FOUNDATION YEAR 2019-2020 GROUP 3
MATHEMATICS Sem.1
PROGRAM and EXERCISES

PROGRAM:
Lesson 1. Basic notions of mathematical logic.
Lesson 2. Basic operations on sets.
Lesson 3. The notion of a function, main types of functions.
Lesson 4. Composition of functions. Inverse function.
Lesson 5. Basic types of numerical sets. Prime numbers. Decomposition of integer numbers.
Lesson 6. Fractions, decimals and arithmetical operations on them.
Lesson 7. Elementary algebraic identities (square of a sum/difference, difference of squares, sum, difference of cubes).
Lesson 8. Short summary and question session. Test1.
Lesson 9. Linear function, the main properties and graphs. Linear equations and inequalities.
Lesson 10. Quadratic function, the main properties and graphs.
Lesson 11. Quadratic equations and inequalities.
Lesson 12. The notion of absolute value. Functions involving modules, graphs.
Lesson 13. Equations and inequalities with module.
Lesson 14. Systems of linear equations or inequalities with two variables.
Lesson 15. Systems of quadratic equations or inequalities with two variables.
Lesson 16. Short summary and question session. Test2.
Lesson 17. Basic planary sets, properties and computing areas.
Lesson 18. Basic solids, properties and computing volumes.
Lesson 19. The notion of polynomials. Operations on polynomials.
Lesson 20. Bezout theorems. Factorization of polynomials.
Lesson 21. Solving polynomial equations and inequalities.
Lesson 22. Solving polynomial equations and inequalities with modules.
Lesson 23. Rational functions. The homographic functions and their graphs.
Lesson 24. Rational equations and inequalities.
Lesson 25. Short summary and question session. Test3.
Lesson 26. The notion of a sequence. Main properties of numerical sequences.
Lesson 27. Arithmetic and geommetric sequences.
Lesson 28. The total sum of a geometric sequence. Applications.
Lesson 29. The notion of a limit of a numerical sequence.
Lesson 30. Final exam.

EXERCISES:

Assume the notation for sets:

$\mathbb{N} = \{1, 2, 3, ...\}$ - set of all natural numbers,
$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ - set of all integer numbers,
$\mathbb{Q} = \{\frac{k}{m} : k, m \in \mathbb{Z}, m \neq 0\}$ - set of all rational numbers,
$\mathbb{Q}'$ - set of all irrational numbers, e.g. $\sqrt{2} \in \mathbb{Q}'$,
$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}' = (-\infty, +\infty)$ - set of all real numbers, $\mathbb{Q} \cap \mathbb{Q}' = \emptyset$.
$\mathbb{C}$ - set of all complex numbers.
There are inclusions between the above sets: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset R \supset Q'$, $\mathbb{R} \subset C$.

Lesson 1.
1.1 Evaluate the following logical sentences:
a) $1 = 0 \Rightarrow 2 + 2 = 5$
b) $1 = 0 \Rightarrow 2 + 2 = 4$
c) $1 + 1 = 2 \Rightarrow 2 + 2 = 5$
d) $1 + 1 = 2 \Rightarrow 2 + 2 = 4$
e) $\forall_{x \in \mathbb{R}} (x^2 - 1 = 0) \Rightarrow (x = 1)$
f) $\forall_{x \in \mathbb{R}} (x^2 - 1 = 0) \Rightarrow (x = -1 \vee x = 1)$
g) $\forall_{x \in \mathbb{R}} (x^2 + 1 = 0) \Rightarrow (x = 2)$.
Lesson 2.
2.1 Give the full list of elements for the following sets:
$A = \{x \in \mathbb{N} : x \text{ is a divisor of } 12\}$
$B = \{x \in \mathbb{Z} : 0 \leq x \leq 6\}$
2.2 Which of the following sets coincide
$A = \{x \in \mathbb{R} : x^2 - 4 = 0\}$
$B = \mathbb{E}ven$
$C = \{x \in \mathbb{R} : |x| = 2\}$
2.3 How many subsets has a set composed of n elements, for $n = 1, 2, 3, 4, 5$ and, in general, for $n \in \mathbb{N}$.
Lesson 3.

Lesson 4.
4.1 Give some examples of:
a) injective functions from $X = \{1, 2, 3\}$ into $Y = \{a, b, c, d, e\}$
b) injective functions from $X = \{1, 2, 3\}$ into $Y = \mathbb{N}$
c) surjective functions from $X = \{1, 2, 3, 4\}$ onto $Y = \{a, b, c\}$
d) surjective functions from $X = \mathbb{N}$ onto $Y = \{a, b, c\}$
e) bijective functions from $X = \{1, 2, 3\}$ onto $Y = \{a, b, c\}$

f) bijective functions from $X = \mathbb{R}$ onto $Y = (-1, 1)$
4.2 Let $m, n \in \mathbb{N}$ be the numerosities of the finite sets X and Y, correspondingly. What is the number of all functions from X into Y? Give some examples for some small values of m and n.

Lesson 5.
5.1 Find the following sets: $\mathbb{N} \cap (\mathbb{R} \setminus \mathbb{Q})$, $\mathbb{Z} \cap (\mathbb{R} \setminus \mathbb{Q})$, $\mathbb{N} \cup \mathbb{Q}$, $\mathbb{N} \setminus \mathbb{Q}$, $\mathbb{Q} \setminus \mathbb{Z}$.
5.2 Prove that: $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7} \notin \mathbb{Q}$.
5.3 Remind the definition of a prime number and give several examples of prime numbers.
5.4 Prove that $\sqrt{p} \notin \mathbb{Q}$, for any prime number p.
5.5 Given sets: $\mathbb{P}rime$ - set of all prime numbers, $\mathbb{E}ven$ - set of all even numbers, $\mathbb{O}dd$ - set of all odd numbers, $\mathbb{D}iv_3$ - set of all integer numbers divisible by 3. Give several elements for each of the sets: $\mathbb{P}rime \cap \mathbb{O}dd$, $\mathbb{E}ven \cap \mathbb{D}iv_3$, $\mathbb{O}dd \cap \mathbb{D}iv_3$, $\mathbb{E}ven \setminus \mathbb{D}iv_3$, $\mathbb{O}dd \setminus \mathbb{D}iv_3$.
5.6 Let $p, q \in \mathbb{E}ven$ and $k, m \in \mathbb{O}dd$. Evaluate the parity type for the numbers: $p + q$, $p - q$, $p \cdot q$, $k + m$, $k - m$, $p + k$, $p - k$, $k + m$, $k - m$, $k \cdot m$.
5.7 Prove that, for any $n \in \mathbb{N}$, the number $n + (n + 1) + (n + 2)$ is divisible by 3.
5.8 Write the extended formulas for the expressions: $(a + b)^n$ and $(a - b)^n$, where $n \in \mathbb{N}$
5.9 Assume that for two numbers $x, y \in \mathbb{R}$ there is $x + y \in \mathbb{Q}$ and $x - y \in \mathbb{Q}$. Prove that in this case there is also $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$.
5.10 Compute: 4% of 58, 20% of 40, 0.20% of 150.
5.11 Find the number $x \in \mathbb{R}$ if:
a) its 5% equals 14
b) its 0.2% equals 1.4
c) its 128% equals 512.

Lesson 6.
6.1 Without making calculations, replace the dots ... by the appropriate inequality or equality symbol to compare the following numbers:
a) $2\frac{1}{3} + 3\frac{1}{4} ... 2\frac{1}{4} + 3\frac{1}{4}$, b) $-5\frac{2}{3} - 2\frac{1}{7} ... -4\frac{1}{2} - 2\frac{1}{7}$

c) $4\frac{5}{8} \cdot 2\frac{1}{4} ... 6\frac{3}{4} \cdot 2\frac{1}{4}$, d) $8\frac{2}{3} : (-\frac{1}{2}) ... \frac{1}{8} : (-\frac{1}{2})$
6.2 Compute (to the simplest form):
a) $\frac{8 \cdot 4\frac{1}{4} - 11\frac{1}{5} : 9\frac{1}{3} - (-2\frac{1}{3}) : \frac{5}{3}}{14 : 2\frac{2}{9} + 8\frac{2}{5} \cdot 1\frac{2}{7}}$, b) $\frac{0.1}{(140\frac{7}{30} - 138\frac{5}{12}) : 18\frac{1}{6}}$
6.3 What percentage of the number a is the number b:
a) $a = 14$, $b = 112$
b) $a = 125$, $b = 50$

c) $a = 0.15$, $b = 0.75$
6.4 The price of the goods was reduced by 20%, and then the new price was increased by 20%. Is the final price equal to the initial price?
6.5 Calculate the arithmetic mean and the corresponding standard deviation of the numbers: 2, 3, 2, 5, 4, 4, 3, 3, 2, 5.
6.6 Formulate the concepts of weighted averages: arithmetic, geometric and harmonic. Calculate the given average types on your own example and compare their values.
6.7 Formulate the concepts of weighted averages: arithmetic, geometric and harmonic. Calculate the given average types on your own example and compare their values.
6.8 The first half of the time the vehicle was traveling at a speed of 40km/h, and the other half time at a speed of 60km/h. Calculate the average speed over the entire time. The obtained result compare with the result in task 6.7.
Lesson 7.
7.1 Check the following formulas for $n = 2, 3, 4, 5$ and then prove these formulas for any natural $n \in \mathbb{N}$:
a) $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
b) $(a-b)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} b^k$
c) $a^n - b^n = (a-b) \sum_{k=0}^{n-1} a^{n-k-1} b^k$
Lesson 8.
Lesson 9.
9.1 Remind the definition, graph and the interpretation of the coefficients in the linear function formula.
9.2 Find the linear function whose graph goes through points:
a) $(-1, 1)$ and $(2, 7)$
b) $(0, 0)$ and $(1, 1)$
9.3 The graph of the linear function $f(x)$ includes points A and B. Check if point C also belongs to the graph of this function:
a) $A = (1, 2)$, $B = (-1, 4)$, $C = (2, 6)$
b) $A = (2, 1)$, $B = (4, 0)$, $C = (3, \frac{1}{2})$
c) $A = (\frac{1}{3}, 3)$, $B = (-\frac{3}{4}, 3)$, $C = (1, 3)$
9.4 Sketch a graph of the linear function $f(x) = \frac{x}{2}$ and then transform it by symmetry:
a) with respect to the x-axis
b) with respect to the y-axis
c) with respect to the line $y = 2$
d) with respect to the line $x = 2$
e) with respect to the origin of the coordinate system
9.5 Repeat Exercise 8.4 for functions $g(x) = x - 4$ and $h(x) = 3 - x$.

9.6 Write a linear function formula whose graph is:
a) parallel to the graph of $y = 2x + 1$ and passes through the point $(1, 5)$
b) parallel to the graph of $y = -x + 5$ and passes through the point $(-1, 4)$
c) perpendicular to the graph of $y = \frac{1}{5}x + 3$ and passes through the point $(1, 5)$
9.7 The graph of the linear function passes through point A and is inclined to the x-axis at the angle α. Write the appropriate formula for this function.
a) $A = (1, 5)$, $\alpha = \frac{2}{3}\pi$
b) $A = (-1, -1)$, $\alpha = \frac{1}{3}\pi$
c) $A = (-2, 4)$, $\alpha = \frac{1}{4}\pi$
9.8 Given a linear function $f(x)$. Write its inverse function formula:
a) $f(x) = x$
b) $f(x) = 3x$
c) $f(x) = -x + 2$
d) $f(x) = \frac{1}{3}x - 4$
9.9 Sketch the graph of the function: a) $y = |x|$
b) $y = |x| + 1$
c) $y = |x - 2| + 3$
$d)$y=|x|+|x+1|
e) $y = \frac{|x|}{x}$
9.10 Find the value of m such that the line $y = mx + 5$ is:
a) parallel to the line $3x + 4y = 5$
b) perpendicular to the line $5x - 3y = 2$
9.11 Solve the following equations: a) $3x + 4(3 - x) - (3x + 2) = 3$
b) $|x + 2| = 2(3 - x)$
c) $|x - 3| + |x + 4| = 9$
d) $2|x| + |x - 1| + |x + 1| = 4$
9.12 Solve the following inequalities:
a) $3x - [7 - (5 - 4x)] - (x - 8) \geq 0$
b) $2x + 5 \leq 3x + 4 < 4x + 2$
c) $|x - 1| + |x + 3| < 5$
d) $|2x + 6| + |3x - 12| + |x| \leq 20$
9.13 Compute the area bounded by graphs of the functions $f(x) = |\frac{x}{2} + 2| - 2$ and $g(x) = 2$.
9.14 At what time between 4 and 5 am both clock hands meet?
9.15 Two taps pour water into the tub. The first tap fills the tub in 3 hours and the second tap in 5 hours. After what time will the bathtub be filled when both taps pour together?
9.16 Is it possible to read precisely each time if both clock hands are identical?

Lesson 10. Quadratic function of $x \in \mathbb{R}$ is defined by: $y = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$, $a \neq 0$. By a root of a quadratic function we mean any $x_0 \in \mathbb{R}$ such that $ax_0^2 + bx_0 + c = 0$. Assume $\Delta = b^2 - 4ac$. Then the number of roots depends on Δ. If $\Delta < 0$ there are no roots. If $\Delta = 0$ there is exactly one root $x_0 = \frac{-b}{2a}$. In the case when $\Delta > 0$ there are two different roots $x_1 = \frac{-b-\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b+\sqrt{\Delta}}{2a}$. The graph of a quadratic function is a parabola with its arms oriented upward if $a > 0$ and downward if $a < 0$. The vertex of the parabola is $V = (\frac{-b}{2a}, \frac{-\Delta}{4a})$. We can present a square function also as $y = a(x + \frac{b}{2a})^2 - \frac{\Delta}{4a}$ (canonical form). If $\Delta > 0$, we get $y = a(x - x_1)(x - x_2)$ and if $\Delta = 0$, we get $y = a(x - x_0)^2$. In the case $\Delta = 0$ one can also use the convention that $x_1 = x_2 = x_0$ (i.e. one but double root). Within this convention we can write two Viete formulas: $x_1 + x_2 = \frac{-b}{a}$ and $x_1 \cdot x_2 = \frac{c}{a}$.
10.1 The graphs of $y = x^2 + c$, $c \in \mathbb{R}$, form a family of parabolas. Sketch a few of them. Find the value of c for which
a) the point $A = (-1, -1)$ belongs to the graph,
b) number 3 is a root.
10.2 The graphs of $y = x^2 + bx$, $b \in \mathbb{R}$, form a family of parabolas. Sketch a few of them. Find the value of b for which
a) the point $A = (-2, 5)$ belongs to the graph
b) number 4 is a root,
c) the quadratic function has exactly one root.
10.3 The graphs of $y = x^2 + 3x + c$, $c \in \mathbb{R}$, form a family of parabolas. Sketch a few of them. Find the value of c for which
a) $(-2, -1)$ belongs to the graph,
b) the graph is tangent to the x-axis.
10.4 Find b and c in $y = x^2 + bx + c$ if a) numbers 2 and -3 are the roots,
b) $y_{min} = 5$ at $x = -2$.
10.5 For which $m \in \mathbb{R}$ a function $y = mx^2 + 3x + 4$
a) has two different roots,
b) has exactly one root
c) has no roots.
Notice that the topic does not ask for the function to be necessarily square.
10.6 For $x \in [0, 2]$ find the maximal value of the following function
a) $y = -2x^2 + x - 1$,
b) $y = -x^2 - 3x + 10$,
c) $y = 2x^2 - x + 1$,
d) $y = -x^2 + x$.
10.7 Find the minimal value of the following function
a) $y = x^2 + 4x - 2$ for $x \in [-1, 2]$,
b) $y = 2x^2 - 1.5x + 0.6$ for $x \in [-2, -1]$,

c) $y = x^2 - 1$ for $x \in [0, 1]$.

10.8
10.9
10.10

Lesson 11.

Lesson 12.