Rozwiązania - kolokwium TCiWdTD 28.11.2011

Zad. 1.

$$\begin{aligned} I(t) &= \int_0^{2t} \frac{x}{\sqrt{2t-x}}dx = \left| \begin{array}{c} x = 2r \\ dx = 2dr \end{array} \right| = 4\int_0^{t} \frac{r}{\sqrt{2t-2r}}dr = \left| \begin{array}{c} r = tu \\ dr = tdu \end{array} \right| = \\ &= 4t^2 \int_0^1 \frac{u}{\sqrt{2t-2tu}}du = 2t\sqrt{2t}\int_0^1 u(1-u)^{-\frac{1}{2}}du = 2t\sqrt{2t}B\left(2,\frac{1}{2}\right). \end{aligned}$$

Zatem

$$I(1) = 2\sqrt{2}B\left(2,\frac{1}{2}\right) = 2\sqrt{2}\frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} = 2\sqrt{2}\frac{\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{8\sqrt{2}}{3}.$$

Zad. 2. Funkcję $f(x) = 2 - x$ przedłużamy do funkcji nieparzystej na przedziale $[-1;1]$. Wówczas

$$b_n = 2\int_0^1 (2-x)\sin n\pi x dx = \frac{2}{n\pi}\left(2-(-1)^n\right) \text{ dla } n = 1,2,3\ldots$$

tzn.

$$2 - x = \frac{2}{\pi}\sum_{n=1}^{+\infty}\frac{1}{n}\left(2-(-1)^n\right)\sin n\pi x \quad \text{dla } x \in [0;1].$$

Z okresowości sumy szeregu wynika, że

$$\lim_{x\to 1^+} S(x) = \lim_{x\to -1^+} f(x) = -1.$$

Zad. 3.

$$\begin{aligned} \mathcal{F}[f](y) &= \int_{-\infty}^{0} e^{-ixy+x}dx + \int_0^{+\infty} e^{-ixy-x}dx = \frac{1}{1-iy}\left. e^{x(1-iy)}\right|_{-\infty}^{0} + \frac{1}{-1-iy}\left. e^{x(-1-iy)}\right|_{0}^{+\infty} = \\ &= \frac{1}{1-iy} + \frac{1}{1+iy} = \frac{2}{1+y^2}. \end{aligned}$$

$$\begin{aligned} \mathcal{F}^2[f](y) &= \mathcal{F}\left[\frac{2}{1+x^2}\right](y) = \int_{-\infty}^{+\infty} e^{-ixy}\frac{2}{1+x^2}dx = \left| \begin{array}{c} x = -t \\ dx = -dt \end{array} \right| = \int_{-\infty}^{+\infty} e^{ity}\frac{2}{1+t^2}dt = \\ &= 2\pi\mathcal{F}^{-1}\left[\frac{2}{1+t^2}\right](y) = 2\pi\exp(-|y|) = 2\pi f(y), \end{aligned}$$

zatem

$$\mathcal{F}^{2n+1}[f](x) = (2\pi)^n\mathcal{F}[f](x) = (2\pi)^n\frac{2}{1+x^2}.$$

Zad. 4. Niech $Y(s) = L[y](s)$. Wówczas

$$\begin{aligned} s^2Y(s) - s - 2 - 2sY(s) + 2 + Y(s) &= \frac{2}{s-2} \\ Y(s) &= \frac{s^2-2s+2}{(s-2)(s-1)^2}. \end{aligned}$$

Odwracając transformatę dowolną metodą otrzymujemy

$$y(t) = 2\exp(2t) - t\exp(t) - \exp(t).$$

Zad. 5. Niech $Y(s) = L[y](s)$. Wówczas

$$\begin{aligned} Y(s) - Y(s)\frac{1}{s^2} &= \frac{1}{s-1} - \frac{2}{s} \\ Y(s) &= \frac{-s^2+2s}{(s-1)^2(s+1)} \end{aligned}$$

a stąd

$$y(t) = \frac{1}{2}t\exp(t) - \frac{1}{4}\exp(t) - \frac{3}{4}\exp(-t).$$