A generic construction of free algebras in varieties of Hilbert algebras and Brouwerian semilattices

Tomasz Kowalski[1] and Katarzyna Słomczyńska[2]

[1] Department of Logic, Jagiellonian University
tomasz.s.kowalski@uj.edu.pl
[2] Department of Mathematics, University of National Education Commission, Kraków
irena.korwin-slomczynska@up.edu.pl

Brouwerian semilattices (Br) are among the most investigated classes of *varieties of logic*. In the seminal Köhler [4] in particular, a construction of the free Brouwerian semilattice is presented, based in part on an earlier de Bruijn [3]. It leads to a recursive formula for the number of meet-irreducible elements, which was used to compute the exact number of elements of the 3-generated free Browerian semilattice. Of the more recent works, Bezhanishvili et al. [1] use a colouring technique to construct free algebras in *nuclear Brouwerian semilattices*, which expand Brouwerian semilattices with a strong form of a nucleus (an S4-like modality).

Hilbert algebras (Hi), implicative subreducts of Brouwerian semilattices, are less studied. A number of observations can be found in de Bruijn [3] and Urqhardt [7]. More recently, finite Hilbert algebras (but not the free algebra) were studied in Celani and Cabrer [2]. All these works use techniques based either on a syntactic analysis of terms, or on dualities and Kripke frames.

We take a different route: a purely algebraic one, motivated by Słomczyńska [6]. Beginning from an n-generated free algebra $\mathbf{F}_{\mathcal{V}}(n)$—which obviously exists and is finite—in some subvariety $\mathcal{V}$ of Hi (or of its variant Ho expanded by zero), we analyse the structure of $\mathbf{F}_{\mathcal{V}}(n)$ given by its subdirectly irreducible factors, that is, its completely meet-irreducible congruences. The poset $\mathbf{Cm}\,\mathbf{F}_{\mathcal{V}}(n)$ of these congruences is isomorphic to the poset $\mathbf{Cm}\,\mathbf{F}_{\widehat{\mathcal{V}}}(n)$ of completely meet-irreducible congruences of the free algebra $\mathbf{F}_{\widehat{\mathcal{V}}}(n)$, where $\widehat{\mathcal{V}}$ is a natural counterpart of $\mathcal{V}$ in Br (or Bo, the zero-expanded variant). Since the lattices of subvarieties of Hi and Br, and respectively of Ho and Bo, are isomorphic, that gives us skeletons (Kripke frames) for all free algebras in a uniform way. The frames have a multilayer structure, corresponding to the higher and higher subdirectly irreducibles (with height measured by chains of meet-irreducible congruences). The crucial difference between varieties with zero and the varieties without zero occurs only at the first layer, higher up the construction of the frame is the same.

The next step, where the distinction between Brouwerian semilattices and Hilbert algebras comes into play, is the choice of some antichains in the frames. If we take all antichains, we obtain a Brouwerian semilattice, if we omit some, we lose some meets, so we obtain a *partial* Brouwerian semilattice. Carefully selecting the right antichains we arrive at our description of $\mathbf{F}_{\mathcal{V}}(n)$. Algorithm 1 below shows how to build $\mathbf{Cm}\,\mathbf{F}_{\mathcal{V}}(n) \cong \mathbb{P}(n) = \bigcup_{i=1}^{n} P_i(n)$, generically, for $\mathcal{V} = \mathsf{Br}, \mathsf{Hi}, \mathsf{Bo}, \mathsf{Ho}$. The last ingredient we need to build the free Hilbert algebra (without or with zero) is a function selecting the right antichains. For any generator x, we put

$$S(x) := \big\{(L,i) \in \mathbb{P}(n)\big\} : x \notin L \text{ and } x \in L' \text{ for all } (L',j) > (L,i)\big\},$$

with the notation explained in Algorithm 1. Finally, we can state

Theorem 1. *For any $n > 0$, we have*

$$\mathbf{F}_{\mathsf{Hi}}(n) \cong \left(\bigcup_{i=1}^{n} \mathcal{P}(S(x_i)); \to, \emptyset\right) \qquad \mathbf{F}_{\mathsf{Ho}} \cong \left(\bigcup_{i=1}^{n+1} \mathcal{P}(S(x_i)) \cup \mathcal{P}(S(0)); \to, \emptyset, P_1(n)\right)$$

where $\mathcal{P}$ *is the power set operation,* 1, 0 *are interpreted as* $\emptyset$, $P_1(n)$ *and* $\to$ *is defined suitably.*

We deal with subvarieties generically by appropriately restricting G in Algorithm 1.

Algorithm 1 $\mathbb{P}(n)$ construction

Require: $J \subseteq X$ with $|X| = n$, $1 \leq i \leq n$ $\qquad\triangleright$ X is the set of generators. Points are pairs (J, i)
  Let $\text{Proj}(U) := \{J : (J, i) \in U \text{ for some } i\}$
  $P_1(n) \leftarrow \{(J, 1) : J \subset X\}$; $\leq \leftarrow =$ $\qquad\triangleright$ If $\mathcal{V} = \mathsf{Ho}, \mathsf{Bo}$, then $P_1(n) \leftarrow \{(J, 1) : J \subseteq X\}$
  function $\text{BuildNextLayer}(\bigcup_{i=1}^{k-1} P_i(n))$
    $P_k(n) \leftarrow \emptyset$
    for $G \in \text{Up}\left(\bigcup_{r=1}^{k-1} P_r(n)\right)$, $G \cap P_{k-1}(n) \neq \emptyset$, $L \subset \bigcap \text{Proj}(G)$ **do**
      $P_k(n) \leftarrow P_k(n) \cup \{(L, k)\}$
      for $(J, j) \in G$ **do**
        $\leq \leftarrow \leq \cup \langle (L, k), (J, j) \rangle$
      end for
    end for
  end function
  for $k \in \{2, \dots, n\}$ **do** $\qquad\triangleright$ If $\mathcal{V} = \mathsf{Ho}, \mathsf{Bo}$, then $k \in \{2, \dots, n+1\}$
    $\text{BuildNextLayer}(\bigcup_{i=1}^{k-1} P_i(n))$
  end for

Combining our description of $\mathbf{F}_{\mathcal{V}}(n)$ with results in Słomczyńska [5] yields simple proofs of well known results on hereditary structural completeness of Hi, Br, and Bo, and structural incompleteness of Ho. As a byproduct we also get

Theorem 2. *Let* $\mathcal{V} \subseteq \mathsf{Ho}$*, and let* $t_1, \dots, t_k, r$ *be terms. If* $t_i(1, \dots, 1) = 1$ *for all* $i \in \{1, \dots, k\}$*, then*

$$\mathbf{F}_{\mathcal{V}}(n) \models \bigwedge_{i=1}^{k} t_i = 1 \implies r = 1 \iff \mathbf{F}_{\mathcal{V}}(n) \models t_1 \to (\dots \to (t_k \to r) \dots) = 1.$$

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