(X_{n-1}) = A_n X_{n-1} + B_n, \tag{2.2}$$

for $n \geq 1$. Note that (X_n) is a Markov chain. It is easy to check that

$$X_n = \left(\sum_{k=1}^n B_k \prod_{j=k+1}^n A_j\right) + X_0 \prod_{i=1}^n A_i.$$

We define another sequence of affine mappings, starting with $S_0(t) = t$ for all $t \in \mathbb{R}$, and continuing inductively by

$$S_n(t) = S_{n-1} \circ \Psi_n(t) = S_{n-1}(A_n t + B_n),$$

for $n \geq 1$. Then we have

$$S_n(t) = \left(\sum_{k=1}^{n} B_k \prod_{j=1}^{k-1} A_j\right) + t \prod_{i=1}^{n} A_i,$$

with the convention that $\prod_{j=k}^{m} A_j = 1$ if $m < k$. Re-indexing of the sequence $(A_n, B_n)_{n\geq 1}$ easily shows that

$$X_n \stackrel{d}{=} S_n(X_0) \tag{2.3}$$

for each $n \geq 1$.

The following follows from the "principle" stated on page 264 of [18].

LEMMA 2.1. *If for each $t \in \mathbb{R}$ the sequence $(S_n(t))$ converges almost surely to a limit, say S, which does not depend on t, then the law of S is the unique solution to* (2.1). *Moreover, (X_n) converges to S in distribution, for any X_0.*

The only natural candidate for the limit S is the series

$$S := \sum_{k=1}^{\infty} B_k \prod_{j=1}^{k-1} A_j. \tag{2.4}$$

If $\mathbb{P}(A = 0) > 0$, then $N = \inf\{n \geq 1 : A_n = 0\}$ is a.s. finite and

$$S_n(t) = \sum_{k=1}^{N} B_k \prod_{j=1}^{k-1} A_j$$

for all $n \geq N$. Thus, the condition $\mathbb{P}(A = 0) > 0$ ensures the a.s. convergence of $(S_n(t))$ for all $t \in \mathbb{R}$.

For $x > 0$, let $f_A(x) = \int_0^x \mathbb{P}(|A| < e^{-t})\, dt$. The following theorem characterizes almost sure convergence of $(S_n(X_0))$. It follows from a more general result in [13], Theorem 2.1.

THEOREM 2.2. *Suppose that $\mathbb{P}(B = 0) < 1$ and $\mathbb{P}(A = 0) = 0$. Then,*

$$\sum_{n=1}^{\infty} |B_n| \prod_{j=1}^{n-1} |A_j| < \infty \quad a.s. \tag{2.5}$$

is equivalent to

$$\prod_{j=1}^{n} A_j \to 0 \quad (n \to \infty a.s.) \quad and \quad \int_{(1,\infty)} \frac{\log b}{f_A(\log b)} \mathbb{P}_{|B|}(db) < \infty. \tag{2.6}$$

Each of the above equivalent conditions (2.5) *and* (2.6) *implies that, a.s.,*

$$S_n(X_0) \to S, \quad n \to \infty. \tag{2.7}$$

Conversely, if

$$\mathbb{P}(Ac + B = c) < 1 \quad for\ all\ c \in \mathbb{R}, \tag{2.8}$$

and (2.6) *does not hold, then*

$$\left|S_n(X_0)\right| \xrightarrow{\mathbb{P}} \infty, \quad n \to \infty.$$

According to [13], Corollary 4.1, or [6], Theorem 2.1.3, a sufficient condition for (2.6) is

$$\mathbb{E}[\log|A|] < 0 \quad \text{and} \quad \mathbb{E}[\log^+|B|] < \infty. \tag{2.9}$$

If (2.8) does not hold, that is, there exists $c \in \mathbb{R}$ such that $Ac + B = c$, a.s., then the law of $X \equiv c$ is the unique solution to (2.1).

The following result follows from Lemma 2.1 and Theorem 2.1; or from [13], Theorem 3.1.

COROLLARY 2.3. *Assume that the nondegeneracy condition* (2.8) *is satisfied.*

(*i*) *If* $\mathbb{P}(A = 0) > 0$ *or* (2.6) *holds, then for every* X_0,

$$X_n \xrightarrow{d} X, \quad n \to \infty,$$

where the law of X *is the unique solution to* (2.1).

(*ii*) *If* $\mathbb{P}(A = 0) = 0$ *and* (2.6) *fails, then for every* X_0,

$$|X_n| \xrightarrow{\mathbb{P}} \infty, \quad n \to \infty.$$

We say that a real random variable Y is stochastically majorized by Z, and we write $Y \le_{st} Z$, if $\mathbb{P}(Y \le x) \ge \mathbb{P}(Z \le x)$ for all $x \in \mathbb{R}$.

LEMMA 2.4. *Consider* (X_n) *defined in* (2.2). *If* $A \ge 0$, *a.s., and* $X_1 \ge_{st} X_0$, *then for all* $n \ge 1$,

$$X \ge_{st} X_{n+1} \ge_{st} X_n \ge_{st} X_0.$$

PROOF. It is enough to show that $X_{n+1} \ge_{st} X_n$ for all $n \ge 1$. We proceed by induction. Suppose that $X_n \ge_{st} X_{n-1}$. Since $A_{n+1} \ge 0$ a.s., we have

$$X_{n+1} = A_{n+1}X_n + B_{n+1} \ge_{st} A_{n+1}X_{n-1} + B_{n+1} \stackrel{d}{=} X_n. \qquad \square$$

If both A and B are nonnegative and $X_0 = 0$ then $X_1 = B_1 \ge 0$ and, therefore, the assumptions of Lemma 2.4 are satisfied. In this case, for all $x \ge 0$ and $n \ge 1$,

$$\mathbb{P}(X \le x) \le \mathbb{P}(X_n \le x). \tag{2.10}$$

3. Local dependence measure and Legendre-type transformation.

From now on, we will assume that the coefficients A and B of the stochastic fixed-point equation (2.1) are nonnegative, that is, $A, B \ge 0$, a.s.

The concept of a regularly varying function is well known. For the definition and a review of properties of regularly varying function needed in this project, see [4] or [9], Section 2.

DEFINITION 3.1 ([9]). We say that a nonnegative random variable X has an inverse exponential decay of the left tail with degree $\rho > 0$ if

$$\lim_{x \to 0^+} \frac{-\log \mathbb{P}(X < x)}{H(x)} = \lambda, \tag{3.1}$$

for a regularly varying function H with index $-\rho$ at zero and $\lambda \in [0, \infty]$. We call such a random variable $\mathrm{IED}_H^\rho(\lambda)$-random variable.

Sometimes we will write $f(x) \sim g(x)$, $x \to \infty$, to indicate that $\lim_{x\to\infty} f(x)/g(x) = 1$ (the same notation will apply in the case when x goes to a different limit).

REMARK 3.2. We will argue that if H is regularly varying with index $-\rho < 0$ at 0 then there exists a continuous strictly decreasing regularly varying function $\widetilde{H}$ with index $-\rho < 0$ at 0, such that

$$\lim_{x\to 0^+} H(x)/\widetilde{H}(x) = 1.$$

To see this, first apply the smooth variation theorem ([4], Theorem 1.8.2), which states that for any regularly varying function f there exists a smooth function f_1 with $f(x) \sim f_1(x)$. So, we have $H(x) \sim H_1(x)$ for some smooth H_1.

Then, the monotone equivalent $\widetilde{H}$ to H_1 can be constructed using [4], Theorem 1.5.3. By continuity of H_1, the function $\widetilde{H}$ will be also continuous. For more details, see [4], Theorem 1.5.4, or [5], Corollary 4.2. The function $\widehat{H}(x) := \widetilde{H}(x)/\log(e + x)$ is continuous, strictly decreasing, and regularly varying with index $-\rho < 0$ at 0. We also have $\widehat{H}(x) \sim H(x)$ when $x \to 0$.

Without loss of generality, we will assume from now on that every regularly varying function H is continuous and strictly monotone. Its inverse H^{-1} is regularly varying with index $-1/\rho$ at ∞.

The ultimate goal of this project is to develop an effective tool for the analysis of the lower tail of the solution to (2.1). The random variables A and B in that formula are not necessarily independent. We will quantify their dependence using the "local dependence measure" (LDM) defined below.

DEFINITION 3.3. We will say that, for a pair of nonnegative random variables (A, B), a function $g : [0,\infty) \to [0,\infty]$ is their (ρ, H)-local dependence measure ((ρ, H)-LDM) if for a regularly varying function H with index $-\rho < 0$ at 0,

$$g(y) = \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(\varepsilon Ay + B < \varepsilon)}{H(\varepsilon)}. \tag{3.2}$$

REMARK 3.4. If $g(0) > 0$, then (3.2) implies

$$\lim_{\varepsilon\to 0^+} \frac{\log \mathbb{P}(\varepsilon Ay + B < \varepsilon)}{\log \mathbb{P}(B < \varepsilon)} = \frac{g(y)}{g(0)}, \qquad y \ge 0.$$

Similar conditions for the distribution of a pair (A, B) were considered in literature in related context. In particular, in [7], Theorem 2.1, it is assumed that there exists a finite function f such that

$$\lim_{x\to\infty} \frac{\mathbb{P}(Ay + B > x)}{\mathbb{P}(B > x)} = f(y), \qquad y \in \mathbb{R}. \tag{3.3}$$

If A and B above are independent, A has a finite moment generating function and $x \mapsto \mathbb{P}(e^B > x)$ is regularly varying with index $-\alpha \le 0$ at ∞, then by the Breiman lemma (see [11]) we have $f(y) = \mathbb{E}[e^{\alpha Ay}]$. However, if A and B are not independent, yet (3.3) holds, then f may be of a different form (see [7], Remark 2.3). Another condition of similar nature was stated in [19], (9).

In Section 5 we will show that if random variables A and B are positively quadrant dependent, then g defined in (3.2) can be given explicitly. If A and B are not positively quadrant dependent then the form of g may vary significantly (see Example 3.7 and Proposition 7.7).

LEMMA 3.5. *If g is (ρ, H)-LDM, then $g : [0,\infty) \to [0,\infty]$ is a nondecreasing function.*

PROOF. If $y_1 \le y_2$ then $\mathbb{P}(\varepsilon A y_2 + B < \varepsilon) \le \mathbb{P}(\varepsilon A y_1 + B < \varepsilon)$. This and (3.2) imply that $g(y_1) \le g(y_2)$. $\square$

REMARK 3.6. If (3.2) holds for $y = 0$ then B is an $\mathrm{IED}_H^{\rho}(g(0))$-random variable.

EXAMPLE 3.7. Some of our results hold only if the LDM g is continuous at 0. In general, g need not be continuous at 0. We demonstrate this by means of an example. Here we do not make any claims concerning continuity of g on $(0, \infty)$.

Let $A = V/U$ and $B = U$, where V and U are positive continuous random variables such that $\mathbb{P}(V < v) = e^{-1/v}$ for $v > 0$, and

$$\mathbb{P}(U \in du, V \in dv) = \big(c_1 e^{-\lambda_1/u} \mathbf{1}_{(v<1)} + c_2 e^{-\lambda_2/u} \mathbf{1}_{(v \ge 1)}\big) \mathbf{1}_{(u \le 1)}\, du \mathbb{P}(V \in dv),$$

where $\lambda_1 > \lambda_2 > 0$, and c_1 and c_2 are positive normalizing constants.

For $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(\varepsilon A y + B < \varepsilon) &= \mathbb{P}\big(\varepsilon (V/U) y + U < \varepsilon\big) = \mathbb{P}\left(V < \frac{U(\varepsilon - U)}{\varepsilon y}\right) \\ &= \mathbb{P}\left(V < \frac{U(\varepsilon - U)}{\varepsilon y}, U \in (0, \varepsilon)\right). \end{aligned} \tag{3.4}$$

For fixed $y > 0$ and small enough $\varepsilon > 0$ we have $u(\varepsilon - u)/(\varepsilon y) < \varepsilon / y < 1$ for $u \in (0, \varepsilon)$. Hence we obtain from (3.4), using the substitution $u = \varepsilon t$,

$$\begin{aligned} &\mathbb{P}(\varepsilon A y + B < \varepsilon) \\ &\quad = \int_0^{\varepsilon} \int_0^{u(\varepsilon - u)/\varepsilon y} c_1 e^{-\lambda_1/u} \mathbb{P}_V(dv)\, du = \int_0^{\varepsilon} c_1 e^{-\lambda_1/u} \mathbb{P}\left(V < \frac{u(\varepsilon - u)}{\varepsilon y}\right) du \\ &\quad = \varepsilon \int_0^1 c_1 e^{-\lambda_1/\varepsilon t} \mathbb{P}\left(V < \frac{\varepsilon t (1-t)}{y}\right) dt = \varepsilon \int_0^1 c_1 \exp\left(-\frac{\lambda_1}{\varepsilon t}\right) \exp\left(-\frac{y}{\varepsilon t (1-t)}\right) dt \\ &\quad = \varepsilon c_1 \int_0^1 \exp\left(-\frac{\lambda_1 + y}{\varepsilon t} - \frac{y}{\varepsilon (1-t)}\right) dt. \end{aligned}$$

This and Lemma 7.5 imply that

$$\begin{aligned} g(y) &= -\lim_{\varepsilon \to 0^+} \varepsilon \log \mathbb{P}(\varepsilon A y + B < \varepsilon) \\ &= -\lim_{\varepsilon \to 0^+} \varepsilon \log(\varepsilon c_1) - \lim_{\varepsilon \to 0^+} \varepsilon \log\left(\int_0^1 \exp\left(-\frac{\lambda_1 + y}{\varepsilon t} - \frac{y}{\varepsilon(1-t)}\right) dt\right) \\ &= (\sqrt{\lambda_1 + y} + \sqrt{y})^2. \end{aligned}$$

Hence $g(0^+) = \lambda_1$. Since $\lambda_2 < \lambda_1$, by Lemmas 7.5 and 7.6,

$$\begin{aligned} g(0) &= -\lim_{\varepsilon \to 0^+} \varepsilon \log \mathbb{P}(B < \varepsilon) = -\lim_{\varepsilon \to 0^+} \varepsilon \log \mathbb{P}(U < \varepsilon) \\ &= -\lim_{\varepsilon \to 0^+} \varepsilon \log\left(\int_0^{\varepsilon} (c_1 \mathbb{P}(V < 1) e^{-\lambda_1/u} + c_2 \mathbb{P}(V \ge 1) e^{-\lambda_2/u})\, du\right) \\ &= \lambda_2 < \lambda_1 = g(0^+). \end{aligned}$$

DEFINITION 3.8. For a function $g : [0, \infty) \to [0, \infty]$, we let

$$\phi_\rho(\lambda) = \inf_{y > 0} \left\{ g(y) + \frac{\lambda}{y^\rho} \right\}, \qquad \lambda \ge 0. \tag{3.5}$$

The Legendre-type transform $\phi_\rho(\lambda)$ will play a key role in our analysis. We will illustrate its significance with a couple of results, before deriving its basic properties.

THEOREM 3.9. *Suppose that $g:[0,\infty)\to[0,\infty]$ is the (ρ,H)-LDM for random variables (A,B), and let X be an independent $\mathrm{IED}_H^{\rho}(\lambda)$-random variable. If $g(0^+)=g(0)$ or $\lambda>0$ then $AX+B$ is an $\mathrm{IED}_H^{\rho}(\phi_\rho(\lambda))$-random variable.*

PROOF. First we will show that

$$\limsup_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(AX+B<\varepsilon)}{H(\varepsilon)}\le\inf_{y>0}\left\{g(y)+\frac{\lambda}{y^\rho}\right\}=\phi_\rho(\lambda). \tag{3.6}$$

For any $y>0$,

$$\begin{aligned}\mathbb{P}(AX+B<\varepsilon)&\ge\mathbb{P}(AX+B<\varepsilon,X<\varepsilon y)\ge\mathbb{P}(\varepsilon Ay+B<\varepsilon,X<\varepsilon y)\\&=\mathbb{P}(\varepsilon Ay+B<\varepsilon)\mathbb{P}(X<\varepsilon y).\end{aligned}$$

This implies that

$$\begin{aligned}&\limsup_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(AX+B<\varepsilon)}{H(\varepsilon)}\\&\quad\le\lim_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(\varepsilon Ay+B<\varepsilon)}{H(\varepsilon)}+\lim_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(X<\varepsilon y)}{H(\varepsilon y)}\frac{H(\varepsilon y)}{H(\varepsilon)}\\&\quad\le g(y)+\frac{\lambda}{y^\rho}.\end{aligned}$$

Since y is an arbitrary number in $(0,\infty)$, we obtain (3.6).

We will consider three cases: (i) $\lambda=0$, (ii) $\lambda>0$ and $\phi_\rho(\lambda)<\infty$, and (iii) $\lambda>0$ and $\phi_\rho(\lambda)=\infty$.

(i) Consider $\lambda=0$. By Lemma 3.12(iv) (proved below) and the assumption that $g(0^+)=g(0)$,

$$\phi_\rho(0)=g(0)=\lim_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(B<\varepsilon)}{H(\varepsilon)}\le\liminf_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(AX+B<\varepsilon)}{H(\varepsilon)}.$$

This and (3.6) prove the theorem in the case $\lambda=0$.

(ii) Under the assumption that $\lambda>0$ and $\phi_\rho(\lambda)<\infty$, there exists $a>0$ such that $\lambda/a^\rho\ge\phi_\rho(\lambda)$. Hence

$$\liminf_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(AX+B<\varepsilon,X<\varepsilon a)}{H(\varepsilon)}\ge\liminf_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(X<\varepsilon a)}{H(\varepsilon)}=\frac{\lambda}{a^\rho}\ge\phi_\rho(\lambda).$$

Since, by Lemma 3.5, g is nondecreasing, we have $\sup_{y>0}g(y)\ge\phi_\rho(\lambda)$. Therefore, for $\delta>0$ there exists $b>0$ such that $g(b)\ge\phi_\rho(\lambda)-\delta$. Thus,

$$\begin{aligned}\liminf_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(AX+B<\varepsilon,X\ge\varepsilon b)}{H(\varepsilon)}&\ge\liminf_{\varepsilon\to0^+}\frac{-\log\mathbb{P}(\varepsilon Ab+B<\varepsilon)}{H(\varepsilon)}\\&=g(b)\ge\phi_\rho(\lambda)-\delta.\end{aligned}$$

We conclude that for $\eta>0$ and small $\varepsilon>0$

$$\begin{aligned}&\mathbb{P}(AX+B<\varepsilon,X<\varepsilon a)\le\exp\bigl(-H(\varepsilon)\bigl(\phi_\rho(\lambda)-\eta\bigr)\bigr),\\&\mathbb{P}(AX+B<\varepsilon,X\ge\varepsilon b)\le\exp\bigl(-H(\varepsilon)\bigl(\phi_\rho(\lambda)-\eta\bigr)\bigr).\end{aligned}\tag{3.7}$$

For $y,h>0$ and small $\varepsilon>0$,

$$\begin{aligned}&\mathbb{P}\bigl(AX+B<\varepsilon,\varepsilon y\le X<\varepsilon(y+h)\bigr)\\&\quad\le\mathbb{P}(\varepsilon Ay+B<\varepsilon)\mathbb{P}\bigl(X<\varepsilon(y+h)\bigr)\\&\quad\le\exp\bigl(-H(\varepsilon)\bigl(g(y)-\eta\bigr)\bigr)\exp\bigl(-H(\varepsilon)\bigl(\lambda/(y+h)^\rho-\eta\bigr)\bigr)\\&\quad\le\exp\bigl(-H(\varepsilon)\bigl(g(y)+\lambda/y^\rho-\lambda/y^\rho+\lambda/(y+h)^\rho-2\eta\bigr)\bigr).\end{aligned}$$

For $y > 0$, by definition, $\phi_\rho(\lambda) \le g(y) + \lambda/y^\rho$. We have $1/y^\rho - 1/(y+h)^\rho \le h\rho/y^{\rho+1}$. Hence,

$$\mathbb{P}\big(AX + B < \varepsilon, \varepsilon y \le X < \varepsilon(y+h)\big) \le \exp\big(-H(\varepsilon)\big(\phi_\rho(\lambda) - \lambda h\rho/y^{\rho+1} - 2\eta\big)\big).$$

From this we obtain

$$\begin{aligned}
&\mathbb{P}(AX + B < \varepsilon, \varepsilon a \le X < \varepsilon b)\\
&\quad = \sum_{k=1}^{n} \mathbb{P}\big(AX + B < \varepsilon, \varepsilon(a + h_{k-1}) \le X < \varepsilon(a + h_k)\big)\\
&\quad \le \sum_{k=1}^{n} \exp\big(-H(\varepsilon)\big(\phi_\rho(\lambda) - \lambda h\rho/(a + h_{k-1})^{\rho+1} - 2\eta\big)\big)\\
&\quad \le n \exp\big(-H(\varepsilon)\big(\phi_\rho(\lambda) - \lambda h\rho/a^{\rho+1} - 2\eta\big)\big),
\end{aligned} \tag{3.8}$$

where $h_0 = 0$, and $h := h_k - h_{k-1} = (b-a)/n$ for $k = 1, \ldots, n$. Using (3.7) and (3.8) we get

$$\mathbb{P}(AX + B < \varepsilon) \le (n+2) \exp\big(-H(\varepsilon)\big(\phi_\rho(\lambda) - \lambda h\rho/a^{\rho+1} - 2\eta\big)\big).$$

Hence

$$\liminf_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(AX + B < \varepsilon)}{H(\varepsilon)} \ge \phi_\rho(\lambda) - \lambda h\rho/a^{\rho+1} - 2\eta.$$

By first letting $\eta \downarrow 0$ and then $n \uparrow \infty$ (so that $h \downarrow 0$), we get

$$\liminf_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(AX + B < \varepsilon)}{H(\varepsilon)} \ge \phi_\rho(\lambda).$$

This and (3.6) prove the theorem in this case.

(iii) If $\phi_\rho(\lambda) = \infty$, then $g(y) = \infty$ for all $y > 0$. We have

$$\begin{aligned}
\mathbb{P}(AX + B < \varepsilon) &= \mathbb{P}(AX + B < \varepsilon, X < \varepsilon y) + \mathbb{P}(AX + B < \varepsilon, X \ge \varepsilon y)\\
&\le \mathbb{P}(X < \varepsilon y) + \mathbb{P}(\varepsilon A y + B < \varepsilon) \le 2 \max\big\{\mathbb{P}(X < \varepsilon y), \mathbb{P}(\varepsilon A y + B < \varepsilon)\big\}.
\end{aligned}$$

Thus,

$$\frac{-\log \mathbb{P}(AX + B < \varepsilon)}{H(\varepsilon)} \ge \frac{-\log 2}{H(\varepsilon)} + \min\left\{\frac{-\log \mathbb{P}(X < \varepsilon y)}{H(\varepsilon)}, \frac{-\log \mathbb{P}(\varepsilon A y + B < \varepsilon)}{H(\varepsilon)}\right\}.$$

The right-hand side converges to $\min\{\lambda/y^\rho, g(y)\} = \lambda/y^\rho$ when $\varepsilon \to 0^+$. We can make λ/y^ρ arbitrarily large by choosing y small enough. This shows that $AX + B$ is an $\mathrm{IED}_H^\rho(\infty)$-random variable. Since $\phi_\rho(\lambda) = \infty$, this completes the proof. □

REMARK 3.10. The condition $g(0) = g(0^+)$ is necessary for the statement of Theorem 3.9 to hold for $\lambda = 0$. To see this, note that if $X \equiv 0$ then X is an $\mathrm{IED}_H^\rho(0)$ random variable, $AX + B = B$ and $AX + B = B$ is $\mathrm{IED}_H^\rho(g(0))$. However, if we pick A and B as in Example 3.7 then $\phi_\rho(0) = g(0^+) > g(0)$, by Lemma 3.12(iv).

COROLLARY 3.11. *Suppose that* $g : [0, \infty) \to [0, \infty]$ *is the* (ρ, H)*-LDM for random variables* (A, B). *If* $\lambda \in (0, \infty)$ *and* X *is an* $\mathrm{IED}_H^\rho(\lambda)$*-random variable, whose distribution is a solution to* (2.1) *then* $\lambda = \phi_\rho(\lambda)$.

PROOF. The claim follows from Theorem 3.9. □

We will now investigate basic properties of ϕ_ρ.

LEMMA 3.12. *Assume that g is an LDM. The function $\phi_\rho : [0,\infty) \to [0,\infty]$ defined in* (3.5) *is nondecreasing and concave. Moreover*:

(i) *If there exists $y_0 > 0$ such that $g(y_0) < \infty$, then $\phi_\rho(\lambda) < \infty$ for all $\lambda \ge 0$.*
(ii) *If g is bounded by M then ϕ_ρ is also bounded by M.*
(iii) *If there exists $\lambda \ge 0$ such that $\phi_\rho(\lambda) > \lambda$ then ϕ_ρ has at most one positive fixed point, that is, $\phi_\rho(\lambda) = \lambda$ for at most one $\lambda > 0$.*
(iv) *We have $\phi_\rho(0) = g(0^+) \ge g(0)$.*
(v) *If there exists $\lambda_0 \ge 0$ such that $\phi_\rho(\lambda_0) < \infty$, then $\phi_\rho(\lambda) < \infty$ for all $\lambda \ge 0$.*

PROOF. It follows directly from the definition (3.5) that ϕ_ρ is nondecreasing. Moreover, as the infimum of a family of affine functions, ϕ_ρ is concave.

(i) Note that $\phi_\rho(\lambda) \le g(y_0) + \lambda y_0^{-\rho}$ for $\lambda \ge 0$.
(ii) Since $\sup_{y>0} g(y) \le M$, the definition (3.5) shows that $\phi_\rho(\lambda) \le M + \lambda y^{-\rho}$ for every $y > 0$. The claim follows by letting $y \to \infty$ in (3.5).
(iii) Suppose that $\kappa \ge 0$, $\phi_\rho(\kappa) > \kappa$ and $\lambda_2 > \lambda_1 > 0$ are fixed points. We will argue that $\lambda_1 > \kappa$. Indeed, by concavity of ϕ_ρ we have

$$\phi_\rho\bigl(\alpha\kappa + (1-\alpha)0\bigr) \ge \alpha\phi_\rho(\kappa) + (1-\alpha)\phi_\rho(0) > \alpha\kappa + (1-\alpha)0$$

for all $\alpha \in (0,1)$. It follows that there are no fixed points on $(0,\kappa]$. Thus λ_1 is a convex combination of κ and λ_2, that is, there exists $\alpha \in (0,1)$ such that $\lambda_1 = \alpha\lambda_2 + (1-\alpha)\kappa$. Again, by concavity of ϕ_ρ we get

$$\phi_\rho(\lambda_1) \ge \alpha\phi_\rho(\lambda_2) + (1-\alpha)\phi_\rho(\kappa) > \alpha\lambda_2 + (1-\alpha)\kappa = \lambda_1.$$

Hence λ_1 is not a fixed point. This contradiction proves the claim.
(iv) This follows from Lemma 3.5 and (3.5).
(v) If there exists $\lambda_0 \ge 0$ such that $\phi_\rho(\lambda_0) < \infty$ then $g(y_0) < \infty$ for some $y_0 > 0$. Thus, by (i) we obtain the assertion. $\square$

LEMMA 3.13. *Assume that g is an LDM. If ϕ_ρ is finite then it is continuous.*

PROOF. By Lemma 3.12, ϕ_ρ is a concave function. A classical result in (convex) analysis says that a real-valued concave function defined on an interval is continuous on the interior of that interval. It will suffice to show that $\phi_\rho(0) = \phi_\rho(0^+)$. By Lemma 3.12(iv) we have $\phi_\rho(0) = g(0^+)$. By the definition of ϕ_ρ we have for all $y > 0$,

$$\phi_\rho(0^+) = \lim_{\lambda\to 0^+} \phi_\rho(\lambda) \le \lim_{\lambda\to 0^+} \left(g(y) + \frac{\lambda}{y^\rho} \right) = g(y).$$

By monotonicity of g and ϕ_ρ, proved in Lemmas 3.5 and 3.12,

$$g(0^+) = \phi_\rho(0) \le \phi_\rho(0^+) \le \inf_{y>0} g(y) = g(0^+).$$

This implies that $\phi_\rho(0) = \phi_\rho(0^+)$. $\square$

DEFINITION 3.14. Let

$$\lambda^* = \inf_{y>1} \left\{ \frac{y^\rho}{y^\rho - 1} g(y) \right\}.$$

LEMMA 3.15. *Assume that g is an LDM.*
(i) Suppose $c \ge 0$. Then $\phi_\rho(c) \ge c$ if and only if $c \le \lambda^*$.
(ii) If $\lambda^* < \infty$ then $\phi_\rho(\lambda^*) = \lambda^*$.

PROOF. (i) We have $\phi_\rho(c) = \inf_{y>0}\{g(y) + cy^{-\rho}\} \ge c$ if and only if

$$g(y) + cy^{-\rho} \ge c$$

for all $y > 0$. The above inequality is always satisfied for $y \le 1$. Thus, $\phi_\rho(c) \ge c$ if and only if

$$\frac{y^\rho}{y^\rho - 1} g(y) \ge c$$

for all $y > 1$, which is equivalent to $c \le \lambda^*$.

(ii) By (i), we have

$$\{c \ge 0 : \phi_\rho(c) \ge c\} = [0, \lambda^*]. \tag{3.9}$$

Thus, if $\lambda^* < \infty$, for any sequence $\lambda_n \downarrow \lambda^*$, we have $\phi_\rho(\lambda_n) < \lambda_n$. This and the continuity of ϕ_ρ imply that $\phi_\rho(\lambda^*) \le \lambda^*$. This inequality and part (i) applied to $c = \lambda^*$ yield (ii). □

PROPOSITION 3.16. *Suppose that there exists $\lambda \ge 0$ such that $\phi_\rho(\lambda) > \lambda$ and $\lambda^* < \infty$. Consider any $\lambda_1 \in [0, \lambda^*]$ and let $\lambda_n = \phi_\rho(\lambda_{n-1})$ for $n \ge 2$. Then (λ_n) is nondecreasing and converges to λ^*.*

PROOF. By Lemma 3.15(i), the assumption that $\lambda_1 \in [0, \lambda^*]$ implies that $\lambda_1 \le \phi_\rho(\lambda_1) = \lambda_2$. Since ϕ_ρ is a nondecreasing function, by Lemma 3.15(ii) we obtain that $\lambda_2 \le \lambda^*$. Arguing inductively, we can show that (λ_n) is a nondecreasing sequence which is bounded by λ^*. Thus (λ_n) converges to a limit $\mu \le \lambda^*$. By the definition of λ_n and continuity of ϕ_ρ we get

$$\mu = \lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \phi_\rho(\lambda_{n-1}) = \phi_\rho\Big(\lim_{n\to\infty} \lambda_{n-1}\Big) = \phi_\rho(\mu).$$

This and Lemmas 3.12(iii) and 3.15(ii) imply that $\mu = \lambda^*$. □

COROLLARY 3.17. *Suppose that g is the (ρ, H)-LDM for (A, B). Recall X_n's defined in (2.2) and suppose that $X_0 = 0$.*

(i) *If $g(0) > 0$ then for every $n \ge 0$, X_n is an $\mathrm{IED}_H^\rho(\lambda_n)$-random variable, where $\lambda_0 = 0$, $\lambda_1 = g(0)$ and $\lambda_n = \phi_\rho(\lambda_{n-1})$ for $n \ge 2$.*

(ii) *If $g(0) > 0$ and $\lambda^* < \infty$ then the sequence (λ_n) in (i) is nondecreasing and converges to λ^*.*

(iii) *If $\phi_\rho(0) = 0$, then (X_n) is a sequence of $\mathrm{IED}_H^\rho(0)$-random variables.*

PROOF. (i) We have $X_0 = 0$, $X_1 = B_1$, and $X_2 = A_2 B_1 + B_2$, so, by Remark 3.6 and Theorem 3.9 (since $g(0) > 0$),

$$\lim_{x\to 0^+} \frac{-\log \mathbb{P}(X_0 < x)}{H(x)} = 0,$$

$$\lim_{x\to 0^+} \frac{-\log \mathbb{P}(X_1 < x)}{H(x)} = g(0),$$

$$\lim_{x\to 0^+} \frac{-\log \mathbb{P}(X_2 < x)}{H(x)} = \phi_\rho\big(g(0)\big).$$

Part (i) follows from Theorem 3.9, by induction.

(ii) The assumption that $g(0) > 0$ implies that $\phi_\rho(0) > 0$ by Lemma 3.12(iv). Hence, the assumption of Proposition 3.16 is satisfied and we can apply Proposition 3.16 with $\lambda_1 = 0$ to conclude that $\phi_\rho(0) \le \lambda^*$. We combine this observation with Lemma 3.12(iv) to obtain $g(0) \le g(0^+) = \phi_\rho(0) \le \lambda^*$. Thus, in the notation of part (i), $\lambda_1 = g(0) \in [0, \lambda^*]$. Part (ii) now follows from Proposition 3.16.

(iii) If $\phi(0) = 0$ then $g(0^+) = g(0) = 0$ by Lemma 3.12(iv). The claim now follows from Theorem 3.9. □

4. Solutions to the fixed-point equation are IED. This section is devoted to the proof of the following result.

THEOREM 4.1. *Suppose that the* (ρ, H)*-LDM for* (A, B) *exists and assume that there exists* $\lambda \geq 0$ *such that* $\phi_\rho(\lambda) > \lambda$. *If the law of* X *is a solution to* (2.1) *then* X *is an* $\mathrm{IED}_H^\rho(\lambda^*)$ *random variable.*

The proof of the theorem will consist of several lemmas. All lemmas in this section are based on the same assumptions as those in Theorem 4.1. The following result was motivated by [14], Lemma 3. A similar idea was applied in [7], Lemma 6.2.

LEMMA 4.2. *Suppose that* $\kappa > 0$ *satisfies*

$$\phi_\rho(\kappa) > \kappa. \tag{4.1}$$

There exists a nonnegative random variable Z_κ, *independent of* (A, B), *such that*

$$-\log \mathbb{P}(Z_\kappa < \varepsilon) \sim \kappa H(\varepsilon), \quad \varepsilon \downarrow 0, \tag{4.2}$$

and

$$AZ_\kappa + B \geq_{st} Z_\kappa. \tag{4.3}$$

PROOF. By Remark 3.2 we may assume without loss of generality that H is continuous, monotone and $\lim_{t\to\infty} H(t) = 0$. Let Z_0 be a random variable independent from (A, B), with the distribution defined by $\mathbb{P}(Z_0 < \varepsilon) = e^{-\kappa H(\varepsilon)}$ for $\varepsilon > 0$. By Theorem 3.9,

$$\lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(AZ_0 + B < \varepsilon)}{H(\varepsilon)} = \phi_\rho(\kappa), \tag{4.4}$$

that is, $\mathbb{P}(AZ_0 + B < \varepsilon) = \exp(-H(\varepsilon)(\phi_\rho(\kappa) + o(1)))$. This, (4.1) and (4.4) imply that there exists $\varepsilon_0 > 0$ such that

$$\mathbb{P}(AZ_0 + B < \varepsilon) \leq \mathbb{P}(Z_0 < \varepsilon) \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Let Z_κ be a random variable independent from (A, B), with the distribution defined by $\mathbb{P}(Z_\kappa \in \cdot) = \mathbb{P}(Z_0 \in \cdot \mid Z_0 < \varepsilon_0)$. Then we have for $\varepsilon \in (0, \varepsilon_0)$,

$$\mathbb{P}(AZ_\kappa + B < \varepsilon) = \mathbb{P}(AZ_0 + B < \varepsilon \mid Z_0 < \varepsilon_0) \leq \frac{\mathbb{P}(AZ_0 + B < \varepsilon)}{\mathbb{P}(Z_0 < \varepsilon_0)}$$

$$\leq \frac{\mathbb{P}(Z_0 < \varepsilon)}{\mathbb{P}(Z_0 < \varepsilon_0)} = \mathbb{P}(Z_\kappa < \varepsilon).$$

For $\varepsilon \geq \varepsilon_0$ the above inequality holds trivially, since then $\mathbb{P}(Z_\kappa < \varepsilon) = 1$. Thus, (4.3) is satisfied. Finally, note that $\log \mathbb{P}(Z_\kappa < \varepsilon) \sim \log \mathbb{P}(Z_0 < \varepsilon) = -\kappa H(\varepsilon)$ as $\varepsilon \to 0^+$. This proves (4.2). $\square$

LEMMA 4.3. *Assume that there exists* $\lambda \geq 0$ *such that* $\phi_\rho(\lambda) > \lambda$. *We have*

$$\liminf_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(X < \varepsilon)}{H(\varepsilon)} \geq \lambda^*.$$

PROOF. By continuity of ϕ_ρ, there exists $\kappa > 0$ such that $\phi_\rho(\kappa) > \kappa$. If we apply Lemma 2.4 with X_0 equal to Z_κ from Lemma 4.2 then we obtain $X \geq_{st} X_0 = Z_\kappa$, for any $\kappa > 0$ such that $\kappa < \phi_\rho(\kappa)$. Then, by (4.2),

$$\liminf_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(X < \varepsilon)}{H(\varepsilon)} \geq \liminf_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(Z_\kappa < \varepsilon)}{H(\varepsilon)} = \kappa. \tag{4.5}$$

By (3.9) and Lemma 3.12(iii), $\sup\{\kappa : \phi_\rho(\kappa) > \kappa\} = \lambda^*$. This observation and (4.5) imply the lemma. $\square$

LEMMA 4.4.

$$\textit{If}\quad s := \limsup_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(X<\varepsilon)}{H(\varepsilon)} < \infty \quad \textit{then}\quad s \le \lambda^*.$$

PROOF. We have

$$\mathbb{P}(X<\varepsilon) = \mathbb{P}(AX+B<\varepsilon) \ge \mathbb{P}(X<\varepsilon y)\mathbb{P}(\varepsilon Ay+B<\varepsilon), \tag{4.6}$$

and this gives us

$$s \le \limsup_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(X<\varepsilon y)}{H(\varepsilon y)} \cdot \frac{H(\varepsilon y)}{H(\varepsilon)} + \limsup_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(\varepsilon Ay+B<\varepsilon)}{H(\varepsilon)} \le \frac{s}{y^\rho} + g(y).$$

Hence, for $y>1$ we have $s \le g(y)y^\rho/(y^\rho-1)$ and thus $s\le\lambda^*$. □

LEMMA 4.5. *Assume that* $\lambda^*<\infty$. *Then*

$$\limsup_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(X<\varepsilon)}{H(\varepsilon)} < \infty.$$

PROOF. Since $\lambda^*<\infty$, there exists $y>1$ such that $g(y)<\infty$. Then, for any $\eta>0$, there exists ε_0 such that for all $\varepsilon\le\varepsilon_0$,

$$\frac{-\log \mathbb{P}(\varepsilon Ay+B<\varepsilon)}{H(\varepsilon)} \le g(y)+\eta. \tag{4.7}$$

It follows from (4.6) that

$$-\log \mathbb{P}(X<\varepsilon) + \log \mathbb{P}(X<\varepsilon y) \le -\log \mathbb{P}(\varepsilon Ay+B<\varepsilon).$$

Substituting εy^k for ε in the last formula yields

$$-\log \mathbb{P}(X<\varepsilon y^k) + \log \mathbb{P}(X<\varepsilon y^{k+1}) \le -\log \mathbb{P}(\varepsilon y^k Ay+B<\varepsilon y^k).$$

If we further assume that $\varepsilon y^k\le\varepsilon_0$, by (4.7), we arrive at

$$-\log \mathbb{P}(X<\varepsilon y^k) + \log \mathbb{P}(X<\varepsilon y^{k+1}) \le (g(y)+\eta)H(\varepsilon y^k).$$

The telescoping sum argument gives

$$-\log \mathbb{P}(X<\varepsilon) + \log \mathbb{P}(X<\varepsilon y^{n+1}) \le (g(y)+\eta)\sum_{k=0}^{n} H(\varepsilon y^k),$$

provided $\varepsilon y^n\le\varepsilon_0$. This condition is satisfied if we set $n=n_\varepsilon=\lfloor\log(\varepsilon_0/\varepsilon)/\log(y)\rfloor$. With this choice of n we also have $\varepsilon y^{n_\varepsilon+1}\ge\varepsilon_0$. Thus, we obtain

$$\limsup_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(X<\varepsilon)}{H(\varepsilon)} \le (g(y)+\eta)\limsup_{\varepsilon\to 0^+}\sum_{k=0}^{n_\varepsilon} \frac{H(\varepsilon y^k)}{H(\varepsilon)}. \tag{4.8}$$

By Potter bounds [4], Theorem 1.5.6, with $C=2$ and $\delta=\rho/2$ we have $H(s)/H(t)\le C(s/t)^{-\rho+\delta}=C(s/t)^{-\rho/2}$ whenever $t\le s\le\varepsilon_0$ (we may have to decrease ε_0, if necessary). Thus, $H(\varepsilon y^k)/H(\varepsilon)\le 2y^{-k\rho/2}$ for all $k=0,\dots,n_\varepsilon$ and all $\varepsilon\le\varepsilon_0$. This ensures convergence of the series on the right-hand side of (4.8). □

PROOF OF THEOREM 4.1. Theorem 4.1 follows from Lemmas 4.3, 4.4 and 4.5 in the case when $\lambda^*<\infty$. If $\lambda^*=\infty$, then the assertion of Lemma 4.4 is satisfied. Therefore, Theorem 4.1 holds in either case. □

5. Positive quadrant dependent coefficients. We will now illustrate the concepts of LDM g and its transform ϕ_ρ by applying them to certain classes of vectors (A, B). In this section we will find a formula for LDM g in the case when the A and B are positively quadrant dependent and B is an $\mathrm{IED}_H^\rho(\lambda)$ random variable. The equation (2.1) with coefficients satisfying these assumptions was studied in [9] using different methods. We will show how the results in [9] relate to the LDM g and its transform ϕ_ρ.

DEFINITION 5.1. We call random variables A and B positively quadrant dependent if

$$\mathbb{P}(A > a, B > b) \geq \mathbb{P}(A > a)\mathbb{P}(B > b), \tag{5.1}$$

for all $a, b \in \mathbb{R}$.

If two random variables are independent then they are also positively quadrant dependent. For the proof of the following lemma, see [9], Lemma 7.3.

LEMMA 5.2. *Random variables A and B are positively quadrant dependent if and only if*

$$\mathbb{P}(A \leq a, B \leq b) \geq \mathbb{P}(A \leq a)\mathbb{P}(B \leq b) \tag{5.2}$$

for all $a, b \in \mathbb{R}$.

PROPOSITION 5.3. *Suppose that B is an $\mathrm{IED}_H^\rho(\gamma)$-random variable, (A, B) are positively quadrant dependent, and let*

$$a = \operatorname{ess\,inf}(A) = \sup\{x \in \mathbb{R} : \mathbb{P}(A < x) = 0\}.$$

Then

$$g(y) = \begin{cases} \gamma(1 - ay)^{-\rho} & y \in [0, 1/a); \\ \infty & y \geq 1/a. \end{cases}$$

PROOF. Since A is nonnegative, $a \geq 0$. The definition of a implies that

$$\mathbb{P}(\varepsilon Ay + B < \varepsilon) \leq \mathbb{P}(\varepsilon ay + B < \varepsilon).$$

This, the assumption that B is an $\mathrm{IED}_H^\rho(\gamma)$-random variable, and Definition 3.1 show that, for $y \in [0, 1/a)$,

$$\begin{aligned} g(y) &= \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(\varepsilon Ay + B < \varepsilon)}{H(\varepsilon)} \geq \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(\varepsilon ay + B < \varepsilon)}{H(\varepsilon)} \\ &= \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(B < \varepsilon(1 - ay))}{H(\varepsilon)} = \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(B < \varepsilon(1 - ay))}{H(\varepsilon(1 - ay))} \frac{H(\varepsilon(1 - ay))}{H(\varepsilon)} \\ &= \gamma(1 - ay)^{-\rho}. \end{aligned}$$

With the convention that $\log 0 = -\infty$, we get for $y \geq 1/a$,

$$\begin{aligned} g(y) &= \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(\varepsilon Ay + B < \varepsilon)}{H(\varepsilon)} \geq \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(\varepsilon ay + B < \varepsilon)}{H(\varepsilon)} \\ &\geq \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(B < 0)}{H(\varepsilon)} = \infty. \end{aligned}$$

To obtain the upper bound, consider any $y < 1/a$ and find $\delta_0 > 0$ such that for $\delta \in (0, \delta_0)$ we have $y < 1/(a+\delta)$. Then for $\delta \in (0, \delta_0)$,

$$\begin{aligned}\mathbb{P}(\varepsilon A y + B \leq \varepsilon) &\geq \mathbb{P}(\varepsilon A y + B \leq \varepsilon, A \in [a, a+\delta]) \\ &\geq \mathbb{P}(\varepsilon(a+\delta) y + B \leq \varepsilon, A \in [a, a+\delta]) \\ &\geq \mathbb{P}(\varepsilon(a+\delta) y + B \leq \varepsilon)\mathbb{P}(A \in [a, a+\delta]),\end{aligned}$$

where the last inequality follows from Lemma 5.2. By definition of a, we have $\mathbb{P}(A \in [a, a+\delta)) > 0$, so

$$\begin{aligned}g(y) &= \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(\varepsilon A y + B < \varepsilon)}{H(\varepsilon)} \\ &\leq \lim_{\varepsilon\to 0^+} \frac{-\log(\mathbb{P}(\varepsilon(a+\delta) y + B \leq \varepsilon)\mathbb{P}(A \in [a, a+\delta]))}{H(\varepsilon)} \\ &= \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(\varepsilon(a+\delta) y + B \leq \varepsilon)}{H(\varepsilon)} \\ &= \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(B \leq \varepsilon(1-(a+\delta) y))}{H(\varepsilon(1-(a+\delta) y))} \frac{H(\varepsilon(1-(a+\delta) y))}{H(\varepsilon)} \\ &= \gamma(1-(a+\delta) y))^{-\rho}.\end{aligned}$$

Letting $\delta \to 0^+$, we obtain $g(y) \leq \gamma(1-ay)^{-\rho}$, for $y \in [0, 1/a)$. $\square$

PROPOSITION 5.4. *Under assumptions of Proposition 5.3*,

$$\phi_\rho(\lambda) = \left(\gamma^{\frac{1}{1+\rho}} + a^{\frac{\rho}{1+\rho}} \lambda^{\frac{1}{1+\rho}}\right)^{1+\rho},$$

and

$$\lambda^* = \begin{cases} \gamma\left(1 - a^{\frac{\rho}{1+\rho}}\right)^{-(1+\rho)} & \textit{for } a < 1; \\ \infty & \textit{for } a \geq 1. \end{cases} \tag{5.3}$$

PROOF. We will prove the result for $\gamma > 0$. The case $\gamma = 0$ requires only minor modifications. Since $g(y)$ takes finite values only on the interval $[0, 1/a)$, we need to find the minimum of the function

$$y \mapsto g(y) + \frac{\lambda}{y^\rho} = \frac{\gamma}{(1-ay)^\rho} + \frac{\lambda}{y^\rho}$$

on the interval $(0, 1/a)$. One can show that that minimum is attained at

$$y_1 = \frac{\lambda^{\frac{1}{1+\rho}}}{(\gamma a)^{\frac{1}{1+\rho}} + a\lambda^{\frac{1}{1+\rho}}} = \frac{1}{a} \cdot \frac{a\lambda^{\frac{1}{1+\rho}}}{(\gamma a)^{\frac{1}{1+\rho}} + a\lambda^{\frac{1}{1+\rho}}} \in (0, 1/a).$$

Straightforward calculations yield the formulas for $\phi_\rho(\lambda) = \frac{\gamma}{(1-ay_1)^\rho} + \frac{\lambda}{y_1^\rho}$ and $\lambda^* = \phi_\rho(\lambda^*)$ given in the proposition. $\square$

We will illustrate the meaning of λ^* by two results borrowed from [9]; they were stated in that paper as Theorems 7.6 and 7.8. The versions given below include λ^*, the parameter introduced only in this paper. The versions given in [9] and these in the present paper are equivalent due to (5.3).

THEOREM 5.5. *Assume that*:

(i) *A and B are nonnegative and positively quadrant dependent*.

(ii) $\mathbb{E}[\log A] < 0$ *and* $\mathbb{E}[\log^+ B] < \infty$.
(iii) B *is an* $\mathrm{IED}_H^\rho(\gamma)$*-random variable.*

Then:

(a) *The random variable* S *defined in* (2.4) *is* $\mathrm{IED}_H^\rho(\lambda^*)$.
(b) *The equation* (2.1) *has a unique solution with the same distribution as that of* S.

THEOREM 5.6. *Suppose that*:

(i) A *and* B *are nonnegative and positively quadrant dependent random variables.*
(ii) *There exists* $\beta \in (0, 1)$ *such that* $A \le \beta$, *a.s.*
(iii) $\mathbb{E}[(\log^+ B)^s] < \infty$ *for all* $s > 0$.
(iv) B *is an* $\mathrm{IED}_H^\rho(\gamma)$*-random variable.*

If the sequence (X_n) *is defined as in* (2.2) *then*

$$\liminf_{n\to\infty} \frac{X_n}{H^{-1}(\log n)} = (\lambda^*)^{1/\rho} \quad a.s.$$

The last two theorems were proved in [9] using techniques tailored for the assumption that A and B were positive quadrant dependent. Part (b) of Theorem 5.5. is a special case of Theorem 4.1. In the next section, we will prove Theorem 6.1, which is a much more general version of Theorem 5.6.

6. Local dependence measure and logarithmic lower envelope.

Recall the sequence (X_n) defined in (2.2) and set $X_0 = 0$.

THEOREM 6.1. *Assume that* $\mathbb{E}[\log A] < 0$ *and* $\mathbb{E}[\log^+ B] < \infty$. *Suppose that* g *is the* (ρ, H)*-LDM for* (A, B), $g(0) > 0$ *and* $\lambda^* \in (0, \infty)$. *Then*

$$\liminf_{n\to\infty} \frac{X_n}{H^{-1}(\log n)} = (\lambda^*)^{1/\rho}.$$

The proof of the theorem will consist of several lemmas. All lemmas in this section implicitly make the same assumptions as those in Theorem 6.1.

LEMMA 6.2. (*i*) *For every* $\varepsilon > 0$,

$$\left\{X_n \le H^{-1}\left(\frac{(1+\varepsilon)\log n}{\lambda^*}\right)\right\}$$

happens finitely often almost surely.

(*ii*) *We have*

$$\liminf_{n\to\infty} \frac{X_n}{H^{-1}(\log n)} \ge (\lambda^*)^{1/\rho} \quad a.s.$$

PROOF. (i) For any $\varepsilon > 0$ there exists $\delta \in (0, 1)$ such that $\gamma := (1-\delta)(1+\varepsilon) > 1$. Recall the notation from Corollary 3.17. The corollary shows that $\lambda_n \uparrow \lambda^*$. Hence there exist n_0 and $x_0 > 0$ such that $\mathbb{P}(X_{n_0} \le x) \le e^{-\lambda^*(1-\delta)H(x)}$ for all $x \in (0, x_0)$. By Lemma 2.4, for $n \ge n_0$ and $x \in (0, x_0)$,

$$\mathbb{P}(X_n \le x) \le e^{-\lambda^*(1-\delta)H(x)}.$$

It follows that, for large n,

$$\mathbb{P}\left(X_n \le H^{-1}\left(\frac{(1+\varepsilon)\log n}{\lambda^*}\right)\right) \le e^{-(1-\delta)(1+\varepsilon)\log n} = n^{-\gamma}.$$

Hence,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(X_n \le H^{-1}\left(\frac{(1+\varepsilon)\log n}{\lambda^*}\right)\right) < \infty,$$

and the claim follows by the Borel–Cantelli lemma.

(ii) Part (i) implies that for every $\varepsilon > 0$, a.s.,

$$\liminf_{n\to\infty} \frac{X_n}{H^{-1}((1+\varepsilon)(\log n)/\lambda^*)} \ge 1.$$

But H^{-1} is regularly varying with index $-1/\rho$ at infinity and thus

$$H^{-1}\left(\frac{(1+\varepsilon)\log n}{\lambda^*}\right) \sim \left(\frac{\lambda^*}{1+\varepsilon}\right)^{1/\rho} H^{-1}(\log n).$$

Hence, a.s.,

$$\liminf_{n\to\infty} \frac{X_n}{H^{-1}(\log n)} \ge \left(\frac{\lambda^*}{1+\varepsilon}\right)^{1/\rho}.$$

Part (ii) follows by letting $\varepsilon \to 0$. $\square$

LEMMA 6.3. *For all $n \ge 1$, $y > 0$ and $\varepsilon > 0$ we have, a.s.,*

$$\mathbb{P}(X_n < \varepsilon \mid X_0) \ge \mathbf{1}_{[0,\varepsilon y^n)}(X_0) \prod_{k=0}^{n-1} \mathbb{P}\left(\varepsilon y^k A y + B < \varepsilon y^k\right). \tag{6.1}$$

PROOF. We have

$$\begin{aligned}
&\mathbb{P}(X_n < \varepsilon \mid X_0) \\
&\quad \ge \mathbb{P}(A_n X_{n-1} + B_n < \varepsilon, X_{n-1} < \varepsilon y \mid X_0) \\
&\quad \ge \mathbb{P}(\varepsilon A_n y + B_n < \varepsilon, X_{n-1} < \varepsilon y \mid X_0) = \mathbb{P}(\varepsilon A_n y + B_n < \varepsilon)\mathbb{P}(X_{n-1} < \varepsilon y \mid X_0) \\
&\quad = \mathbb{P}(\varepsilon A y + B < \varepsilon)\mathbb{P}(X_{n-1} < \varepsilon y \mid X_0).
\end{aligned}$$

The assertion follows by induction. $\square$

We state, without formal proofs, three simple results, for reference. Recall that $\lambda^* = \inf_{y>1}\{\frac{g(y)y^\rho}{y^\rho - 1}\}$.

LEMMA 6.4. *Assume that $\lambda^* \in (0, \infty)$. For any $\alpha > 0$, there exists $y_* > 1$ such that*

$$\lambda^* \le \frac{g(y_*)y_*^\rho}{y_*^\rho - 1} \le \lambda^*(1+\alpha).$$

LEMMA 6.5. *For any $\alpha > 0$ and $y > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\mathbb{P}(\varepsilon A y + B < \varepsilon) \ge e^{-(1+\alpha)g(y)H(\varepsilon)}.$$

Recall that $H(\varepsilon y) \sim y^{-\rho} H(\varepsilon)$ as $\varepsilon \to 0^+$. The following result is an application of Potter bounds to function H (see [4], Theorem 1.5.6).

LEMMA 6.6. *For any $\alpha > 0$ and $\eta \in (0, \rho)$, there exists $\varepsilon_1 > 0$ such that if $0 < \varepsilon \le \varepsilon y \le \varepsilon_1$ then*

$$\frac{H(\varepsilon y)}{H(\varepsilon)} \le (1+\alpha) y^{-\rho+\eta}.$$

LEMMA 6.7. *For any $\delta > 0$ and $n \ge 1$, there exist $y_* > 1$ and $\tilde{\varepsilon} > 0$ such that*

$$\mathbb{P}(X_n < \varepsilon \mid X_0) \ge \mathbf{1}_{[0,\varepsilon y_*^n)}(X_0) \exp\bigl(-(1+\delta)\lambda^* H(\varepsilon)\bigr),$$

provided $\varepsilon y_*^{n-1} < \tilde{\varepsilon}$.

PROOF. Fix $\alpha > 0$ and let $y_* > 1$ be as in Lemma 6.4. By Lemma 6.5 there exists $\varepsilon_0 > 0$ such that

$$\mathbb{P}(\varepsilon A y_* + B < \varepsilon) \ge \exp\bigl(-(1+\alpha) g(y_*) H(\varepsilon)\bigr)$$

for all $\varepsilon \in (0, \varepsilon_0)$. Thus, by Lemma 6.3, we obtain

$$\mathbb{P}(X_n < \varepsilon \mid X_0) \ge \mathbf{1}_{[0,\varepsilon y_*^n)}(X_0) \exp\left(-(1+\alpha) g(y_*) \sum_{l=0}^{n-1} H\bigl(\varepsilon y_*^l\bigr)\right),$$

provided $\varepsilon y_*^{n-1} < \varepsilon_0$. By Lemma 6.6, for $\eta \in (0, \rho)$,

$$H\bigl(\varepsilon y_*^k\bigr) \le (1+\alpha) y_*^{-k(\rho-\eta)} H(\varepsilon), \qquad k = 0, 1, \dots, n-1,$$

as long as $\varepsilon y_*^{n-1} < \varepsilon_1$. Hence, if $\varepsilon y_*^{n-1} < \tilde{\varepsilon} := \min\{\varepsilon_0, \varepsilon_1\}$, then

$$\mathbb{P}(X_n < \varepsilon \mid X_0) \ge \mathbf{1}_{[0,\varepsilon y_*^n)}(X_0) \exp\left(-(1+\alpha)^2 g(y_*) \sum_{k=0}^{n-1} y_*^{-k(\rho-\eta)} H(\varepsilon)\right). \tag{6.2}$$

By Lemma 6.4, for sufficiently small $\eta > 0$,

$$\begin{aligned} g(y_*) \sum_{k=0}^{n-1} y_*^{-k(\rho-\eta)} &= g(y_*) \frac{y_*^{\rho-\eta}}{y_*^{\rho-\eta} - 1} \bigl(1 - y_*^{-n(\rho-\eta)}\bigr) \le (1+\alpha) g(y_*) \frac{y_*^{\rho}}{y_*^{\rho} - 1} \\ &\le (1+\alpha)^2 \lambda^*. \end{aligned}$$

This and (6.2) show that

$$\mathbb{P}(X_n < \varepsilon \mid X_0) \ge \mathbf{1}_{[0,\varepsilon y_*^n)}(X_0) \exp\bigl(-(1+\alpha)^4 \lambda^* H(\varepsilon)\bigr).$$

The lemma follows if we take $(1+\alpha)^4 = 1 + \delta$. □

We will need the following version of the Borel–Cantelli lemma.

LEMMA 6.8.

(i) *Suppose that $(\mathcal{F}_n)$ is a filtration such that $\mathcal{F}_0 = \{\varnothing, \Omega\}$, and $A_n \in \mathcal{F}_n$ for $n \ge 0$. Then*

$$\{A_n \quad i.o.\} = \left\{\sum_{n=1}^{\infty} \mathbb{P}(A_n \mid \mathcal{F}_{n-1}) = \infty\right\}.$$

(ii) *Suppose that (X_n) is a Markov process with respect to a filtration $(\mathcal{F}_n)$ such that $\mathcal{F}_0 = \{\varnothing, \Omega\}$, and $A_n \in \sigma(X_n)$ for $n \ge 1$. Then*

$$\{A_n \quad i.o.\} = \left\{\sum_{n=1}^{\infty} \mathbb{P}(A_n \mid X_{n-1}) = \infty\right\}.$$

PROOF. For (i), see [10], Theorem 5.1.2. Part (ii) is an easy corollary of (i). □

We state the following well-known Kronecker's lemma without proof.

LEMMA 6.9. *If $a_n \uparrow \infty$ and $\sum_{n=1}^{\infty} x_n/a_n$ converges then* $\lim_{n\to\infty} \frac{1}{a_n} \sum_{m=1}^{n} x_m = 0$.

We will need the following result on the ergodicity for subsequences of the iterated stochastic sequence.

LEMMA 6.10. *Suppose that X is a solution to* (1.1). *For any bounded uniformly continuous functions f on $\mathbb{R}$ and any increasing integer sequence (n_k), a.s.,*

$$L(f) := \limsup_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} f(X_{n_k}) \geq \mathbb{E}\big[f(X)\big] \geq l(f) := \liminf_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} f(X_{n_k}). \tag{6.3}$$

Moreover, $L(f)$ and $l(f)$ are constants a.s.

PROOF. For $r \geq 1$, we define

$$X_n^r := \begin{cases} 0 & n \leq r; \\ A_n X_{n-1}^r + B_n & n > r. \end{cases}$$

We have assumed that $\mathbb{E}[\log A] < 0$ so $\lim_{n\to\infty} \prod_{j=r+1}^{n} A_j = 0$, a.s. Therefore, when $n \to \infty$, a.s.,

$$X_n - X_n^r = \left(\prod_{j=r+1}^{n} A_j\right) X_r \to 0.$$

Hence $\lim_{n\to\infty} f(X_n) - f(X_n^r) = 0$, a.s., and it follows that, a.s.,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} f(X_{n_k}) - f\big(X_{n_k}^r\big) = 0.$$

This implies that, a.s.,

$$\limsup_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} f(X_{n_k}) = \limsup_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} f\big(X_{n_k}^r\big), \tag{6.4}$$

$$\liminf_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} f(X_{n_k}) = \liminf_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} f\big(X_{n_k}^r\big). \tag{6.5}$$

For every fixed $r > 0$, the random variables on the right hand sides of (6.4) and (6.5) are measurable with respect to the σ-field $\mathcal{G}_r := \sigma((A_n, B_n) : n > r)$. Thus the same applies to the random variables on the left hand sides of (6.4) and (6.5). Hence, these random variables are measurable with respect to the σ-field $\mathcal{G}_\infty := \bigcap_{r=1}^{\infty} \mathcal{G}_r$. By the Kolomogorov 0–1 law, random variables on both sides of (6.4) and (6.5) are constant, a.s.

By Corollary 2.3(i), $X_n \to X$ in distribution. This implies that $\lim_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$. We combine this observation with Fatou's lemma (f need not be nonnegative, but it is bounded) to obtain,

$$\begin{aligned} \liminf_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} f(X_{n_k}) &= \mathbb{E}\left[\liminf_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} f(X_{n_k})\right] \leq \lim_{m\to\infty} \mathbb{E}\left[\frac{1}{m} \sum_{k=1}^{m} f(X_{n_k})\right] \\ &= \mathbb{E}\big[f(X)\big]. \end{aligned}$$

This proves the inequality on the right hand side of (6.3). The inequality on the left hand side follows by applying the claim to $-f$ in place of f. □

LEMMA 6.11. (*i*) *For every* $\varepsilon > 0$,

$$\left\{X_n \le H^{-1}\left(\frac{\log n}{\lambda^*(1+\varepsilon)}\right)\right\}$$

happens infinitely often almost surely.

(*ii*) *Almost surely,*

$$\liminf_{n\to\infty} \frac{X_n}{H^{-1}(\log n)} \le (\lambda^*)^{1/\rho}.$$

PROOF. Fix any $\varepsilon > 0$. Let $(k_n)_n$ be a strictly increasing sequence of integers. Since $(X_{k_{n+1}-k_n} \mid X_0) \stackrel{d}{=} (X_{k_{n+1}} \mid X_{k_n})$ for any $\delta > 0$ and $n \ge 1$, by Lemma 6.7 there exist $y_* > 1$ and $\tilde{\varepsilon} > 0$ such that, a.s., for $t > 0$,

$$\mathbb{P}(X_{k_{n+1}} < t \mid X_{k_n}) \ge \mathbf{1}_{[0, t y_*^{k_{n+1}-k_n})}(X_{k_n}) e^{-(1+\delta)\lambda^* H(t)} \tag{6.6}$$

provided

$$t y_*^{k_{n+1}-k_n-1} < \tilde{\varepsilon}.$$

By Lemma A.1 we can choose the sequence (k_n), so it satisfies for each $n \ge 1$,

$$H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\varepsilon)}\right) y_*^{k_{n+1}-k_n-1} < \tilde{\varepsilon},$$

$$H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\varepsilon)}\right) y_*^{k_{n+1}-k_n} \ge c,$$

where $c \in (0, \tilde{\varepsilon} y_*)$. Then, taking $t = H^{-1}(\frac{\log k_{n+1}}{\lambda^*(1+\varepsilon)})$ in (6.6), we have, a.s.,

$$\mathbb{P}\left(X_{k_{n+1}} < H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\varepsilon)}\right) \mid X_{k_n}\right) \ge \mathbf{1}_{[0,c)}(X_{k_n}) \frac{1}{k_{n+1}^{\gamma}}, \tag{6.7}$$

where $\gamma = \frac{1+\delta}{1+\varepsilon}$. Take $\delta < \varepsilon$ so that $\gamma < 1$. By Lemma A.1, there exists $K > 0$ such that $k_n^{\gamma} \le K n$ for all n.

We have, a.s.,

$$\begin{aligned} &\limsup_{m\to\infty} \frac{1}{k_{m+1}^{\gamma}} \sum_{n=0}^{m} \mathbf{1}_{[0,c)}(X_{k_n}) \\ &\quad \ge \limsup_{m\to\infty} \frac{K^{-1}}{m+1} \sum_{n=0}^{m} \mathbf{1}_{[0,c)}(X_{k_n}) \ge \limsup_{m\to\infty} \frac{K^{-1}}{m+1} \sum_{n=0}^{m} f_c(X_{k_n}) \\ &\quad \ge \mathbb{E}[f_c(X)]/K \ge \mathbb{P}(X < c/2)/K > 0, \end{aligned} \tag{6.8}$$

where the first inequality on the second line of (6.8) follows from Lemma 6.10 applied to the function

$$f_c(x) = \begin{cases} 1 & x < c/2; \\ 2(c-x)/c & x \in [c/2, c]; \\ 0 & x > c. \end{cases}$$

The last inequality in (6.8) follows from Theorem 4.1 because we assumed that $\lambda^* \in (0, \infty)$ in Theorem 6.1.

Kronecker's lemma (Lemma 6.9) and (6.8) imply that

$$\sum_{n=0}^{\infty} \mathbf{1}_{[0,c)}(X_{k_n}) \frac{1}{k_{n+1}^{\gamma}} = \infty \quad \text{a.s.}$$

Hence, in view of (6.7), a.s.,

$$\sum_{n=1}^{\infty} \mathbb{P}\left(X_{k_{n+1}} < H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\varepsilon)} \right) \mid X_{k_n} \right) = \infty.$$

This and Lemma 6.8(b) imply part (i) of the present lemma.

Recall that H^{-1} is a regularly varying function at ∞ with index $-1/\rho$ to see that part (ii) of the lemma follows from part (i). □

PROOF OF THEOREM 6.1. The theorem follows from Lemmas 6.2 and 6.11. □

7. Application to Fleming–Viot-type process. This section is devoted to the proof of Theorem 7.1, a version of the law of iterated logarithm for a Fleming–Viot-type process. This result was the primary motivation for introducing and analyzing the "local dependence measure."

Fleming–Viot-type processes were originally defined in [8]. The specific model discussed below appeared in [3, 15]. Any Fleming–Viot process has a unique spine, that is, a trajectory inside the branching tree that never hits the boundary of the domain where the process is confined; this was proved under strong assumptions in [15], Theorem 4, and later in the full generality in [1].

We will now define a Fleming–Viot process and other elements of the model. Informally, the process consists of two independent Brownian particles starting at the same point in $(0, \infty)$. At the time when one of them hits 0, it is killed and the other one branches into two particles. The new particles start moving as independent Brownian motions and the scheme is repeated.

On the formal side, let $(W_1(t) : t \geq 0)$ and $(W_2(t) : t \geq 0)$ be two independent Brownian motions starting from $W_1(0) = W_2(0) = 1$. Let

$$\begin{aligned} T_0 &= 0, \\ Y_0 &= 1, \\ \tau_j &= \inf\{t \geq 0 : W_j(t) = 0\}, \qquad j = 1, 2, \\ T_1 &= \min(\tau_1, \tau_2), \\ Y_1 &= \max\bigl(W_1(T_1), W_2(T_1)\bigr), \end{aligned}$$

and for $k \geq 2$,

$$\begin{aligned} T_k &= \inf\bigl\{t > T_{k-1} : \min\bigl(W_1(t) - W_1(T_{k-1}) + Y_{k-1}, W_2(t) - W_2(T_{k-1}) + Y_{k-1}\bigr) = 0\bigr\}, \\ Y_k &= \max\bigl(W_1(T_k) - W_1(T_{k-1}) + Y_{k-1}, W_2(T_k) - W_2(T_{k-1}) + Y_{k-1}\bigr). \end{aligned}$$

It follows from [2], Theorem 5.4, or [15], Theorem 1, that $T_k \to \infty$, a.s. Hence, for any $t \geq 0$ we can find j such that $t \in [T_{j-1}, T_j)$. Then we set

$$\begin{aligned} \mathcal{Y}(t) &= \bigl(Y_1(t), Y_2(t)\bigr) \\ &= \bigl(W_1(t) - W_1(T_{j-1}) + Y_{j-1}, W_2(t) - W_2(T_{j-1}) + Y_{j-1}\bigr). \end{aligned} \tag{7.1}$$

This completes the definition of $\{\mathcal{Y}(t), t \geq 0\}$, an example of a Fleming–Viot process. If Z is the spine of our process, we have $Z(T_k) = Y_k$ for all k.

The specific Fleming–Viot process defined above was one of the models analyzed in [15]. The distribution of Y_1 was given in [15], Prop. 10.

The following is the main result of this section.

THEOREM 7.1. *Almost surely,*

$$\limsup_{n\to\infty} \frac{Y_n}{\sqrt{2T_n \log\log T_n}} = 1. \tag{7.2}$$

The main technical challenge in the proof of Theorem 7.1 comes from the fact that Y_n's and T_n's are not independent. In the last remark on page 360 of [15], the authors pointed out that the sequence $(Y_n/Y_{n-1})_{n\geq 2}$ is i.i.d. and, therefore, $(Y_n)_{n\geq 1}$ is relatively easy to analyze. A direct consequence is that the law of large numbers implies that $\log Y_n/n \to \mathbb{E}[\log Y_1] > 0$. Hence, $Y_n \to \infty$, a.s.

However, as we will see in Lemma 7.10, T_n depends on both T_{n-1} and Y_{n-1}, since the sequence in (7.16) is i.i.d. This is where the LDM theory for dependent random variables developed in the first six sections is crucially used.

We note that the law of iterated logarithm stated in (7.2) indicates (but does not prove) that the spine $Z(t)$ satisfies the same law of iterated logarithm as the three-dimensional Bessel process, which is known to have the same distribution as the one-dimensional Brownian motion conditioned not to hit 0. Hence, it is possible that the spine $Z(t)$ is distributed, at least in an asymptotic or approximate sense, as the driving Brownian motion $W_1(t)$ conditioned not to return to 0. We plan to investigate this question in a forthcoming paper.

The remaining part of this section will be devoted to the proof of Theorem 7.1, presented as a sequence of lemmas. The formulas in the first of the lemmas are taken from [16], Chapter 2, Remark 8.3 and Problem 8.6.

LEMMA 7.2. *If $W_1(0) = 1$ then for $y, t > 0$,*

$$\mathbb{P}(\tau_1 \in dt) = \frac{1}{\sqrt{2\pi t^3}} e^{-1/2t}\, dt,$$

$$\mathbb{P}\big(W_1(t) \in dy, \tau_1 > t\big) = \frac{1}{\sqrt{2\pi t}}\left(\exp\left(-\frac{(1-y)^2}{2t}\right) - \exp\left(-\frac{(1+y)^2}{2t}\right)\right) dy.$$

LEMMA 7.3. *If $W_1(0) = W_2(0) = 1$ then for $y, t > 0$,*

$$\mathbb{P}\big(W_1(\tau_2) \in dy, \tau_2 \leq t, \tau_1 > \tau_2\big) = \frac{1}{\pi}\left[\frac{\exp(-((1-y)^2+1)/(2t))}{(1-y)^2+1} - \frac{\exp(-((1+y)^2+1)/(2t))}{(1+y)^2+1}\right] dy.$$

PROOF. We use Lemma 7.2 as follows,

$$\begin{aligned}
&\mathbb{P}\big(W_1(\tau_2) \in dy, \tau_2 \leq t, \tau_1 > \tau_2\big)\\
&= \int_0^t \mathbb{P}\big(W_1(s) \in dy, \tau_1 > s\big)\mathbb{P}(\tau_2 \in ds)\\
&= \int_0^t \frac{1}{\sqrt{2\pi s}}\left(\exp\left(-\frac{(1-y)^2}{2s}\right) - \exp\left(-\frac{(1+y)^2}{2s}\right)\right) dy \frac{1}{\sqrt{2\pi s^3}} e^{-1/2s}\, ds\\
&= \int_0^t \frac{1}{2\pi s^2}\left(\exp\left(-\frac{(1-y)^2+1}{2s}\right) - \exp\left(-\frac{(1+y)^2+1}{2s}\right)\right) dy\, ds.
\end{aligned}$$

Now easy integration yields the formula stated in the lemma. $\square$

LEMMA 7.4. *If* $W_1(0) = W_2(0) = 1$ *then for* $y, t > 0$,

$$\mathbb{P}(Y_1 \in dy, T_1 \in dt) = \frac{1}{\pi t^2}\left(\exp\left(-\frac{(1-y)^2+1}{2t}\right) - \exp\left(-\frac{(1+y)^2+1}{2t}\right)\right) dt\, dy. \tag{7.3}$$

PROOF. It follows from the definition that, a.s.,

$$(Y_1, T_1) = \big(W_1(\tau_2), \tau_2\big)\mathbf{1}(\tau_1 > \tau_2) + \big(W_2(\tau_1), \tau_1\big)\mathbf{1}(\tau_2 > \tau_1),$$

so for Borel sets C,

$$\begin{aligned}\mathbb{P}(Y_1 \in C, T_1 \le t) &= \mathbb{P}\big(W_1(\tau_2) \in C, \tau_2 \le t, \tau_1 > \tau_2\big) + \mathbb{P}\big(W_2(\tau_1) \in C, \tau_1 \le t, \tau_2 > \tau_1\big) \\ &= 2\mathbb{P}\big(W_1(\tau_2) \in C, \tau_2 \le t, \tau_1 > \tau_2\big).\end{aligned}$$

The claim now follows from Lemma 7.3. □

Let $A = Y_1^{-2}$ and $B = T_1 Y_1^{-2}$. Lemma 7.4 and a standard calculation, left to the reader, show that for $a, b > 0$,

$$\mathbb{P}(A \in da, B \in db) = \frac{1}{2\pi b^2\sqrt{a}}\left[\exp\left(-\frac{(a^{1/2}-\frac{1}{2})^2+\frac{1}{4}}{b}\right) - \exp\left(-\frac{(a^{1/2}+\frac{1}{2})^2+\frac{1}{4}}{b}\right)\right] db\, da. \tag{7.4}$$

LEMMA 7.5. *Suppose that* μ *is a finite positive measure on* $[a, b]$, *it is absolutely continuous with respect to Lebesgue measure, and* $\mu(I) > 0$ *for every interval* $I \subset [a, b]$ *of strictly positive length. Assume that* f *is a continuous function on the interval* $[a, b]$. *Then*

$$\lim_{\varepsilon\to 0^+} \varepsilon \log \int_a^b e^{-f(x)/\varepsilon}\mu(dx) = -f_{\min},$$

where $f_{\min} = \inf_{x\in[a,b]} f(x)$.

PROOF. For $\varepsilon > 0$,

$$\int_a^b e^{-f(x)/\varepsilon}\mu(dx) \le e^{-f_{\min}/\varepsilon}\mu\big([a, b]\big). \tag{7.5}$$

Suppose that f attains the minimum at $x_0 \in [a, b]$. For any $\delta > 0$ there is an interval $I_\delta \subset [a, b]$ with strictly positive length, containing x_0, and such that for all $x \in I_\delta$ we have $f(x) \le f_{\min} + \delta$. Then

$$e^{-(f_{\min}+\delta)/\varepsilon}\mu(I_\delta) \le \int_a^b e^{-f(x)/\varepsilon}\mu(dx). \tag{7.6}$$

Since

$$\limsup_{\varepsilon\to 0^+} \varepsilon \log \mu\big([a, b]\big) = \liminf_{\varepsilon\to 0^+} \varepsilon \log \mu(I_\delta) = 0,$$

estimates (7.5) and (7.6) yield

$$-(f_{\min} + \delta) \le \liminf_{\varepsilon\to 0^+} \varepsilon \log \int_a^b e^{-f(x)/\varepsilon}\mu(dx) \le \limsup_{\varepsilon\to 0^+} \varepsilon \log \int_a^b e^{-f(x)/\varepsilon} \le -f_{\min}.$$

The claim follows by letting $\delta \downarrow 0$. □

The next lemma is elementary so we leave the proof to the reader.

LEMMA 7.6. *Assume that* $\lambda_1 > \lambda_2 \geq 0$, *and* f_1 *and* f_2 *are nonnegative functions such that*

$$\lim_{\varepsilon\to 0^+} \varepsilon \log f_j(\varepsilon) = -\lambda_j,$$

for $j = 1, 2$. *Then*

$$\lim_{\varepsilon\to 0^+} \varepsilon \log\bigl(f_2(\varepsilon) \pm f_1(\varepsilon)\bigr) = -\lambda_2.$$

Let $H_1(x) = x^{-1}$.

PROPOSITION 7.7. *The random vector* (A, B) *with density* (7.4) *has* $(1, H_1)$*-LDM given by*

$$g(x) = \frac{1}{2} - \frac{1}{x + 2 + \sqrt{4 + x^2}}.$$

PROOF. It has been proved in [9], Prop. 8.1, that $g(0) = 1/4$.

We will compute $g(x)$ for $x > 0$. In the following calculation we use formula (7.4), and the substitution $a = u^2$ on the last line.

$$\begin{aligned}
&\mathbb{P}(\varepsilon Ax + B < \varepsilon)\\
&\quad= \int_0^{1/x}\int_0^{\varepsilon-\varepsilon ax} \frac{1}{2\pi b^2\sqrt{a}}\biggl[\exp\biggl(-\frac{(a^{1/2}-\frac{1}{2})^2+\frac{1}{4}}{b}\biggr)\\
&\qquad - \exp\biggl(-\frac{(a^{1/2}+\frac{1}{2})^2+\frac{1}{4}}{b}\biggr)\biggr]\,db\,da\\
&\quad= \int_0^{1/x} \frac{1}{2\pi\sqrt{a}}\biggl[\frac{\exp(-\frac{(a^{1/2}-1/2)^2+1/4}{\varepsilon(1-ax)})}{(a^{1/2}-1/2)^2+1/4} - \frac{\exp(-\frac{(a^{1/2}+1/2)^2+1/4}{\varepsilon(1-ax)})}{(a^{1/2}+1/2)^2+1/4}\biggr]\,da\\
&\quad= \int_0^{1/\sqrt{x}} \frac{1}{\pi}\biggl[\frac{\exp(-\frac{(u-1/2)^2+1/4}{\varepsilon(1-u^2x)})}{(u-1/2)^2+1/4} - \frac{\exp(-\frac{(u+1/2)^2+1/4}{\varepsilon(1-u^2x)})}{(u+1/2)^2+1/4}\biggr]\,du.
\end{aligned}\tag{7.7}$$

If we define measures μ_1 and μ_2 by

$$\mu_1\bigl([x_1, x_2]\bigr) = \int_{x_1}^{x_2} \frac{1}{\pi}\frac{1}{(u-1/2)^2+1/4}\,du,$$

$$\mu_2\bigl([x_1, x_2]\bigr) = \int_{x_1}^{x_2} \frac{1}{\pi}\frac{1}{(u+1/2)^2+1/4}\,du,$$

then (7.7) can be written as

$$\begin{aligned}
&\mathbb{P}(\varepsilon Ax + B < \varepsilon)\\
&\quad= \int_0^{1/\sqrt{x}} \exp\biggl(-\frac{(u-1/2)^2+1/4}{\varepsilon(1-u^2x)}\biggr)\mu_1(du)\\
&\qquad - \int_0^{1/\sqrt{x}} \exp\biggl(-\frac{(u+1/2)^2+1/4}{\varepsilon(1-u^2x)}\biggr)\mu_2(du).
\end{aligned}\tag{7.8}$$

The function $u \mapsto \frac{(u-1/2)^2+1/4}{1-u^2x}$ attains the minimum value of

$$\frac{1}{2} - \frac{1}{x+2+\sqrt{4+x^2}},$$

at $\frac{2}{\sqrt{4+x^2}+2+x} \in (0, 1/\sqrt{x})$. Thus Lemma 7.5 implies that

$$\lim_{\varepsilon\to 0^+} \varepsilon \log \int_0^{1/\sqrt{x}} \exp\left(-\frac{(u-1/2)^2+1/4}{\varepsilon(1-u^2x)}\right)\mu_1(du) = -\frac{1}{2} + \frac{1}{x+2+\sqrt{4+x^2}}. \tag{7.9}$$

The function $u \mapsto \frac{(u+1/2)^2+1/4}{1-u^2x}$ is increasing on $[0, 1/x]$, so it achieves the minimum of $1/2$ at 0. Lemma 7.5 yields

$$\lim_{\varepsilon\to 0^+} \varepsilon \log \int_0^{1/\sqrt{x}} \exp\left(-\frac{(u+1/2)^2+1/4}{\varepsilon(1-u^2x)}\right)\mu_2(du) = -1/2.$$

This, (7.8), (7.9) and Lemma 7.6 imply that

$$\lim_{\varepsilon\to 0^+} \varepsilon \log \mathbb{P}(\varepsilon Ax + B < \varepsilon) = -\frac{1}{2} + \frac{1}{x+2+\sqrt{4+x^2}}.$$

The proposition now follows from (3.2). $\square$

Recall Definitions 3.8 and 3.14.

PROPOSITION 7.8. *We have*

$$\phi_1(\lambda) = \begin{cases} \frac{1}{4}(2\sqrt{\lambda-\lambda^2}+1) & \text{if } \lambda \in [0, 1/2), \\ 1/2 & \text{if } \lambda \geq 1/2. \end{cases} \tag{7.10}$$

The fixed point of ϕ_1 is equal to $\lambda^* = 1/2$.

PROOF. For a fixed $\lambda \in [0, 1/2)$ the function

$$x \mapsto \frac{1}{2} - \frac{1}{x+2+\sqrt{x^2+4}} + \frac{\lambda}{x}$$

attains the minimum of $\frac{1}{4}(2\sqrt{\lambda-\lambda^2}+1)$ at $x = \frac{4\sqrt{\lambda(1-\lambda)}}{1-2\lambda}$. For $\lambda \geq 1/2$, it attains the minimum of $1/2$ at $x = \infty$. This proves (7.10). It is easy to check that $\phi_1(1/2) = 1/2$ and there are no other fixed points. $\square$

LEMMA 7.9. *If X is an $\mathrm{IED}^1_{H_1}(\lambda)$-random variable with $\lambda > 0$, then*

$$\lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}(X^{-1/2} \geq t) = -\lambda.$$

PROOF. Recall that $H_1(x) = x^{-1}$ and use the definition of IED random variables. $\square$

Let

$$(\Theta_n, \Lambda_n) = \left(\frac{Y_{n+1}}{Y_n}, \frac{T_{n+1}-T_n}{Y_n^2}\right)$$

for $n \geq 0$. Then

$$\frac{T_n}{Y_n^2} = \frac{T_{n-1}+Y_{n-1}^2\Lambda_{n-1}}{\Theta_{n-1}^2 Y_{n-1}^2} = \frac{1}{\Theta_{n-1}^2}\frac{T_{n-1}}{Y_{n-1}^2} + \frac{\Lambda_{n-1}}{\Theta_{n-1}^2}.$$

If we set

$$X_0 = 0, \tag{7.11}$$

$$X_n = T_n / Y_n^2, \tag{7.12}$$

$$A_n = \Theta_{n-1}^{-2}, \tag{7.13}$$

$$B_n = \Lambda_{n-1} / \Theta_{n-1}^2, \tag{7.14}$$

for $n \geq 1$ then

$$X_n = A_n X_{n-1} + B_n. \tag{7.15}$$

LEMMA 7.10. *The sequence $(\Theta_n, \Lambda_n)_{n\geq 0}$ is i.i.d. with elements distributed as (Y_1, T_1). The sequence (A_n, B_n) is i.i.d. and its elements are distributed as (A, B) in (7.4).*

PROOF. Recall the definition (7.1). By the strong Markov property and the scaling property of Brownian motion, for every $k \geq 1$,

$$\left(\frac{\mathcal{Y}(T_k + tY_k^2)}{Y_k}, t \geq 0\right)$$

has the same distribution as $(\mathcal{Y}(t), t \geq 0)$ and is independent of $(\mathcal{Y}(t), t \in [0, T_k])$. Hence,

$$(\Theta_n, \Lambda_n)_{n\geq 0} := \left(\frac{Y_{n+1}}{Y_n}, \frac{T_{n+1} - T_n}{Y_n^2}\right)_{n\geq 0} \tag{7.16}$$

is an i.i.d. sequence with elements distributed as (Y_1, T_1).

The sequence $(A_n, B_n)_{n\geq 1}$ is i.i.d. because $(\Theta_n, \Lambda_n)_{n\geq 0}$ is i.i.d. Since $(\Theta_n, \Lambda_n) \stackrel{d}{=} (Y_1, T_1)$ for all n, it follows that (A_n, B_n) are distributed as (A, B) in (7.4). $\square$

LEMMA 7.11. *We have*

$$\lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}\big(X_1^{-1/2} \geq t\big) = -1/4. \tag{7.17}$$

PROOF. By Remark 3.6 random variable B_1 is $\mathrm{IED}_{H_1}^1(\lambda_1)$, where $\lambda_1 = g(0)$. It follows from Proposition 7.7 that $g(0) = 1/4$ so $X_1 = B_1$ is $\mathrm{IED}_{H_1}^1(1/4)$. Lemma 7.9 now yields (7.17). $\square$

We will need the following version of the results by Kesten [17] and Goldie [12], formulated in [6], Theorem 2.4.4.

THEOREM 7.12. *Assume that (A, B) satisfy the following conditions.*

(i) *$A \geq 0$, a.s., and the law of $\log A$ conditioned on $\{A > 0\}$ is nonarithmetic, that is, it is not supported on $a\mathbb{Z}$ for any $a > 0$.*
(ii) *There exists $\alpha > 0$ such that $\mathbb{E}[A^\alpha] = 1$, $\mathbb{E}[|B|^\alpha] < \infty$ and $\mathbb{E}[A^\alpha \log^+ A] < \infty$.*
(iii) *$\mathbb{P}(Ax + B = x) < 1$ for every $x \in \mathbb{R}$.*

Then the equation $X \stackrel{d}{=} AX + B$ has a solution. There exist constants c_+, c_- such that $c_+ + c_- > 0$ and

$$\mathbb{P}(X > x) \sim c_+ x^{-\alpha} \quad \text{and} \quad \mathbb{P}(X < -x) \sim c_- x^{-\alpha} \quad \text{when } x \to \infty. \tag{7.18}$$

The constants c_+, c_- are given by

$$c_+ = \frac{1}{\alpha m_\alpha}\mathbb{E}\big[(AX+B)_+^\alpha - (AX)_+^\alpha\big], \qquad c_- = \frac{1}{\alpha m_\alpha}\mathbb{E}\big[(AX+B)_-^\alpha - (AX)_-^\alpha\big],$$

where $m_\alpha = \mathbb{E}[A^\alpha \log A]$.

COROLLARY 7.13. *There exists $c_1 > 0$ such that for all $x \geq 0$,*

$$\mathbb{P}\left(\frac{Y_n}{\sqrt{T_n}} \leq x\right) \leq c_1 x. \tag{7.19}$$

PROOF. Recall (7.11)–(7.15). Suppose that the law of X is the solution to (2.1). By Lemma 2.4,

$$\mathbb{P}(X_n \geq x) \leq \mathbb{P}(X \geq x). \tag{7.20}$$

We will now verify the assumptions of Theorem 7.12. Assumptions (i) and (iii) clearly hold in view of (7.4). We will show that assumption (ii) holds for $\alpha = 1/2$.

It has been proved in [9], Prop. 8.1, that

$$\begin{aligned} \mathbb{P}(A \in da) &= \frac{4}{\pi(4a^2+1)}\,da, \quad a > 0, \\ \mathbb{P}(B > x) &\sim \frac{1}{\pi x} \quad \text{as } x \to \infty. \end{aligned} \tag{7.21}$$

These formulas imply that

$$\begin{aligned} \mathbb{E}\big[A^{1/2}\big] &= \int_0^\infty a^{1/2}\frac{4}{\pi(4a^2+1)}\,da = 1, \\ \mathbb{E}\big[A^{1/2}\log^+ A\big] &= \int_0^\infty a^{1/2}(\log^+ a)\frac{4}{\pi(4a^2+1)}\,da < \infty, \\ \mathbb{E}\big[|B|^{1/2}\big] &< \infty. \end{aligned}$$

The assumptions of Theorem 7.12 are verified so we obtain

$$\mathbb{P}(X \geq x) \sim c_+ x^{-1/2},$$

as $x \to \infty$. This and (7.20) give

$$\mathbb{P}\left(\frac{Y_n}{\sqrt{T_n}} \leq x^{-1/2}\right) = \mathbb{P}\big(X_n^{-1/2} \leq x^{-1/2}\big) \leq \mathbb{P}\big(X^{-1/2} \leq x^{-1/2}\big) \sim c_+ x^{-1/2}.$$

This implies the lemma. $\square$

PROOF OF THEOREM 7.1.. We can apply Theorem 6.1 and Proposition 7.8 to see that, a.s.,

$$\liminf_{n\to\infty}(\log n)\frac{T_n}{Y_n^2} = \liminf_{n\to\infty}(\log n)X_n = \lambda^* = \frac{1}{2}.$$

Hence, a.s.,

$$\limsup_{n\to\infty}\frac{Y_n}{\sqrt{2T_n\log n}} = 1. \tag{7.22}$$

We will show that $\frac{\log\log T_n}{\log n} \to 1$ a.s. It follows from (7.16) that $Y_n = \prod_{j=1}^{n-1}\Theta_j$.

It was pointed out in [15], page 360, that the logarithm $\log Y_1$ is integrable and it was determined numerically that $\mathbb{E}[\log Y_1] \approx 0.34$. In the following calculation we use the fact that $Y_1 = \Theta_0 = A_1^{-1/2}$, formula (7.21), and substitutions $2a = x$ and $y = 1/x$,

$$\begin{aligned}
\mathbb{E}[\log Y_1] &= -\mathbb{E}[\log A]/2 = -\frac{1}{2}\int_0^\infty \frac{4\log a}{\pi(4a^2+1)}\,da = -\int_0^\infty \frac{\log(x/2)}{\pi(x^2+1)}\,dx \\
&= -\int_0^1 \frac{\log x}{\pi(x^2+1)}\,dx - \int_1^\infty \frac{\log x}{\pi(x^2+1)}\,dx + \int_0^\infty \frac{\log 2}{\pi(x^2+1)}\,dx \\
&= -\int_0^1 \frac{\log x}{\pi(x^2+1)}\,dx + \int_0^1 \frac{\log y}{\pi(y^2+1)}\,dy + \frac{\log 2}{\pi}\arctan x|_0^\infty = \frac{\log 2}{2}.
\end{aligned}$$

Thus, by the law of large numbers, a.s.,

$$\lim_{n\to\infty} \frac{\log Y_n}{n} = \mathbb{E}[\log Y_1] = \frac{\log 2}{2}. \tag{7.23}$$

Consider any $\varepsilon > 0$. By Lemmas 2.4 and 7.11, for large n,

$$\mathbb{P}\left(\frac{T_n}{Y_n^2} \le e^{-n\varepsilon}\right) = \mathbb{P}(X_n \le e^{-n\varepsilon}) \le \mathbb{P}(X_1 \le e^{-n\varepsilon}) < \exp(-(1/8)e^{n\varepsilon}). \tag{7.24}$$

By Corollary 7.13,

$$\mathbb{P}\left(\frac{Y_n^2}{T_n} \le e^{-n\varepsilon}\right) \le c_1 e^{-n\varepsilon/2}.$$

This and (7.24) imply that

$$\sum_{n=1}^\infty \mathbb{P}\left(\left|\frac{\log T_n - 2\log Y_n}{n}\right| > \varepsilon\right) = \sum_{n=1}^\infty \left[\mathbb{P}\left(\frac{Y_n^2}{T_n} \le e^{-n\varepsilon}\right) + \mathbb{P}\left(\frac{T_n}{Y_n^2} \le e^{-n\varepsilon}\right)\right] < \infty.$$

By the Borel–Cantelli lemma, only a finite number of events $\{|\frac{\log T_n - 2\log Y_n}{n}| > \varepsilon\}$ occur, a.s. Since this holds for every rational $\varepsilon > 0$, we have $\frac{\log T_n - 2\log Y_n}{n} \to 0$ a.s. We combine this observation with (7.23) to obtain

$$\lim_{n\to\infty} \frac{\log T_n}{n} = \log 2 \quad \text{a.s.}$$

This implies that

$$\lim_{n\to\infty} \frac{\log\log T_n}{\log n} = 1 \quad \text{a.s.}$$

It follows from this and (7.22) that, a.s.,

$$\limsup_{n\to\infty} \frac{Y_n}{\sqrt{2T_n \log\log T_n}} = 1,$$

so the proof is complete. $\square$

APPENDIX

This section is a part of the proof of Theorem 6.1. Because of the specialized nature of this material we relegated it to an appendix.

LEMMA A.1. *Assume that* $\tilde{\varepsilon}, \lambda^*, \varepsilon > 0$ *and* $y_* > 1$. *Suppose that* H *is regularly varying at* 0 *with index* $-\rho < 0$ *and* H^{-1} *is its inverse. There exists a strictly increasing sequence* (k_n) *of integers such that for each* $n \geq 1$,

$$H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\varepsilon)}\right) y_*^{k_{n+1}-k_n-1} < \tilde{\varepsilon}, \tag{A.1}$$

$$H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\varepsilon)}\right) y_*^{k_{n+1}-k_n} \geq c, \tag{A.2}$$

where $c \in (0, \tilde{\varepsilon} y_*)$.

Moreover, for any $\gamma \in (0,1)$ *there exists* $K > 0$ *such that for all* $n \geq 1$,

$$k_n^\gamma \leq Kn. \tag{A.3}$$

Recall that H^{-1} is regularly varying at infinity with index $-1/\rho$. Let $f(x)$ be defined for $x > 1$ by

$$f(x) = \log \tilde{\varepsilon} - \frac{1}{\log y_*} \log H^{-1}\left(\frac{\log x}{\lambda^*(1+\varepsilon)}\right),$$

so that for any $k_n, k_{n+1} > 1$,

$$H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\varepsilon)}\right) y_*^{k_{n+1}-k_n} = \tilde{\varepsilon} y_*^{k_{n+1}-k_n-f(k_{n+1})}. \tag{A.4}$$

LEMMA A.2. *For any* $\delta > 0$, *there exist* $C_1, C_2 \in \mathbb{R}$ *such that*

$$C_1 + \frac{1/\rho - \delta}{\log y_*} \log\log x \leq f(x) \leq C_2 + \frac{1/\rho + \delta}{\log y_*} \log\log x$$

for sufficiently large x.

PROOF. By regular variation of H^{-1}, we have for any $\delta > 0$,

$$\lim_{x\to\infty} x^{1/\rho+\delta} H^{-1}(x) = \infty,$$

$$\lim_{x\to\infty} x^{1/\rho-\delta} H^{-1}(x) = 0.$$

Thus, there exists $x_1 > 0$ such that

$$x^{-1/\rho-\delta} \leq H^{-1}(x) \leq x^{-1/\rho+\delta}$$

for $x > x_1$. The assertion follows by using above inequalities in the definition of f. □

Suppose that $a_0 > 0$ is such that $f(a_0) \geq 1$ and let

$$a_{n+1} = a_n + f(a_n), \quad n \geq 0.$$

LEMMA A.3. *The following claims hold for the sequence* (a_n).

(i) $a_{n+1} \geq a_n + 1$ *for* $n \geq 0$.

(ii) *For any* $\gamma \in (0,1)$, *there exists* $K > 0$ *such that*

$$a_n^\gamma \leq Kn, \quad n \geq 1.$$

(iii) $a_{n+1}/a_n \to 1$ *as* $n \to \infty$.

PROOF. (i) Since f is nondecreasing, we have

$$a_{n+1} - a_n = f(a_n) \geq f(a_0) \geq 1.$$

(ii) We use induction and Lemma A.2. Fix $\gamma \in (0,1)$ and $\delta > 0$. Let C_2 be as in Lemma A.2. Suppose that K is so large that,

$$C_2 + \frac{1/\rho+\delta}{\gamma \log y_*}\log K + \frac{1/\rho+\delta}{\log y_*}\frac{1}{1-\gamma} \leq \frac{K^{1/\gamma}}{\gamma}. \tag{A.5}$$

Make K larger if necessary so that $a_1^\gamma \leq K$.

For the induction step, assume that $a_n^\gamma \leq Kn$ for some n. Note that $\log n \leq \frac{1}{1/\gamma - 1} n^{1/\gamma-1}$. We use this inequality and (A.5) to see that,

$$\begin{aligned}
a_{n+1} &= a_n + f(a_n) \leq (Kn)^{1/\gamma} + f\big((Kn)^{1/\gamma}\big) \\
&\leq (Kn)^{1/\gamma} + C_2 + \frac{1/\rho+\delta}{\log y_*}\log\log(Kn)^{1/\gamma} \\
&\leq (Kn)^{1/\gamma} + C_2 + \frac{1/\rho+\delta}{\log y_*}\log(Kn)^{1/\gamma} \\
&= (Kn)^{1/\gamma} + \left[C_2 + \frac{1/\rho+\delta}{\gamma\log y_*}\log K\right] + \frac{1/\rho+\delta}{\gamma\log y_*}\log n \\
&\leq (Kn)^{1/\gamma} + \left[C_2 + \frac{1/\rho+\delta}{\gamma\log y_*}\log K\right]n^{1/\gamma-1} + \frac{1/\rho+\delta}{\gamma\log y_*}\frac{1}{1/\gamma-1}n^{1/\gamma-1} \\
&\leq (Kn)^{1/\gamma} + \frac{K^{1/\gamma}}{\gamma}n^{1/\gamma-1} \leq \big(K(n+1)\big)^{1/\gamma},
\end{aligned}$$

where the last inequality follows by convexity of $x \mapsto x^{1/\gamma}$. Part (ii) follows by induction.

(iii) By the definition of (a_n) and Lemma A.2 we have

$$\frac{a_{n+1}}{a_n} = 1 + \frac{f(a_n)}{a_n} \to 1. \qquad \square$$

PROOF OF LEMMA A.1. Let

$$k_n = \lceil a_n \rceil.$$

Since $a_n \leq k_n < a_n + 1 \leq a_{n+1} \leq k_{n+1} < a_{n+1} + 1$, we have

$$\begin{aligned}
y_* &= y_*^{(a_{n+1}+1)-a_n-f(a_n)} > y_*^{k_{n+1}-k_n-f(k_{n+1})} > y_*^{a_{n+1}-(a_n+1)-f(k_{n+1})} \\
&= y_*^{-1+f(a_n)-f(k_{n+1})}.
\end{aligned} \tag{A.6}$$

By Lemma A.3(ii) $a_{n+1}/a_n \to 1$, by Lemma A.3(i) $a_n \to \infty$, and by Lemma A.2 $f(x) \to \infty$ as $x \to \infty$, so $k_{n+1}/a_n \to 1$ as $n \to \infty$. It follows that

$$f(a_n) - f(k_{n+1}) = \frac{1}{\log y^*}\log\frac{H^{-1}(\frac{\log k_{n+1}}{\lambda^*(1+\varepsilon)})}{H^{-1}(\frac{\log a_n}{\lambda^*(1+\varepsilon)})} \to 0, \quad n \to \infty.$$

Hence $y_*^{-1+f(a_n)-f(k_{n+1})}$, that is, the right-hand side of (A.6), converges to $1/y_*$ as $n \to \infty$. Thus, by (A.4) and (A.6), for large n,

$$\tilde{\varepsilon} y_* > H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\varepsilon)}\right) y_*^{k_{n+1}-k_n} \geq \frac{\tilde{\varepsilon}}{2y_*}.$$

Since $\tilde{\varepsilon}/(2y_*) < \tilde{\varepsilon} y_*$, this implies (A.1)–(A.2).

The bound (A.3) follows from Lemma A.3(ii). $\square$

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