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Affine spaces and algebras of subalgebras

A. PILITOWSKA, A. ROMANOWSKA AND J. D. H. SMITH

Dedicated to the memory of Alan Day

1. Introduction

The algebras forming the main topic of this paper are *modes*, i.e. they are *idempotent*, in the sense that each singleton is a subalgebra, and they are *entropic*, i.e. each operation, as a mapping from a direct power of the algebra into the algebra, is actually a homomorphism. The two properties may be expressed algebraically by means of identities

$$x \cdots x\omega = x,$$

$$x_{11} \cdots x_{1n}\omega \cdots \qquad x_{m1} \cdots x_{mn}\omega\omega' = x_{11} \cdots x_{m1}\omega' \cdots x_{1n} \cdots x_{mn}\omega'\omega,$$

that are satisfied in each mode (A, Ω), for any n-ary operation ω and m-ary operation ω' in Ω. Such algebras are studied in detail in [12]. (See also [11]–[16].) Given a mode (A, Ω), as a set A with a set Ω of operations $\omega: A^{\omega\tau} \to A$ on it, one may form the set $(A, \Omega)S$ or AS of non-empty subalgebras of (A, Ω). This set AS carries an Ω-algebra structure under the complex products

$$\omega: AS^{\omega\tau} \to AS; (X_1, \ldots, X_{\omega\tau}) \mapsto \{x_1 \cdots x_{\omega\tau}\omega \mid x_i \in X_i\},$$

and it turns out that the algebra (AS, Ω) is again a mode, preserving many of the algebraic properties of (A, Ω) [12, 146], [13], [14].

One of the more important examples of modes is given by affine spaces (or affine modules) over ring R ([1], [2], [11], [12], [18]–[21]). Affine spaces can be described as modes $(E, \underline{R}, P)$ with binary operations $\underline{r}$ for each r in R and one ternary Mal'cev (parallelogram-completion) operation P satisfying certain identities

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[12]. In some cases some of the operations $\{P\} \cup \underline{R}$ are defined by means of others and it is possible to give much more simple description of affine spaces.

In [11] modes of subspaces of affine spaces over fields are investigated. The structure of certain reducts of such modes yields a direct, invariant passage from affine to projective geometry. The present paper continues the study set out in [11] and investigates modes of subspaces of affine spaces over arbitrary commutative rings with unity. In Section 2, we present our approach to studying affine and projective geometries and relationship between them. The main Section 3 is devoted to modes of subspaces of affine spaces. It is shown that certain reducts of such modes may be constructed as Płonka sums of reducts of affine spaces over the corresponding projective space (Theorem 3.9). Some consequences of this fact are pointed out. In Section 4, we apply Theorem 3.9 to describe the structure of modes of subalgebras for several classes of modes, among them for quasigroup modes, pairs of orthogonal quasigroup modes, Mal'cev groupoid modes, and affine spaces over fields.

The notation and terminology of the paper is basically as in the book [12]. We refer the reader to the book for all undefined notions and results. The meanings of "term operation" and "derived operation" are the same. We use "Polish" notation for operations, e.g. $\omega(x_1, \ldots, x_n)$ denotes a word (term) with variables $x_1, \ldots, x_n$ and then $x_1 \cdots x_n \omega$ denotes the induced derived (or term) operation in an algebra. Algebras and varieties are equivalent if they have the same derived operations.

2. Affine spaces, projective spaces and Ω-semilattices

The algebraic approach to affine geometry here is that of [8], [12, 2.5] and [11]. Let R be a commutative ring with unity, and let $(E, +, R)$ be a module over R. For each element r of R, define a binary operation

$$\underline{r}: E \times E \to E; \ (x, y) \mapsto xy\underline{r} = x(1 - r) + yr$$

so that $(E, \underline{R})$ becomes an algebra with the set $\underline{R} = \{\underline{r} \mid r \in R\}$ of binary operations. On E, define the Mal'cev operation

$$P: E \times E \times E \to E; \ (x, y, z) \mapsto x - y + z.$$

Then the algebra $(E, \underline{R}, P)$ with the ternary operation P and set $\underline{R}$ of binary operations has as its derived operations (those obtained from successive compositions of the basic operations P and $\underline{r}$ for r in R) precisely the affine combinations $x_1 r_1 + \cdots + x_n r_n$ with $r_1 + \cdots + r_n = 1$ of elements $x_1, \ldots, x_n$ of E. It follows that the algebra $(E, \underline{R}, P)$ is equivalent to the *full idempotent reduct* $(E, \{x_1 r_1 + \cdots + x_n r_n \mid r_1, \ldots, r_n \in R, \Sigma_{i=1}^n r_i = 1\})$ of the module E in the sense of [20, 4]. Note that the algebra $(E, \underline{R}, P)$ has the affine group as its group of automorphisms, and may thus be identified with the affine geometry (cf. [8], [7]). Carrying out this identification, we will refer to the algebra $(E, \underline{R}, P)$ as an *affine space* over R or an *affine R-space*. (Note, however, that such algebras have also been called *affine modules* [1], [2], [18], ..., [21].) The affine space $(E, \underline{R}, P)$ is said to be *faithful* if the module $(E, +, R)$ is faithful. As was shown by Ostermann and Schmidt ([8], cf. [1], [12]), the class of affine spaces over the ring R forms a variety. According to [12], this variety is equivalent to the variety $\underline{\underline{R}}$ of Mal'cev modes $(E, \underline{R}, P)$, algebras with the ternary Mal'cev operation P and one binary operation $\underline{r}$ for each r in R, satisfying the identities defining modes and for all p, q, r, s in R the identities

(A1) $xyx\,P = yx\underline{2}$,

(A2) $xy\underline{p}\,xy\underline{q}\,\underline{r} = xy\underline{pqr}$,

(A3) $xy\underline{p}\,xy\underline{q}\,xy\underline{r}P = xy\underline{pqrP}$,

(A4) $xy\underline{0} = yx\underline{1} = x$.

If 2 is invertible in R, the identities (A1)–(A4) reduce to (A2) and (A4). The free $\underline{\underline{R}}$-algebra on two generators is in fact the affine space $(R, \underline{R}, P)$. Each subvariety of the variety $\underline{\underline{R}}$ is a variety of affine spaces over a homomorphic image of the ring R. Thus the lattice of subvarieties of $\underline{\underline{R}}$ is dually isomorphic to the congruence lattice of the ring R.

Projective space is considered here as the set $L(E) = (E, +, R)S$ of submodules of the R-module $(E, +, R)$, together with the semilattice operation $+$, where for submodules U and V of E, $U + V = \{u + v \mid u \in U, v \in V\}$ is the sum of U and V. The inclusion structure is recovered from $(L(E), +)$ via $U \leq V$ iff $U + V = V$.

Obviously each semilattice $(H, +)$ is a mode as well. We now recall two approaches to semilattices needed later in the paper. Let Ω be a non-empty domain of operations with an arity mapping $\tau: \Omega \to \{n \in \mathbb{N} \mid n > 1\}$. A semilattice $(H, +)$ may be considered as an Ω-algebra (H, Ω), a so-called *Ω-semilattice*, on defining $h_1 \cdots h_{\omega\tau}\omega = h_1 + \cdots + h_{\omega\tau}$ for h_i in H. The semilattice operation $+$ is then recovered as $h + k = hk \cdots k\omega$ for any ω in Ω. The semilattice $(H, +)$ may also be considered as a (small) category (H) with a set H of objects, and with a unique morphism $h \to k$ precisely when $h + k = k$, i.e. $h \leq k$.

3. Algebras of affine subspaces

For an affine space $(E, \underline{R}, P)$, consider the set $(E, \underline{R}, P)S$ or ES of non-empty subalgebras of $(E, \underline{R}, P)$. The set ES forms an algebra under the complex products

$$\underline{r}: ES \times ES \to ES; \ (X, Y) \mapsto \{xy\underline{r} \mid x \in X, y \in Y\}$$

for r in R and

$$P: ES \times ES \times ES \to ES; \ (X, Y, Z) \mapsto \{xyzP \mid x \in X, y \in Y, z \in Z\}.$$

It turns out that the algebra $(ES, \underline{R}, P)$ is again a mode preserving many of the algebraic properties of $(E, \underline{R}, P)$ [12, 146]. In particular $(ES, \underline{R}, P)$ satisfies each linear identity (i.e. each identity for which the arguments appear at most once on each side) satisfied by $(E, \underline{R}, P)$.

In [11], for the case where R is a field, the internal structure of the $(ES, \underline{R}, P)$ was described using the concept of a Płonka sum ([9], [10], [12, 236]). This notion depends on viewing a semilattice $(H, +)$ both as a category and as an Ω-algebra. Let (Ω) denote the (concrete) category of Ω-algebras and homomorphisms between them. Let $F: (H) \to (\Omega)$ be a functor. Then the *Płonka sum* of the Ω-algebras (hF, Ω) for h in H over the semilattice $(H, +)$ by the functor F is the disjoint union $HF = \dot{\bigcup} (hF \mid h \in H)$ of the underlying sets hF, equipped with the Ω-algebra structure given, for an n-ary operation ω in Ω and $h_1, \ldots, h_n$, $h = h_1 + \cdots + h_n$ in H, by

$$\omega: h_1F \times \cdots \times h_nF \to hF; \ (x_1, \ldots, x_n) \mapsto x_1(h_1 \to h)F \cdots x_n(h_n \to h)F\omega.$$

The *canonical projection* of the Płonka sum HF is the homomorphism $\pi_F: (HF, \Omega) \to (H, \Omega)$ with restrictions $\pi_F: hF \to \{h\}$. The subalgebras $(hF, \Omega) = (\pi_F^{-1}h, \Omega)$ of (HF, Ω) are referred to as the *Płonka fibres*. Recall that for Ω-algebras in an idempotent irregular variety $\mathfrak{V}$, the identities satisfied by their Płonka sums are precisely the regular identities holding in the fibres. On the other hand, the regularisation $\tilde{\mathfrak{V}}$ of $\mathfrak{V}$, the variety defined by all the regular identities true in $\mathfrak{V}$, consists precisely of Płonka sums of $\mathfrak{V}$-algebras (cf. [9] and [12]).

In [11], for R a field, one considered a certain subset Ω_R of $\{P\} \cup \underline{R}$. It was shown that the algebra $((E, \underline{R}, P)S, \Omega_R)$ is a Płonka sum over the Ω_R-semilattice $(L(E), \Omega_R)$ corresponding to the projective geometry $(L(E), +)$. The corresponding Płonka fibres were shown to be quotient spaces of the affine space E. Certain considerations of convexity suggested particular choices for the set Ω_R of operations of the reducts. In the more general case considered in this paper, the rôle of Ω_R is taken by the subset $\underline{J}^0_R \cup \{P\}$ of $\underline{R} \cup \{P\}$, where J^0_R comprises the set of units r of R for which $1 - r$ is also invertible. The motivation for choosing the set J^0_R is not only a desire to have an invariant passage from affine to projective geometry as in [11], but also to obtain an adequate description of the structure of $(ES, \underline{R}, P)$. We will prove that for each commutative ring R with unity, the algebra $((E, \underline{R}, P)S, \underline{J}^0_R, P)$ is a Płonka sum of cosets of submodules of the module E over the $\underline{J}^0_r$-semilattice $(L(E), \underline{J}^0_R) = (L(E), +)$.

LEMMA 3.1. *Let* $J^0_R = \{r \in R \mid r^{-1}, (1-r)^{-1} \in R\}$. *Then for all* r *in* J^0_R, *the elements* r^{-1}, $(1-r)^{-1}$ *and* $1 - r$ *of* R *also lie in* J^0_R.

Proof. For r in J^0_R, one has

$$(1 - r^{-1}) \cdot r(r-1)^{-1} = 1 \quad \text{and} \quad (1-(1-r)^{-1}) \cdot r^{-1}(r-1) = 1. \qquad \square$$

LEMMA 3.2. *Let* $U, V, W, U_1, \ldots, U_n$ *be submodules of the module* E, *and let* r *be in* R. *Then in* $(E, \underline{R}, P)S$, *one has*:

(i) $(x + U)(y + V)\underline{r} = xy\underline{r} + UV\underline{r}$;

(ii) $(x + U)(y + V)(z + W)P = xyzP + UVWP = xyzP + (U + V + W)$;

(iii) *for a derived operation* w *of* $\underline{\underline{R}}$,

$$(x_1 + U_1) \cdots (x_n + U_n)w = x_1 \cdots x_n w + U_1 \cdots U_n w. \qquad \square$$

COROLLARY 3.3. *Let* U *be a submodule of* E. *Then for each* $\underline{R} \cup \{P\}$-*word* $w(x_1, \ldots, x_n)$,

$$(x_1 + U) \cdots (x_n + U)w = x_1 \cdots x_n w + U. \qquad \square$$

COROLLARY 3.4. *For a submodule* U *of* E, *the set* $\{x + U \mid x \in E\}$ *is a subalgebra of* $((E, \underline{R}, P)S, \underline{R}, P)$, *and is an affine space over* R. $\square$

LEMMA 3.5. *For submodules* U, V *of* E *and for* r *in* J^0_R, *one has*

$$UV\underline{r} = U + V.$$

Proof. If r is in J^0_R, then $U = \{u(1-r)^{-1}(1-r) \mid u \in U\} \subseteq U(1-r)$. Since obviously $U(1-r) \subseteq U$, it follows that $U(1-r) = U$. Similarly, one shows that $Vr = V$. Hence $UV\underline{r} = U(1-r) + Vr = U + V$. $\square$

Note that if r is not in J^0_R, the subset $UV\underline{r}$ does not necessarily equal $U + V$. First observe that Ur may be different from U. Indeed, in $\mathbb{Z}_6$, $\{0, 3\}$ and $\{0, 2, 4\}$ are subgroups of $\mathbb{Z}_6$ and $\{0, 3\}4 = \{0\} \neq \{0, 3\}$. Similarly $\{0, 2, 4\}3 = \{0\} \neq \{0, 2, 4\}$. Now in $(\mathbb{Z}_6, \underline{\mathbb{Z}}_6, P)S$, $\{0, 3\}\{0, 2, 4\}\underline{3} = \{0, 3\}4 + \{0, 2, 4\}3 = \{0\} + \{0\} = \{0\} \neq \mathbb{Z}_6 = \{0, 3\} + \{0, 2, 4\}$. The operation $\underline{3}$ is not in J_R^0. In fact, for each $(\mathbb{Z}_{2n}, \underline{\mathbb{Z}}_{2n}, P)$, the set J_R^0 is empty. Indeed, if for k, l in $\mathbb{Z}_{2n}$, $kl = 1$, then both k and l are odd. But in this case, both $1 - k$ and $1 - l$ are even, and hence non-invertible. Note however that for submodules U, V of E and r in R, Ur is a submodule of U and $UV\underline{r}$ is a submodule of $U + V$.

LEMMA 3.6. *For each r in J_R^0 the mapping*

$$\pi: (E, \underline{R}, P)S \to L(E): x + U \mapsto U$$

is an $\{\underline{r}\} \cup \{P\}$-homomorphism.

Proof. Follows by 3.2(i), 3.2(ii) and 3.5. □

LEMMA 3.7. *For an affine space $(E, \underline{R}, P)$ and r a unit of R, the reducts $(E, \underline{r}, \underline{r}^{-1})$ and (E, P) have no non-trivial semilattice quotients.*

Proof. The identity $x = yyx\underline{r}\ \underline{r}^{-1}$ fails in every non-trivial $\{\underline{r}, \underline{r}^{-1}\}$-semilattice. The identity $x = xyyP$ fails in every non-trivial $\{P\}$-semilattice. □

PROPOSITION 3.8. *For each r in J_R^0 and for the functor F: $(L(E)) \to (\{\underline{r}\})$ with $UF = \pi^{-1}(U)$ and $(U \to V)F: \pi^{-1}(U) \to \pi^{-1}(V)$; $x + U \mapsto x + V$, the algebra $((E, \underline{R}, P)S, \underline{r})$ is the Płonka sum of $\underline{r}$-reducts of affine R-spaces over the projective space $(L(E), +)$ by the functor F.*

Proof. By 3.4, 3.6 and 3.7, π is an $\underline{r}$-homomorphism onto the semilattice replica $(L(E), +)$ of $((E, \underline{R}, P)S, \underline{r})$. The fibres $(\pi^{-1}(U), \underline{r}) = (\{x + U \mid x \in E\}, \underline{r})$ are $\underline{r}$-reducts of the affine spaces $(\{x + U \mid x \in E\}, \underline{R}, P)$. Further, by 3.2 and 3.5, $(x + U)(y + V)\underline{r} = xy\underline{r} + (U + V) = (x + (U + V))(y + (U + V))\underline{r} = (x + U)(U \to U + V)F(y + V)(V \to U + V)F\underline{r}$. Thus $((E, \underline{R}, P)S, \underline{r})$ is a Płonka sum as claimed. □

THEOREM 3.9. *For an affine space $(E, \underline{R}, P)$ in $\underline{R}$, each algebra $((E, \underline{R}, P)S, \Omega)$ of affine subspaces of $(E, \underline{R}, P)$, where $\Omega \subseteq J_R^0 \cup \{P\}$, is a Płonka sum of Ω-reducts of affine R-spaces $(E/U, \underline{R}, P)$ over the projective space $(L(E), +) = ((E, +, R)S, +)$ by the functor F: $(L(E)) \to (\Omega)$ with $UF = \{x + U \mid x \in E\}$ and $(U \to V)F$: $UF \to VF$; $x + U \mapsto x + V$.*

Proof. The fact that $((E, \underline{R}, P)S, J_R^0, P)$ is the Płonka sum over $(L(E), +)$ by the functor F follows directly by Proposition 3.8 since

$$(x+U)(y+V)(z+W)P = xyzP + (U+V+W)$$
$$= (x+(U+V+W))(y+(U+V+W))(z+(U+V+W))P$$
$$= (x+U)(U \to U+V+W)F(y+V)(V \to U+V+W)$$
$$\times F(z+W)(W \to U+V+W)FP. \qquad \square$$

Note that by 3.7, if Ω contains at least one element $\underline{r}$ together with $\underline{r}^{-1}$, or if it contains P, then the Płonka fibres in 3.9 are semilattice-indecomposable.

Let $\mathfrak{J}_R$ be the variety generated by the $\underline{J}_R^0 \cup \{P\}$-reducts of the affine spaces $(E, \underline{R}, P)$. Then Theorem 3.9 shows that the algebras $((E, \underline{R}, P)S, \underline{J}_R^0, P)$ are in the regularisation $\tilde{\mathfrak{J}}_R$ of $\mathfrak{J}_R$. Note however that for r not in J_R^0, a regular identity involving $\underline{r}$ that holds in $(E, \underline{R}, P)$ does not necessarily hold in $(ES, \underline{R}, P)$. Consider the affine space $(\mathbb{Z}_6, \mathbb{Z}_6, P)$. As shown before, the set $J_{\mathbb{Z}_6}^0$ is empty. So $\underline{2}$ is not in $J_{\mathbb{Z}_6}^0$. For $p = 2$, $q = 1$ and $r = 2$, the identity (A2) takes the form $xy\underline{2}y\underline{2} = xy\underline{0}$. But in $\mathbb{Z}_6 S$, $\{0, 3\}\{0, 2, 4\}\underline{2}\{0, 2, 4\}\underline{2} = \mathbb{Z}_6 \neq \{0, 3\} = \{0, 3\}\{0, 2, 4\}\underline{0}$. It follows that in general the algebra $(ES, \underline{R}, P)$ is not in the regularisation $\tilde{\underline{\underline{R}}}$ of the variety $\underline{\underline{R}}$, and Theorem 3.9 does not describe the full $\underline{R} \cup \{P\}$-structure of ES. However, subsequent corollaries show that in certain situations the operations $\underline{J}_R^0 \cup \{P\}$ provide an adequate description of the structure of the algebra of subalgebras of an affine space $(E, \underline{R}, P)$.

COROLLARY 3.10. *Let R be a subring of a commutative ring R', and let $(E, \underline{R}', P)$ be an affine R'-space. For $\Omega \subseteq \underline{J}_R^0 \cup \{P\}$, let $\mathfrak{V}$ be a variety of Ω-algebras equivalent to the variety $\underline{\underline{R}}$ of affine R-spaces. Then for (E, Ω) in $\mathfrak{V}$:*

(i) *the algebra $((E, \underline{R}', P)S, \Omega)$ is a Płonka sum of quotients of (E, Ω) over the semilattice $((E, +, R')S, +)$;*

(ii) *the algebra $((E, \Omega)S, \Omega)$ is a Płonka sum of quotients of (E, Ω) over the semilattice $((E, +, R)S, +)$.* $\square$

Note that the Płonka fibres in 3.10 are equivalent to affine R-spaces, and the algebras $((E, \Omega)S, \Omega)$ are in the regularisation $\tilde{\mathfrak{V}}$ of $\mathfrak{V}$.

COROLLARY 3.11. *Let R be a subring of a commutative ring R', and let $(E, \underline{R}', P)$ be an affine R'-space. For $\Omega \subseteq \underline{J}_R^0$, let $\mathfrak{V}$ be a variety of $\Omega \cup \{P\}$-algebras equivalent to the variety $\underline{\underline{R}}$ of affine R-spaces. Then for (E, Ω, P) in $\mathfrak{V}$:*

(i) *the algebra $((E, \underline{R}', P)S, \Omega)$ is a Płonka sum of quotients of (E, Ω) over the semilattice $((E, +, R')S, +)$;*

(ii) *the algebra $((E, \Omega, P)S, \Omega, P)$ is a Płonka sum of quotients of (E, Ω, P) over the semilattice $((E, +, R)S, +)$.* $\square$

4. Examples and applications

In this section we give a list of examples of varieties of modes for which Theorem 3.9 and its corollaries describe the structure of their modes of subalgebras. The sets J_R^0 introduced in the previous section again turn out to be important.

It is convenient to use some notation introduced in [12, 2.4]. Let $\mathfrak{V}$ be a variety of modes. Then the free $\mathfrak{V}$-algebra $\{0, 1\}V$ on two generators is a semigroup under the multiplication $\cdot$ given by $xy(\mu \cdot v) = x\, xy\mu v$. Also $\{0, 1\}V$ has a unary operation given by $xy\mu' = yx\mu$. For r, s in a ring R, one has $\underline{r} \cdot \underline{s} = \underline{rs}$ and $\underline{r}' = \underline{1-r}$. Transferring notation between rings R and free algebras $\{0, 1\}V$, the operation μ^n is defined by $xy\mu^n = x \cdots xy\mu \cdots \mu$ and the ring element r' is defined as $1 - r$.

The following lemma, a direct corollary of Theorem 3.9, will be used frequently in discussion of our examples. Let $\mathfrak{V}$ be a variety of Ω-algebras equivalent to a variety $\underline{\underline{R}}$ of affine R-spaces. For each $\mathfrak{V}$-algebra (A, Ω), let $\mathfrak{V}(A)$ be the smallest subvariety of $\mathfrak{V}$ containing (A, Ω). Then there is quotient $R(A)$ of the ring R such that the varieties $\mathfrak{V}(A)$ and $R(A)$ are equivalent. The algebra (A, Ω) is equivalent to the faithful affine space $(A, \underline{R(A)}, P)$.

LEMMA 4.1. *Let $\mathfrak{V}$ be a variety of Ω-algebras equivalent to a variety $\underline{\underline{R}}$ of affine R-spaces. Let (A, Ω) be in $\mathfrak{V}$. If $\Omega \subseteq J^0_{R(A)} \cup \{P\}$, then the algebra $((A, \Omega)S, \Omega)$ is a Płonka sum of $\mathfrak{V}(A)$-algebras, equivalent to affine $R(A)$-spaces, over the semilattice* $((A, +, R)S, +) = ((A, +, R(A))S, +)$. □

Examples

A. Quasigroup modes

Recall that a *quasigroup* is an algebra (A, μ, ρ, λ) of type $\{\mu, \rho, \lambda\} \times \{2\}$ satisfying the identities $xy\mu y\rho = x = xy\rho y\mu$ and $xxy\mu\lambda = y = xxy\lambda\mu$. A groupoid (A, μ) is said *to be a quasigroup* if, for each pair (a, b) of elements of A, the equations $ax\mu = b$ and $ya\mu = b$ have unique solutions x and y (written as $x = ab\lambda$ and $y = ba\rho$ respectively). The variety $\mathfrak{V}$ of quasigroup modes is a Mal'cev variety with the Mal'cev operation defined by $xyzP = xy\rho yz\lambda\mu$. So there is a ring R such that $\mathfrak{V}$ is equivalent to the variety $\underline{\underline{R}}$ of affine R-spaces. The ring R is the localization of the ring $\mathbb{Z}[X]$ at the multiplicative subset $\{X^k(1-X)^l \mid k, l \in \mathbb{N}\}$ of $\mathbb{Z}[X]$ (cf. [2]). A quasigroup (A, μ, ρ, λ) in $\mathfrak{V}$ is equivalent to the faithful affine $R(A)$-space $(A, \underline{R(A)}, P)$. So there are m, r, l in $R(A)$ with $\mu = \underline{m}$, $\rho = \underline{r}$ and $\lambda = \underline{l}$. For each element x of A, one has $x = 00x\underline{m}\underline{l} = xml$. The faithfulness of A then yields $ml = 1$, i.e. m and l are invertible. Similarly $x = x0\underline{m}0\underline{r} = xm'r'$ yields $m'r' = 1$, i.e. m' and

r' are invertible. Thus m is element of $J^0_{R(A)}$ with $\underline{r} = \underline{m}'^{-1\prime}$ and $\underline{l} = \underline{m}^{-1}$. By Lemma 3.1, $l = m^{-1}$, $r' = m'^{-1}$ and m' are in $J^0_{R(A)}$, and hence $r = r'' = m'^{-1\prime}$ is in $J^0_{R(A)}$, as well. As a consequence of Lemma 4.1, one has:

PROPOSITION 4.2. *For each quasigroup mode* (A, μ, ρ, λ), *the algebra* (AS, μ, ρ, λ) *is a Plonka sum of quasigroups in* $\mathfrak{B}(A)$. □

B. *MOQ-modes (pairs of mutually orthogonal quasigroup modes)*

A *MOQ-mode* is a mode $(A, \mu, \rho, \lambda, \mu^0, \rho^0, \lambda^0, \alpha, \beta)$ of the type $\{\mu, \rho, \lambda, \mu^0, \rho^0, \lambda^0, \alpha, \beta\} \times \{2\}$, where the reducts (A, μ, ρ, λ) and $(A, \mu^0, \rho^0, \lambda^0)$ are both quasigroups and the operations α and β satisfy the identities $xy\mu xy\mu^0\alpha = x$ and $xy\mu xy\mu^0\beta = y$ (cf. [3]). The latter identities mean that the quasigroups (A, μ) and (A, μ^0) are mutually orthogonal, i.e. for a and b in A, the equations $a = xy\mu$ and $b = xy\mu^0$ have a unique solution in x and y. The variety $\mathfrak{M}$ of MOQ-modes is obviously a Mal'cev variety. So there is a ring R such tath $\mathfrak{M}$ and $\underline{\underline{R}}$ are equivalent. Each MOQ-mode A is equivalent to the faithful affine $R(A)$-space $(A, \underline{R(A)}, P)$. By Example A, there are m and n in $J^0_{R(A)}$ such that $\mu = \underline{m}$, $\rho = \underline{m}'^{-1\prime}$, $\lambda = \underline{m}^{-1}$, $\mu^0 = \underline{n}$, $\rho^0 = \underline{n}'^{-1\prime}$, $\lambda^0 = \underline{n}^{-1}$. Now the quasigroups (A, μ) and (A, μ^0) are mutually orthogonal iff the pair of equations

$$a = xy\mu = xy\underline{m} = xm' + ym$$
$$b = xy\mu^0 = xy\underline{n} = xn' + yn$$

has a unique solution. This happens when the determinant

$$\begin{vmatrix} m' & m \\ n' & n \end{vmatrix} = m'n - n'm = n - mn + nm - m = n - m$$

is invertible in $R(A)$. In this case we have the solution

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} m' & m \\ n' & n \end{bmatrix}^{-1} \begin{bmatrix} xy\mu \\ xy\mu^0 \end{bmatrix}$$
$$= (n-m)^{-1} \begin{bmatrix} n & -m \\ -n' & m' \end{bmatrix} \begin{bmatrix} xy\mu \\ xy\mu^0 \end{bmatrix},$$

whence

$$x = xy\mu\, xy\mu^0 \underline{(n-m)^{-1}(-m)}$$
$$y = xy\mu\, xy\mu^0 \underline{(n-m)^{-1}m'}.$$

Consequently $\alpha = \underline{(n-m)^{-1}(-m)}$ and $\beta = \underline{(n-m)^{-1}m'}$. Now since

$$[(n-m)^{-1}(-m)]' = \frac{n-m+m}{n-m} = \frac{n}{n-m}$$

and

$$([n-m)^{-1}m']' = \frac{n-m-m'}{n-m} = \frac{n-m-1+m}{n-m}$$

$$= \frac{-n'}{n-m},$$

it follows that α and β are in $\underline{J}^0_{R(A)}$. Similarly as in Example A we get the following:

PROPOSITION 4.3. *For each MOQ-mode A, the algebra $(AS, \mu, \rho, \lambda, \mu^0, \rho^0, \lambda^0, \alpha, \beta)$ is a Plonka sum of MOQ-modes in $\mathfrak{M}(A)$.* □

C. Mal'cev varieties of groupoid modes

If $\mathfrak{V}$ is a Mal'cev variety of groupoid modes, then $\mathfrak{V}$ is equivalent to a variety $\underline{\underline{R}}$ and the groupoid multiplication may be identified with $\underline{r}$ for some r in R. Note that in this case the ring R is generated by r. For any $\mathfrak{V}$-groupoid $(A, \underline{r})$ there is a faithful affine space $(A, \underline{R(A)}, P)$ equivalent to $(A, \underline{r})$. Now the Mal'cev operation P can be represented as

$$\begin{aligned} xyzP &= \sum_i a_i(1-r)^{m_i} r^{n_i} x \\ &+ \sum_j b_j(1-r)^{m_j} r^{n_j} y \\ &+ \sum_k c_k(1-r)^{m_k} r^{n_k} z \end{aligned}$$

(cf. e.g. [17]). Without loss of generality we may assume that $n_i \neq 0$. Then in particular

$$\begin{aligned} x = xyyP = x - y + y &= \sum_i a_i(1-r)^{m_i} r^{n_i} x \\ &= \omega(r) r x. \end{aligned}$$

The faithfulness of $(A, \underline{R(A)}, P)$ then yields $\omega(r)r = 1$, implying that r is a unit of $R(A)$. Similarly, one shows that $1 - r$ is a unit, as well. Hence $r \in J^0_{R(A)}$. As before, Lemma 4.1 implies the following:

PROPOSITION 4.4. *Let $\mathfrak{V}$ be a Mal'cev variety of groupoid modes. Then for each $\mathfrak{V}$-groupoid $(A, \underline{r})$, the algebra $(AS, \underline{r})$ is a Płonka sum of $\mathfrak{V}(A)$-groupoids.* □

Note that the groupoids $(A, \underline{r})$ are in fact equivalent to quasigroups $(A, \underline{r}, \underline{r}'^{-1'}, \underline{r}^{-1})$. It follows that varieties $\mathfrak{V}$ of Mal'cev groupoid modes are equivalent to varieties of quasigroups. As an example of such a variety consider the variety $\mathfrak{V}_{n,k}$, for n, $k \geq 2$, of groupoid modes defined by the identities

$$(\cdots((xy_1)y_2)\cdots)y_k = x = y_{k+1}(\cdots(y_{k+n-1}(y_{k+n}x))\cdots), \tag{4.5}$$

where $y_1 = \cdots = y_k = \cdots = y_{k+n} = y$. The variety $\mathfrak{V}_{n,k}$ is a Mal'cev variety with the Mal'cev operation defined by

$$xyzP = (\cdots((xy_1)y_2)\cdots)y_{k-1} \cdot y_k(\cdots(y_{k+n-2}z)\cdots) \tag{4.6}$$

for $y_1 = \cdots = y_{k+n-2} = y$. So $\mathfrak{V}_{n,k}$ is equivalent to some $\underline{\underline{R}}_{n,k}$, and we can identify the groupoid multiplication with $\underline{r}$ for a generator r of the ring $R_{n,k}$. Then (4.5) and (4.6) may be written as

$$yx\underline{r}'^k = x = yx\underline{r}^n, \tag{4.5'}$$

$$xyzP = yx\underline{r}'^{(k-1)}yz\underline{r}^{n-1}\underline{r}. \tag{4.6'}$$

The identities (4.5′) imply that $r^n = 1$ and $(1 - r)^k = 1$ in the ring $R_{n,k}$. Since in particular $(R_{n,k}, \underline{r})$ is a member of $\mathfrak{V}_{n,k}$, it follows that $R_{n,k}$ is the ring $\mathbb{Z}[X]/\langle X^n - 1, (1 - X)^k - 1\rangle$. The varieties $\mathfrak{V}_{n,k}$ contain as subvarieties many well-known varieties of Mal'cev groupoid modes. Among them are the varieties $\mathfrak{G}(n, k)$ of groupoids studied by Mitschke, Werner [6], equivalent to affine spaces over their rings $R(n, k) = \mathbb{Z}[X]/\langle X^n - 1, X^k + X - 1\rangle$. Each $\mathfrak{G}(n, k)$ is a subvariety of $\mathfrak{V}_{n,n/(n,k)}$. The varieties $\mathfrak{G}(q)$ of groupoids equivalent to affine spaces over finite fields $GF(q)$, described by Ganter, Werner [4], are subvarieties of $\mathbb{G}(q - 1, k)$, where $r + r^k = 1$ and r is a primitive element of $GF(q)$. Finally, any variety $\underline{\underline{2m+1}}$ of commutative groupoid modes (see [5], [11] and [12]) is equivalent to the variety $\underline{\underline{\mathbb{Z}}}_{2m+1}$ of affine spaces, and is contained in $\mathfrak{V}_{n,n}$ for some n. The groupoid multiplication is given by $\underline{r} = \underline{n+1}$. Here $1 - r = r$.

D. Affine spaces over fields of characteristic zero

Let K be a field of characteristic zero. The field K contains the field $\mathbb{Q}$ of rationals as its prime subfield. The reduct (E, $\underline{\mathbb{Q}}$, P) of the affine K-space $(E, \underline{K}, P)$ is a rational affine space. Given such an algebra $(E, \underline{\mathbb{Q}}, P)$, one may consider its reduct $(E, \underline{I}^0)$ obtained by admitting only those operations $\underline{k}$ for which k lies in the open unit interval $I^0 = \{x \in Q \mid 0 < x < 1\}$. The subalgebras $(X, \underline{I}^0)$ of $(E, \underline{I}^0)$ are precisely the $\mathbb{Q}$-convex subsets X of E. The smallest variety containing the class of all $\mathbb{Q}$-convex sets is the variety of rational barycentric algebra [11]. The following proposition illustrates Corollary 3.11(i). Note that, as shown in [11], the algebraic structures $(E, \underline{K}, P)$ and $(E, \underline{\mathbb{Q}}, P)$ may be replaced by the structures $(E, \underline{K})$ and $(E. \underline{\mathbb{Q}})$, respectively, and that $I^0 \subset J^0_{\mathbb{Q}}$.

PROPOSITION 4.7 [11]. *For an affine K-space* $(E. \underline{K})$, *the algebra* $((E, \underline{K})S, \underline{I}^0)$ *is a Płonka sum of* $\mathbb{Q}$*-convex sets* $(E/U, \underline{I}^0)$ *over the semilattice* $((E, +, K)S, +)$. □

The $I^0 \cup \{P\}$-reducts $(E, \underline{I}^0, P)$ of the affine K-spaces $(E, \underline{K}, P)$ are in fact equivalent to affine $\mathbb{Q}$-spaces, and form a variety equivalent to the variety $\underline{\underline{\mathbb{Q}}}$. The following proposition illustrates Corollary 3.11(ii).

PROPOSITION 4.8. *For an affine K-space* $(E, \underline{K})$, *the algebra* $((E, \underline{I}^0, P)S, \underline{I}^0, P)$ *is a Płonka sum of quotients of* $(E, \underline{I}^0, P)$ *over the semilattice* $((E, +, \mathbb{Q})S, +)$. □

E. Affine spaces over fields of odd characteristic

If the characteristic of the field K is odd, then again the algebraic structure $(E, \underline{K}, P)$ may be replaced by the structure $(E, \underline{K})$. The field K contains the inverse $1/2$ of $2 = 1 + 1$, and in reducts $(E, \underline{1/2})$, the operation $\underline{1/2}$ generates all the operations $\underline{r}$ for r in the open unit interval $\mathbb{D}^0 = \{x \in \mathbb{D} \mid 0 < x < 1\}$ in the set $\mathbb{D} = \{m2^{-n} \mid m, n \in \mathbb{Z}\}$ of dyadic rationals. The operations $\underline{1/2}$ and P generate all the operation $\underline{r}$ for r in $\mathbb{D}$. The reducts $(E, \underline{1/2}, P)$ are equivalent to $(E, \underline{\mathbb{D}}^0, P)$ and to $(E, \underline{\mathbb{D}}, P)$. The reducts $(E, \underline{1/2})$ are quasigroups and lie in irregular varieties $\underline{\underline{2m+1}}$ of commutative groupoid modes. Similarly as in Example D, the following proposition illustrates Corollary 3.11.

PROPOSITION 4.9 [11]. *For an affine space* $(E, \underline{K})$ *over a field of odd characteristic, there is an irregular variety* $\underline{\underline{2m+1}}$ *of commutative groupoid modes such that* *the algebra* $(ES, \underline{1/2})$ *is a Płonka sum of* $\underline{\underline{2m+1}}$*-groupoids over the semilattice* $((E, +, K)S, +)$. □

PROPOSITION 4.10. *For an affine space* $(E, \underline{K})$ *over a field of odd characteristic, the algebra* $((E, \underline{1/2}, P)S, \underline{1/2}, P)$ *is a Płonka sum of quotients of* $(E, \underline{1/2}, P)$ *over the semilattice* $((E, +, \mathbb{D})S, +)$. □

F. Affine spaces over fields of characteristic two

Let K be a field of characteristic two. The reducts (E, P) of affine K-spaces (E, K, P) are in an arithmetical variety $\mathfrak{Mi}$, called the variety of *minority modes* (A, P), defined by the entropic law for P and the identities

$$yxyP = xyyP = yyxP = x.$$

The following illustrates Corollary 3.10.

PROPOSITION 4.11 [11]. *Let K be a field of characteristic* 2. *Then for an affine* K*-space* $(E, \underline{K}, P)$, *the ternary mode* $((E, \underline{K}, P)S, P)$ *is a Płonka sum of minority modes over the semilattice* $((E, +, K)S, +)$. □

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Pilitowska and Romanowska:
Warsaw Technical University
Institute of Mathematics
Warsaw 00661
Poland

Department of Mathematics
Iowa State University
Ames, IA 50011
U.S.A.