A semigroup approach to the space-fractional diffusion

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Motivation

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^\alpha u = 0 & \text{in } Q_{s,T}, \\ u_x(0,t) = 0, \quad u(t,s(t)) = 0 & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{for } 0 < x < s(0) = b, \\ \dot{s}(t) = -(D^\alpha u)(s(t),t) & \text{for } t \in (0,T), \end{cases} \tag{1}$$

$$Q_{s,T} := \{(x,t) : 0 < x < s(t), 0 < t < T\}.$$

$$(D^\alpha u)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-p)^{-\alpha} u'(p) dp.$$

- V. Voller, *Fractional Stefan Problems*, 2017.
- V. Voller, *On a fractional derivative form of the Green-Ampt infiltration model*, 2011.

Fractional operators

Let $\alpha \in (0,1)$. For integrable function u we define the fractional integral

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and the Caputo derivative

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For u absolutely continuous we have

$$(D^{\alpha}u)(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x (x-p)^{-\alpha}u'(p)dp.$$

Let us define the operator of integration on $L^p(0, L)$ for $p \in [1, \infty]$ by

$$(If)(x) = \int_0^x f(p)dp \quad \text{for } f \in L^p(0, L). \tag{2}$$

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Proposition

Let $L > 0$, $p \in [1,\infty]$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$. If $u, v \in L^1(0,L)$, then

$$(\lambda E + I)v(t) = u(t) \iff v(t) = \lambda^{-1}u(t) - \lambda^{-2}\int_0^t u(s)e^{\frac{s-t}{\lambda}}ds. \tag{3}$$

Furthermore, $I + \lambda E : L^p(0,L) \longrightarrow L^p(0,L)$ is an isomorphism and there holds the following estimate

$$\|(\lambda E + I)^{-1}\|_{B(L^p(0,L))} \leq (1+\sqrt{2})|\lambda|^{-1} \quad \text{for} \quad \lambda \in \Sigma, \tag{4}$$

where

$$\Sigma = \{z \in \mathbb{C} : \quad \operatorname{Re} z > |\operatorname{Im} z|\}.$$

Definition

[1, Definition 1.1.1] We say that A is non-negative if $(-\infty, 0) \subseteq \rho(A)$ *and there exists* $M > 0$ *such that*

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Let A be a non-negative operator. We define for $0 < \operatorname{Re} \alpha < 1$ operator J^{α} as follows $D(J^{\alpha}) = D(A)$

$$J^{\alpha} u = \frac{\sin \alpha\pi}{\pi} \int_0^{\infty} \lambda^{\alpha - 1} (\lambda + A)^{-1} A u d\lambda.$$

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[1, Definition 5.1.2] Let A be an unbounded and positive operator (nonnegative and $0 \in \rho(A)$). We define for $\operatorname{Re}\alpha > 0$

$$A^\alpha = ((A^{-1})^\alpha)^{-1}.$$

Here, the domain of A^α consists of $u \in X$ such that $u \in R((A^{-1})^\alpha)$.

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$$(\lambda E+I)^{-1}Iu(t) = \lambda^{-1}\int_0^t u(\tau)d\tau - \lambda^{-2}\int_0^t\int_0^s u(\tau)d\tau e^{\frac{s-t}{\lambda}}ds = \lambda^{-1}\int_0^t u(\tau)e^{\frac{\tau-t}{\lambda}}d\tau.$$

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Applying the Fubini theorem and then the substitution $\frac{t-\tau}{\lambda} = p$ we arrive at

$$J^\alpha u = \frac{\sin \alpha\pi}{\pi} \int_0^t u(\tau)(t-\tau)^{\alpha-1} \int_0^\infty p^{-\alpha} e^{-p} dp d\tau$$

$$= \frac{\sin \alpha\pi}{\pi} \int_0^t u(\tau)(t-\tau)^{\alpha-1} d\tau \Gamma(1-\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^t u(\tau)(t-\tau)^{\alpha-1} d\tau.$$

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Proposition

Let $\mathrm{Re}\,\alpha, L > 0$, $p \in [1, \infty]$. Then the operator I^α defined as an operator acting on $L^p(0, L)$ coincides with the fractional power of integration operator defined by (2).

Let us define the operator of differentiation

$$\frac{\partial}{\partial x} : D(\frac{\partial}{\partial x}) := {}_0W^{1,p}(0,L) \to L^p(0,L), \qquad \frac{\partial}{\partial x} u := u'. \tag{5}$$

We will show that this operator is positive.

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belonging to $D(\frac{\partial}{\partial x})$. We multiply the equation by $e^{\lambda x}$.

$$\frac{\partial}{\partial x}(ue^{\lambda x}) = ve^{\lambda x}.$$

Since $u(0) = 0$, we get

$$u = \int_0^x e^{-\lambda(x-p)} v(p) dp$$

and by the Young inequality for convolution

$$\|u\|_{L^p(0,L)} \leq \|v\|_{L^p(0,L)} \left\| e^{-\lambda x} \right\|_{L^1(0,L)} \leq \frac{\|v\|_{L^p(0,L)}}{\mathrm{Re}\,\lambda} \text{ for } \mathrm{Re}\,\lambda > 0.$$

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Obviously, zero belongs to the resolvent set of $\frac{\partial}{\partial x}$ and $(\frac{\partial}{\partial x})^{-1} = I$, where I is an integration operator defined in (2).

$$D((\frac{\partial}{\partial x})^{\alpha}) = \{u \in L^p(0,L) : u \in R(I^{\alpha})\}$$

and

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[1, Theorem 7.1.1] Let $\alpha, \beta \in \mathbb{C}$ and let A be a non-negative and injective operator. If $u \in D(A^{\alpha+\beta}) \cap D(A^{\beta})$, then $A^{\beta}u \in D(A^{\alpha})$ and $A^{\alpha}A^{\beta}u = A^{\alpha+\beta}u$.

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If $u \in D(I^{-\alpha})$, then $u \in D(I^{1-\alpha}) = L^p(0,L)$. We apply Proposition with parameters $\alpha = -1$ and $\beta = 1-\alpha$ and we obtain that $I^{1-\alpha}u \in D(I^{-1})$ and $I^{-\alpha}u = I^{-1}I^{1-\alpha}u$.

$$D((\frac{\partial}{\partial x})^\alpha) = \{u \in L^p(0,L) : u \in R(I^\alpha)\}$$

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$$I^{-1}I^{1-\alpha}u = \frac{\partial}{\partial x} I^{1-\alpha}u = \partial^\alpha u.$$

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Summing up the results, we obtain that

$$(\frac{\partial}{\partial x})^{\alpha}u = \partial^{\alpha}u \text{ for every } u \in D((\frac{\partial}{\partial x})^{\alpha}) = D(I^{-\alpha}) = R(I^{\alpha}).$$

Theorem

[1, Theorem 12.1.9] Let $\frac{\partial}{\partial x}$ *be defined by (5) and* $p \in (1, \infty)$. *Then,*

$$\left\| \left(\frac{\partial}{\partial x}\right)^{i\tau} \right\|_{L^p(0,L)} \leq c(1+|\tau|)e^{\frac{\pi|\tau|}{2}} \quad \text{for} \quad \tau \neq 0.$$

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Proposition

For $L > 0, \alpha \in (0,1), p \in (1,\infty)$ *the operators* $I^\alpha : L^p(0,L) \longrightarrow {}_0H^{\alpha,p}(0,L)$ *and* $\partial^\alpha : {}_0H^{\alpha,p}(0,L) \longrightarrow L^p(0,L)$ *are isomorphism and the following inequalities hold*

$$c^{-1}\|u\|_{{}_0H^{\alpha,p}(0,L)} \leq \|\partial^\alpha u\|_{L^p(0,L)} \leq c\|u\|_{{}_0H^{\alpha,p}(0,L)} \quad \text{for } u \in {}_0H^{\alpha,p}(0,L),$$

$$c^{-1}\|I^\alpha f\|_{{}_0H^{\alpha,p}(0,L)} \leq \|f\|_{L^p(0,L)} \leq c\|I^\alpha f\|_{{}_0H^{\alpha,p}(0,L)} \quad \text{for } f \in L^p(0,L).$$

Here by ${}_0H^{\alpha,p}(0,L)$ *we denote the fractional Lebesgue space defined by*

$${}_0H^{\alpha,p}(0,L) := [L^p(0,L), {}_0W^{1,p}(0,L)]_\alpha$$

and c *denotes a positive constant dependent on* α, p, L*.*

Proposition

[1, Theorem 3.1.8 and Corollary 5.1.12] Let $\operatorname{Re}\alpha > 0$ *and* A *be an non-negative operator. Then,* J^{α} *is closable and* $A^{\alpha} = \overline{J^{\alpha}}$ *if and only if* A *is densely defined.*

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Let $L > 0$ *and* $0 < \alpha < 1$*. Let us discuss the operator* $\frac{\partial}{\partial x}$ *defined in (5). Then, the Balakrishnan operator* J^α *of* $\frac{\partial}{\partial x}$ *coincides with the Caputo derivative* D^α*. Furthermore, the operator* ∂^α *defined on* ${}_0H^{\alpha,p}(0,L)$ *is the closure of* D^α *defined on* ${}_0W^{1,p}(0,L)$*.*

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$$J^\alpha u = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(\lambda + \frac{\partial}{\partial x})^{-1} \frac{\partial}{\partial x} u d\lambda$$

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Let $L > 0$ *and* $0 < \alpha < 1$*. Let us discuss the operator* $\frac{\partial}{\partial x}$ *defined in (5). Then, the Balakrishnan operator* J^α *of* $\frac{\partial}{\partial x}$ *coincides with the Caputo derivative* D^α*. Furthermore, the operator* ∂^α *defined on* ${}_0H^{\alpha,p}(0,L)$ *is the closure of* D^α *defined on* ${}_0W^{1,p}(0,L)$*.*

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$$J^\alpha u = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} (\lambda + \frac{\partial}{\partial x})^{-1} \frac{\partial}{\partial x} u d\lambda$$

$$= \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} \int_0^x e^{-\lambda(x-p)} u'(p) dp d\lambda$$

$$= \frac{\sin \alpha\pi}{\pi} \int_0^x u'(p) \int_0^\infty \lambda^{\alpha-1} e^{-\lambda(x-p)} d\lambda dp.$$

Applying substitution $\lambda(x-p) = w$ we get

$$J^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-p)^{-\alpha} u'(p) dp = D^\alpha u.$$

Definition

$$
{}_0H^\alpha(0,1) = \begin{cases} H^\alpha(0,1) & \textit{for} \quad \alpha \in (0,\frac{1}{2}), \\ \{u \in H^{\frac{1}{2}}(0,1): \quad \int_0^1 \frac{|u(t)|^2}{t} dt < \infty\} & \textit{for} \quad \alpha = \frac{1}{2}, \\ \{u \in H^\alpha(0,1): \quad u(0) = 0\} & \textit{for} \quad \alpha \in (\frac{1}{2},1). \end{cases}
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Theorem

The operators $\partial^\alpha : {}_0H^\alpha(0,1) \to L^2(0,1)$, $I^\alpha : L^2(0,1) \to {}_0H^\alpha(0,1)$ *are isomorphism and*

$$ c_\alpha^{-1} \|u\|_{{}_0H^\alpha(0,1)} \le \|\partial^\alpha u\|_{L^2(0,1)} \le c_\alpha \|u\|_{{}_0H^\alpha(0,1)} \quad \text{for } u \in {}_0H^\alpha(0,1), $$

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$$ \frac{\partial}{\partial x} D^\alpha u = \frac{\partial}{\partial x} I^{1-\alpha} u_x = \partial^\alpha u_x. $$

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\frac{\partial}{\partial x} D^\alpha u = \frac{\partial}{\partial x} I^{1-\alpha} u_x = \partial^\alpha u_x.
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We define the domain of $\frac{\partial}{\partial x} D^\alpha$ by

$$
D(\frac{\partial}{\partial x} D^\alpha) \equiv \mathcal{D}_\alpha := \{u \in H^{1+\alpha}(0,1) : u_x \in {}_0H^\alpha(0,1), \quad u(1) = 0\}.
$$

Theorem

Operator $\frac{\partial}{\partial x} D^{\alpha} : \mathcal{D}_{\alpha} \subseteq L^2(0,1) \rightarrow L^2(0,1)$ *generates an analytic semigroup.*

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- $\frac{\partial}{\partial x} D^{\alpha} : \mathcal{D}_{\alpha} \subseteq L^2(0,1) \to L^2(0,1)$ generates a C_0 semigroup of contractions.

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  - $\mathrm{Re}(-\frac{\partial}{\partial x} D^{\alpha} u, u) \geq 0$
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  - $R(\lambda I - \frac{\partial}{\partial x} D^\alpha) = L^2(0,1)$ for $\lambda > 0$.

Proposition

Let us discuss $\frac{\partial}{\partial x} D^\alpha : \mathcal{D}_\alpha \to L^2(0,1)$. *Then, for every* $\lambda \in \mathbb{C}$ *belonging to the sector*

$$\vartheta_\alpha := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \frac{\pi(\alpha+1)}{2}\} \cup \{0\} \tag{6}$$

there holds

$$R(\lambda E - \frac{\partial}{\partial x} D^\alpha) = L^2(0,1).$$

$$\sigma(\frac{\partial}{\partial x} D^\alpha) = \{\lambda \in \mathbb{C} : E_{\alpha+1}(\lambda) = 0\}.$$

The solution to

$$\lambda u - \frac{\partial}{\partial x} D^{\alpha} u = g$$

is given by

$$u(x) = (E_{\alpha+1}(\lambda))^{-1}(g{*}y^{\alpha}E_{\alpha+1,\alpha+1}(\lambda y^{\alpha+1}))(1)E_{\alpha+1}(\lambda x^{\alpha+1}) - g{*}x^{\alpha}E_{\alpha+1,\alpha+1}(\lambda x^{\alpha+1}).$$

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- An analytic extension of C_0- semigroup on a sector of complex plane.

Proposition

For $u \in \mathcal{D}_{\alpha}$ we have

$$\mathrm{Re}(-\frac{\partial}{\partial x} D^{\alpha} u, u) \geq c_{\alpha} \|u\|^2_{H^{\frac{1+\alpha}{2}}(0,1)} \tag{7}$$

and

$$\left|(-\frac{\partial}{\partial x} D^{\alpha} u, u)\right| \leq b_{\alpha} \|u\|^2_{H^{\frac{1+\alpha}{2}}(0,1)}, \tag{8}$$

where c_{α}, b_{α} are positive constant which depends only on α.

Theorem

Let us consider problem

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^\alpha u = 0 & in\ (0,1)\times(0,T), \\ u_x(0,t) = 0, \quad u(1,t) = 0 & for\ t\in(0,T), \\ u(x,0) = u_0(x) & in\ (0,1). \end{cases}$$

If we assume that $u_0 \in L^2(0,1)$, *then there exists exactly one solution which belongs to* $C([0,T];L^2(0,1)) \cap C((0,T];\mathcal{D}_\alpha) \cap C^1((0,T];L^2(0,1))$. *Furthermore, there exists a positive constant* $c = c(T)$, *such that the following estimate holds for every* $t \in (0,T]$

$$\|u(\cdot,t)\|_{L^2(0,1)} + t\,\|u_t(\cdot,t)\|_{L^2(0,1)} + t\left\|\frac{\partial}{\partial x}D^\alpha u(\cdot,t)\right\|_{L^2(0,1)} \le c\,\|u_0\|_{L^2(0,1)}.$$

Nevertheless, $u \in C^\infty((0,T];L^2(0,1))$ *and for every* $t\in(0,T]$, *for very* $k\in\mathbb{N}$ *we have* $u(\cdot,t)\in D((\frac{\partial}{\partial x}D^\alpha)^k)$. *The last property implies that* $u(\cdot,t) \in C^\infty(0,1)$ *for every* $t\in(0,T]$, *however* u *has a singularity of the form* $x^{\alpha+1}$ *at the left endpoint of the interval.*

Case with Dirichlet boundary conditions

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^{\alpha} u = f & \text{in } (0,1) \times (0,T), \\ u(0,t) = 0, \quad u(1,t) = 0 & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1), \end{cases} \tag{9}$$

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Let us introduce

$$\overline{\mathcal{D}}_\alpha := \{u = w - w(1)x^\alpha, \text{ where } w \in {}_0H^{1+\alpha}(0,1)\}.$$

We equip $\overline{\mathcal{D}}_\alpha$ with the following norm

$$\|u\|_{\overline{\mathcal{D}}_\alpha} = \|w\|_{H^{1+\alpha}(0,1)} \text{ for } \alpha \in (0,1) \setminus \{\frac{1}{2}\}$$

and

$$\|u\|_{\overline{\mathcal{D}}_\alpha} = \left(\|w\|^2_{H^{\frac{3}{2}}(0,1)} + \int_0^1 \frac{|w_x(x)|^2}{x} dx \right)^{\frac{1}{2}} \text{ for } \alpha = \frac{1}{2}.$$

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Theorem

The operator $\frac{\partial}{\partial x} D^\alpha : \overline{\mathcal{D}}_\alpha \subseteq L^2(0,1) \to L^2(0,1)$ is a densely defined sectorial operator, thus it generates an analytic semigroup.

Case with prescribed flux at the left boundary

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^{\alpha} u = f & \text{in } (0,1) \times (0,T), \\ (D^{\alpha} u)(0,t) = h(t), \quad u(1,t) = 0 & \text{for } t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } (0,1). \end{cases} \tag{10}$$

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Proposition

Let F be an absolutely continuous function and $f := F'$. Then we denote

$$
(D^\alpha F)(0) := \lim_{x \to 0} \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-p)^{-\alpha} f(p) dp.
$$

1. *If $(D^\alpha F)(0)$ exists and $(D^\alpha F)(0) = c$, then $\lim_{y \to 0} \frac{F(y)}{y^\alpha} = \frac{c}{\Gamma(1+\alpha)}$,*
2. *if the limit $\lim_{y \to 0} \frac{f(y)}{y^{\alpha - 1}}$ exists and $\lim_{y \to 0} \frac{f(y)}{y^{\alpha-1}} = \frac{c}{\Gamma(\alpha)}$, then $(D^\alpha F)(0) = c$.*

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Hence, it is natural to search for a solution to (10) in the form

$$u = \frac{h(t)}{\Gamma(1+\alpha)} x^\alpha + v, \quad v_x \in {}_0H^\alpha(0,1). \tag{11}$$

Theorem

Let $b, T > 0$ *and* $\alpha \in (0,1)$. *Let us assume that*

- $u_0 \in H^{1+\alpha}(0,b)$, $u_0' \in {}_0H^{\alpha}(0,b)$, $u_0(b) = 0$ *and* $u_0 \geq 0$, $u_0 \not\equiv 0$.
- $$\exists M > 0 \quad \forall x \in [0,b] \quad u_0(x) \leq \frac{M\Gamma(2-\alpha)}{b^{1-\alpha}}(b-x).$$

Then, there exists exactly one (u,s) *a solution to the system (1), s.t.*

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- $u_x \in C(\overline{Q_{s,T}})$ *in the case* $\alpha \in (\frac{1}{2},1)$ *and* $u_x \in C(\overline{Q_{s,T}} \setminus (\{t=0\} \times [0,b]))$ *in the case* $\alpha \in (0,\frac{1}{2}]$.

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- *There exists* $\beta \in (\alpha,1)$, *such that for every* $t \in (0,T]$ *and every* $0 < \varepsilon < \omega < s(t)$ *we have* $u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega)$.

The idea of the proof

- The solution to parabolic type problem in non-cylindrical domain with given boundary s, where

$$s \in C^{0,1}[0,T], \quad 0 < \dot{s} \leq M \quad \text{for a.a.} \quad t \in (0,T). \tag{12}$$

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  - Existence and regularity of the solution by means of evolution operator theory.

$$t \mapsto A(t) := \frac{1}{s^{1+\alpha}(t)} \frac{\partial}{\partial x} D^{\alpha} \in C^{0,1}([0, T]; B(\mathcal{D}_{\alpha}, L^2(0, 1)))$$

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$$v(x,t) = G(t,0)v_0(x) + \int_0^t G(t,\sigma)\frac{\dot{s}(\sigma)}{s(\sigma)}xv_x(x,\sigma)d\sigma.$$

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$$s \in C^{0,1}[0,T], \quad 0 < \dot{s} \le M \quad \text{for a.a.} \quad t \in (0,T). \tag{12}$$

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$$\begin{cases} v_t - x\frac{\dot{s}(t)}{s(t)}v_x - \frac{1}{s^{1+\alpha}(t)}\frac{\partial}{\partial x}D^\alpha v = 0 & \text{for } 0 < x < 1, 0 < t < T, \\ v_x(0,t) = 0, \quad v(1,t) = 0 & \text{for } t \in (0,T), \\ v(x,0) = v_0(x) & \text{for } 0 < x < 1. \end{cases} \tag{13}$$

  - Existence and regularity of the solution by means of evolution operator theory.

$$t \mapsto A(t) := \frac{1}{s^{1+\alpha}(t)}\frac{\partial}{\partial x}D^\alpha \in C^{0,1}([0,T]; B(\mathcal{D}_\alpha, L^2(0,1)))$$

  and $A(t)$ is sectorial for every $t \in [0,T]$ and $A(t)$ have common domain.

$$v(x,t) = G(t,0)v_0(x) + \int_0^t G(t,\sigma)\frac{\dot{s}(\sigma)}{s(\sigma)}xv_x(x,\sigma)d\sigma.$$

  - Higher spatial regularity in the interior of the domain.

- The solution to parabolic type problem in non-cylindrical domain with given boundary s, where

$$s \in C^{0,1}[0,T], \quad 0 < \dot{s} \leq M \quad \text{for a.a.} \quad t \in (0,T). \tag{12}$$

  - Transformation to cylindrical domain $p = \frac{x}{s(t)}$, $v(p,t) := u(s(t)p,t) = u(x,t)$

$$\begin{cases} v_t - x\frac{\dot{s}(t)}{s(t)}v_x - \frac{1}{s^{1+\alpha}(t)}\frac{\partial}{\partial x}D^{\alpha}v = 0 & \text{for } 0 < x < 1, 0 < t < T, \\ v_x(0,t) = 0, \quad v(1,t) = 0 & \text{for } t \in (0,T), \\ v(x,0) = v_0(x) & \text{for } 0 < x < 1. \end{cases} \tag{13}$$

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  - Higher spatial regularity in the interior of the domain.
    We note that $v_x(t,1)$ need not to vanish.

- The maximum principles

Lemma

Let $f \in AC[0,L]$ and for every $\varepsilon \in (0,L)$ $f \in W^{1,\frac{1}{1-\beta}}(\varepsilon, L)$ for some $\beta \in (0,1]$. Then, if f attains its maximum at the point $x_0 \in (0,L]$, then for every $\alpha \in (0,\beta)$ there holds the inequality $(D^{\alpha} f)(x_0) \geq 0$. Furthermore, if f is not constant on $[0,x_0]$, then $(D^{\alpha} f)(x_0) > 0$.

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Lemma

Let $f \in AC[0,L]$ and $f' \in W^{1,\frac{1}{1-\beta}}(\varepsilon, L)$ for every $\varepsilon > 0$ and for fixed $\beta \in (0,1)$. If f attains its maximum at $x_0 \in (0,L)$, then $(\frac{\partial}{\partial x} D^\alpha f)(x_0) \leq 0$ for every $\alpha \in (0,\beta)$. Furthermore, if f is not constant on $[0,x_0]$, then $(\frac{\partial}{\partial x} D^\alpha f)(x_0) < 0$.

- Space-fractional version of the Hopf's, lemma, i.e. $D^{\alpha}u(s(t),t) < 0$.

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Proposition

Let u be a nonnegative solution to $u_t - \frac{\partial}{\partial x}D^{\alpha}u = 0$ in $Q_{s,T}$, where s satisfies (12). We assume that u has the following regularity $u \in C(\overline{Q_{s,T}})$, $u_t \in C(Q_{s,T})$, $u(\cdot,t) \in AC[0,s(t)]$ for every $t \in (0,T)$, $\frac{\partial}{\partial x}D^{\alpha}u \in C(Q_{s,T})$. Furthermore, for every $t \in (0,T)$, for every $0 < \varepsilon < \omega < s(t)$ we have $u(\cdot,t) \in W^{2,\frac{1}{1-\beta}}(\varepsilon,\omega)$ for some $\beta \in (\alpha,1]$. Let $t_0 \in (0,T]$ be fixed. Then if $u(s(t_0),t_0) = 0$, then either $(D^{\alpha}u)(s(t_0),t_0) < 0$ or $u \equiv 0$ on Q_{s,t_0}.

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- Estimates

$$(D^{\alpha}u)(s(t),t) \geq -M, \qquad 0 \leq u(x,t) \leq M\Gamma(2-\alpha)s^{\alpha-1}(t)(s(t)-x).$$

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$$\Sigma := \{s \in C^{0,1}[0,T], \;\; 0 < \dot{s} \leq M, \;\; s(0) = b\}.$$

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$$(Ps)(t) = b - \int_0^t (D^{\alpha}u)(s(\tau),\tau)d\tau = \dots$$

$$\dots = b + \int_0^b u_0(x)dx - \int_0^{s(t)} u(x,t)dx.$$

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Proposition

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$P : \Sigma \to \Sigma$ and P is continuous in maximum norm.

Theorem

Let (u^i, s_i) be a solution to (1) corresponding to b_i and u_0^i for $i = 1, 2$. If $b_1 \leq b_2$ and $u_0^1 \leq u_0^2$, then for every $t \in [0, T]$ we have $s_1(t) \leq s_2(t)$.

$$
\begin{cases}
u_t - \frac{\partial}{\partial x} D^{\alpha} u = 0 & \text{in } \{(x,t) : 0 < x < s(t), \quad 0 < t < \infty\}, \\
u(0,t) = c_1, \quad u(t, s(t)) = 0 & \text{for } t \in (0,\infty), \\
\dot{s}(t) = -(D^{\alpha} u)(s(t), t) & \text{for } t \in (0,\infty),
\end{cases}
\tag{14}
$$

where we assume that $s(0) = 0$ and $c_1 > 0$.

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^\alpha u = 0 & \text{in } \{(x,t) : 0 < x < s(t), \quad 0 < t < \infty\}, \\ u(0,t) = c_1, \quad u(t, s(t)) = 0 & \text{for } t \in (0,\infty), \\ \dot{s}(t) = -(D^\alpha u)(s(t),t) & \text{for } t \in (0,\infty), \end{cases} \tag{14}$$

where we assume that $s(0) = 0$ and $c_1 > 0$.

- similarity variable $\xi = xt^{-\frac{1}{\alpha+1}}$,

$$\begin{cases} u_t - \frac{\partial}{\partial x} D^\alpha u = 0 & \text{in } \{(x,t) : 0 < x < s(t), \quad 0 < t < \infty\}, \\ u(0,t) = c_1, \quad u(t, s(t)) = 0 & \text{for } t \in (0,\infty), \\ \dot{s}(t) = -(D^\alpha u)(s(t), t) & \text{for } t \in (0,\infty), \end{cases} \tag{14}$$

where we assume that $s(0) = 0$ and $c_1 > 0$.

- similarity variable $\xi = xt^{-\frac{1}{\alpha+1}}$,
- $$F(\xi) = F(xt^{-\frac{1}{\alpha+1}}) := u(x,t),$$
- $$\partial^\alpha F'(\xi) = -\frac{\xi}{\alpha+1} F'(\xi).$$

Proposition

Let us consider the problem for fixed $c_1 > 0$, $R > 0$, $c_2 < 0$.

$$\begin{cases} \partial^\alpha F'(\xi) = -\frac{\xi}{\alpha+1} F'(\xi) & \textit{for} \quad 0 < \xi < R, \\ F(0) = c_1, \quad I^{1-\alpha} F'(0) = c_2, \end{cases} \tag{15}$$

There exists exactly one solution to (15) which belongs to

$$X_{R,c_1,c_2} := \{v \in C^1((0,R]) : \xi^{1-\alpha} v' \in C([0,R]), \quad v(0) = c_1, \quad I^{1-\alpha} v'(0) = c_2\}.$$

Furthermore, the solution is given by the formula

$$F(\xi) = c_1 + \frac{c_2}{\Gamma(\alpha+1)} \left[\xi^\alpha + \Gamma(\alpha+1)\xi^\alpha \sum_{k=1}^{\infty} \left(\frac{-\xi^{1+\alpha}}{1+\alpha} \right)^k \frac{\prod_{i=1}^k (i\alpha + i - 1)}{\Gamma((\alpha+1)(k+1))} \right], \tag{16}$$

where the series is uniformly convergent on $[0,R]$. *Finally, if we define*

$$u(x,t) := F(xt^{-\frac{1}{1+\alpha}}), \tag{17}$$

then $u(0,t) = c_1$ *and* u *satisfies* $(14)_1$ *on* $\{(x,t) : 0 < x < Rt^{\frac{1}{\alpha+1}}, 0 < t < \infty\}$.

In the next lemma we obtain the family $(u^R, s^R)_{R>0}$ of solutions to $(14)_1$ and $(14)_3$.

Proposition

For every $c_1 > 0$ and every $R > 0$ the functions

$$s^R(t) = Rt^{\frac{1}{1+\alpha}}, \tag{18}$$

$$u^R(x,t) = c_1 + \frac{\tilde{c}_2}{\Gamma(\alpha+1)}[x^\alpha t^{-\frac{\alpha}{\alpha+1}}$$

$$+\Gamma(\alpha+1)x^\alpha t^{-\frac{\alpha}{\alpha+1}} \sum_{k=1}^{\infty} \left(\frac{-x^{1+\alpha}}{(1+\alpha)t}\right)^k \frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)(k+1))}] \tag{19}$$

where

$$\tilde{c}_2 = -\frac{R}{(1+\alpha)\left[1+\sum_{k=1}^{\infty}\left(\frac{-R^{1+\alpha}}{1+\alpha}\right)^k \frac{\prod_{i=1}^{k}(i\alpha+i-1)}{\Gamma((\alpha+1)k+1)}\right]} \tag{20}$$

satisfy the equation $(14)_3$. Moreover, u^R is a solution to $(14)_1$ with $s(t) = s^R(t)$ and $u^R(0,t) = c_1$.

It remains to choose $R > 0$ such that the pair (u^R, s^R) given by Lemma (12) satisfies $u^R(s^R(t), t) = 0$.

Theorem

For every $c_1 > 0$ there exists $c_0 > 0$ such that the pair $(u, s) := (u^{c_0}, s^{c_0})$, where (u^{c_0}, s^{c_0}) come from Lemma 12 with $R = c_0$, satisfies the system (14). Furthermore,

$$\forall x > 0 \quad u(x, \cdot), u_t(x, \cdot), u_x(x, \cdot) \in C([s^{-1}(x), \infty)) \tag{21}$$

$$\forall t > 0 \quad u(\cdot, t), u_t(\cdot, t) \in C([0, s(t)]), \quad u_x(\cdot, t) \in C((0, s(t)]) \tag{22}$$

and

$$\forall t > 0 \quad \frac{\partial}{\partial x} D^\alpha u(\cdot, t) \in C([0, s(t)]). \tag{23}$$

Finally, $u > 0$, $u_t > 0$, $u_x < 0$ on $\{(x, t) : \quad 0 < x < s(t), \quad 0 < t < \infty\}$.

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