ON COMPLEX ALGEBRAS OF SUBALGEBRAS

KIRA ADARICHEVA, AGATA PILITOWSKA, AND DAVID STANOVSKÝ

ABSTRACT. Let $\mathcal{V}$ be a variety of algebras. We establish a condition (so called *generalized entropic property*), equivalent to the fact that for every algebra $\mathbf{A} \in \mathcal{V}$, the set of all subalgebras of $\mathbf{A}$ is a subuniverse of the complex algebra of $\mathbf{A}$. We investigate the relationship between the generalized entropic property and the entropic law. Further, provided the generalized entropic property is satisfied in $\mathcal{V}$, we study the identities satisfied by the complex algebras of subalgebras of algebras from $\mathcal{V}$.

Dedicated to the 70th birthday of George Grätzer

1. Introduction

For an algebra $\mathbf{A} = (A, F)$, we define *complex operations* on the set $\mathcal{P}(A)$ of all non-empty subsets of the set A by

$$f(A_1, \ldots, A_n) = \{f(a_1, \ldots, a_n) : a_i \in A_i\}$$

for every $\emptyset \neq A_1, \ldots, A_n \subseteq A$ and every n-ary $f \in F$. The set $f(A_1, \ldots, A_n)$ is called the *complex product* of the subsets A_i and the algebra $\mathbf{Cm}\,\mathbf{A} = (\mathcal{P}(A), F)$ is called the *complex algebra* of $\mathbf{A}$. Complex algebras (called also *globals* or *powers* of algebras) were studied by several authors, for instance G. Grätzer and H. Lakser [6], S. Whitney [7], A. Shafaat [19], C. Brink [2], I. Bošnjak and R. Madarász [1].

The notation of complex operations is used widely. In groups, for instance, a coset xN is the complex product of the singleton $\{x\}$ and the subgroup N. For a lattice $\mathbf{L}$, the set of its ideals forms a lattice $\mathbf{Id}\,\mathbf{L}$ under the set inclusion. If $\mathbf{L}$ is distributive, then joins and meets in $\mathbf{Id}\,\mathbf{L}$ are precisely the complex operations obtained from joins and meets of $\mathbf{L}$, so $\mathbf{Id}\,\mathbf{L}$ is a subalgebra of $\mathbf{Cm}\,\mathbf{L}$.

Consider the set CSub$\mathbf{A}$ of all nonempty subalgebras of an algebra $\mathbf{A} = (A, F)$. This set may or may not be closed under the complex operations. For instance, if $\mathbf{A}$ is an Abelian group, it is, but for most groups, it is not. In the case when CSub$\mathbf{A}$ is closed under the complex operations, $\mathbf{CSub}\,\mathbf{A} = (\mathrm{CSub}\mathbf{A}, \mathrm{F})$ is a subalgebra of $\mathbf{Cm}\,\mathbf{A}$ and we call it the *complex algebra of subalgebras*. We will also say in this case that $\mathbf{A}$ *has the complex algebra of subalgebras* or that $\mathbf{CSub}\,\mathbf{A}$ *exists*. Complex algebras of subalgebras were introduced and studied by A. Romanowska and J. D. H. Smith in [15]. A very natural setting for considering the complex algebras of subalgebras is the variety of *modes* (idempotent entropic algebras). Research on complex algebras of submodes was carried out by A. Romanowska

Key words and phrases. complex algebra, complex algebra of subalgebras, mode, entropic, medial, linear identity.

2000 *Mathematics Subject Classification.* 06B20, 06B05.

While working on this paper, the authors were partially supported by the INTAS grant #03-51-4110. The third author was also supported by the research project MSM 0021620839 financed by MŠMT ČR and by the grant #201/05/0002 of the Grant Agency of the Czech Republic.

and J. D. H. Smith in [16], [17], and by the second author of this paper in [12], [13]. In [14], the complex algebras of subalgebras were considered also in the non-idempotent case.

We are concerned with the following question: *In which varieties does every algebra have the complex algebra of subalgebras?* In Section 2 we establish the *generalized entropic property* for a variety, equivalent to the fact that every algebra has the complex algebra of subalgebras. The generalized entropic property appears to be a weak version of the entropic law, so it is natural to ask about their relationship.

The relationship is investigated in Sections 3 and 4. In general, the generalized entropic property and the entropic law are inequivalent. We provide several examples when they are inequivalent: an idempotent algebra with many binary operations (Example 3.1), a non-idempotent groupoid (Example 4.1) or unary algebras (Example 4.3). On the other hand, the generalized entropic property and the entropic law are equivalent under several additional assumptions, e.g., in groupoids with a unit element, in commutative idempotent groupoids, or in idempotent semigroups. We provide several partial results which support the conjecture that the two conditions are equivalent for idempotent groupoids (Theorem 3.3 and other).

In Sections 5 and 6, we continue the research started by the second author in [13] and investigate which identities are satisfied by complex algebras of subalgebras. We describe how such identities can look like in Theorem 5.3. We are not able to decide the validity of a conjecture stated in [13] saying that the variety generated by complex algebras of subalgebras for algebras from an idempotent variety $\mathcal{V}$ coincides with $\mathcal{V}$ if and only if the latter has a basis of linear and idempotent identities. We show that a similar statement for non-idempotent varieties is false, according to Example 5.10.

Notation and terminology. We denote by $\mathbf{F}_{\mathcal{V}}(X)$ the free algebra over a set X in a variety $\mathcal{V}$ and we assume the standard representation of the free algebra by terms modulo the identities of $\mathcal{V}$. The notation $t(x_1, \ldots, x_n)$ means that the term t contains no other variables than $x_1, \ldots, x_n$ (but not necessarily all of them) and we say that t is n-ary; equivalently, we write $t \in \mathbf{F}(\{x_1, \ldots, x_n\})$. We call a term t *linear*, if every variable occurs in t at most once. An identity $t \approx u$ is called *linear*, if both terms t, u are linear. An identity $t \approx u$ is called *regular*, if t, u contain the same variables.

An algebra $\mathbf{A} = (A, F)$ is called *entropic* if it satisfies for every n-ary $f \in F$ and m-ary $g \in F$ the identity

$$g(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1m}, \ldots, x_{nm})) \approx f(g(x_{11}, \ldots, x_{1m}), \ldots, g(x_{n1}, \ldots, x_{nm}))$$

(in other words, if all operations of $\mathbf{A}$ commute). Note that a *groupoid*, i.e., a binary algebra, with the operation denoted usually multiplicatively, is entropic if it satisfies the identity

$$xy \cdot uv \approx xu \cdot yv,$$

called sometimes the *mediality* [8]. A variety $\mathcal{V}$ is called *entropic* if every algebra in $\mathcal{V}$ is entropic. An algebra (A, F) is *idempotent* if $f(a, \ldots, a) = a$ for every $f \in F$ and every $a \in A$. Idempotent entropic algebras are called *modes*. The monograph by A. Romanowska and J.D.H. Smith [18] provides the most full up-to-date account of results about modes.

2. Generalized entropic property

In this section we introduce and discuss the central notion of this paper, the generalized entropic property.

Definition 2.1. We say that a variety $\mathcal{V}$ (respectively, an algebra $\mathbf{A}$) satisfies the *generalized entropic property* if for every n-ary operation f and m-ary operation g of $\mathcal{V}$ (of $\mathbf{A}$), there exist m-ary terms $t_1, \dots, t_n$ such that the identity

$$g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1m}, \dots, x_{nm})) \approx f(t_1(x_{11}, \dots, x_{1m}), \dots, t_n(x_{n1}, \dots, x_{nm}))$$

holds in $\mathcal{V}$ (in $\mathbf{A}$).

It was proved by T. Evans in [5], that every groupoid in a variety $\mathcal{V}$ has the complex algebra of subalgebras if and only if $\mathcal{V}$ satisfies generalized entropic property. We prove the statement for an arbitrary signature. The "if" part of it first appeared in [13], where the generalized entropic property was presented as a "complex condition".

Proposition 2.2. *Let $\mathcal{V}$ be a variety of algebras of signature σ. Then* $\mathbf{CSub\,A}$ *exists for all $\mathbf{A} \in \mathcal{V}$ if and only if $\mathcal{V}$ has the generalized entropic property.*

Proof. Let $\mathcal{V}$ have the generalized entropic property. Let $\mathbf{A} \in \mathcal{V}$, $\mathbf{A}_1, \dots, \mathbf{A}_n \in \mathrm{CSub}\mathbf{A}$, and let $f \in \sigma^n$, $g \in \sigma^m$. In order to prove that $\mathbf{CSub\,A}$ exists, it suffices to show that $f(A_1, \dots, A_n)$ is closed under g. If $x_1, \dots, x_m \in f(A_1, \dots, A_n)$, then there are $a_{ij} \in A_i$, for $1 \le i \le n$, $1 \le j \le m$, such that $x_j = f(a_{1j}, \dots, a_{nj})$ for all $1 \le j \le m$. Applying the generalized entropic property, we get that

$$\begin{gathered} g(x_1, \dots, x_m) = g(f(a_{11}, \dots, a_{n1}), \dots, f(a_{1m}, \dots, a_{nm})) = \\ f(t_1(a_{11}, \dots, a_{1m}), \dots, t_n(a_{n1}, \dots, a_{nm})) \in f(A_1, \dots, A_n) \end{gathered}$$

for some terms $t_1, \dots, t_n$.

Conversely, let $\mathbf{CSub\,A}$ exists for any $\mathbf{A} \in \mathcal{V}$. Let $f \in \sigma^n$, $g \in \sigma^m$. Choose pairwise distinct variables x_{ij}, $1 \le i \le n$, $1 \le j \le m$, and put $X_i = \{x_{ij} : 1 \le j \le m\}$. Let X be an infinite set which contains X_i for all $1 \le i \le n$. Let also $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(X)$ and let $\mathbf{F}_i = \mathbf{F}_{\mathcal{V}}(X_i)$ for all $1 \le i \le n$; then $\mathbf{F}_1, \dots, \mathbf{F}_n \in \mathbf{CSub\,F}$. Since $f(F_1, \dots, F_n) \in \mathrm{CSub}\mathbf{F}$, it is closed under g. In particular,

$$g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1m}, \dots, x_{nm})) \in f(F_1, \dots, F_n).$$

Thus, $g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1m}, \dots, x_{nm})) = f(b_1, \dots, b_n)$ for some $b_i \in F_i$, $1 \le i \le n$. This means that there are terms t_i, $1 \le i \le n$, such that $b_i = t_i(x_{i1}, \dots, x_{im})$. Since the equality

$$g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1m}, \dots, x_{nm})) = f(t_1(x_{11}, \dots, x_{1m}), \dots, t_n(x_{n1}, \dots, x_{nm}))$$

holds on a $\mathcal{V}$-free algebra of infinite rank, it also holds in $\mathcal{V}$, whence $\mathcal{V}$ has the generalized entropic property. □

Example 2.3. *For an algebra* $\mathbf{A}$*, existence of* $\mathbf{CSub\,A}$ *does not imply that* $\mathbf{A}$ *has the generalized entropic property.*

Consider the following 3-element groupoid $\mathbf{G}_1$:

$\cdot$	a	b	c
a	a	c	c
b	c	b	c
c	a	b	c

Notice that $\mathbf{G}_1$ is not entropic, because $c = aa{\cdot}ba \neq ab{\cdot}aa = a$. It is straightforward to see that $\mathbf{CSub}\,\mathbf{G}_1$ is a subgroupoid of $\mathbf{Cm}\,\mathbf{G}_1$, with the multiplication table

$\cdot$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{a\}$	$\{a\}$	$\{c\}$	$\{c\}$	$\{a,c\}$	$\{c\}$	$\{a,c\}$
$\{b\}$	$\{c\}$	$\{b\}$	$\{c\}$	$\{c\}$	$\{b,c\}$	$\{b,c\}$
$\{c\}$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{a,c\}$	$\{a\}$	$\{b,c\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{b,c\}$	$\{a,c\}$	$\{b\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{a,b,c\}$	$\{a,c\}$	$\{b,c\}$	$\{c\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$

There is a groupoid $\mathbf{F}$ in the variety $\mathrm{V}(\mathbf{G}_1)$, namely $\mathbf{F} = \mathbf{F}_{\mathrm{V}(\mathbf{G}_1)}(x,y,z)$, such that $\mathbf{CSub}\,\mathbf{F}$ is not a subgroupoid of $\mathbf{Cm}\,\mathbf{F}$. To see this, consider the subgroupoid $\mathbf{A}$ of $\mathbf{F}$ generated by x, y and $\mathbf{B} = (\{z\}, \cdot)$. One can check that $A = \{x, y, xy, yx\}$, $x \approx (xz)x$ and $((yx)z)y, x \in (AB)A$, but $(((yx)z)y)x \notin (AB)A$. Hence the set $(AB)A$ is not a subgroupoid of $\mathbf{F}$ and thus $\mathbf{G}_1$ does not satisfy the generalized entropic property. (Since c is a left unit of $\mathbf{G}_1$, the latter statement follows also from Corollary 3.9.)

3. Generalized entropic property vs. entropy: the idempotent case

The entropic law is a special case of the generalized entropic property, where the terms $t_1, \dots, t_n$ are equal to g. We would like to investigate how far is the generalized entropic property from entropy. Generally, these two laws are not equivalent. In this section we consider a case for idempotent algebras. We start with an example of idempotent and non-entropic algebras which satisfy the generalized entropic property. We also provide several sufficient conditions implying that an idempotent groupoid satisfying the generalized entropic property is entropic. The main result, Theorem 3.3, is applied several times in the following propositions and examples.

Example 3.1. *There are idempotent non-entropic algebras possessing the generalized entropic property.*

Let $\mathbf{R} = (R, +, \cdot, -, 0, 1)$ be a ring with unit 1, let $\mathbf{G}$ be a submonoid of $(R, \cdot, 1)$ which is a group, and let $X \subseteq G$ be closed under the group conjugation as well as under the map $x \mapsto 1 - x$. Consider a left module $\mathbf{M}$ over $\mathbf{R}$; for every $r \in R$, define a binary operation $\underline{r}$ on M by

$$\underline{r}(x, y) = (1 - r)x + ry.$$

Then the groupoid $(M, \underline{r})$ is idempotent and entropic for every $r \in R$. Consider the algebra $\underline{\mathbf{M}}_X = (M, \underline{X})$, where $\underline{X} = \{\underline{r} : r \in X\}$. If $r, t \in X$, we put $t_1 = (1-r)^{-1}t(1-r)$ and $t_2 = r^{-1}tr$. Then

$$\underline{t}(\underline{r}(x_1, x_2), \underline{r}(y_1, y_2)) \approx (1-t)(1-r)x_1 + (1-t)rx_2 + t(1-r)y_1 + try_2 \approx$$
$$\approx (1-r)(1-t_1)x_1 + r(1-t_2)x_2 + (1-r)t_1y_1 + rt_2y_2 \approx \underline{r}(\underline{t_1}(x_1, y_1), \underline{t_2}(x_2, y_2)).$$

Thus, $\underline{\mathbf{M}}_X$ has the generalized entropic property. Substituting x_1, y_1, y_2 with 0 and x_2 with 1 in the above equations, we get that $\underline{\mathbf{M}}_X$ is entropic if and only if $rt = tr$ for all $r, t \in X$.

Therefore, choosing $\mathbf{R}$ to be a non-commutative division ring and taking $X = R \smallsetminus \{0, 1\}$, we get that $\underline{\mathbf{M}}_X$ is an (infinite) idempotent non-entropic algebra which possesses the generalized entropic property.

Our next example is finite. Let $\mathbf{R}$ be the ring of 2×2 matrices over a field $\mathbb{F}$, let $G=\{A\in R : |A|=1\}$, and let $X=\{A\in G : tr(A)=1\}$, where $tr(A)$ denotes the *trace* of A. Since both, determinant and trace, are invariant under the matrix conjugation and $A\in X$ if and only if $E-A\in X$, where E denotes the unit matrix, we again get that $\underline{\mathbf{M}}_X$ is idempotent and has the generalized entropic property. If $\mathbb{F}=\mathrm{GF}(2)$, then X has two elements and those elements commute. Therefore in this case, $\underline{\mathbf{M}}_X$ is entropic. If $\mathbb{F}=\mathrm{GF}(3)$, then X has nine elements, and not all of its elements commute, so that $\underline{\mathbf{M}}_X$ is not entropic.

Finally, we note that similar examples can be obtained with operations of an arbitrary arity $n\geq 2$; consider the operations

$$\underline{(r_2,\ldots,r_n)}(x_1,\ldots,x_n)=(1-r_2-\cdots-r_n)x_1+r_2x_2+\cdots+r_nx_n.$$

Because the algebra $(M,\underline{r})$ is entropic, for any r, one might think about the following conjecture:

Conjecture 3.2. *Every idempotent algebra (A,f) with the generalized entropic property is entropic.*

In the sequel, we prove several special cases of the conjecture for groupoids. A groupoid satisfies the generalized entropic property, if there are binary terms t,s such that the identity

$$xy\cdot uv\approx t(x,u)s(y,v) \tag{G1}$$

holds. An immediate consequence of the generalized entropic property in idempotent groupoids are the following important identities that can be treated as the laws of *pseudo-distributivity*:

$$xy\cdot xz\approx xs(y,z), \tag{G2}$$
$$yx\cdot zx\approx t(y,z)x, \tag{G3}$$
$$x\cdot yz\approx t(x,y)s(x,z), \tag{G4}$$
$$yz\cdot x\approx t(y,x)s(z,x). \tag{G5}$$

(G2) states that, for every a, the *left translation* $L_a : x\mapsto ax$ is a homomorphism $(G,s)\to(G,\cdot)$ and (G3) states that the *right translation* $R_a : x\mapsto xa$ is a homomorphism $(G,t)\to(G,\cdot)$.

The main partial result towards Conjecture 3.2 is the following theorem.

Theorem 3.3. *If an idempotent groupoid $\mathbf{G}$ satisfies the generalized entropic property for some terms t,s and at least one of t,s is linear, then $\mathbf{G}$ is entropic.*

Proof. If t is linear, one of Lemmas 3.4–3.7, applies. If s is linear, consider the dual groupoid $\mathbf{G}^\partial$ (with the operation defined by $x\bullet y=yx$); this groupoid satisfies the generalized entropic property with the role of t,s interchanged, hence both $\mathbf{G}^\partial$ and $\mathbf{G}$ are entropic by one of Lemmas 3.4–3.7; note that entropy is a self-dual identity. □

Lemma 3.4. *If an idempotent groupoid $\mathbf{G}$ satisfies the generalized entropic property for the term $t(x,y)=x$ and an arbitrary term s, then $\mathbf{G}$ is entropic.*

Proof. The generalized entropic property states that $xy\cdot uv\approx xs(y,v)$. Since the value of $xy\cdot uv$ does not depend on u, we have $xy\cdot uv\approx xy\cdot vv\approx xy\cdot v$. Hence, with $x=y$, we obtain $x\cdot uv\approx xv$. Applying this identity to the term $xs(y,v)$, we get $xs(y,v) \approx xw$, where $w \in \{y,v\}$ is the rightmost variable in the term $s(y,v)$. So, we have $xy \cdot uv \approx xy \cdot v \approx xs(y,v) \approx xw$. If $w = y$, then $xv \approx xx \cdot vv \approx xx \approx x$ by identifying: $x = y$ and $u = v$. Thus the entropy holds. If $w = v$ then $xy \cdot uv$ does not depend on y and u, hence we can interchange them and the entropy holds again. □

Lemma 3.5. *If an idempotent groupoid* $\mathbf{G}$ *satisfies the generalized entropic property for the term* $t(x,y) = y$ *and an arbitrary term* s*, then* $\mathbf{G}$ *is entropic.*

Proof. The generalized entropic property says that $xy \cdot uv \approx us(y,v)$. Since the value of $xy \cdot uv$ does not depend on x, we have $xy \cdot uv \approx yy \cdot uv \approx y \cdot uv$. Hence, with $u = v$ we obtain $xy \cdot u \approx yu$. Applying this identity to the term $s(x,y)z$, we get $s(x,y)z \approx wz$, where $w \in \{x,y\}$ is the rightmost variable in the term $s(x,y)$. So, we have $s(x,y) \approx s(x,y)s(x,y) \approx ws(x,y) = t(x,w)s(x,y)$, and thus $s(x,y) \approx x \cdot wy$ by the generalized entropic property. So we may assume that the rightmost variable of $s(x,y)$ is y, i.e., $w = y$. Consequently, $s(x,y) \approx xy$ and thus $xy \cdot uv \approx u \cdot yv \approx y \cdot uv \approx xu \cdot yv$ by (G1), (G4) and (G1). □

Lemma 3.6. *If an idempotent groupoid* $\mathbf{G}$ *satisfies the generalized entropic property for the term* $t(x,y) = xy$ *and an arbitrary term* s*, then* $\mathbf{G}$ *is entropic.*

Proof. Note that (G3) is the right distributivity. Hence $r(x,y)z \approx r(xz,yz)$, for every term r.

Claim 1. $s(x,z) \cdot xz \approx s(x,z)$.

Using right distributivity in s, twice the generalized entropic property and again the right distributivity in s, we obtain

$$\begin{aligned} s(x,z) \cdot xz &\approx s(x \cdot xz, z \cdot xz) \approx s(xs(x,z), zx \cdot z)) \approx s(xs(x,z), zs(x,z)) \\ &\approx s(x,z)s(x,z) \approx s(x,z). \end{aligned}$$

Claim 2. $xs(y,z) \approx x \cdot yz$.

Using several times the generalized entropic property and the idempotent law, we get

$$\begin{aligned} xs(y,z) &\approx xy \cdot xz \approx (xy)(xz \cdot xz) \approx (x \cdot xz)s(y,xz) \approx (xs(x,z))s(y,xz) \\ &\approx (xy)(s(x,z) \cdot xz) \approx (xy)s(x,z) \approx x \cdot yz, \end{aligned}$$

where the last but one equality follows from Claim 1.

Finally, it follows from Claim 2 that $xy \cdot uv \approx xu \cdot s(y,v) \approx xu \cdot yv$. □

Lemma 3.7. *If an idempotent groupoid* $\mathbf{G}$ *satisfies the generalized entropic property for the term* $t(x,y) = yx$ *and an arbitrary term* s*, then* $\mathbf{G}$ *is entropic.*

Proof. Note that (G3) is read as $xy \cdot z \approx yz \cdot xz$ that can be treated as the *right anti-distributivity*. One can check by induction that $r(x,y)z \approx r^\partial(xz,yz)$ for every term r, where r^∂ denotes the term dual to r (this is the term that results when reading r from right to left; inductively, $x^\partial = x$ and $(r_1r_2)^\partial = r_2^\partial r_1^\partial$).

Claim 1. $s(x,z) \cdot xz \approx s(x,z)$.

Using the right anti-distributivity in s, then three times the generalized entropic property and again the right anti-distributivity in s, we get

$$\begin{aligned} s(x,z) \cdot xz &\approx s^\partial(x \cdot xz, z \cdot xz) \approx s^\partial(xs(x,z), xz \cdot z)) \approx s^\partial(xs(x,z), zx \cdot z)) \\ &\approx s^\partial(xs(x,z), zs(x,z)) \approx s(x,z)s(x,z) \approx s(x,z). \end{aligned}$$

Claim 2. $s(x, y) \approx xy$.

Using twice Claim 1 and three times the generalized entropic property, we obtain

$$s(x, y) \approx s(x, y)(xy) \approx (s(x, y) \cdot xy)(xy) \approx (xs(x, y))s(xy, y) \approx (x \cdot xy)s(xy, y) \approx (xy \cdot xy)(xy) \approx xy.$$

Hence the groupoid $\mathbf{G}$ satisfies $xy \cdot uv \approx ux \cdot yv$. Consider the dual groupoid $\mathbf{G}^\partial$; it satisfies $xy \cdot uv \approx xu \cdot vy$ and thus it is entropic by the preceding lemma. Since entropy is a self-dual identity, $\mathbf{G}$ is entropic too. □

Theorem 3.3 has several interesting consequences.

Corollary 3.8. *Let $\mathcal{V}$ be a variety of idempotent groupoids such that every binary term is equivalent to a linear term in $\mathcal{V}$. If $\mathcal{V}$ satisfies the generalized entropic property, then $\mathcal{V}$ is entropic.*

All groupoids with the property that every binary term is equivalent to a linear term were characterized by J. Dudek [4], see also [3]. The groupoid $\mathbf{G}_1$ from Example 2.3 can be found in the list of these groupoids.

We say that an element $e \in G$ is a *one-sided unit* of a groupoid $\mathbf{G}$, if $ex = x$ for all $x \in G$, or $xe = x$ for all $x \in G$.

Corollary 3.9. *Let $\mathbf{G}$ be an idempotent groupoid with a one-sided unit. If $\mathbf{G}$ satisfies the generalized entropic property, then it is entropic.*

Proof. Assume that e is a left unit in $\mathbf{G}$. Then

$$xy \approx ex \cdot ey \approx t(e, e)s(x, y) \approx es(x, y) \approx s(x, y)$$

in $\mathbf{G}$ and thus Theorem 3.3 applies. If e is a right unit, proceed dually. □

For example, the element c is a left unit in the groupoid $\mathbf{G}_1$ from Example 2.3. Since $\mathbf{G}_1$ is non-entropic, it cannot satisfy the generalized entropic property.

The following observation will also become useful in the sequel.

Lemma 3.10. *If an idempotent algebra $\mathbf{A} = (A, F)$ satisfies the generalized entropic property such that, for each pair $f, g \in F$, the terms $t_1, \dots, t_n$ are equal, then $\mathbf{A}$ is entropic.*

Proof. Let $t = t_1 = \dots = t_n$. Then

$$g(x_1, \dots, x_m) \approx g(f(x_1, \dots, x_1), \dots, f(x_m, \dots, x_m)) \approx f(t(x_1, \dots, x_m), \dots, t(x_1, \dots, x_m)) \approx t(x_1, \dots, x_m).$$

□

We apply our previous results in several well-known classes of groupoids. Recall that idempotent semigroups are also called *bands*.

Proposition 3.11. *A band satisfying the generalized entropic property is entropic.*

Proof. In bands, any binary term is equivalent to one of x, y, xy, yx, xyx, yxy: by the idempotency, neither a variable can appear at two consecutive places, nor xy can appear more then once in a row. So, if t or s is equivalent to one of the first four (linear) terms, we can apply Theorem 3.3. If t, s are equivalent to the same term, then we can use Lemma 3.10. Hence we are left with two cases:

$$xyuv \approx uxuyvy \qquad \text{and} \qquad xyuv \approx xuxvyv.$$

If the first identity holds, then we get $xv \approx xvx$ by substitution $x = y = u$, and $xv \approx vxv$ by substitution $y = u = v$. So, we have the commutativity, hence the entropy follows.

In the latter case, $xuw \approx xxuw \approx xuxwxw \approx xuxw$, where the last equality follows from the idempotency, and, similarly, $wvyv \approx wyv$. Thus $xuxvyv$ is equal to $(xux)(vyv) \approx (xu)(vyv) \approx (xu)(yv)$. □

Proposition 3.12. *An idempotent commutative groupoid satisfying the generalized entropic property is entropic.*

Proof. Using (G3), the commutativity and (G2), we obtain

$$t(x,y)z \approx xz \cdot yz \approx zx \cdot zy \approx zs(x,y) \approx s(x,y)z.$$

Consequently,

$$s(x,u) \approx s(x,u)s(x,u) \approx t(x,u)s(x,u) \approx xx \cdot uu \approx xu.$$

Similarly for t. □

A groupoid $\mathbf{G}$ is called *left* (respectively, *right*) *cancellative*, if $zx = zy$ implies $x = y$ ($xz = yz$ implies $x = y$), for all $x, y, z \in G$. For instance, quasigroups are both left and right cancellative.

Proposition 3.13. *An idempotent left or right cancellative groupoid satisfying the generalized entropic property is entropic.*

Proof. Assume the left cancellativity. Then $x \cdot xy \approx xx \cdot xy \approx t(x,x)s(x,y) \approx x \cdot s(x,y)$ and so by the left cancellativity we get $s(x,y) \approx xy$. Apply Theorem 3.3. In the case of the right cancellativity proceed dually. □

Next, we apply Corollary 3.8 to show that there are varieties which do not possess the generalized entropic property although its generating algebras have their complex algebras of subalgebras.

Example 3.14. *The variety generated by equivalence algebras does not possess the generalized entropic property, although every equivalence algebra has the complex algebra of subalgebras.*

Let A be a set and let $\alpha \subseteq A \times A$ be an equivalence relation on A. The *equivalence algebra* $\mathbf{A}(\alpha)$ is a groupoid with the multiplication defined as follows (see, for example, [9]):

$$x \cdot y = \begin{cases} x, & \text{if } (x,y) \in \alpha, \\ y, & \text{otherwise.} \end{cases}$$

It is easy to see that a homomorphic image and a subalgebra of an equivalence algebra is again an equivalence algebra. In fact, any subset of an equivalence algebra is a subalgebra. Hence, every equivalence algebra has the complex algebra of subalgebras.

Consider the variety $\mathcal{E}$ generated by equivalence algebras. It is not entropic, since in the equivalence algebra on the set $\{a, b, c\}$, corresponding to the equivalence with two blocks $\{a, b\}$ and $\{c\}$, we have $a = (ca)b \neq (cb)(ab) = b$. It is not difficult to check that the two-generated free algebra in $\mathcal{E}$ has only four elements: x, y, xy, yx. Hence, by Corollary 3.8, the variety $\mathcal{E}$ does not satisfy the generalized entropic property.

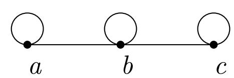
Example 3.15. *The variety generated by all graph algebras does not have the generalized entropic property, although every graph algebra has the complex algebra of subalgebras.*

Let $G = (V, E)$ be a graph with a set V of vertices and a set $E \subseteq V \times V$ of edges. Its *graph algebra* $\mathbf{A}(G) = (V \cup \{0\}, \cdot)$ is a groupoid with the multiplication defined as follows:

$$x \cdot y = \begin{cases} x, & \text{if } (x, y) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

As shown in [11], any subalgebra with 0 and any homomorphic image of a graph algebra is a graph algebra. In fact, any subset with 0 of a graph algebra is clearly a subalgebra. Moreover, for any two subalgebras $\mathbf{A}$ and $\mathbf{B}$ of the graph algebra $\mathbf{A}(G)$, if all $a \in A$ and $b \in B$ are connected by the edge, then $\mathbf{AB} = \mathbf{A}$. On the other side, if there are $a \in A$ and $b \in B$ such that $(a, b) \notin E$, then $0 \in AB$. Thus every graph algebra has the complex algebra of subalgebras.

Consider the variety $\mathcal{G}_I$ generated by idempotent graph algebras. It is not entropic, since in the graph algebra corresponding to the graph



we have $b = (bc)a \neq (ba)(ca) = 0$. Similarly to Example 3.14, the two-generated free algebra in $\mathcal{G}_I$ has only four elements: x, y, xy, yx. By Corollary 3.8, the variety $\mathcal{G}_I$ does not satisfy the generalized entropic property.

We finish this section with an observation.

Proposition 3.16. *Every idempotent groupoid with the generalized entropic property satisfies the identity*

$$xy \cdot uv \approx (xy \cdot uy)(xv \cdot uv) \approx (xy \cdot xv)(uy \cdot uv).$$

Proof. Using (G1) and (G2) we obtain $xy \cdot uv \approx t(x, u)s(y, v) \approx t(x, u)y \cdot t(x, u)v$ and now the first identity follows from (G3). Similarly, using (G1), (G3) and (G2) we obtain $xy \cdot uv \approx t(x, u)s(y, v) \approx xs(y, v) \cdot us(y, v) \approx (xy \cdot xv)(uy \cdot uv)$. □

The converse is false. It can be checked that the following groupoid $\mathbf{G}_2$

$\cdot$	a	b	c
a	a	b	a
b	c	b	c
c	c	b	c

satisfies the identities from Proposition 3.16, but it fails the generalized entropic property.

4. GENERALIZED ENTROPIC PROPERTY VS. ENTROPY: THE NON-IDEMPOTENT CASE

We start with an observation that the generalized entropic property and the entropic law are generally inequivalent for non-idempotent groupoids.

Example 4.1. *There is a variety of non-idempotent and non-entropic groupoids which has the generalized entropic property.*

Let $\mathcal{V}_A$ denote the variety of groupoids satisfying the identity

$$(x_1x_2)(x_3x_4) \approx (x_3x_1)(x_2x_4).$$

Clearly, the generalized entropic property holds in $\mathcal{V}_A$. It follows from Lemma 4.2 that $\mathcal{V}_A$ is not entropic: in our case $A = \{(1, 3, 2)\}$, and the subgroup generated by A in the symmetric group $\mathbf{S}_4$ does not contain the transposition $(2, 3)$.

Lemma 4.2. *Let $A \subseteq S_4$ be a set of permutations on four elements and let $\mathcal{V}_A$ be the variety of groupoids satisfying the identities*

$$x_1x_2 \cdot x_3x_4 \approx x_{\pi 1}x_{\pi 2} \cdot x_{\pi 3}x_{\pi 4}$$

for every $\pi \in A$. Then $\mathcal{V}_A$ is entropic, if and only if the transposition $(2, 3)$ is in the subgroup generated by A in $\mathbf{S}_4$.

Proof. Generally, two terms p, q are equivalent in a variety $\mathcal{V}$, if and only if there is a sequence $p = w_1, w_2, \dots, w_n = q$ such that, for each i, w_i has a subterm which is a substitution instance of some term which appears in an equation ε from the base of $\mathcal{V}$, and w_{i+1} is derived from w_i by replacing this subterm by the same substitution instance of the other side of ε. In $\mathcal{V}_A$, starting with the term $x_1x_2 \cdot x_3x_4$, we cannot make any proper substitution, hence w_{i+1} is always obtained from w_i by permuting variables of w_i by some $\pi \in A$. Hence, if we use permutations $\pi_1, \dots, \pi_{n-1}$, we arrive in the term $x_{\pi 1}x_{\pi 2} \cdot x_{\pi 3}x_{\pi 4}$, where $\pi = \pi_{n-1} \cdots \pi_1$. So, the entropy can be obtained iff $(2, 3)$ is generated by permutations from A. □

The two conditions are inequivalent also for unary algebras. (They haven't appeared in the previous section, because idempotency is a rather trivial property in there.)

Example 4.3. *There is a non-idempotent and non-entropic unary algebra which satisfies the generalized entropic property.*

Let $\mathbf{A} = (A, F)$ be a unary algebra, i.e., F contains only unary operations. Clearly, $\mathbf{A}$ is entropic iff $fg \approx gf$, for all $f, g \in F$.

Let B be a subset of the symmetric group over a set X such that $B = B^{-1}$. Put $\mathbf{B} = (X, \{f : f \in B\})$. Then for every $f, g \in B$ we can always find a term t such that $fg = gt$ (namely, $t = g^{-1}fg$), so $\mathbf{B}$ satisfies the generalized entropic property. On the other hand, if $fg \neq gf$ for at least one pair $f, g \in B$, then $\mathbf{B}$ is not entropic.

On the other hand, there are several important classes, where the generalized entropic property is equivalent with the entropic law, regardless idempotency. For instance, this is true for groupoids with a unit element. The following statement covers a more general setting. We say that an element e is a *unit* for an operation f, if

$$f(x, e, \dots, e) \approx f(e, x, e, \dots, e) \approx \dots \approx f(e, \dots, e, x) \approx x$$

for every $x \in A$. We say that e is a *unit* for an algebra (A, F), if it is a unit for each operation $f \in F$.

Lemma 4.4. *Let $\mathbf{A} = (A, F)$ be an algebra with a one-element subalgebra $\{e\}$ and assume that e is a unit for an n-ary operation $f \in F$. If (A, F) satisfies the generalized entropic property, then f commutes with each operation $g \in F$.*

Proof. The generalized entropic property says that

$$g(f(x_{11},\dots,x_{n1}),\dots,f(x_{1m},\dots,x_{nm}))\approx f(t_1(x_{11},\dots,x_{1m}),\dots,t_n(x_{n1},\dots,x_{nm}))$$

for some terms $t_1,\dots,t_n$. Hence

$$\begin{aligned} g(x_1,\dots,x_m) &\approx g(f(e,\dots,x_1,\dots,e),f(e,\dots,x_2,\dots,e),\dots,f(e,\dots,x_m,\dots,e))\\ &\approx f(t_1(e,\dots,e),\dots,t_k(x_1,\dots,x_m),\dots,t_n(e,\dots,e))\\ &\approx f(e,\dots,t_k(x_1,\dots,x_m),\dots,e)\approx t_k(x_1,\dots,x_m), \end{aligned}$$

for every $k\le n$. So

$$g(f(x_{11},\dots,x_{n1}),\dots,f(x_{1m},\dots,x_{nm}))\approx f(g(x_{11},\dots,x_{1m})\dots,g(x_{n1},\dots,x_{nm})).$$

□

As a consequence, we get

Proposition 4.5. *Let* **A** *be an algebra with a unit element* e*. If* **A** *satisfies the generalized entropic property, then* **A** *is entropic.*

Adjoining an outside unit element is quite a standard operation when dealing with algebras. The following example shows that such an extended algebra may fail the generalized entropic property.

Example 4.6. *There is a groupoid* $(G\cup\{e\},\cdot)$*, with a unit element* e*, which does not satisfy the generalized entropic property although* $(G,\cdot)$ *does.*

Consider a groupoid **G** satisfying the generalized entropic property and possessing elements a,b such that $ab\neq ba$. Let $\mathbf{G}^*$ denote the groupoid obtained from **G** by adjoining a unit element e. Then $\mathbf{G}^*$ is not entropic, because $ea\cdot be=ab\neq ba=eb\cdot ae$. Hence, although **G** itself satisfies the generalized entropic property, by Proposition 4.5 the groupoid $\mathbf{G}^*$ does not.

A *loop* is an algebra $\mathbf{A}=(A,\cdot,/,\backslash,e)$ such that the identities

$$\begin{aligned} x\backslash(xy)\approx y,&\qquad (yx)/x\approx y,\\ x(x\backslash y)\approx y,&\qquad (y/x)x\approx y,\\ &xe\approx ex\approx x \end{aligned}$$

hold in **A**. In other words, loops can be considered as "non-associative groups". On the other hand, groups can be regarded as loops with $x/y=xy^{-1}$ and $y\backslash x=y^{-1}x$.

Proposition 4.7. *Let* $\mathcal{V}$ *be a variety of loops. The following conditions are equivalent:*

1. $\mathcal{V}$ *satisfies the generalized entropic property;*
2. $\mathcal{V}$ *is entropic;*
3. $\mathcal{V}$ *is a variety of Abelian groups.*

Proof. (1) ⇒ (2). Let $\mathbf{A}\in\mathcal{V}$. It follows from Lemma 4.4 that $(A,\cdot)$ is entropic. And it is easy to check that, for any loop **A**, if $(A,\cdot)$ is entropic, then **A** is entropic.

(2) ⇒ (3). If **A** is an entropic loop and $x,y,z\in A$, then

$$xy\cdot z=xy\cdot ez=xe\cdot yz=x\cdot yz$$

(hence **A** is a group) and

$$xy=(xy)(x^{-1}x)=(xx^{-1})(yx)=yx.$$

(3) ⇒ (1). It is well known that the complex product of two subgroups is a subgroup. □

We finish this section with a result on commutative groupoids. A term r is called **G**-*symmetric*, if **G** satisfies $r(x,y) \approx r(y,x)$.

Proposition 4.8. *If a commutative groupoid* **G** *satisfies the generalized entropic property for some terms t, s and at least one of t, s is linear or* **G***-symmetric, then* **G** *is entropic.*

Proof. Because of commutativity, we can assume that the linear or **G**-symmetric term is t. If t is **G**-symmetric, then, using several times the commutativity and the generalized entropic property, we get

$$xy \cdot uv \approx yx \cdot uv \approx t(y,u) \cdot s(x,v) \approx t(u,y) \cdot s(x,v) \approx ux \cdot yv \approx xu \cdot yv.$$

If t is linear, then either $t(x,y) \in \{xy, yx\}$ (so t is **G**-symmetric and the first case applies), or $t(x,y) = x$, or $t(x,y) = y$. First, assume $xy \cdot uv \approx xs(y,v)$. Consequently, the term $xy \cdot uv$ does not depend on u and we can compute using the commutativity:

$$xy \cdot uv \approx xy \cdot yv \approx yv \cdot xy \approx yv \cdot yx \approx yv \cdot ux \approx ux \cdot yv \approx xu \cdot yv.$$

Next, if $xy \cdot uv \approx us(y,v)$, then $xy \cdot uv$ does not depend on x and a similar computation does the job. □

5. Identities in complex algebras of subalgebras

Let $\mathcal{V}$ be a variety. We will denote by $\mathbf{Cm}\,\mathcal{V}$ the variety generated by complex algebras of algebras in $\mathcal{V}$, i.e.,

$$\mathbf{Cm}\,\mathcal{V} = \mathrm{V}(\{\mathbf{Cm}\,\mathbf{A} : \mathbf{A} \in \mathcal{V}\}).$$

Further, if $\mathcal{V}$ satisfies the generalized entropic property, we let $\mathbf{CSub}\,\mathcal{V}$ be the variety generated by complex algebras of subalgebras of algebras in $\mathcal{V}$, i.e.,

$$\mathbf{CSub}\,\mathcal{V} = \mathrm{V}(\{\mathbf{CSub}\,\mathbf{A} : \mathbf{A} \in \mathcal{V}\}).$$

Evidently, $\mathbf{CSub}\,\mathcal{V} \subseteq \mathbf{Cm}\,\mathcal{V}$, because $\mathbf{CSub}\,\mathbf{A}$ is a subalgebra of $\mathbf{Cm}\,\mathbf{A}$. Also $\mathcal{V} \subseteq \mathbf{Cm}\,\mathcal{V}$, because every algebra $\mathbf{A}$ can be embedded into $\mathbf{Cm}\,\mathbf{A}$ by $x \mapsto \{x\}$. And if $\mathcal{V}$ is idempotent, then $\mathcal{V} \subseteq \mathbf{CSub}\,\mathcal{V}$, by the same embedding. On the other hand, we do not have $\mathcal{V} \subseteq \mathbf{CSub}\,\mathcal{V}$ in general, for instance, for the variety of Abelian groups ($\mathbf{CSub}\,\mathcal{V}$ is defined due to Proposition 4.7), because in this case $\mathbf{CSub}\,\mathcal{V}$ is idempotent, while $\mathcal{V}$ is not.

In [6], G. Grätzer and H. Lakser proved the following theorem.

Theorem 5.1. *Let $\mathcal{V}$ be a variety. Then $\mathbf{Cm}\,\mathcal{V}$ satisfies precisely those identities resulting through identification of variables from the linear identities true in $\mathcal{V}$.*

Corollary 5.2. *Let $\mathcal{V}$ be a variety. Then $\mathcal{V} = \mathbf{Cm}\,\mathcal{V}$, if and only if $\mathcal{V}$ has a base consisting of linear identities.*

We investigate the question raised in [12]: *What are the identities satisfied by* $\mathbf{CSub}\,\mathcal{V}$ (provided it is defined)? In particular, *when $\mathcal{V} = \mathbf{CSub}\,\mathcal{V}$?*

It follows from Theorem 5.1 that $\mathbf{CSub}\,\mathcal{V}$ satisfies the linear identities valid in $\mathcal{V}$. If $\mathcal{V}$ is idempotent, $\mathbf{CSub}\,\mathcal{V}$ is also idempotent, and the idempotency is not linear. Moreover $\mathbf{CSub}\,\mathcal{V}$ can still be idempotent, while $\mathcal{V}$ is not—recall the example with

Abelian groups. We are going to prove an analogue of Theorem 5.1, characterizing the identities satisfied by $\mathbf{CSub}\,\mathcal{V}$. First, we have to introduce the notion of a *semilinear precursor*.

An identity $t \approx s$ is called *semilinear*, if at least one of the terms t, s is linear. The *linearization* of a term $t(x_1, \ldots, x_n)$ is the term t^*, resulting from t by replacement of the j-th occurence of a variable x_i by the variable x_{ij}, for all $1 \le i \le n$ and $1 \le j \le k_i$, where k_i is the number of occurences of the variable x_i in t.

Let t, s be terms and let k_i, l_i denote the number of occurences of the variable x_i in t, s. If x_i does not occur in the term t, we redefine $k_i = 1$.

The identity $t^* \approx \tilde{s}$ is called a *semilinear precursor* for the (ordered) pair (t, s), if there are terms $r_{ij}(x_{i1}, \ldots, x_{ik_i})$, $1 \le i \le n$, $1 \le j \le l_i$ such that

$$\tilde{s} = s^*(r_{11}(\overline{x_1}), \ldots, r_{1l_1}(\overline{x_1}), \ldots, r_{n1}(\overline{x_n}), \ldots, r_{nl_n}(\overline{x_n}))$$

(where $\overline{x_i}$ denotes the tuple $(x_{i1}, \ldots, x_{ik_i})$). For example, the semilinear precursors for the pair $(xy \cdot xz, yz \cdot x)$ are precisely the identities of the form $x_1 y \cdot x_2 z \approx p(y)q(z) \cdot r(x_1, x_2)$, where p, q are unary terms and r is a binary term. The semilinear precursors for the pair $(yz \cdot x, xy \cdot xz)$ are precisely the identities of the form $yz \cdot x \approx p_1(x)q(y) \cdot p_2(x)r(z)$, where p_1, p_2, q, r are unary terms. (In both examples, instead of double indices we used different letters for variables and terms.)

Indeed, the identity $t \approx s$ results from any of its semilinear precursors through identification of the variables $x_{i1}, \ldots, x_{ik_i}$ and replacement of the unary subterms $r_{ij}(x_i, \ldots, x_i)$ by a single variable. In particular, the identity $t \approx s$ is a consequence of each semilinear precursor for the pair (t, s) and idempotency.

Theorem 5.3. *Let $\mathcal{V}$ be a variety satisfying the generalized entropic property. Then $\mathbf{CSub}\,\mathcal{V}$ satisfies the identity $t \approx s$, if and only if there are semilinear precursors for the pair (t, s) and for the pair (s, t), both satisfied in $\mathcal{V}$.*

Proof. First, assume that $t(x_1, \ldots, x_n) \approx s(x_1, \ldots, x_n)$ holds in $\mathbf{CSub}\,\mathcal{V}$ and denote k_i, l_i the number of occurences of the variable x_i in t, s. Again, if x_i does not occur in the term t, we redefine $k_i = 1$.

Let $\mathbf{A}_i$, $i = 1, \ldots, n$, be the subalgebra generated by the set $\{x_{i1}, \ldots, x_{ik_i}\}$ in $\mathbf{F}_{\mathcal{V}}(X)$, the free algebra in $\mathcal{V}$ over the set $X = \{x_{ij} : 1 \le i \le n, 1 \le j \le k_i\}$. Since $t(A_1, \ldots, A_n) = s(A_1, \ldots, A_n)$, we have

$$t^*(x_{11}, \ldots, x_{1k_1}, \ldots, x_{n1}, \ldots, x_{nk_n}) \in s(A_1, \ldots, A_n).$$

It means that there are terms $r_{ij}(x_{i1}, \ldots, x_{ik_i}) \in A_i$, $1 \le i \le n$, $1 \le j \le l_i$ such that

$$t^*(x_{11}, \ldots, x_{1k_1}, \ldots, x_{n1}, \ldots, x_{nk_n}) \approx$$
$$s^*(r_{11}(\overline{x_1}), \ldots, r_{1l_1}(\overline{x_1}), \ldots, r_{n1}(\overline{x_n}), \ldots, r_{nl_n}(\overline{x_n})).$$

In other words, the above identity is a semilinear precursor for the pair (t, s) and it is satisfied in $\mathcal{V}$, because the identity holds in a free algebra. To get a semilinear precursor for the pair (s, t), consider the same procedure with the role of t, s interchanged.

We prove the converse. Let $t(x_1, \ldots, x_n), s(x_1, \ldots, x_n)$ be terms and assume there are semilinear precursors $t^* \approx \tilde{s}$ and $s^* \approx \tilde{t}$ satisfied in $\mathcal{V}$. Let $\mathbf{A} \in \mathcal{V}$ and take arbitrary subalgebras $\mathbf{A}_1, \ldots, \mathbf{A}_n$ of $\mathbf{A}$. To prove the inclusion $t(A_1, \ldots, A_n) \subseteq s(A_1, \ldots, A_n)$, let $a \in t(A_1, \ldots, A_n)$. It means that there are $a_{i1}, \ldots, a_{ik_i} \in A_i$

$(i = 1, \dots, n)$ such that
$$a = t^*(a_{11}, \dots, a_{nk_n}).$$
The algebra $\mathbf{A}$ satisfies $t^* \approx \tilde{s}$, so
$$a = s^*(r_{11}(\overline{a_1}), \dots, r_{1l_1}(\overline{a_1}), \dots, r_{n1}(\overline{a_n}), \dots, r_{nl_n}(\overline{a_n})).$$
Since $r_{ij}(a_{i1}, \dots, a_{ik_i}) \in A_i$ for every i, j, we see that
$$a \in s^*(A_1, \dots, A_1, \dots, A_n \dots, A_n) = s(A_1, \dots, A_n).$$
The other inclusion $s(A_1, \dots, A_m) \subseteq t(A_1, \dots, A_m)$ follows similarly from the identity $s^* \approx \tilde{t}$. Hence $t \approx s$ holds in $\mathbf{CSub}\,\mathcal{V}$. □

Corollary 5.4. *Let $\mathcal{V}$ be a variety satisfying the generalized entropic property. Then $\mathbf{CSub}\,\mathcal{V} \subseteq \mathcal{V}$, if and only if for every identity $t \approx s$ valid in $\mathcal{V}$ there is a semilinear precursor for the pair (t, s) valid in $\mathcal{V}$.*

Corollary 5.5. *Let $\mathcal{V}$ be an idempotent variety satisfying the generalized entropic property. Then $\mathcal{V} = \mathbf{CSub}\,\mathcal{V}$, if and only if for every identity $t \approx s$ valid in $\mathcal{V}$ there is a semilinear precursor for the pair (t, s) valid in $\mathcal{V}$.*

Corollary 5.6. *Let $\mathcal{V}$ be an idempotent variety satisfying the generalized entropic property and assume that $t(x_1, \dots, x_n)$ is a linear term and $s(x_1, \dots, x_n, y_1, \dots, y_m)$ is a term such that the variables $x_1, \dots, x_n$ occur in it at most once. Then $\mathbf{CSub}\,\mathcal{V}$ satisfies the identity $t \approx s$, if and only if $\mathcal{V}$ satisfies the linear identity $t \approx s^*$.*

It follows from Theorem 5.1 that $\mathbf{CSub}\,\mathcal{V}$ satisfies all linear identities true in $\mathcal{V}$. This is in accordance with Theorem 5.3. For every pair (t, s) of linear terms there is a semilinear precursor $t \approx s$ (indeed, $t^* = t$ and $s^* = s$), so if $t \approx s$ holds in $\mathcal{V}$, it is satisfied in $\mathbf{CSub}\,\mathcal{V}$ too.

The following couple of examples illustrates applications of Theorem 5.3.

Example 5.7. *If $\mathcal{V}$ is a variety of Abelian groups, then $\mathbf{CSub}\,\mathcal{V}$ is idempotent, i.e., satisfies the identity $x + x \approx x$.*

First, we find a semilinear precursor for the pair $(x, x+x)$: for $s(x) = x + x$ we have $s^*(x, y) = x + y$ and we can put $\tilde{s}(x) = s^*(x, 0)$ (0 is a constant term in any variables); indeed, $x \approx x + 0$ holds in $\mathcal{V}$. Next, we find a semilinear precursor for the pair $(x + x, x)$: for $t(x) = x + x$ we have $t^*(x, y) = x + y$, so we can substitute in $s(x) = s^*(x) = x$ the term $x + y$ for the variable x; indeed, $x + y \approx x + y$ holds in $\mathcal{V}$.

Example 5.8. *If $\mathcal{V}$ is a variety of idempotent entropic groupoids satisfying $x(xy) \approx y$, then this identity can not hold in $\mathbf{CSub}\,\mathcal{V}$.*

Assume the contrary. Put $t(x, y) = x(xy)$, $s(y) = y$ and assume that there is a semilinear precursor $t^* \approx \tilde{s}$ true in $\mathcal{V}$. It means, there is a unary term u such that the identity $x_1(x_2y) = u(y)$ holds in $\mathcal{V}$. Because of idempotency, we can assume $u(y) = y$. It is easy to find a groupoid in $\mathcal{V}$ which fails the property:

$\cdot$	0	1	2
0	0	1	2
1	2	1	0
2	0	1	2

Unfortunately, Theorem 5.3 does not help us to decide, whether the following conjecture from [12] is true.

Conjecture 5.9. *Let $\mathcal{V}$ be an idempotent variety satisfying the generalized entropic property. Then $\mathcal{V} = \mathbf{CSub}\,\mathcal{V}$, if and only if $\mathcal{V}$ has a base consisting of linear identities and the identities $f(x, \ldots, x) \approx x$, for all basic operations f.*

Note that the backward implication is true for any idempotent variety.

All known idempotent varieties with $\mathcal{V} = \mathbf{CSub}\,\mathcal{V}$ have a linear and idempotent base. For instance, the variety of all modes of a given type, the variety of commutative binary modes, the variety of differential groupoids (groupoid modes satisfying $x(yz) \approx xy$), the variety of normal semigroups (semigroup modes) and any subvariety of this variety (in particular, varieties of semilattices), left ($xy \approx x$) and right ($xy \approx y$) zero bands, rectangular bands ($xyz \approx xz$) and left ($xyz \approx xzy$) and right ($zyx \approx yzx$) normal bands or the variety of barycentric algebras [18].

We also note that at the moment we do not know any example of a non-entropic idempotent variety $\mathcal{V}$ with $\mathcal{V} = \mathbf{CSub}\,\mathcal{V}$. Indeed, the only examples (known to us) of non-entropic idempotent algebras with the generalized entropic property were shown in Example 3.1. For instance, it is straightforward to check that for $\mathbb{F} = \mathrm{GF}(3)$, $\underline{\mathbf{M}}_X$ satisfies the identity $\underline{r}(x, \underline{r}(y, x)) \approx y$, where $\underline{r}$ is any basic operation from $\underline{X}$, but this identity fails in $\mathbf{CSub}\,\underline{\mathbf{M}}_X$.

Example 5.10. *The Conjecture 5.9 is false if the assumption of idempotency is dropped.*

Consider the variety $\mathcal{V}$ of entropic groupoids with $(xx)y \approx xy$ and $y(xx) \approx yx$. Clearly, $\mathcal{V}$ satisfies the generalized entropic property. It follows from Theorem 5.1 that $\mathbf{CSub}\,\mathcal{V}$ is entropic, and it is easy to check that $\mathbf{CSub}\,\mathcal{V}$ satisfies the two identities. Hence, $\mathbf{CSub}\,\mathcal{V} \subseteq \mathcal{V}$. For any algebra $\mathbf{A} \in \mathcal{V}$, we can embed $\mathbf{A}$ into $\mathbf{CSub}\,\mathbf{A}$ by $x \mapsto \{x, xx\}$ (a straightforward caculation). Therefore, $\mathcal{V} = \mathbf{CSub}\,\mathcal{V}$. We prove that $\mathcal{V}$ cannot be based by linear identities.

All identities of $\mathcal{V}$ are regular, i.e., have the same variables on both sides, because the basis of $\mathcal{V}$ consists of regular identities. Evidently, regular linear identities are *balanced*, which means that the number of each variable symbol, counting repetitions, is the same on both sides. It is easy to see that consequences of balanced indentities are balanced. Since the identities $(xx)y \approx xy$ and $y(xx) \approx yx$ are not balanced, they cannot be deduced from any set of linear identities of $\mathcal{V}$.

6. Stronger conjecture fails

In this section we are interested in varieties that do not necessarily possess the generalized entropic property. Our aim is to disprove an analogue of Conjecture 5.9: There is a variety generated by an idempotent algebra $\mathbf{A}$ such that $\mathbf{CSub}\,\mathbf{A}$ exists, $\mathrm{V}(\mathbf{CSub}\,\mathbf{A}) = \mathrm{V}(\mathbf{A})$ and $\mathrm{V}(\mathbf{A})$ has no base of linear and idempotent identities. The rest of the section is fully devoted to such example.

Consider, again, the groupoid $\mathbf{G}_1$ from Example 2.3.

$\cdot$	a	b	c
a	a	c	c
b	c	b	c
c	a	b	c

We already noticed that $\mathbf{CSub}\,\mathbf{G}_1$ exists, though $\mathbf{G}_1$ does not satisfy the generalized entropic property. We show that the groupoids $\mathbf{G}_1$ and $\mathbf{CSub}\,\mathbf{G}_1$ generate the same variety (Lemma 6.5), but $\mathrm{V}(\mathbf{G}_1)$ has no base of linear and idempotent identities. In fact, we prove that all linear identities satisfied by $\mathbf{G}_1$ are regular (Lemma 6.3) and thus the non-regular identities

$$(xy)x \approx x \qquad \text{and} \qquad (yx)x \approx x,$$

valid in $\mathbf{G}_1$, are not consequences of idempotent and linear identities of $\mathbf{G}_1$.

Every term t can be written in the form

$$t = t_1(t_2(\dots t_{k-1}(t_k x)\dots)),$$

where $t_1, \dots, t_k$ are terms and x is a variable. The variable x will be called the *focal of* t and denoted by $fc(t)$.

Lemma 6.1. *If $\mathbf{G}_1$ satisfies an identity $t \approx u$, then $fc(t) = fc(u)$.*

Proof. Assume $fc(t) \neq fc(u)$. Assign the element c to $fc(t)$ and the element a to all other variables of t and u. Then the value of t is c and the value of u is a, because a, c are right zeros in the subgroupoid $\{a, c\}$. Hence $t \not\approx u$ in $\mathbf{G}_1$. □

Lemma 6.2. *If $\mathbf{G}_1$ satisfies a linear identity $t \approx u$, $t = t_1(t_2(\dots t_{k-1}(t_k x)\dots))$ and $u = u_1(u_2(\dots u_{m-1}(u_m x)\dots))$, then*

1. *$\{fc(t_i) : i \leq k\} = \{fc(u_j) : j \leq m\}$. In particular, $m = k$.*
2. *For every $i \leq k$ there exists $j \leq k$ such that $t_i = u_j$ is a linear identity of $\mathbf{G}_1$.*

Proof. To prove (1), we can assume that $fc(t) = fc(u) = x$ and there exists $y = fc(t_i)$ that does not belong to $\{fc(u_j) : j \leq m\}$. Then we assign $x = a$, $y = b$ and the rest of variables will be c. It will follow that all variables in $\{fc(u_j) : j \leq k\}$ will be assigned c, hence all u_j are equal to c and $u = a$. On the other hand, $t_i = b$, while the rest of t_p, $p \neq i$, are c. Hence $t = c$ and $t \neq u$ under such assignment of variables.

To show (2), for any t_i we pick u_j with the same focal y. Suppose that $t_i \neq u_j$ for some assignment of variables. Then y is assigned to a or b.

If $y = a$, then $\{t_i, u_j\} = \{a, c\}$ under such assignment. Say, $t_i = a$ and $u_j = c$. Let $fc(t) = fc(u)$ be assigned to b and all $fc(t_p)$, $p \neq i$, and $fc(u_q)$, $q \neq j$, to c. Under such assignment we get that $t = c$ and $u = b$, a contradiction with $t = u$ in $\mathbf{G}_1$. The case of $y = b$ is shown similarly by interchanging a and b. □

Lemma 6.3. *Every linear identity of $\mathbf{G}_1$ is regular.*

Proof. Let $r(t, u)$ be the number of distinct variables in the identity $t \approx u$ (e.g., $r(xy, (xz)y) = 3$). We argue by induction on $r(t, u)$. If $r(t, u) = 1$, then $t = u = x$ and the statement is true.

Suppose we know that every linear identity $t' \approx u'$ with $r(t', u') \leq n$ is regular and consider a linear identity $t \approx u$ with $r(t, u) = n+1$. Then, according to Lemma 6.2, $t = t_1(t_2(\dots t_{k-1}(t_k x)\dots))$ and $u = u_1(u_2(\dots u_{k-1}(u_k x)\dots))$, for some k and some terms t_i, u_i such that for every $i \leq k$ there exists $j \leq k$ with $t_i \approx u_j$ satisfied in $\mathbf{G}_1$. This is indeed a linear identity and $r(t_i, u_j) \leq n$, because x does not occur in $t_i \approx u_j$. By induction hypothesis, $t_i \approx u_j$ is regular. Hence the set of variables occuring in t is a subset of the set of variables occuring in u. Similarly, applying Lemma 6.2 on the identity $u \approx t$, we obtain that the latter set is a subset of the former one. Consequently, the identity $t \approx u$ is regular. □

As a byproduct we also get a description of linear identities satisfied in $\mathbf{G}_1$. For this, we define *focally equivalent* terms $t \equiv_f u$, recursively by the length of t, u:

(1) If one of t, u has only one variable x then $t \equiv_f u$ if and only if $t = u = x$.
(2) If both t, u have more than one variable and $t = t_1(t_2(\dots t_{k-1}(t_k x)\dots))$, $u = u_1(u_2(\dots u_{l-1}(u_l y)\dots))$, then $t \equiv_f u$ if and only if $k = l$, $x = y$, for every $i \le k$ there is $j \le k$ such that $t_i \equiv_f u_j$ and for every $i \le k$ there is $j \le k$ such that $u_i \equiv_f t_j$.

Corollary 6.4. $\mathbf{G}_1$ *satisfies a linear identity* $t \approx u$*, iff the terms* t, u *are focally equivalent.*

Proof. Apply induction and Lemmas 6.1 and 6.2. □

Lemma 6.5. $\mathbf{G}_1$ *and* $\mathbf{CSub}\,\mathbf{G}_1$ *generate the same variety.*

Proof. Since an idempotent algebra always embeds into its algebra of subalgebras (provided it exists), it is sufficient to find an embedding of the groupoid $\mathbf{CSub}\,\mathbf{G}_1$ into the product $\mathbf{G}_1 \times \mathbf{G}_1$. We notice that there are two homomorphisms from $\mathbf{CSub}\,\mathbf{G}_1$ onto $\mathbf{G}_1$:

$$\begin{aligned} f_1(\{a\}) = f_1(\{a,c\}) = f_1(\{a,b,c\}) &= a, \\ f_1(\{b\}) &= b, \\ f_1(\{c\}) = f_1(\{b,c\}) &= c \end{aligned}$$

and

$$\begin{aligned} f_2(\{b\}) = f_2(\{b,c\}) = f_2(\{a,b,c\}) &= b, \\ f_2(\{a\}) &= a, \\ f_2(\{c\}) = f_2(\{a,c\}) &= c. \end{aligned}$$

It is easy to check that $ker(f_1) \cap ker(f_2) = 0$, hence $\mathbf{CSub}\,\mathbf{G}_1$ is a subdirect power of $\mathbf{G}_1$. □

Acknowledgments. The paper was initiated during the first author's visit to Warsaw University of Technology in summer of 2003. The support and welcoming atmosphere created by Prof. A.Romanowska and the group of her collaborators are greatly appreciated. The work on this project was further inspired by the INTAS workshop at Charles University in Prague in summer of 2004 and the visits of the last and first authors to Warsaw in 2005.

Several times we helped us with the automated theorem prover Otter [10]. A number of proofs in Section 3 are translated and simplified versions of Otter's ones.

We want to thank Peter Jones for his interest in this work, in particular providing us with an alternative direct proof of Proposition 3.11. Many helpful suggestions were also communicated to us by Jonathan Smith, Michał Stronkowski, Marina Semenova and George Grätzer.

References

[1] I. Bošnjak, R. Madarász, *On power structures*, Algebra and Discr. Math. 2 (2003), 14–35.
[2] C. Brink, *Power structures*, Algebra Univers. 30 (1993), 177–216.
[3] P. Đapić, J. Ježek, P. Marković, R. McKenzie, D. Stanovský, *∗-linear equational theories of groupoids*, Algebra Universalis 56/3-4 (2007), 357-397.
[4] J. Dudek, *Small idempotent clones I.*, Czech. Math. J. 48/1 (1998), 105–118.

[5] T. Evans, *Properties of algebras almost equivalent to identities*, J. London Math. Soc. 35(1962), 53–59.
[6] G. Grätzer, H. Lakser, *Identities for globals (complex algebras) of algebras*, Colloq. Math. 56 (1988), 19–29.
[7] G. Grätzer, S. Whitney, *Infinitary varieties of structures closed under the formation of complex structures*, Colloq. Math. 48 (1984), 1–5.
[8] J. Ježek, T. Kepka, *Medial groupoids,* Rozpravy ČSAV 93/2 (1983).
[9] J. Ježek, R. McKenzie, *The variety generated by equivalence algebras*, Algebra Univers. 45 (2001), 211–219.
[10] W.W. McCune, *Otter: An Automated Deduction System.* Available at `http://www-unix.mcs.anl.gov/AR/otter/`
[11] S. Oates-Williams, *Graphs and universal algebras*, Lecture Notes Math. 884(1981), 351–354.
[12] A. Pilitowska, *Modes of submodes*, PhD thesis, Warsaw University of Technology, 1996.
[13] A. Pilitowska, *Identities for classes of algebras closed under the complex structures*, Discuss. Math. Algebra and Stochastic Methods 18 (1998), 85–109.
[14] A. Pilitowska, *Enrichments of affine spaces and algebras of subalgebras*, Discuss. Math. Algebra and Stochastic Methods 19 (1999), 207–225.
[15] A. Romanowska, J.D.H. Smith, *Modal Theory—an Algebraic Approach to Order, Geometry, and Convexity*, Heldermann Verlag, Berlin, 1985.
[16] A. Romanowska, J.D.H. Smith, *Subalgebra systems of idempotent entropic algebras*, J. Algebra 12(1989), 247–262.
[17] A. Romanowska, J.D.H. Smith, *On the structure of the subalgebra systems of idempotent entropic algebras*, J. Algebra 12(1989), 263–283.
[18] A. Romanowska, J.D.H. Smith, *Modes*. World Scientific, 2002.
[19] A. Shafaat, *On varieties closed under the construction of power algebras*, Bull. Austral. Math. Soc. 11 (1974), 213–218.

Harold Washington College, 30 East Lake St., Chicago, IL 60601, USA
E-mail address: `kadaricheva@ccc.edu`

Warsaw University of Technology, Plac Politechniki 1, 00-661 Warsaw, Poland
E-mail address: `apili@mini.pw.edu.pl`

Charles University, Sokolovská 83, 186 00 Prague, Czech Republic
E-mail address: `stanovsk@karlin.mff.cuni.cz`