

Anharmonic action: Algebra and logic on the real projective line and the Bloch sphere

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The non-compact real line \mathbb{R} and complex plane \mathbb{C} are compactified by the addition of a single point ∞ “at infinity,” creating the *real projective line* $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ and the *complex projective line* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Topologically, $\widehat{\mathbb{R}}$ is a circle S^1 , while $\widehat{\mathbb{C}}$ is a sphere S^2 , the *Bloch sphere*. The Bloch sphere is fundamental to quantum computation, and thus by restriction also to classical computation, which is now understood to be an important fragment of quantum computation.

Algebraically, \mathbb{R} and \mathbb{C} carry field structure, with the distinguished constants 0 and 1. Notoriously, fields are not amenable to universal algebra, since 0 does not participate in the multiplicative group structure. The carrier of the multiplicative group structure, namely the complement of 0 in the field, may equally well be described as the complement of $\{\infty, 0\}$ in the projective line. Furthermore, the group structure may be construed as a quasigroup structure, where the right and left divisions appear as residuations (a realization due to Lambek).

The group of projective transformations of a projective line acts in triply transitive fashion. Thus, the natural action of the symmetric group Σ_3 on the set $\{0, 1, \infty\}$ extends uniquely to a so-called *anharmonic action* on the projective lines $\widehat{\mathbb{R}}$ and $\widehat{\mathbb{C}}$. This group is generated by the conformal (i.e., angle-preserving) transformation $J: x \mapsto x^{-1}$ of *inversion* (with $0^{-1} = \infty$) and the affine transformation $K: x \mapsto (1-x)$ of *complementation* (which fixes ∞). These involutions are accompanied by the conformal transformation $JKJ = KJK$, the third involution in the anharmonic group Σ_3 .

Anharmonic action provides a powerful tool to move algebra structure around isomorphically on the projective lines. For example, the usual Boolean algebra structure on $\{0, 1\}$, with K as its complementation, translates to a Boolean algebra structure on $\{\infty, 0\}$, with J as its complementation. On the other hand, J is the inversion in the usual multiplicative group of the field, whose carrier complements the underlying set of this latter Boolean algebra. Summarizing, the projective line decomposes as the disjoint union of a Boolean algebra and a group, in three mutually isomorphic ways, where the complementations in the Boolean algebras appear as inversions in the groups.