Stochastic Processes and their Applications 190 (2025) 104740

Contents lists available at ScienceDirect

Stochastic Processes and their Applications

journal homepage: www.elsevier.com/locate/spa

Perpetuities with light tails and the local dependence measure

Julia Le Bihan, Bartosz Kołodziejek *

Faculty of Mathematics and Information Science, Warsaw University of Technology, Koszykowa 75, Warsaw, 00-662, Poland

ARTICLE INFO

MSC:
primary 60H25
60G10
37M10
secondary 60J05

Keywords:
Tail estimates
Stochastic fixed-point equation
Iterated random sequence
Perpetuity

ABSTRACT

This work investigates the tail behavior of solutions to the affine stochastic fixed-point equation of the form $X \overset{d}{=} AX + B$, where X and (A, B) are independent. Focusing on the light-tail regime, following (Burdzy et al., 2022), we introduce a local dependence measure along with an associated Legendre-type transform. These tools allow us to effectively describe the logarithmic right-tail asymptotics of the solution X.

Moreover, we extend our analysis to a related recursive sequence $X_n = A_n X_{n-1} + B_n$, where $(A_n, B_n)_n$ are i.i.d. copies of (A, B). For this sequence, we construct deterministic scaling $(f_n)_n$ such that $\limsup_{n\to\infty} X_n/f_n$ is a.s. positive and finite, with its non-random explicit value provided.

1. Introduction

We study the tail behavior of a solution X to the stochastic fixed-point equation

$$X \overset{d}{=} AX + B, \qquad X \text{ and } (A, B) \text{ are independent.} \tag{1}$$

When such a solution exists, it is referred to as a perpetuity. Our primary focus is on the light-tail regime. Throughout the paper, we assume that A and B are almost surely nonnegative and nonzero with positive probability, thus excluding trivial cases. For comprehensive background material, we refer to [1].

The influence of the joint distribution of (A, B) on the tail behavior of X is notably diverse. When conditions ensure a light-tailed solution (see [2–6]), the analysis of (1) differs fundamentally from that in the heavy-tailed setting (cf. [7–12]). Roughly, the distinction between the light- and heavy-tailed cases hinges upon whether $\mathbb{P}(A \le 1) = 1$ with light-tailed B or $\mathbb{P}(A > 1) > 0$. In particular, for light-tailed distributions, the dependence structure between A and B significantly affects the tail asymptotics. Conversely, for heavy-tailed solutions, marginal distributions predominantly determine asymptotics, with dependence structure influencing only multiplicative constants (see e.g. [8,13,14]).

Our methodology builds upon the framework established by [6], which analyzed the left-tail behavior (near 0^+) of X. Following these ideas, we introduce a local dependence measure (LDM) along with its associated Legendre-type transform. These tools form the cornerstone of our analysis and effectively describe the logarithmic right-tail asymptotics of X.

Specifically, we assume the following limit exists for all $y > 0$: (a LDM)

$$g(y) = \lim_{t\to\infty} \frac{\log \mathbb{P}(Aty + B > t)}{\log \mathbb{P}(B > t)},$$

* Corresponding author.
E-mail address: bartosz.kolodziejek@pw.edu.pl (B. Kołodziejek).

https://doi.org/10.1016/j.spa.2025.104740
Received 26 March 2025; Received in revised form 20 June 2025; Accepted 30 June 2025
Available online 17 July 2025

where the function $t \mapsto -\log \mathbb{P}(B > t)$ is regularly varying with index $\rho > 0$. Our first main result (Theorem 7.2) shows that if a solution to (1) exists and the function g satisfies a technical condition (termed admissibility), then

$$\lim_{t\to\infty} \frac{\log \mathbb{P}(X > t)}{\log \mathbb{P}(B > t)} = \lambda^*,$$

where λ^* is the unique nonzero fixed point of the function $\phi_\rho(\lambda) = \inf_{y>0}\{y^\rho \lambda + g(y)\}$. Moreover, we show that $\lambda^* = \inf_{y\in(0,1)} \left\{ \frac{g(y)}{1-y^\rho} \right\}$.

It is natural, given (1), to consider the recursive sequence

$$X_n = A_n X_{n-1} + B_n, \quad n \geq 1, \tag{2}$$

where $(A_n, B_n)_{n\geq 1}$ are i.i.d. copies of (A, B), and X_0 is independent of this sequence. This sequence forms a Markov chain whose stationary distribution solves (1). Our second main result (Theorem 8.1) identifies an upper envelope for $(X_n)_{n\geq 0}$ when $X_0 = 0$. We construct a deterministic scaling sequence $(f_n)_n$ such that

$$\limsup_{n\to\infty} \frac{X_n}{f_n}$$

is almost surely positive and finite, with its deterministic explicit value provided. To the best of our knowledge, this constitutes the first characterization of the upper envelope for a sequence defined by (2). The proof extends techniques from [6], which derived a lower envelope under related but distinct assumptions.

Our study is inspired by several previous investigations into the tail behavior of X. A common strategy in earlier work (cf. [11,15–17]) involves bounding the tail probability of X from above and below as

$$\mathbb{P}(X_n > t) \leq \mathbb{P}(X > t) \leq \mathbb{P}(X_n' > t),$$

where both $(X_n)_{n\geq 0}$ and $(X_n')_{n\geq 0}$ satisfy (2) for some special X_0 and X_0'. Thus, understanding $\mathbb{P}(X > t)$ effectively reduces to analyzing the tail asymptotics of $\mathbb{P}(AY + B > t)$, with Y independent of (A, B) but not necessarily satisfying (1).

1.1. Relation to the literature

We note that similar ideas to the definition of LDM have appeared in related literature. In [17, Theorem 1], for instance, it is assumed that there exists a finite function f such that

$$f(y) = \lim_{t\to\infty} \frac{\mathbb{P}(Ay + B > t)}{\mathbb{P}(B > t)}. \tag{3}$$

If A and B are independent, A has a finite moment generating function, and $t \mapsto \mathbb{P}(B > \log t)$ is regularly varying with index $-\alpha \leq 0$, then by the Breiman Lemma (see e.g. [18]) one obtains explicit form of $f(y) = \mathbb{E}[\exp(\alpha y A)]$. However, if A and B are not independent, yet (3) holds, the form of f may differ (see [17, Remark 2.3]). Under (3) with $\mathbb{P}(B > t) \sim a\, t^c e^{-bt}$, $\mathbb{E}[e^{bB}\mathbf{1}_{A=1}] < 1$ and some technical assumptions, [17] established that

$$\lim_{t\to\infty} \frac{\mathbb{P}(X > t)}{\mathbb{P}(B > t)} = \frac{\mathbb{E}[f(X)]}{1 - \mathbb{E}[e^{bB}\mathbf{1}_{A=1}]}.$$

A closely related scenario is presented in [14], where it is assumed that $t \mapsto \mathbb{P}(A > t)$ is regularly varying with index $-\alpha < 0$, along with the conditions $\mathbb{E}[A^\alpha] < 1$ and $\limsup_{t\to\infty} \mathbb{P}(B > t)/\mathbb{P}(A > t) < \infty$. In this setting, f is defined as

$$f(y) = \lim_{t\to\infty} \frac{\mathbb{P}(Ay + B > t)}{\mathbb{P}(A > t)},$$

and, under further technical conditions on the distribution of A, the tail asymptotics of X are described by

$$\lim_{t\to\infty} \frac{\mathbb{P}(X > t)}{\mathbb{P}(A > t)} = \frac{\mathbb{E}[f(X)]}{1 - \mathbb{E}[A^\alpha]}.$$

In the special case of independence between A and B, explicit computation of f becomes possible.

Another result obtained under similar conditions is Theorem 3 from [16], which considers a related stochastic fixed point-equation of the form $X \stackrel{d}{=} \max\{AX, B\}$, where X and the pair (A, B) are independent.

Thus, our contribution can be viewed as an advancement of this line of research, relaxing independence assumptions and thereby enabling analysis of a broader class of models.

Additionally, logarithmic asymptotics of the tail behavior of X were considered in [19], under the assumption that the functions

$$t \mapsto -\log \mathbb{P}(1/(1-A) > t) \quad \text{and} \quad t \mapsto -\log \mathbb{P}(B > t)$$

are regularly varying (or when either A or B is bounded), again requiring independence of A and B. For general dependent A and B, however, only a lower bound on the tail was provided. As noted, the light tails heavily depend on the dependence structure of (A, B). This relation seems to be captured quite well by a function h, which was defined in [19] by

$$h(t) = \inf_{s\geq 1} \left\{ -s \log \mathbb{P}\left(\frac{1}{1-A} > s, B > \frac{t}{s} \right) \right\}, \qquad t > 0.$$

There it was shown (see [19, Theorem 5.1]) that

$$\liminf_{t\to\infty} \frac{\log \mathbb{P}(X > t)}{h(t)} \geq -c$$

with an explicit constant c depending on the index of regular variation of h. In Section 9 we present an example demonstrating that this lower bound is generally not optimal.

In contrast, our paper addresses this gap by deriving precise logarithmic asymptotics within a general framework built upon the local dependence measure. Our results extend and complement previous findings in light-tailed scenarios where the tail behavior of X is predominantly governed by the distribution of B rather than A.

In [6] a less common problem in the theory of (1) was considered. If A and B are nonnegative, then X, if exists, is also nonnegative. Thus, one may investigate the tail behavior of X near the left endpoint of the support. One of the main results in [6] establishes that if the limit

$$g(y) = \lim_{\varepsilon\to 0^+} \frac{\log \mathbb{P}(A\varepsilon y + B < \varepsilon)}{\log \mathbb{P}(B < \varepsilon)}$$

exists for $y \in [0, \infty)$, where $\varepsilon \mapsto -\log \mathbb{P}(B < \varepsilon)$ is a regularly varying function with index $\rho < 0$ at 0, then—under some technical assumptions (analogous to the admissibility condition in the present paper)—one has

$$\lim_{\varepsilon\to 0^+} \frac{\log \mathbb{P}(X < \varepsilon)}{\log \mathbb{P}(B < \varepsilon)} = \inf_{y>1} \left\{ \frac{y^\rho}{y^\rho - 1} g(y) \right\}.$$

Another result in [6] concerns the lower envelope of the sequence (2) with $X_0 = 0$; specifically, there exists an explicit deterministic scaling sequence $(h_n)_n$ such that

$$\liminf_{n\to\infty} \frac{X_n}{h_n}$$

is deterministic, positive and finite.

There are clear analogies between the statements and proofs in the present paper and those in [6]. However, aside from Lemmas 6.6 and 8.5, which are taken directly from [6], all other results in our paper have independent proofs. Notably, our analysis reveals quantitatively different behavior for $\rho \in (0, 1]$ and $\rho > 1$ (see, e.g., Theorem 4.2), a phenomenon that was not observed in [6]. While [6] applied the left-tail results to a Fleming–Viot-type process, in Section 9 we illustrate our findings using a new family of distributions.

We also note that the assumptions in the present paper and in [6] are compatible. Consequently, one can readily construct a distribution of (A, B) such that both the left and right tail behaviors of X satisfying (1) are asymptotically available and the sequence $X_n = A_n X_{n-1} + B_n$ exhibits explicit lower and upper envelopes. For example, assume that the random pair (A, B) is positively quadrant dependent (see Section 4) and that for $\rho, \sigma > 0$ one has

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}(B > t)}{t^\rho} = \lambda_+ \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(B < \varepsilon)}{\varepsilon^{-\sigma}} = \lambda_-$$

together with

$$a_- = \operatorname{ess\,inf}(A) \geq 0 \qquad \text{and} \qquad a_+ = \operatorname{ess\,sup}(A) < 1.$$

Under these conditions, since $\mathbb{E}[\log A] < 0$, $\mathbb{E}[\max\{\log B, 0\}] < \infty$, Theorems 4.2 and 7.2 in this paper together with [6, Theorem 4.1 and Proposition 5.4] yield

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}(X > t)}{t^\rho} = c_+ \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+} \frac{-\log \mathbb{P}(X < \varepsilon)}{\varepsilon^{-\sigma}} = c_-,$$

where

$$c_+ = \lambda_+ \begin{cases} \left(1 - a_+^{\rho/(\rho-1)}\right)^{\rho-1}, & \text{if } \rho > 1, \\ 1, & \text{if } \rho \in (0,1), \end{cases} \qquad \text{and} \qquad c_- = \lambda_- \left(1 - a_-^{\sigma/(1+\sigma)}\right)^{-(1+\sigma)}.$$

Moreover, if $X_0 = 0$, then by Theorem 8.1 and [6, Theorem 6.1],

$$\limsup_{n\to\infty} \frac{X_n}{(\log n)^{1/\rho}} = c_+^{-1/\rho} \qquad \text{and} \qquad \liminf_{n\to\infty} \frac{X_n}{(\log n)^{-1/\sigma}} = c_-^{1/\sigma}.$$

1.2. Organization of the paper

The remainder of the paper is organized as follows. In the next section, we give a short review of the theory of regularly varying functions. Section 3 introduces the local dependence measure and its Legendre-type transform, and establishes their key properties. We show that when A and B are independent (or more generally, positively quadrant dependent), the LDM admits an explicit representation. In Section 5, we derive the asymptotics of $t \mapsto \log \mathbb{P}(AX + B > t)$ under independence between X and (A, B), where X does not necessarily satisfy (1). This is a result that is instrumental in controlling the asymptotics of $\mathbb{P}(X_n > t)$. Section 6 discusses the existence, uniqueness, and basic properties of solutions to (1). Section 7 is devoted to our first main result, the precise logarithmic tail asymptotics of the solution to (1). In Section 8 we prove our second main result: the identification of the upper envelope for the sequence $(X_n)_{n\geq 0}$ with $X_0 = 0$. Finally, in Section 9 we describe a family of distributions of pairs (A, B) for which the LDM is computable and provide its general form. We use our earlier results to demonstrate that the lower bound in [19] is not optimal in general.

2. Regular variation

We write $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$. A measurable function $f : (0,\infty) \to (0,\infty)$ is said to be regularly varying of index $\rho \in \mathbb{R}$ if and only if $f(\lambda x) \sim \lambda^\rho f(x)$ for all $\lambda > 0$.

The class of regularly varying functions of index $\rho \in \mathbb{R}$ is denoted by $\mathcal{R}^\rho$. In particular, elements of the class $\mathcal{R}^0$ are called slowly varying functions.

We gather all the necessary properties of regularly varying functions in the lemma below.

Lemma 2.1. *Let $f \in \mathcal{R}^\rho$ with $\rho > 0$. Then:*

(i) $f(x) = x^\rho \ell(x)$ *for some slowly varying function* $\ell \in \mathcal{R}^0$.
(ii) $\lim_{x\to\infty} f(x) = \infty$.
(iii) For any $C > 1$ and $\delta > 0$, there exists K such that for $x, y \geq K$,

$$\frac{f(y)}{f(x)} \leq C \max\left\{ \left(\frac{y}{x}\right)^{\rho+\delta}, \left(\frac{y}{x}\right)^{\rho-\delta} \right\}.$$

(iv) There exists $g \in \mathcal{R}^{1/\rho}$ such that

$$f(g(x)) \sim g(f(x)) \sim x$$

and g is determined uniquely to within asymptotic equivalence $\sim$.
(v) There exists $\tilde{f} \in \mathcal{R}^\rho$ such that $\tilde{f}$ continuous, strictly increasing and $\tilde{f}(x) \sim f(x)$.

Proof. (i) follows directly from the definition of $\mathcal{R}^\rho$. (ii) is proved in [20, Proposition 1.5.1], (iii) is known as the Potter bounds, [20, Theorem 1.5.6]. (iv) is proved in [20, Theorem 1.5.12]. For (v), fix $X > 0$ so large that f is locally bounded on [X, ∞) and define $\tilde{f}(x) = \rho \int_X^x t^{-1} f(t)\mathrm{d}t$ for $x \in (X,\infty)$. Extend the definition of $\tilde{f}$ to $(0, X]$ in a way that it is continuous and strictly increasing on $(0,\infty)$. By Karamata's Theorem, [20, Theorem 1.5.11], we have $\tilde{f}(x) \sim f(x)$. □

We note that Lemma 2.1(v) also follows from [20, Theorem 1.8.2].

3. Exponential decay and local dependence measure

Definition 3.1. Let f be a function defined on a neighborhood of infinity such that $f(t) \to \infty$ as $t \to \infty$. We say that a nonnegative random variable X has an exponential-f-decay tail if

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}(X > t)}{f(t)} = \lambda,$$

where $\lambda \in [0,\infty]$. We call such a random variable an $\mathrm{ED}_f(\lambda)$-random variable.

We note that if X is a bounded random variable, then it is $\mathrm{ED}_f(\infty)$ for any f.

Definition 3.2. Let (A, B) be a pair of nonnegative random variables, and let $f \in \mathcal{R}^\rho$ with $\rho > 0$. A function $g : [0,\infty) \to [0,\infty]$ is said to be the local dependence measure of (A, B) (denoted LDM_f^ρ), if the limit

$$g(y) = \lim_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)}, \qquad y \in [0,\infty)$$

exists.

In general, for an arbitrary pair (A, B) and a function f, the existence of g on $[0,\infty)$ is not guaranteed. Nevertheless, we will show that $g(y) = 0$ for $y > a_+^{-1}$, where

$$a_+ = \operatorname{ess\,sup}(A).$$

Our standing assumptions are that $\mathbb{P}(A \geq 0, B \geq 0) = 1$ and $a_+ \in (0,\infty)$. Certain results additionally require $a_+ \in (0, 1]$; whenever this stronger condition is needed, we state it explicitly.

Furthermore, if A and B are positively quadrant dependent (PQD) and $g(0) < \infty$, then g exists on $[0,\infty)$ and its explicit form can be derived—it depends only on a_+ and $g(0)$ (see Section 4 for details). Another family of distributions for which an explicit expression for g is available is presented in Section 9.

Assume that $\tilde{f} \in \mathcal{R}^\rho$ with $\rho > 0$, is such that $\tilde{f}$ continuous, strictly increasing and $\tilde{f}(x) \sim f(x)$. Clearly, we have $\mathrm{ED}_f(\lambda) = \mathrm{ED}_{\tilde{f}}(\lambda)$ and $\mathrm{LDM}_f^\rho = \mathrm{LDM}_{\tilde{f}}^\rho$. In view of Lemma 2.1, without loss of generality, we will assume from now on that every regularly varying function f is continuous and strictly increasing.

Lemma 3.3. *Assume that (A, B) are nonnegative random variables and $f \in \mathcal{R}^\rho$ with $\rho > 0$. Then:*

(i) For every $y > a_+^{-1}$,

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} = 0.$$

(ii) If $g(0)$ exists (possibly infinite), then for every $y \in [0, a_+^{-1})$,

$$\liminf_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \geq g(0)(1 - a_+ y)^\rho.$$

(iii) If $y > x \geq 0$ and both $g(x)$ and $g(y)$ exist, then $g(y) \leq g(x)$,

Proof.

(i) Assume $y > a_+^{-1}$. Clearly, $\mathbb{P}(A > 1/y) > 0$. Notice that $\{A > 1/y\} = \{Aty > t\} \subset \{Aty + B > t\}$ for $t > 0$ and therefore,

$$0 \leq \liminf_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \leq \limsup_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \leq \lim_{t\to\infty} \frac{-\log \mathbb{P}(A > 1/y)}{f(t)} = 0.$$

(ii) Since $\mathbb{P}(A \leq a_+) = 1$, we have

$$\mathbb{P}(Aty + B > t) \leq \mathbb{P}(a_+ ty + B > t) = \mathbb{P}(B > t(1 - a_+ y)).$$

Thus, for $y \in [0, 1/a_+)$,

$$\liminf_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \geq \lim_{t\to\infty} \frac{-\log \mathbb{P}(B > t(1 - a_+ y))}{f(t)} = \lim_{t\to\infty} \frac{-\log \mathbb{P}(B > t(1 - a_+ y))}{f(t(1 - a_+ y))} \cdot \frac{f(t(1 - a_+ y))}{f(t)}$$
$$= g(0)(1 - a_+ y)^\rho.$$

(iii) Let $y > x \geq 0$. We note that since A is a nonnegative random variable, for all $t > 0$, we have the inclusion $\{Atx + B > t\} \subset \{Aty+B > t\}$. By the monotonicity of $\mathbb{P}$ and the assumption that both $g(x)$ and $g(y)$ are well defined we get that $g(y) \leq g(x)$. ☐

Remark 3.4. Lemma 3.3 implies several key properties of the local dependence measure g:

- On the interval (a_+^{-1}, ∞), the limit always exists and equals 0.
- If g is well defined on $[0, a_+^{-1})$, then it is bounded below by $g(0)(1 - a_+ y)^\rho$ on this interval.
- If g exists on the entire $[0, \infty)$, then it is a nonincreasing function.

In Section 4, we show that the lower bound is attained for positively quadrant dependent A and B.

Definition 3.5. For a function $g : (0, \infty) \to [0, \infty]$, let $\phi_\rho : [0, \infty) \to [0, \infty]$ be a Legendre-type transform defined by

$$\phi_\rho(\lambda) = \inf_{y>0} \{y^\rho \lambda + g(y)\}, \qquad \lambda \in [0, \infty).$$

From this point forward, unless explicitly stated otherwise, we assume that g exists on $[0, \infty)$.

Lemma 3.6. *Let g be the LDM_f^ρ for (A, B), where $f \in \mathcal{R}^\rho$ with $\rho > 0$. Let ϕ_ρ be its Legendre-type transform. Then:*

(i) ϕ_ρ is finite, nondecreasing, concave.
(ii) $\phi_\rho(0) = \phi_\rho(0^+) = 0$.
(iii) ϕ_ρ is continuous.
(iv) $\sup_{y>0} \{g(y)\} \geq \phi_\rho(\lambda)$.
(v) If there exists $\lambda > 0$ such that $\phi_\rho(\lambda) > \lambda$, then $\phi_\rho(\lambda) = \lambda$ for at most one $\lambda > 0$.
(vi) With $\gamma = g(0) < \infty$ and $a_+ \in (0, \infty)$,

$$\lambda a_+^{-\rho} \geq \phi_\rho(\lambda) \geq \begin{cases} \left(\gamma^{1/(1-\rho)} + a_+^{\rho/(\rho-1)} \lambda^{1/(1-\rho)}\right)^{1-\rho}, & \text{if } \rho > 1, \\ \min\left\{\gamma, \lambda a_+^{-\rho}\right\}, & \text{if } 0 < \rho \leq 1. \end{cases}$$

Proof.

(i) Finiteness follows from Lemma 3.3(i), monotonicity is obvious, ϕ_ρ as the point-wise infimum of a family of affine functions is concave.

(ii) We have $\phi_\rho(0) = \inf_{y>0} \{g(y)\} = 0$ by Lemma 3.3(i). Moreover, by the definition of ϕ_ρ we get $\phi_\rho(\lambda) \leq y^\rho \lambda + g(y)$ for any $y > 0$. Thus, by taking y sufficiently large (recall Lemma 3.3(i)) and letting $\lambda \downarrow 0^+$, we obtain $\phi_\rho(0^+) \leq 0$.

(iii) From (ii) we get continuity in 0. Continuity on $(0,\infty)$ is a consequence of the fact that ϕ_ρ is concave, which was stated in (i).

(iv) By the definition of $\phi_\rho(\lambda)$, we have $\phi_\rho(\lambda) \le y^\rho\lambda + g(y)$ for all $y > 0$. Letting $y \downarrow 0^+$ (noting that g is a nonincreasing function, so its limit as $y \to 0^+$ exists, although it may be infinite), we obtain $\phi_\rho(\lambda) \le g(0^+) = \sup_{y>0}\{g(y)\}$.

(v) Define $\psi(\lambda) = \phi_\rho(\lambda) - \lambda$. Then ψ is concave, continuous, and $\psi(0) = 0$. Suppose there exist $0 < \lambda_1 < \lambda_2$ with $\phi_\rho(\lambda_1) = \lambda_1$ and $\phi_\rho(\lambda_2) = \lambda_2$, so that $\psi(0) = \psi(\lambda_1) = \psi(\lambda_2) = 0$. By concavity, $\psi \equiv 0$ on $[0, \lambda_2]$ and $\psi(\lambda) \le 0$ for $\lambda > \lambda_2$. Now, if some λ_0 satisfies $\phi_\rho(\lambda_0) > \lambda_0$ (i.e., $\psi(\lambda_0) > 0$), then we obtain a contradiction. Hence, the equation $\phi_\rho(\lambda) = \lambda$ can have at most one solution for $\lambda > 0$.

(vi) By Lemma 3.3(i), we have $g(y) = 0$ for $y > a_+^{-1}$. Thus,

$$\phi_\rho(\lambda) \le \inf_{y>a_+^{-1}}\{y^\rho\lambda\} = \lambda a_+^{-\rho}.$$

Additionally, by Lemma 3.3 (ii) and the fact that g is non-negative, we have $g(y) \ge \gamma\max\{1 - a_+y, 0\}^\rho$ for $y \ge 0$. Therefore,

$$\phi_\rho(\lambda) \ge \min\left\{\inf_{y\in(0,a_+^{-1}]}\{y^\rho\lambda + \gamma(1-a_+y)^\rho\},\ \inf_{y>a_+^{-1}}\{y^\rho\lambda\}\right\} = \min\left\{\inf_{y\in(0,a_+^{-1}]}\{h_1(y)\}, \lambda a_+^{-\rho}\right\},$$

where $h_1 : (0, a_+^{-1}] \to \mathbb{R}$ is defined by $h_1(y) = \lambda y^\rho + \gamma(1 - a_+y)^\rho$.

If $0 < \rho < 1$, then h_1 is increasing on $(0, K]$ and decreasing on $[K, a_+^{-1}]$, where $K = (a_+ + (\frac{a_+\gamma}{\lambda})^{1/(1-\rho)})^{-1}$. Therefore,

$$\inf_{(0,a_+^{-1}]}\left\{h_1(y)\right\} = \min\left\{h_1(0), h_1(a_+^{-1})\right\} = \min\left\{\gamma, \lambda a_+^{-\rho}\right\}.$$

The case $\rho \in (0,1)$ easily extends to $\rho = 1$.

If $\rho > 1$, then h_1 is decreasing on $(0, K]$ and increasing on $[K, a_+^{-1}]$, where K is as before. Therefore,

$$\inf_{(0,a_+^{-1}]}\left\{h_1(y)\right\} = h_1(K) = \frac{\lambda\gamma}{\left(\lambda^{\frac{1}{\rho-1}} + a_+^{\frac{\rho}{\rho-1}}\gamma^{\frac{1}{\rho-1}}\right)^{\rho-1}} = \left(\gamma^{1/(1-\rho)} + a_+^{\rho/(\rho-1)}\lambda^{1/(1-\rho)}\right)^{1-\rho}.$$

Moreover, it is easy to see that $h_1(K) \le \lambda a_+^{-\rho}$. □

Definition 3.7. For a LDM_f^ρ function g, we define

$$\lambda^* = \inf_{y\in(0,1)}\left\{\frac{g(y)}{1-y^\rho}\right\}. \tag{4}$$

Lemma 3.8.

(i) Assume $c \ge 0$. Then, $\phi_\rho(c) \ge c$ if and only if $c \le \lambda^*$.

(ii) If $\lambda^* < \infty$, then $\phi_\rho(\lambda^*) = \lambda^*$.

(iii) Suppose $a_+ \le 1$. Assume that $\gamma = g(0)$ exists. Then,

$$\lambda^* \ge \begin{cases}\gamma\left(1 - a_+^{\rho/(\rho-1)}\right)^{\rho-1}, & \text{if } \rho > 1 \text{ and } a_+ < 1,\\ \gamma, & \text{if } 0 < \rho \le 1 \text{ or } (a_+ = 1 \text{ and } \gamma = \infty).\end{cases}$$

(iv) Suppose $a_+ \le 1$. Then, $\lambda^* = \infty \iff g(0) = \infty$.

Proof.

(i) We have

$$\phi_\rho(c) \ge c \iff \inf_{y>0}\{y^\rho c + g(y)\} \ge c \iff \forall\, y > 0\ \ y^\rho c + g(y) \ge c$$
$$\iff \forall\, y \in (0,1)\ \ y^\rho c + g(y) \ge c \iff c \le \inf_{y\in(0,1)}\left\{\frac{g(y)}{1-y^\rho}\right\}.$$

(ii) Let $(\lambda_n)_n$ be a sequence such that $\lambda_n \downarrow \lambda^*$. Then, by (i) we have $\phi_\rho(\lambda_n) < \lambda_n$ and by continuity of ϕ_ρ, we obtain $\phi_\rho(\lambda^*) \le \lambda^*$. Setting $c = \lambda^*$ in (i), we obtain the reversed bound.

(iii) By Lemma 3.3 (ii), we have $g(y) \ge \gamma(1 - a_+y)^\rho$ for $y \in [0,1) \subset [0, a_+^{-1})$. Thus,

$$\lambda^* \ge \inf_{y\in(0,1)}\left\{h_2(y)\right\},$$

where $h_2 : (0,1) \to \mathbb{R}$ is defined by $h_2(y) = \gamma(1 - a_+y)^\rho/(1 - y^\rho)$. If $\rho \in (0,1]$, then h_2 is increasing on $(0,1)$ and therefore

$$\inf_{y\in(0,1)}\left\{h_2(y)\right\} = h_2(0^+) = \gamma.$$

If $\rho > 1$, then h_2 is increasing on $(a_+^{1/(\rho-1)}, 1)$ and decreasing on $(0, a_+^{1/(\rho-1)})$. Therefore,

$$\inf_{y\in(0,1)} \{h_2(y)\} = h_2\left(\left(a_+^{1/(\rho-1)}\right)^-\right) = \begin{cases} \gamma\left(1 - a_+^{\rho/(\rho-1)}\right)^{\rho-1}, & \text{if } a_+ < 1 \text{ or } \gamma \neq \infty, \\ \gamma, & \text{if } a_+ = 1 \text{ and } \gamma = \infty. \end{cases}$$

(iv) If $g(0) = \infty$, then by (iii) we obtain $\lambda^* \geq \infty$. If $\lambda^* = \infty$, then by definition of λ^*, we have for any $y \in (0,1)$, $g(y) \geq (1 - y^\rho)\lambda^*$. Since g is nonincreasing, $g(0) = \infty$. □

4. Positive quadrant dependence

Definition 4.1. We say that random variables A and B are positively quadrant dependent (PQD for short) if, for all $a, b \in \mathbb{R}$,

$$\mathbb{P}(A > a, B > b) \geq \mathbb{P}(A > a)\mathbb{P}(B > b).$$

It turns out that for a PQD pair (A, B), the LDM always exists $[0,\infty)$ and, moreover, has explicit form.

Recall that $a_+ = \operatorname{ess\,sup}(A) = \inf\{x \in \mathbb{R} : \mathbb{P}(A > x) = 0\}$ and that A and B are assumed to be a.s. nonnegative.

Theorem 4.2. *Assume that A and B are PQD. Suppose that $\gamma := g(0)$ exists and is finite for some $f \in \mathcal{R}^\rho$ with $\rho > 0$. Then, the local dependence measure g of (A, B) exists on $[0,\infty)$.*

(i) For $y \geq 0$,

$$g(y) = \gamma \max\{1 - a_+ y, 0\}^\rho.$$

(ii) If $a_+ \in (0, 1]$, then for $\lambda \geq 0$,

$$\phi_\rho(\lambda) = \begin{cases} \left(\gamma^{1/(1-\rho)} + a_+^{\rho/(\rho-1)}\lambda^{1/(1-\rho)}\right)^{1-\rho}, & \text{if } \rho > 1, \\ \min\left\{\gamma, \lambda a_+^{-\rho}\right\}, & \text{if } 0 < \rho \leq 1, \end{cases} \quad \text{and} \quad \lambda^* = \begin{cases} \gamma\left(1 - a_+^{\rho/(\rho-1)}\right)^{\rho-1}, & \text{if } \rho > 1, \\ \gamma, & \text{if } 0 < \rho \leq 1. \end{cases}$$

Proof.

(i) By Lemma 3.3 (ii), for $y \geq 0$ we have

$$\liminf_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \geq \gamma \max\{1 - a_+ y, 0\}^\rho.$$

We now show that for $y \in [0, a_+^{-1}]$, the quantity $\gamma(1 - a_+ y)^\rho$ is an upper bound for the superior limit. Fix $\delta \in (0, a_+)$. Then, if $0 \leq y \leq a_+^{-1} < (a_+ - \delta)^{-1}$, we obtain

$$\begin{aligned} \mathbb{P}(Aty + B > t) &\geq \mathbb{P}(Aty + B > t, A > a_+ - \delta) \geq \mathbb{P}((a_+ - \delta)ty + B > t, A > a_+ - \delta) \\ &\geq \mathbb{P}(A > a_+ - \delta)\,\mathbb{P}(B > t(1 - (a_+ - \delta)y)), \end{aligned}$$

where we have used the fact that A and B are PQD. Therefore,

$$\limsup_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \leq \lim_{t\to\infty} \frac{-\log \mathbb{P}(A > a_+ - \delta) - \log \mathbb{P}(B > t(1 - (a_+ - \delta)y))}{f(t)}$$

Since $f(t) \to \infty$ as $t \to \infty$, and from the fact that $\mathbb{P}(A > a_+ - \delta) > 0$, we obtain

$$\limsup_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \leq \lim_{t\to\infty} \frac{-\log \mathbb{P}(B > t(1 - (a_+ - \delta)y))}{f(t(1 - (a_+ - \delta)y))} \cdot \frac{f(t(1 - (a_+ - \delta)y))}{f(t)} = \gamma(1 - (a_+ - \delta)y)^\rho.$$

By letting $\delta \downarrow 0^+$, we conclude that for $0 \leq y \leq a_+^{-1}$,

$$\gamma(1 - a_+ y)^\rho \leq \liminf_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \leq \limsup_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \leq \gamma(1 - a_+ y)^\rho.$$

Thus, g exists on $[0, a_+^{-1}]$ and is equal to $\gamma(1 - a_+ y)^\rho$ on this interval. Finally, by Remark 3.4, for $y > a_+^{-1}$ we have that $g(y)$ exists and equals 0. This completes the proof of (i).

(ii) Since $g(y) = 0$ for $y > a_+^{-1}$, we have

$$\phi_\rho(\lambda) = \min\left\{\inf_{y\in(0,a_+^{-1}]} \{h_1(y)\}, \inf_{y>a_+^{-1}} \{\lambda y^\rho\}\right\},$$

where $h_1 : (0, a_+^{-1}] \to \mathbb{R}$ is defined by $h_1(y) = \lambda y^\rho + \gamma(1 - a_+ y)^\rho$. By the proof of Lemma 3.6 (vi), the explicit formula for ϕ_ρ follows.

We have

$$\lambda^* = \inf_{y\in(0,1)} \left\{ \frac{g(y)}{1-y^\rho} \right\} = \inf_{y\in(0,1)} \{h_2(y)\},$$

where $h_2 : (0,1) \to \mathbb{R}$ is defined by $h_2(y) = \gamma(1-a_+y)^\rho/(1-y^\rho)$. The right hand side of the above equation was already calculated in the proof of Lemma 3.8 (iii). □

5. Tails of $AX + B$

Recall our standing assumption that A and B are a.s. nonnegative random variables and $a_+ := \operatorname{ess\,sup}(A) \in (0,\infty)$. In this section, X is assumed to be independent of (A,B) but does not necessarily satisfy (1).

Theorem 5.1. *Let g be the LDM_f^ρ for the nonnegative random variables (A,B), where $f \in \mathcal{R}^\rho$ with $\rho > 0$. Suppose X is $\mathrm{ED}_f(\lambda)$ with $\lambda \in [0,\infty)$. Then $AX + B$ is $\mathrm{ED}_f(\phi_\rho(\lambda))$.*

Proof. First, we prove that

$$\limsup_{t\to\infty} \frac{-\log \mathbb{P}(AX+B>t)}{f(t)} \le \phi_\rho(\lambda). \tag{5}$$

For any $y > 0$ and $t > 0$, we have

$$\mathbb{P}(AX+B>t) \ge \mathbb{P}(AX+B>t, X>ty) \ge \mathbb{P}(Aty+B>t, X>ty) = \mathbb{P}(Aty+B>t)\,\mathbb{P}(X>ty).$$

Thus,

$$\begin{aligned}\limsup_{t\to\infty} \frac{-\log \mathbb{P}(AX+B>t)}{f(t)} &\le \limsup_{t\to\infty} \frac{-\log \mathbb{P}(Aty+B>t) - \log \mathbb{P}(X>ty)}{f(t)} \\ &\le g(y) + \limsup_{t\to\infty} \frac{-\log \mathbb{P}(X>ty)}{f(ty)} \cdot \frac{f(ty)}{f(t)} = g(y) + \lambda y^\rho.\end{aligned}$$

By taking $\inf_{y>0}$ on both sides, we obtain (5).

Next, we establish the lower bound,

$$\liminf_{t\to\infty} \frac{-\log \mathbb{P}(AX+B>t)}{f(t)} \ge \phi_\rho(\lambda). \tag{6}$$

If $\lambda = 0$, then by Lemma 3.6 (ii), we have $\phi_\rho(\lambda) = 0$, and the above inequality is trivial. Now suppose $\lambda > 0$. Since ϕ_ρ is finite, there exists $a > 0$ such that $\lambda a^\rho \ge \phi_\rho(\lambda)$. We note that

$$\begin{aligned}\liminf_{t\to\infty} \frac{-\log \mathbb{P}(AX+B>t, X>ta)}{f(t)} &\ge \liminf_{t\to\infty} \frac{-\log \mathbb{P}(X>ta)}{f(t)} \\ = \liminf_{t\to\infty} \frac{-\log \mathbb{P}(X>ta)}{f(ta)} \cdot \frac{f(ta)}{f(t)} &= \lambda a^\rho \ge \phi_\rho(\lambda).\end{aligned} \tag{7}$$

Moreover, by Lemma 3.6 (iv), we have $\sup_{y>0}\{g(y)\} \ge \phi_\rho(\lambda)$. Fix $\eta > 0$. There exists $b \in (0,a)$ such that $g(b) \ge \phi_\rho(\lambda) - \eta/2$. We have

$$\liminf_{t\to\infty} \frac{-\log \mathbb{P}(AX+B>t, X\le tb)}{f(t)} \ge \liminf_{t\to\infty} \frac{-\log \mathbb{P}(Atb+B>t)}{f(t)} = g(b) \ge \phi_\rho(\lambda) - \eta/2. \tag{8}$$

Combining (7) and (8), we deduce that there exists $M > 0$ such that for all $t > M$,

$$\mathbb{P}(AX+B>t, X>ta) + \mathbb{P}(AX+B>t, X\le tb) \le 2\exp(-f(t)(\phi_\rho(\lambda)-\eta)).$$

Now, fix $y, h > 0$ such that $h < y$. Then,

$$\mathbb{P}(AX+B>t, t(y-h)<X\le ty) \le \mathbb{P}(Aty+B>t, t(y-h)<X) = \mathbb{P}(Aty+B>t)\,\mathbb{P}(X>t(y-h)). \tag{9}$$

Since

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}(X>t(y-h))}{f(t)} = \lim_{t\to\infty} \frac{-\log \mathbb{P}(X>t(y-h))}{f(t(y-h))} \cdot \frac{f(t(y-h))}{f(t)} = \lambda(y-h)^\rho,$$

for sufficiently large t, we obtain

$$\mathbb{P}(X>t(y-h)) \le \exp(-f(t)(\lambda(y-h)^\rho - \eta)).$$

If $g(y) < \infty$, then, from the definition of g, we conclude that

$$\exists\, M>0 \text{ such that } \forall\, t>M \quad \mathbb{P}(Aty+B>t) \le \exp(-f(t)(g(y)-\eta)).$$

Thus, from (9), we get for sufficiently large t,

$$\mathbb{P}(AX+B>t,\ t(y-h)<X\le ty) \le \exp(-f(t)(g(y)+\lambda(y-h)^\rho - 2\eta)). \tag{10}$$

If $g(y) = \infty$, then for any $G > 0$ there exists $N > 0$ such that for all $t > N$, we obtain a bound

$$\mathbb{P}(AX + B > t,\ t(y-h) < X \le ty) \le \exp(-f(t)(G + \lambda(y-h)^\rho)). \tag{11}$$

We now consider two cases: (i) $\rho \ge 1$ and (ii) $\rho \in (0,1)$.

(i) By the convexity of $(0,\infty) \ni x \mapsto x^\rho$, we obtain

$$\lambda y^\rho - \lambda(y-h)^\rho \le \lambda h \rho y^{\rho-1}.$$

Using (10), the definition of ϕ_ρ, and the above inequality, we conclude that if $g(y) < \infty$, then for all $y > h > 0$ and sufficiently large t,

$$\begin{aligned}\mathbb{P}(AX + B > t, t(y-h) < X \le ty) &\le \exp(-f(t)(g(y) + \lambda y^\rho - \lambda y^\rho + \lambda(y-h)^\rho - 2\eta)) \\ &\le \exp(-f(t)(\phi_\rho(\lambda) - \lambda h\rho y^{\rho-1} - 2\eta)).\end{aligned}$$

The same bound holds when $g(y) = \infty$ by choosing G sufficiently large in (11).

Fix $n \in \mathbb{N}$. Let $h_0 = 0$, $h_k - h_{k-1} = \frac{a-b}{n} = h$ for $k = 1, \dots, n$. Then for sufficiently large t,

$$\begin{aligned}\mathbb{P}(AX + B > t, tb < X \le ta) &= \sum_{k=1}^{n} \mathbb{P}(AX + B > t, t(a-h_k) < X \le t(a-h_{k-1})) \\ &\le \sum_{k=1}^{n} \exp(-f(t)(\phi_\rho(\lambda) - \lambda h\rho(a-h_{k-1})^{\rho-1} - 2\eta)) \\ &\le n \exp(-f(t)(\phi_\rho(\lambda) - \lambda h\rho a^{\rho-1} - 2\eta)).\end{aligned}$$

Thus, for sufficiently large t,

$$\begin{aligned}\mathbb{P}(AX + B > t) &= \mathbb{P}(AX + B > t, X \in t(b,a]) + \mathbb{P}(AX + B > t, X \notin t(b,a]) \\ &\le n \exp(-f(t)(\phi_\rho(\lambda) - \lambda h\rho a^{\rho-1} - 2\eta)) + 2\exp(-f(t)(\phi_\rho(\lambda) - \eta)) \\ &\le (n+2)\exp(-f(t)(\phi_\rho(\lambda) - \lambda h\rho a^{\rho-1} - 2\eta)).\end{aligned}$$

Therefore, for sufficiently large t,

$$\frac{-\log \mathbb{P}(AX + B > t)}{f(t)} \ge \frac{-\log(n+2)}{f(t)} + \phi_\rho(\lambda) - \lambda h\rho a^{\rho-1} - 2\eta.$$

Taking $\liminf_{t\to\infty}$ on both sides, we get

$$\liminf_{t\to\infty} \frac{-\log \mathbb{P}(AX + B > t)}{f(t)} \ge \phi_\rho(\lambda) - \lambda h\rho a^{\rho-1} - 2\eta.$$

By letting $n \uparrow \infty$ (recall that $h = (a-b)/n$) and then $\eta \downarrow 0^+$, we obtain (6).

(ii) For $\rho \in (0,1)$, we have $y^\rho - (y-h)^\rho \le h^\rho$ for $y > h > 0$. Similarly to case (i), for all $y > h > 0$ and sufficiently large t, we get

$$\begin{aligned}\mathbb{P}(AX + B > t, t(y-h) < X \le ty) &\le \exp(-f(t)(g(y) + \lambda y^\rho - \lambda y^\rho + \lambda(y-h)^\rho - 2\eta)) \\ &\le \exp(-f(t)(\phi_\rho(\lambda) - \lambda h^\rho - 2\eta))\end{aligned}$$

and this bound also holds when $g(y) = \infty$. Proceeding as in (i), we obtain that for sufficiently large t,

$$\mathbb{P}(AX + B > t) \le (n+2)\exp(-f(t)(\phi_\rho(\lambda) - \lambda h^\rho - 2\eta)).$$

The remaining part of the proof is analogous. □

6. The equation $X \overset{d}{=} AX + B$

The stochastic fixed-point Eq. (1) associated with the affine recursion $X_n = A_n X_{n-1} + B_n$ has been extensively studied in the literature. We summarize key results on the existence and uniqueness of the solution below, [1].

Theorem 6.1. *Assume that* $\mathbb{P}(A \ge 0, B \ge 0) = 1$*,*

$$\mathbb{E}[\log A] < 0 \quad \text{and} \quad \mathbb{E}[\max\{\log B, 0\}] < \infty.$$

Then there exists a unique solution X *to*

$$X \overset{d}{=} AX + B, \qquad (A,B) \text{ and } X \text{ are independent.}$$

Moreover, X *is given by the a.s. convergent series representation*

$$X \overset{d}{=} \sum_{n=1}^{\infty} B_n \prod_{k=1}^{n-1} A_k,$$

where $(A_n, B_n)_n$ *are independent copies of* (A, B)*.*

Additionally, if $X_n = A_n X_{n-1} + B_n$ *for* $n = 1, 2, \dots$*, where* X_0 *is independent of* $(A_n, B_n)_n$*, then* X_n *converges in distribution to* X*.*

We say that a real random variable Y is stochastically majorized by Z and we write $Y \le_{st} Z$ if $\mathbb{P}(Y \le t) \ge \mathbb{P}(Z \le t)$ for all $t \in \mathbb{R}$.

Lemma 6.2. *Let $(A_n, B_n)_{n\ge1}$ be a sequence of independent copies of a generic pair (A, B). Let X_0 be independent of $(A_n, B_n)_{n\ge1}$ and define $X_n = A_n X_{n-1} + B_n$. Assume that $A \ge 0$ a.s.*

(i) If $X_1 \ge_{st} X_0$, then $X_n \ge_{st} X_{n-1}$ for all $n \ge 1$.
(ii) If $X_1 \le_{st} X_0$, then $X_n \le_{st} X_{n-1}$ for all $n \ge 1$.

Proof. (i) We proceed by induction, assume that $X_n \ge_{st} X_{n-1}$ for $n \ge 1$. Then, since $A_{n+1} \ge 0$ a.s. and X_n is independent of (A_{n+1}, B_{n+1}), we obtain

$$X_{n+1} \overset{d}{=} A_{n+1}X_n + B_{n+1} \ge_{st} A_{n+1}X_{n-1} + B_{n+1} \overset{d}{=} X_n.$$

Point (ii) is proved in the same way. □

Corollary 6.3. *If X_n converges in distribution to X, then $X_1 \ge_{st} X_0$ implies that $X \ge_{st} X_n$ for all $n \in \mathbb{N}$ and if $X_1 \le_{st} X_0$, then $X \le_{st} X_n$ for all $n \in \mathbb{N}$.*

Lemma 6.4. *Under the assumptions of Theorem 6.1, we have*

$$\operatorname{ess\,sup}(X) = \operatorname{ess\,sup}\left(\frac{B}{1-A} \mid A < 1\right).$$

Proof. Since $\mathbb{E}[\log A] < 0$, we obtain $\mathbb{P}(A < 1) > 0$. Denote $x_+ = \operatorname{ess\,sup}(X)$ and

$$x_0 = \operatorname{ess\,sup}\left(\frac{B}{1-A} \mid A < 1\right).$$

Let $g_{a,b}(t) = a\,t + b$. With

$$G(A,B) = \left\{ g_{(a_1,b_1)} \circ \ldots \circ g_{(a_n,b_n)} : (a_i, b_i) \in \operatorname{supp}(A,B), i = 1, \ldots, n, n \ge 1 \right\}$$

by [1, Proposition 2.5.3], we have

$$\operatorname{supp}(X) = \overline{\left\{ \frac{b}{1-a} : g_{a,b} \in G(A,B), a < 1 \right\}}.$$

Thus,

$$\overline{\left\{ \frac{b}{1-a} : (a,b) \in \operatorname{supp}(A,B), a < 1 \right\}} \subset \operatorname{supp}(X)$$

and therefore $x_+ \ge x_0$. If $x_0 = \infty$, then $x_+ = \infty$ and there is nothing to prove.

Assume that $x_0 < \infty$. By [1, Lemma 2.5.1], for every $(a,b) \in \operatorname{supp}(A,B)$ we have

$$a\operatorname{supp}(X) + b \subset \operatorname{supp}(X)$$

and therefore $ax_+ + b \le x_+$ for all $(a,b) \in \operatorname{supp}(A,B)$. Thus, $Ax_+ + B \le x_+$ a.s. and

$$B\mathbf{1}_{A\ge1} \le x_+(1-A)\mathbf{1}_{A\ge1} \le x_0(1-A)\mathbf{1}_{A\ge1} \quad \text{a.s.},$$

where the latter inequality follows from the fact that $x_+ \ge x_0$.

By definition of x_0 we have

$$B\mathbf{1}_{A<1} \le x_0(1-A)\mathbf{1}_{A<1}, \quad \text{a.s.}$$

and therefore $B \le x_0(1-A)$ a.s. Thus,

$$X \overset{d}{=} \sum_{k=1}^{\infty} A_1 \ldots A_{k-1} B_k \le x_0 \sum_{k=1}^{\infty} A_1 \ldots A_{k-1}(1-A_k) = x_0, \quad \text{a.s.},$$

which implies that $\mathbb{P}(X \le x_0) = 1$, i.e., $x_0 \ge x_+$. □

We have established above that the right endpoint of the support of X coincides with the right endpoint of the conditional distribution of $B/(1-A)$ given $A < 1$. Note that if this endpoint is finite, the asymptotic behavior of $t \mapsto \mathbb{P}(X > t)$ as $t \to \infty$ becomes trivial, and this case should thus be excluded from further analysis. It turns out that suitable conditions involving $g(1)$ or λ^* are sufficient to ensure this exclusion, but only under the assumption that $\mathbb{P}(A \in [0,1)) = 1$. In what follows, we explain that restricting our analysis to such distributions does not lead to any significant loss of generality.

By [2, Theorem 4.1], if $\mathbb{P}(A > 1) > 0$, then the tail of X is at least of power-law type:

$$\liminf_{t\to\infty} \frac{\log \mathbb{P}(X > t)}{\log t} > -\infty.$$

Since this behavior lies outside the light-tail regime we consider, we henceforth restrict to the case where $A \in [0,1]$ a.s. Moreover, when $\mathbb{P}(A=1) > 0$, and under suitable nondegeneracy conditions, [21, Theorem 1.7] shows that the moment generating function $\mathbb{E}[\exp(tX)]$ is finite if and only if

$$\mathbb{E}[e^{tB}\mathbf{1}_{A=1}] < 1.$$

In addition, [22, Lemma 5] establishes that

$$\liminf_{t\to\infty} \frac{-\log \mathbb{P}(X>t)}{t} = \sup\{t\in\mathbb{R} : \mathbb{E}[e^{tX}] < \infty\} = \sup\{t\in\mathbb{R} : \mathbb{E}[e^{tB}\mathbf{1}_{A=1}] < 1\} =: t_0.$$

If $t_0 < \infty$, the analysis for the case $\mathbb{P}(A=1) > 0$ is complete. Under $\mathbb{P}(A=1) > 0$, we have $t_0 = \infty$ if and only if $\mathbb{P}(B=0 \mid A=1) = 1$. In this degenerate scenario, one can show that

$$X \overset{d}{=} \tilde{A}X + \tilde{B}, \qquad X \text{ and } (\tilde{A}, \tilde{B}) \text{ are independent},$$

where $(\tilde{A}, \tilde{B}) \overset{d}{=} (A,B) \mid \{A<1\}$. Indeed, for a bounded continuous function f we have

$$\mathbb{E}[f(X)] = \mathbb{E}[f(AX+B)] = \mathbb{E}[f(AX+B)\mathbf{1}_{A=1}] + \mathbb{E}[f(AX+B)\mathbf{1}_{A<1}] = \mathbb{E}[f(X)]\,\mathbb{P}(A=1) + \mathbb{E}[f(\tilde{A}X+\tilde{B})]\,\mathbb{P}(A<1),$$

which immediately implies that $\mathbb{E}[f(X)] = E[f(\tilde{A}X+\tilde{B})]$.

Therefore, to avoid these solved cases, we eventually assume that $\mathbb{P}(A\in[0,1)) = 1$.

Lemma 6.5. *Let g be a local dependence measure of (A,B) with $\mathbb{P}(A\in[0,1), B\geq 0) = 1$ for $f\in\mathcal{R}^\rho$, $\rho>0$. Then:*

(i) If $g(1) < \infty$, then $\operatorname{ess\,sup}\left(\frac{B}{1-A} \mid A<1\right) = \infty$.

(ii) If $\lambda^* < \infty$, then $g(1) < \infty$.

Proof.

(i) First we notice that since $A<1$ a.s., then $\operatorname{ess\,sup}\left(\frac{B}{1-A} \mid A<1\right) = \operatorname{ess\,sup}\left(\frac{B}{1-A}\right)$. Moreover,

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}\left(\frac{B}{1-A} > t\right)}{f(t)} = \lim_{t\to\infty} \frac{-\log \mathbb{P}(At+B>t)}{f(t)} = g(1) < \infty.$$

Then for sufficiently large t we get that $\mathbb{P}(\frac{B}{1-A} > t) > 0$, which proves the assertion.

(ii) Condition $\lambda^* < \infty$ implies that there exists $y_0 \in (0,1)$ such that $g(y_0) < \infty$. Since g is nonincreasing, we have $g(1) \leq g(y_0) < \infty$. □

We will also need the following result, which was proved in [6, Lemma 6.10].

Lemma 6.6. *Suppose that $X \overset{d}{=} AX+B$, where X and (A,B) are independent. For any bounded, uniformly continuous function f on $\mathbb{R}$ and any increasing positive integer sequence $(n_k)_k$, a.s.,*

$$\limsup_{m\to\infty} \frac{1}{m}\sum_{k=1}^{m} f(X_{n_k}) \geq \mathbb{E}[f(X)].$$

7. Tails of $X \overset{d}{=} AX+B$

Throughout this section, we assume that $\mathbb{P}(A\in[0,1), B\geq 0) = 1$ (and thus $\mathbb{E}[\log A] < 0$) and $\mathbb{E}[\max\{\log B, 0\}] < \infty$, which implies that there exists a unique solution X to

$$X \overset{d}{=} AX+B, \qquad X \text{ and } (A,B) \text{ are independent}.$$

Definition 7.1. We say that a LDM_f^ρ function g is admissible if either: there exists $\lambda>0$ such that $\phi_\rho(\lambda) > \lambda$, or $\lambda^* = 0$.

Assume that the nonnegative random variables A and B are PQD, $a_+ = \operatorname{ess\,sup}(A) \in (0,1]$ and that $f(t) = -\log \mathbb{P}(B>t)$ is regularly varying with index $\rho>0$. Then, by Theorem 4.2, g exists and is given by $g(y) = \max\{1-a_+y, 0\}^\rho$ for $y\geq 0$. In this case, one can verify that the admissibility condition holds if and only if $a_+\in(0,1)$ or $\rho>1$; see Theorem 4.2 (ii) for explicit expression for ϕ_ρ and λ^*.

The notion of admissibility is introduced to exclude those distributions of (A,B) and choices of the function f for which our analytical framework fails; see in particular Lemma 7.4, where an asymptotic lower bound on $-\mathbb{P}(X>t)$ is established. However, even without admissibility, we are still able to obtain an asymptotic upper bound for $-\log \mathbb{P}(X>t)$; see Lemmas 7.5 and 7.6. In Section 9, we present an example in which this upper bound is sharper than the bound following from [19, Theorem 5.1].

Recall that our standing assumption is that both A and B are almost surely nonnegative and nonzero with positive probability. Moreover, throughout this section, we assume that g is LDM_f^ρ for random variables (A,B), where $f\in\mathcal{R}^\rho$ and $\rho>0$. The main result in this section is as follows.

Theorem 7.2. *Assume that g is admissible. Then X is $\mathrm{ED}_f(\lambda^*)$, where $\lambda^* \in [0,\infty]$ is defined in (4).*

The proof of Theorem 7.2 relies on several lemmas. Unless stated explicitly, we do not assume that g is admissible in the statements of the following lemmas.

Lemma 7.3. *If there exists $\kappa > 0$ such that $\kappa < \phi_\rho(\kappa)$, then there exists a nonnegative $\mathrm{ED}_f(\kappa)$-random variable Z_κ such that*

$$Z_\kappa \geq_{st} AZ_\kappa + B, \qquad Z_\kappa \text{ and } (A,B) \text{ are independent.} \tag{12}$$

Proof. Let Z_0 be a nonnegative random variable independent of (A,B), with the distribution defined by (recall that, without loss of generality, we assumed that f is strictly increasing and continuous)

$$\mathbb{P}(Z_0 > t) = \exp(-\kappa f(t)), \quad t > 0.$$

Clearly, Z_0 is $\mathrm{ED}_f(\kappa)$. Thus, by Theorem 5.1, we have

$$\lim_{t\to\infty} \frac{-\log \mathbb{P}(AZ_0 + B > t)}{f(t)} = \phi_\rho(\kappa) > \kappa = \lim_{t\to\infty} \frac{-\log \mathbb{P}(Z_0 > t)}{f(t)},$$

which implies that there exists $M > 0$ such that

$$\forall t \geq M \ \ \mathbb{P}(Z_0 > t) \geq \mathbb{P}(AZ_0 + B > t). \tag{13}$$

Let Z_κ be a random variable, independent of (A,B), with the distribution defined by

$$\mathbb{P}(Z_\kappa \in \cdot) = \mathbb{P}(Z_0 \in \cdot \mid Z_0 > M).$$

For $t \geq M$, we obtain

$$\mathbb{P}(AZ_\kappa + B > t) = \mathbb{P}(AZ_0 + B > t \mid Z_0 > M) \leq \frac{\mathbb{P}(AZ_0 + B > t)}{\mathbb{P}(Z_0 > M)} \overset{(13)}{\leq} \frac{\mathbb{P}(Z_0 > t)}{\mathbb{P}(Z_0 > M)} = \mathbb{P}(Z_0 > t \mid Z_0 > M) = \mathbb{P}(Z_\kappa > t),$$

while for $t < M$, we have $\mathbb{P}(Z_\kappa > t) = 1 \geq \mathbb{P}(AZ_\kappa + B > t)$. Thus, (12) holds true.

For $t \geq M$, we have

$$\frac{-\log \mathbb{P}(Z_\kappa > t)}{f(t)} = \frac{-\log \mathbb{P}(Z_0 > t \mid Z_0 > M)}{f(t)} = \frac{-\log \mathbb{P}(Z_0 > t) + \log \mathbb{P}(Z_0 > M)}{f(t)} \xrightarrow{t\to\infty} \kappa. \quad \square$$

Lemma 7.4. *If g is admissible then, $\liminf_{t\to\infty} \left(-f(t)^{-1} \log \mathbb{P}(X > t)\right) \geq \lambda^*$.*

Proof. The inequality holds trivially in the case $\lambda^* = 0$ and in the case where X has a bounded support (so that $\operatorname{ess\,sup}\left(\frac{B}{1-A} \mid A < 1\right) < \infty$, recall Lemma 6.4), in which we have $\lambda^* = \infty$.

Assume that $\lambda^* \in (0,\infty]$ and $\operatorname{ess\,sup}\left(\frac{B}{1-A} \mid A < 1\right) = \infty$, which, by Lemma 6.4, implies that $\mathbb{P}(X > t) > 0$ for any $t \in \mathbb{R}$. Moreover, since $\lambda^* > 0$, the admissibility of g ensures that there exists $\kappa > 0$ such that $\kappa < \phi_\rho(\kappa)$. Let Z_κ be a random variable whose distribution is constructed in Lemma 7.3. Let $X_0 = Z_\kappa$. From (12), we obtain $X_1 \leq_{st} X_0$, which, by Corollary 6.3, implies

$$\forall t \in \mathbb{R} \quad \frac{-\log \mathbb{P}(X > t)}{f(t)} \geq \frac{-\log \mathbb{P}(X_0 > t)}{f(t)}.$$

Taking $\liminf_{t\to\infty}$ on both sides, we get

$$\liminf_{t\to\infty} \frac{-\log \mathbb{P}(X > t)}{f(t)} \geq \liminf_{t\to\infty} \frac{-\log \mathbb{P}(X_0 > t)}{f(t)} = \kappa.$$

Therefore,

$$\liminf_{t\to\infty} \frac{-\log \mathbb{P}(X > t)}{f(t)} \geq \sup\left\{\kappa > 0 : \phi_\rho(\kappa) > \kappa\right\}.$$

By Lemma 3.8, we have

$$\left\{c \geq 0 : \phi_\rho(c) \geq c\right\} = \begin{cases} [0,\lambda^*], & \lambda^* < \infty, \\ [0,\infty), & \lambda^* = \infty. \end{cases}$$

However, by Lemma 3.6(v), the set $\left\{c \geq 0 : \phi_\rho(c) = c\right\}$ contains at most two elements. Therefore,

$$\sup\left\{\kappa > 0 : \phi_\rho(\kappa) > \kappa\right\} = \sup\left\{c \geq 0 : \phi_\rho(c) \geq c\right\} = \lambda^*. \quad \square$$

Lemma 7.5. *If $s = \limsup_{t\to\infty} \left(-f(t)^{-1} \log \mathbb{P}(X > t)\right) < \infty$, then $s \leq \lambda^*$.*

Proof. For all $t > 0$ and $y > 0$, we have

$$\begin{aligned} \mathbb{P}(X > t) &= \mathbb{P}(AX + B > t) \geq \mathbb{P}(AX + B > t, X > ty) \\ &\geq \mathbb{P}(Aty + B > t, X > ty) = \mathbb{P}(Aty + B > t)\mathbb{P}(X > ty). \end{aligned} \tag{14}$$

Thus, for $y > 0$,

$$s \le \limsup_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t) - \log \mathbb{P}(X > ty)}{f(t)}$$
$$\le \limsup_{t\to\infty} \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} + \limsup_{t\to\infty} \frac{-\log \mathbb{P}(X > ty)}{f(ty)} \frac{f(ty)}{f(t)} = g(y) + s\, y^{\rho}.$$

Now, using the assumption that $s < \infty$, we obtain $s(1 - y^{\rho}) \le g(y)$. Therefore

$$s \le \inf_{y\in(0,1)} \left\{ \frac{g(y)}{1 - y^{\rho}} \right\} = \lambda^*. \quad \square$$

Lemma 7.6. *If $\lambda^* < \infty$, then $\limsup_{t\to\infty} \left(-f(t)^{-1} \log \mathbb{P}(X > t)\right) < \infty$.*

Proof. The assumption $\lambda^* < \infty$ implies that there exists $y \in (0, 1)$ such that $g(y) < \infty$. Fix $\eta > 0$. By the definition of g, we conclude that

$$\exists\, M > 0 \ \ \forall\, t \ge M \quad \frac{-\log \mathbb{P}(Aty + B > t)}{f(t)} \le g(y) + \eta.$$

Using (14), we obtain for $t > 0$ and $k = 0, 1, \ldots$,

$$-\log \mathbb{P}(X > ty^k) + \log \mathbb{P}(X > ty^{k+1}) \le -\log \mathbb{P}(Aty^{k+1} + B > ty^k)$$
$$\le f(ty^k)(g(y) + \eta),$$

provided $ty^k \ge M$. Summing these inequalities for $k = 0, \ldots, n \in \mathbb{N}$, we get

$$-\log \mathbb{P}(X > t) + \log \mathbb{P}(X > ty^{n+1}) \le (g(y) + \eta) \sum_{k=0}^{n} f(ty^k), \tag{15}$$

provided $ty^k \ge M$ for $k = 0, \ldots, n$. Set $n = n_t = \left\lfloor \log(\frac{M}{t}) / \log(y) \right\rfloor$. It is easy to verify that

$$ty^{n_t} \ge M \ge ty^{n_t+1}. \tag{16}$$

Thus, (15) implies that

$$\limsup_{t\to\infty} \left\{ \frac{-\log \mathbb{P}(X > t)}{f(t)} + \frac{\log \mathbb{P}(X > ty^{n_t+1})}{f(t)} \right\} \le (g(y) + \eta) \limsup_{t\to\infty} \sum_{k=0}^{n_t} \frac{f(ty^k)}{f(t)}.$$

Using Potter bounds, Lemma 2.1 (iv), for $C = 2$ and $\delta = \rho/2$, we conclude that for sufficiently large t,

$$\frac{f(ty^k)}{f(t)} \le 2y^{k\rho/2}.$$

Since $y \in (0, 1)$, the series $\sum_{k=0}^{\infty} \frac{f(ty^k)}{f(t)}$ is finite as $t \to \infty$. Finally, we observe that

$$\limsup_{t\to\infty} \frac{-\log \mathbb{P}(X > t)}{f(t)} + \liminf_{t\to\infty} \frac{\log \mathbb{P}(X > ty^{n_t+1})}{f(t)} \le \limsup_{t\to\infty} \left\{ \frac{-\log \mathbb{P}(X > t)}{f(t)} + \frac{\log \mathbb{P}(X > ty^{n_t+1})}{f(t)} \right\} < \infty.$$

By (16), we obtain $\mathbb{P}(X > ty^{n_t+1}) \ge \mathbb{P}(X > M)$. Since $\lambda^* < \infty$, by Lemma 6.5, we have $\operatorname{ess\,sup}\left(\frac{B}{1-A} \mid A < 1\right) = \infty$, which, by Lemma 6.4, implies that $\operatorname{ess\,sup}(X) = \infty$. Therefore $\mathbb{P}(X > M) > 0$ and the second term on the left-hand side above is 0. This concludes the proof. $\square$

Proof of Theorem 7.2. If $\lambda^* = \infty$, then X is $\mathrm{ED}_f(\lambda^*)$ by Lemma 7.4. If $\lambda^* < \infty$, then Lemma 7.4 gives the lower bound, which, by Lemma 7.6, is the same as the upper bound of Lemma 7.5. $\square$

8. Upper envelope for $(X_n)_n$

Similarly as in the previous section, we assume that (A, B) satisfy $\mathbb{P}(A \in [0, 1), B \ge 0) = 1$ and $\mathbb{E}[\max\{\log B, 0\}] < \infty$ and that X is the unique solution X to (1). Let $(A_n, B_n)_{n\ge1}$ be a sequence of independent copies of (A, B). We set $X_0 = 0$ and consider the sequence $X_n = A_n X_{n-1} + B_n$ for $n \ge 1$.

In view of Lemma 2.1(v), without loss of generality, we assume that f is continuous and strictly increasing so that f^{-1} is well defined.

Theorem 8.1. *Let g be an admissible LDM_f^{ρ} for (A, B), where $f \in \mathcal{R}^{\rho}$ and $\rho > 0$. Assume that $\lambda^* \in (0, \infty)$. Then, almost surely*

$$\limsup_{n\to\infty} \frac{X_n}{f^{-1}(\log n)} = (\lambda^*)^{-1/\rho}.$$

All lemmas in this section implicitly make the same assumptions as those in Theorem 8.1.

Lemma 8.2. *Almost surely, we have*

$$\limsup_{n\to\infty} \frac{X_n}{f^{-1}(\log n)} \le (\lambda^*)^{-1/\rho}.$$

Proof. Fix $\epsilon > 0$. There exists $\delta \in (0,1)$ such that $\gamma := (1-\delta)(1+\epsilon) > 1$. By Theorem 7.2, X is $\mathrm{ED}_f(\lambda^*)$. Since $\lambda^* < \infty$, there exists $M > 0$ such that

$$\forall\, t \ge M \;\; \mathbb{P}(X > t) \le e^{-\lambda^* f(t)(1-\delta)}$$

Define

$$t_n = f^{-1}\left(\frac{(1+\epsilon)\log n}{\lambda^*}\right), \quad n \in \mathbb{N}.$$

Since $B_1 = X_1 \ge_{st} X_0 = 0$, Corollary 6.3 implies that for all $n \in \mathbb{N}$, it holds that $X \ge_{st} X_n$. As $t_n \to \infty$, for sufficiently large n, we obtain

$$\mathbb{P}\left(X_n > t_n\right) \le \mathbb{P}(X > t_n) \le e^{-(1+\epsilon)(1-\delta)\log n} = n^{-\gamma}.$$

Since $\gamma > 1$, we conclude that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(X_n > t_n\right) < \infty.$$

By the Borel–Cantelli Lemma, we get that almost surely,

$$\limsup_{n\to\infty} \frac{X_n}{f^{-1}\left(\frac{(1+\epsilon)\log n}{\lambda^*}\right)} \le 1.$$

By Lemma 2.1 (iv), f^{-1} is regularly varying with index $\frac{1}{\rho}$, and therefore,

$$f^{-1}\left(\frac{(1+\epsilon)\log n}{\lambda^*}\right) \sim \left(\frac{1+\epsilon}{\lambda^*}\right)^{\frac{1}{\rho}} f^{-1}(\log n).$$

Hence, almost surely,

$$\limsup_{n\to\infty} \frac{X_n}{f^{-1}(\log n)} \le \left(\frac{1+\epsilon}{\lambda^*}\right)^{\frac{1}{\rho}}.$$

By letting $\epsilon \to 0^+$, we obtain the assertion. □

In the following lemma, we assume that X_0 is arbitrary but independent of $(A_n, B_n)_{n\in\mathbb{N}}$.

Lemma 8.3. *For any $\delta > 0$, there exist $y_* \in (0,1)$ and $\tilde{M} > 0$ such that if $t y_*^{n-1} \ge \tilde{M}$, then, a.s.,*

$$P(X_n > t \mid X_0) \ge \mathbf{1}_{(t y_*^n, \infty)}(X_0) \exp(-(1+\delta)\lambda^* f(t)).$$

Before proving the above result, we first present a simple lemma.

Lemma 8.4. *For all $n \ge 1$, $y > 0$ and $t > 0$ we have, a.s.,*

$$\mathbb{P}(X_n > t \mid X_0) \ge \mathbf{1}_{(y^n t, \infty)}(X_0) \prod_{k=0}^{n-1} \mathbb{P}(t\, y^k A y + B > t\, y^k).$$

Proof. Notice that

$$\begin{aligned}\mathbb{P}(X_n > t \mid X_0) &\ge \mathbb{P}(A_n X_{n-1} + B_n > t, X_{n-1} > ty \mid X_0) \ge \mathbb{P}(t\, A_n y + B_n > t, X_{n-1} > ty \mid X_0)\\ &= \mathbb{P}(t\, Ay + B > t)\mathbb{P}(X_{n-1} > ty \mid X_0).\end{aligned}$$

We obtain the assertion by iterating the above inequality. □

Now we are ready to present the proof of Lemma 8.3.

Proof of Lemma 8.3. Fix $\alpha > 0$ and let $y_* \in (0,1)$ be such that

$$\lambda^* \le \frac{g(y_*)}{1 - y_*^{\rho}} \le \lambda^*(1+\alpha). \tag{17}$$

Such y_* exists as a consequence of the definition of $\lambda^* = \inf_{y\in(0,1)}\{g(y)/(1-y^\rho)\}$ and the fact that $\lambda^* > 0$. Clearly $g(y_*) > 0$. By Lemma 8.4, we have, a.s.,

$$\mathbb{P}(X_n > t \mid X_0) \ge \mathbf{1}_{(y_*^n t, \infty)}(X_0) \prod_{k=0}^{n-1} \mathbb{P}\left(t y_*^k A y_* + B > t y_*^k\right).$$

By the definition of g, there exists $M > 0$ such that

$$\forall t \geq M \;\; \mathbb{P}(Aty_* + B > t) \geq \exp(-(1+\alpha)g(y_*)f(t)).$$

Thus, if $ty_*^k \geq M$ for $k = 0, \dots, n-1$, (which is equivalent to $ty_*^{n-1} \geq M$), we obtain

$$\mathbb{P}(X_n > t \mid X_0) \geq \mathbf{1}_{(y_*^n t, \infty)}(X_0) \exp\left(-(1+\alpha)g(y_*) \sum_{k=0}^{n-1} f(ty_*^k) \right). \tag{18}$$

Fix η in $(0, \rho)$. By Lemma 2.1 (iii), there exists $M_1 > 0$ such that for all $z \in (0, 1]$ and for all $t > 0$ such that $tz \geq M_1$, it holds that

$$\frac{f(tz)}{f(t)} \leq (1+\alpha)z^{\rho-\eta}$$

Assume that $ty_*^{n-1} \geq M_1$. Then for $k = 0, \dots, n-1$, we have $ty_*^k \geq ty_*^{n-1} \geq M_1$. Therefore,

$$f(ty_*^k) \leq (1+\alpha)y_*^{k(\rho-\eta)} f(t), \qquad k = 0, \dots, n-1. \tag{19}$$

Notice that for all $n \in \mathbb{N}$ and sufficiently small η, we have

$$\frac{1 - y_*^{n(\rho-\eta)}}{1 - y_*^{\rho-\eta}} \leq \frac{1}{1 - y_*^{\rho-\eta}} \leq \frac{1+\alpha}{1 - y_*^{\rho}}, \tag{20}$$

where the first inequality above follows from the fact that $y_* \in (0, 1)$ and the latter is true for sufficiently small η. Take $\eta \in (0, \rho)$ satisfying the condition above.

Define $\tilde{M} = \max\{M_1, M\}$ and assume $ty_*^{n-1} \geq \tilde{M}$. Then, by (18), (19), (20) and (17), we obtain

$$\begin{aligned}\mathbb{P}(X_n > t \mid X_0) &\overset{(19)}{\geq} \mathbf{1}_{(y_*^n t, \infty)}(X_0) \exp\left(-(1+\alpha)^2 g(y_*) \sum_{k=0}^{n-1} y_*^{k(\rho-\eta)} f(t) \right) = \mathbf{1}_{(y_*^n t, \infty)}(X_0) \exp\left(-(1+\alpha)^2 g(y_*) \frac{1 - y_*^{n(\rho-\eta)}}{1 - y_*^{\rho-\eta}} f(t) \right) \\ &\overset{(20)}{\geq} \mathbf{1}_{(y_*^n t, \infty)}(X_0) \exp\left(-(1+\alpha)^3 g(y_*) \frac{1}{1 - y_*^{\rho}} f(t) \right) \overset{(17)}{\geq} \mathbf{1}_{(y_*^n t, \infty)}(X_0) \exp\left(-(1+\alpha)^4 \lambda^* f(t) \right).\end{aligned}$$

Since, for all $\delta > 0$, it holds that $1 + \delta$ is of the form $(1+\alpha)^4$ for some $\alpha > 0$, this concludes the proof. □

Lemma 8.5. *Let $\tilde{M}, \lambda^*, \epsilon > 0$ and $y_* \in (0, 1)$. Suppose that f is regularly varying with index $\rho > 0$. There exists a strictly increasing sequence $(k_n)_{n=1}^{\infty}$ of positive integers and a positive constant $c > \tilde{M} y_*$ such that, for $n \geq 1$,*

$$c \geq f^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\epsilon)} \right) y_*^{k_{n+1} - k_n} > \tilde{M} y_*. \tag{21}$$

Moreover, for any $\gamma \in (0, 1)$, there exists $K > 0$ such that for all $n \geq 1$,

$$k_n^{\gamma} \leq Kn. \tag{22}$$

Proof. Denote $H(t) = f(1/t)$. By definition, f is regularly varying (at ∞) with index $\rho > 0$ if and only if H is regularly varying at 0 with index $-\rho$, [20, page 8]. Since $f^{-1}(t) = 1/H^{-1}(t)$, (21) can be rewritten as

$$H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\epsilon)} \right) \left(\frac{1}{y_*} \right)^{k_{n+1} - k_n - 1} < \frac{1}{\tilde{M}} \quad \text{and} \quad H^{-1}\left(\frac{\log k_{n+1}}{\lambda^*(1+\epsilon)} \right) \left(\frac{1}{y_*} \right)^{k_{n+1} - k_n} \geq \frac{1}{c},$$

where $c^{-1} < (\tilde{M} y_*)^{-1}$. The existence of a sequence $(k_n)_{n=1}^{\infty}$ satisfying the two above conditions and (22) was established in [6, Lemma A.1]. □

We will use the following version of the Borel–Cantelli lemma, which can be found in [23, Theorem 5.1.2].

Lemma 8.6. *Suppose that $(\mathcal{F}_n)_{n \geq 0}$ is a filtration such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Assume that $A_n \in \mathcal{F}_n$ for all $n \geq 0$. Then,*

$$\limsup_{n \to \infty} A_n = \left\{ \sum_{n=1}^{\infty} \mathbb{P}(A_n \mid \mathcal{F}_{n-1}) = \infty \right\}.$$

Lemma 8.7 (*Kronecker's Lemma*)**.** *Assume that $a_n \uparrow \infty$. If $\sum_{n=1}^{\infty} x_n / a_n$ converges, then $\lim_{n\to\infty} a_n^{-1} \sum_{m=1}^{n} x_m = 0$. Equivalently, if $\lim_{n\to\infty} a_n^{-1} \sum_{m=1}^{n} x_m \neq 0$, then $\sum_{n=1}^{\infty} x_n / a_n$ does not converge.*

Lemma 8.8. *Almost surely, we have*

$$\limsup_{n \to \infty} \frac{X_n}{f^{-1}(\log n)} \geq (\lambda^*)^{-1/\rho}.$$

Proof. Fix $\epsilon > 0$ and define

$$t_n = f^{-1}\left(\frac{\log n}{\lambda^*(1+\epsilon)}\right).$$

Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of the sequence $(X_n)_{n\geq 0}$. By Lemma 8.6, it suffices to show that for a strictly increasing sequence of positive integers $(k_n)_{n\geq 1}$, we have, a.s.,

$$\sum_{n=1}^{\infty} \mathbb{P}(X_{k_n} > t_{k_n} \mid \mathcal{F}_{k_{n-1}}) = \infty,$$

which implies that $\mathbb{P}(\limsup_{n\to\infty} A_{k_n}) = 1$. Since $\limsup_{n\to\infty} A_n \supset \limsup_{n\to\infty} A_{k_n}$, this would prove the assertion.

Since $(X_{k_n})_{n\geq 1}$ is a Markov chain, we have

$$\mathbb{P}(X_{k_{n+1}} > t_{k_{n+1}} \mid \mathcal{F}_{k_n}) = \mathbb{P}(X_{k_{n+1}} > t_{k_{n+1}} \mid X_{k_n}).$$

By Lemma 8.3, applied to the sequence $(Y_n)_{n\in\mathbb{N}\cup\{0\}}$ defined by $Y_i = X_{k_n+i}$, we obtain that for any $\delta > 0$, there exist $y_* \in (0,1)$ and $\tilde{M} > 0$ such that

$$\mathbb{P}(X_{k_{n+1}} > t \mid X_{k_n}) \geq \mathbf{1}_{(t y_*^{k_{n+1}-k_n},\infty)}(X_{k_n}) \exp(-(1+\delta)\lambda^* f(t)), \tag{23}$$

provided $t y_*^{k_{n+1}-k_n-1} \geq \tilde{M}$.

We use the sequence $(k_n)_{n\geq 1}$ from Lemma 8.5. The condition $t_{k_{n+1}} y_*^{k_{n+1}-k_n-1} \geq \tilde{M}$ is satisfied as a consequence of the lower bound in (21). Therefore, using (23) and the upper bound in (21), we obtain, a.s.,

$$\mathbb{P}(X_{k_{n+1}} > t_{k_{n+1}} \mid X_{k_n}) \geq \mathbf{1}_{(c,\infty)}(X_{k_n}) \exp\left(-\tfrac{1+\delta}{1+\epsilon}\log k_{n+1}\right) = \mathbf{1}_{(c,\infty)}(X_{k_n})\frac{1}{k_{n+1}^{\gamma}},$$

where $\gamma := (1+\delta)/(1+\epsilon)$. By decreasing δ if necessary, we ensure that $\gamma < 1$. Our goal is to show that $\sum_{n\geq 1} \mathbf{1}_{(c,\infty)}(X_{k_n}) k_{n+1}^{-\gamma}$ diverges a.s. By Lemma 8.7, applied to $a_n = k_{n+1}^{\gamma}$ and $x_n = \mathbf{1}_{(c,\infty)}(X_{k_n})$, to meet this goal, it suffices to show that, a.s.,

$$\limsup_{m\to\infty} \frac{1}{k_{m+1}^{\gamma}} \sum_{n=1}^{m} \mathbf{1}_{(c,\infty)}(X_{k_n}) > 0. \tag{24}$$

From (22), we have $k_n^{\gamma} \leq Kn$ for some $K > 0$ and all $n \geq 1$. Therefore, a.s.,

$$\limsup_{m\to\infty} \frac{1}{k_{m+1}^{\gamma}} \sum_{n=1}^{m} \mathbf{1}_{(c,\infty)}(X_{k_n}) \geq \limsup_{m\to\infty} \frac{K^{-1}}{m+1} \sum_{n=1}^{m} \mathbf{1}_{(c,\infty)}(X_{k_n}).$$

Let $f_c : \mathbb{R} \to \mathbb{R}$ be defined by $f_c(x) = (x-c)/c\mathbf{1}_{[c,2c]}(x) + \mathbf{1}_{(2c,\infty)}(x)$. Since f_c is bounded and uniformly continuous on $\mathbb{R}$, and since $\mathbf{1}_{(c,\infty)} \geq f_c \geq \mathbf{1}_{(2c,\infty)}$, by Lemma 6.6 we have

$$\limsup_{m\to\infty} \frac{1}{m+1} \sum_{n=1}^{m} \mathbf{1}_{(c,\infty)}(X_{k_n}) \geq \limsup_{m\to\infty} \frac{1}{m+1} \sum_{n=1}^{m} f_c(X_{k_n}) \geq \mathbb{E}[f_c(X)] \geq \mathbb{P}(X > 2c).$$

Since $\mathbb{P}(X > t) > 0$ for any $t \in \mathbb{R}$ (recall that under $\lambda^* < \infty$, by Lemmas 6.4 and 6.5, we have $\operatorname{ess\,sup}(X) = \infty$), (24) follows. Therefore, by Lemma 8.6, for every $\epsilon > 0$, we have almost surely

$$\limsup_{n\to\infty} \frac{X_n}{f^{-1}\left(\frac{\log n}{\lambda^*(1+\epsilon)}\right)} \geq 1.$$

Proceeding as in the proof of Lemma 8.2, we obtain

$$\limsup_{n\to\infty} \frac{X_n}{f^{-1}(\log n)} \geq \left(\frac{1}{\lambda^*(1+\epsilon)}\right)^{\frac{1}{\rho}}.$$

The proof is completed by letting $\epsilon \to 0^+$. □

Proof of Theorem 8.1. This follows directly from Lemmas 8.2 and 8.8. □

9. Example

In this section, we present a family of distributions of (A, B), which admit an explicit LDM, but is not PQD. Thus, allowing us to illustrate our results on explicit example outside PQD pairs. This family is indexed by an absolutely continuous, nondecreasing function $\alpha : [0,1) \mapsto [1,\infty)$.

Let $\rho > 0$, and consider a random vector (A, B) whose law consists of a unique atom and an absolutely continuous component. Specifically, assume that $\mathbb{P}(A = 0, B = 1) = 1 - e^{-1}$, and that

$$\mathbb{P}(A > a, B > b) = \exp(-\alpha(a)\beta(b)), \qquad a \in [0,1), b \geq 1,$$

with

$$\beta(b) = b^\rho \quad \text{and} \quad \operatorname{ess\,sup}(A) = 1.$$

Since $e^{-1} = \mathbb{P}(A > 0, B > 1) = \exp(-\alpha(0)\beta(1))$, we deduce that $\alpha(0) = 1$. In addition, one may verify that $\alpha(1^-) = \infty$, that $\alpha(a) = -\log\mathbb{P}(A > a)$ for $a \ge 0$ and that $\log\mathbb{P}(B > b) = -b^\rho$ for $b \ge 1$.

Our goal is to compute the LDM_f^ρ function for (A, B) with $f(t) = -\log\mathbb{P}(B > t) = t^\rho$. Note that A and B are not positively quadrant dependent, so that the results of Section 4 do not apply. Indeed, if (A, B) were PQD, then we would have for $a \in [0, 1)$ and $b \ge 1$,

$$\exp(-\alpha(a)b^\rho) = \mathbb{P}(A > a, B > b) \ge \mathbb{P}(A > a)\mathbb{P}(B > b) = \exp(-\alpha(a) - b^\rho),$$

which implies that for all $a \in [0, 1)$ and all $b > 1$ it holds that $\alpha(a) \le b^\rho/(b^\rho - 1)$. By letting $b \uparrow \infty$ we obtain $\alpha \equiv 1$, but such α does not define a valid probability distribution.

By Lemma 3.3(i), we have $g(y) = 0$ for $y > 1$. Fix $y \in [0, 1)$ and consider $t > 1/(1 - y)$. Then,

$$\begin{aligned}\mathbb{P}(Aty + B > t) &= \int_0^1 \int_{t(1-ya)}^\infty \frac{\partial^2 \exp(-\alpha(a)\beta(b))}{\partial a\,\partial b}\,\mathrm{d}b\,\mathrm{d}a = -\int_0^1 \frac{\partial \exp(-\alpha(a)\beta(b))}{\partial a}\Big|_{b=t(1-ya)}\mathrm{d}a \\ &= t^\rho \int_0^1 (1 - ya)^\rho e^{-t^\rho(1-ya)^\rho\alpha(a)}\alpha'(a)\mathrm{d}a = t^\rho \int_1^\infty (1 - y\alpha^{-1}(u))^\rho e^{-t^\rho(1-y\alpha^{-1}(u))^\rho u}\mathrm{d}u,\end{aligned}$$

where the substitution $u = \alpha(a)$ was used (with α^{-1} denoting the generalized inverse of α). Since $(1 - y)^\rho \le (1 - y\alpha^{-1}(u))^\rho \le 1$, we obtain

$$(1 - y)^\rho t^\rho \int_1^\infty e^{-t^\rho(1-y\alpha^{-1}(u))^\rho u}\mathrm{d}u \le \mathbb{P}(Aty + B > t) \le t^\rho \int_1^\infty e^{-t^\rho(1-y\alpha^{-1}(u))^\rho u}\mathrm{d}u.$$

By the Laplace method, it follows that g exists and equals

$$g(y) = \lim_{t\to\infty} \frac{-\log\mathbb{P}(Aty + B > t)}{t^\rho} = \inf_{u\in[1,\infty)} \{(1 - y\alpha^{-1}(u))^\rho u\} = \inf_{a\in[0,1)} \{(1 - ya)^\gamma \alpha(a)\}, \qquad y \in [0, 1).$$

We obtain $g(0) = \alpha(0) = 1$ and

$$\phi_\rho(\lambda) = \inf_{y>0} \{y^\rho\lambda + g(y)\} = \begin{cases} \min\{1, \lambda\}, & \rho \in (0, 1], \\ \min\left\{\inf_{a\in[0,1)} \left\{ \left(\alpha(a)^{1/(1-\rho)} + a^{\rho/(\rho-1)}\lambda^{1/(1-\rho)}\right)^{1-\rho} \right\}, \lambda\right\}, & \rho > 1. \end{cases}$$

Moreover, one can easily show that

$$\lambda^* = \inf_{y\in(0,1)} \left\{ \frac{g(y)}{1 - y^\rho} \right\} = \inf_{a\in[0,1)} \inf_{y\in(0,1)} \left\{ \frac{\alpha(a)(1 - ya)^\rho}{1 - y^\rho} \right\} = \begin{cases} \inf_{a\in[0,1)} \{\alpha(a)\} = 1, & \rho \in (0, 1], \\ \inf_{a\in[0,1)} \{\alpha(a)(1 - a^{\rho/(\rho-1)})^{\rho-1}\}, & \rho > 1. \end{cases}$$

We note that, in either case, g is not admissible. Consider the case $\rho > 1$. Then, if X is a solution to (1), by Lemmas 7.5 and 7.6 (which do not require admissibility), we obtain

$$\liminf_{t\to\infty} \frac{\log\mathbb{P}(X > t)}{t^\rho} \ge -\inf_{a\in[0,1)} \left\{\alpha(a)\left(1 - a^{\rho/(\rho-1)}\right)^{\rho-1}\right\}. \tag{25}$$

We are going to relate this result to the findings of [19]. In [19, Theorem 5.1], it was shown that

$$\liminf_{t\to\infty} \frac{\log\mathbb{P}(X > t)}{h(t)} \ge -\left(s\left(1 - \left(1 - \frac{1}{s}\right)^{\gamma/(\gamma-1)}\right)\right)^{\gamma-1}, \tag{26}$$

where

$$h(t) = \inf_{x\ge1} \left\{-x\log\mathbb{P}\left(A > 1 - \frac{1}{x}, B > \frac{t}{x}\right)\right\},$$

γ is the index of regular variation of h and $s = \lim_{t\to\infty} \sigma(t)$, where σ is any function satisfying

$$h(t) = -\sigma(t)\log\mathbb{P}\left(A > 1 - \frac{1}{\sigma(t)}, B > \frac{t}{\sigma(t)}\right) + o(1), \qquad t \to \infty.$$

In our setting, for $t > 1$ one can write

$$h(t) = \inf_{x\in[1,t]} \left\{x\,\alpha\left(1 - \frac{1}{x}\right)\beta\left(\frac{t}{x}\right)\right\} = t^\rho \inf_{x\in[1,t]} \left\{x^{1-\rho}\,\alpha\left(1 - \frac{1}{x}\right)\right\}.$$

Now, we consider two cases:

(a) $\alpha(a) = (1 - a)^{1-\rho}$ with $\rho > 1$,
(b) $\alpha(a) = \exp(a/(1 - a))$ with $\rho > 2$.

In case (a) one immediately obtains $h(t) = t^\rho$ for $t > 1$. In this situation, one can take $\sigma(t) = \min\{s,t\}$ for any $s \geq 1$. Consequently, $\gamma = \rho$ and (26) becomes (after taking $\sup_{s\geq 1}$ of both sides)

$$\liminf_{t\to\infty} \frac{\log \mathbb{P}(X>t)}{t^\rho} \geq -\inf_{s\geq 1}\left\{\left[s\left(1-\left(1-\frac{1}{s}\right)^{\rho/(\rho-1)}\right)\right]^{\rho-1}\right\}.$$

A direct calculation shows that this lower bound agrees with the expression in (25).

Case (b). Then,

$$h(t) = t^\rho \inf_{x\in[1,t]}\left\{x^{1-\rho}e^{x-1}\right\}.$$

For $t > \rho-1 > 1$, the infimum is attained at $x = \rho - 1$, yielding

$$h(t) = t^\rho(\rho-1)^{1-\rho}e^{\rho-2}.$$

Again, one obtains $\gamma = \rho$, and by taking $\sigma(t) = \rho - 1 = s$, (26) gives

$$\begin{aligned}\liminf_{t\to\infty} \frac{\log \mathbb{P}(X>t)}{t^\rho} = \liminf_{t\to\infty} \frac{\log \mathbb{P}(X>t)}{h(t)}\cdot\frac{h(t)}{t^\rho} &\geq -\left((\rho-1)\left(1-\left(1-\frac{1}{\rho-1}\right)^{\rho/(\rho-1)}\right)\right)^{\rho-1}\cdot(\rho-1)^{1-\rho}e^{\rho-2}\\ &= -e^{a/(1-a)}\left(1-a^{\rho/(\rho-1)}\right)^{\rho-1}\Big|_{a=1-(\rho-1)^{-1}}.\end{aligned}$$

On the other hand, by (25), we have

$$\liminf_{t\to\infty} \frac{\log \mathbb{P}(X>t)}{t^\rho} = -\inf_{a\in[0,1)}\left\{e^{a/(1-a)}\left(1-a^{\rho/(\rho-1)}\right)^{\rho-1}\right\} > -e^{a/(1-a)}\left(1-a^{\rho/(\rho-1)}\right)^{\rho-1}\Big|_{a=1-(\rho-1)^{-1}},$$

which gives a sharper lower bound. Thus, the lower bound provided by Theorem 5.1 in [19] is, in general, not optimal.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

We appreciate the reviewer's detailed comments, which have helped to clarify the technical aspects of this paper.

BK: This research was funded in part by National Science Centre, Poland, 2023/51/B/ST1/01535.

References

[1] D. Buraczewski, E. Damek, T. Mikosch, Stochastic models with power-law tails, Springer Series in Operations Research and Financial Engineering, Springer, 2016, p. xv+320, The equation $X = AX + B$.
[2] C.M. Goldie, R. Grübel, Perpetuities with thin tails, Adv. in Appl. Probab. 28 (2) (1996) 463–480.
[3] P. Hitczenko, J. Wesołowski, Perpetuities with thin tails revisited, Ann. Appl. Probab. 19 (6) (2009) 2080–2101.
[4] P. Hitczenko, On tails of perpetuities, J. Appl. Probab. 47 (4) (2010) 1191–1194.
[5] B. Kołodziejek, Logarithmic tails of sums of products of positive random variables bounded by one, Ann. Appl. Probab. 27 (2) (2017) 1171–1189.
[6] K. Burdzy, B. Kołodziejek, T. Tadić, Stochastic fixed-point equation and local dependence measure, Ann. Appl. Probab. 32 (4) (2022) 2811–2840.
[7] H. Kesten, Random difference equations and renewal theory for products of random matrices, Acta Math. 131 (1973) 207–248.
[8] P. Kevei, A note on the Kesten-Grincevičius-Goldie theorem, Electron. Commun. Probab. 21 (2016) Paper No. 51, 12.
[9] P. Kevei, Implicit renewal theory in the arithmetic case, J. Appl. Probab. 54 (3) (2017) 732–749.
[10] C.M. Goldie, Implicit renewal theory and tails of solutions of random equations, Ann. Appl. Probab. 1 (1) (1991) 126–166.
[11] A.K. Grincevićius, One limit distribution for a random walk on the line, Lith. Math. J. 15 (4) (1975) 580–589.
[12] E. Damek, B. Kołodziejek, Stochastic recursions: between Kesten's and Grincevičius-Grey's assumptions, Stochastic Process. Appl. 130 (3) (2020) 1792–1819.
[13] P. Dyszewski, Iterated random functions and slowly varying tails, Stochastic Process. Appl. 126 (2) (2016) 392–413.
[14] E. Damek, P. Dyszewski, Iterated random functions and regularly varying tails, J. Difference Equ. Appl. 24 (9) (2018) 1503–1520.
[15] D.R. Grey, Regular variation in the tail behaviour of solutions of random difference equations, Ann. Appl. Probab. 4 (1) (1994) 169–183.
[16] Z. Palmowski, B. Zwart, Tail asymptotics of the supremum of a regenerative process, J. Appl. Probab. 44 (2) (2007) 349–365.
[17] D. Buraczewski, P. Dyszewski, A. Iksanov, A. Marynych, On perpetuities with gamma-like tails, J. Appl. Probab. 55 (2) (2018) 368–389.
[18] D. Denisov, B. Zwart, On a theorem of Breiman and a class of random difference equations, J. Appl. Probab. 44 (4) (2007) 1031–1046.
[19] B. Kołodziejek, On perpetuities with light tails, Adv. in Appl. Probab. 50 (4) (2018) 1119–1154.
[20] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular variation, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1989, p. xx+494.
[21] G. Alsmeyer, A. Iksanov, U. Rösler, On distributional properties of perpetuities, J. Theoret. Probab. 22 (3) (2009) 666–682.
[22] D.J. Daley, C.M. Goldie, The moment index of minima. II, Statist. Probab. Lett. 76 (8) (2006) 831–837.
[23] T.K. Chandra, The Borel-Cantelli lemma, SpringerBriefs in Statistics, Springer, Heidelberg, 2012, p. xii+106.