ENRICHMENTS OF AFFINE SPACES AND ALGEBRAS OF SUBALGEBRAS

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Abstract. The paper continues the study initiated in [5]. We describe algebras of subalgebras of certain reducts of modules over an arbitrary commutative ring with unity having affine spaces as their reducts.
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Given an algebra (A, Ω) of type $\tau : \Omega \to N$ one may form the set $(A, \Omega)S$ or AS of non-empty subalgebras of (A, Ω). This set AS may carry an Ω-algebra structure under the ω-complex products

$$\omega : AS^{\omega\tau} \to AS; \qquad A_1 \ldots A_{\omega\tau}\omega = \{a_1 \ldots a_{\omega\tau}\omega \mid a_i \in A_i\},$$

for each ω in Ω. But in general, the family AS does not need to be closed under complex operations. However if it is, the algebra (AS, Ω) is called the *complex algebra of subalgebras* of the algebra (A, Ω), or briefly *the algebra of subalgebras*.

An important class of algebras for which the set AS is closed under complex operations is given by modes - idempotent and entropic algebras. The two properties of idempotency and entropicity of a mode (A, Ω) may be expressed algebraically by means of identities:

$$x \ldots x\omega = x,$$

$$x_{11} \ldots x_{1n}\omega \ldots x_{m1} \ldots x_{mn}\omega\omega' = x_{11} \ldots x_{m1}\omega' \ldots x_{1n} \ldots x_{mn}\omega'\omega,$$

that are satisfied in (A, Ω) for any n-ary operation ω and m-ary operation ω' in Ω. Such algebras are studied in detail in [7].

One of the more important examples of modes is given by affine spaces over ring R. Let R be a commutative ring with unity and let $(E, +, R)$ be a module over R. For each element r of R, define a binary operation $\underline{r}$ by

$$\underline{r} : E \times E \to E; \quad (x, y) \mapsto (1 - r)x + ry,$$

and the Mal'cev operation P by

$$P : E \times E \times E \to E; \quad (x, y, z) \mapsto x - y + z.$$

Then the algebra $(E, \underline{R}, P)$ with the ternary operation P and the set $\underline{R} = \{\underline{r} \mid r \in R\}$ of binary operations has as its derived operations (those obtained from successive compositions of the basic operations P and $\underline{r}$ for r in R) precisely the affine combinations $r_1x_1+r_2x_2+\ldots+r_nx_n$ with $r_1+r_2+\ldots+r_n = 1$ of elements $x_1, x_2, \ldots, x_n$ of E. The set E together with all the idempotent term operations (considered as fundamental) is called the *full idempotent reduct* of the R-module $(E, +, R)$ ([12]). It follows that the algebra $(E, \underline{R}, P)$ is equivalent to the full idempotent reduct $(E, \{r_1x_1 + r_2x_2 + \ldots + r_nx_n \mid r_1, r_2, \ldots, r_n \in R, \sum_{i=1}^{n} r_i = 1\})$ of the module $(E, +, R)$. Note that the algebra $(E, \underline{R}, P)$ has the affine group as its group of automorphisms, and may thus be identified with the affine geometry ([2]). Carrying out this identification, we will refer to the algebra $(E, \underline{R}, P)$ as an *affine space over* R or an *affine* R*-space.* (Note, however, that such algebras have also been called affine modules). It is well known that the class of affine spaces over a commutative ring R with unity forms a variety $\underline{\underline{R}}$.

The entropic law plays a special rôle in complex algebras of subalgebras. For an entropic algebra (A, Ω), the set of non-empty subalgebras of (A, Ω) carries an Ω-algebra structure under the complex products. In particular, if (A, Ω) is an affine space over a commutative ring R with unity, then (AS, Ω) is again a mode satisfying all the linear identities true in (A, Ω). In general the algebra (AS, Ω) is not an affine space. In [5] modes of subspaces of affine spaces over an arbitrary commutative ring with unity were investigated. It was shown there that certain reducts of such modes are Ponka sums of affine spaces over corresponding projective spaces. The present paper continues the study set out in [5]. We generalise results of [5] to describe algebras of subalgebras of certain reducts of modules over an arbitrary commutative ring with unity having affine spaces as their reducts.

Such reducts of modules are defined in Section 1. The basic operations are linear combinations $r_1x_1 + \ldots + r_nx_n$ with $r_1 + \ldots + r_n - 1$ in a fixed subset T_R of R containing 0. (This generalises an idea of S. Givant [1].) Since such algebras have as reducts affine R-spaces, we call them T_R*-enrichments of affine* R*-spaces.* We show that they are entropic and Mal'cev algebras. However they are not idempotent. $\{0\}$-enrichments are precisely affine R-spaces. In this case the subalgebras of an affine space $(E, \underline{R}, P)$ are exactly the cosets of submodules of the module $(E, +, R)$. We obtain a similar characterisation of subalgebras of T_R-enrichments of affine R-spaces in Proposition 1.4. This result allows us to describe algebras of subalgebras of T_R-enrichments of affine R-spaces using the concept of Ponka sum. In Section 2 we show that certain reducts of such algebras may be constructed as Ponka sums of reducts of T_R-enrichments of affine R-spaces over corresponding projective spaces. In Section 3 we use the construction of a generalized coherent Lallement sum ([11]) to describe a wider class of reducts of algebras of subalgebras of T_R-enrichments of affine R-spaces.

The notation and terminology of the paper is basically as in the book [7]. We refer the reader to the book for all undefined notions and results. We use

"Polish" notation for words (terms) and operations, e.g. instead of $w(x_1,\ldots,x_n)$ we write $x_1\ldots x_n w$. Moreover, the symbol $x_1\ldots x_n w$ means that $x_1,\ldots,x_n$ are exactly the variables appearing in the word w. Algebras and varieties are equivalent if they have the same derived operations.

1. Enrichments of affine spaces

Let R be a commutative ring with unity and let $(E,+,R)$ be a module over R. Moreover let T_R be a subset of R which includes 0.
For $r_1,\ldots,r_n$ in R, we define an n-ary operation:

$$(r_1,r_2,\ldots,r_n): E^n \to E;$$

$$(x_1,x_2,\ldots,x_n)\mapsto x_1x_2\ldots x_n(r_1,r_2,\ldots,r_n) := r_1x_1+r_2x_2+\ldots+r_nx_n = \sum_{i=1}^{n} r_i x_i.$$

Let $\Re(T_R)$ denote the set $\{(r_1,r_2,\ldots,r_n) \mid r_1,r_2,\ldots,r_n\in R, \sum_{i=1}^{n} r_i - 1 \in T_R\}$. Then $(E,\Re(T_R))$ is an algebra, a reduct of the module $(E,+,R)$. We will refer to the algebra $(E,\Re(T_R))$ as the T_R-*enrichment of the affine* R-*space*.

Example 1.1. For each r in R, $1-r+r-1=0$ is in T_R, hence $(1-r,r)$ and $(1,-r,r)$ are in $\Re(T_R)$, where

$$(1-r,r): E\times E\to E;\quad (x_1,x_2)\mapsto x_1x_2(1-r,r) = (1-r)x_1 + rx_2 = x_1x_2\underline{r},$$

$$(1,-r,r): E\times E\times E\to E;\quad (x_1,x_2,x_3)\mapsto x_1x_2x_3(1,-r,r) = x_1 - rx_2 + rx_3.$$

In particular

$$x_1x_2x_3(1,-1,1) = x_1 - x_2 + x_3 = x_1x_2x_3P$$

is the Mal'cev operation. It follows that the basic operations of the affine space $(E,\underline{R},P)$ are contained in $\Re(T_R)$. □

Example 1.2. The full idempotent reduct

$$(E,\{r_1x_1+r_2x_2+\ldots+r_nx_n \mid r_1,r_2,\ldots,r_n\in R, \sum_{i=1}^{n} r_i = 1\})$$

of the module $(E,+,R)$ is equivalent to the affine R-space $(E,\underline{R},P)$. It follows that for $T_R=\{0\}$, the $\{0\}$-enrichment $(E,\Re(\{0\}))$ of the affine R-space $(E,\underline{R},P)$ is equivalent to the affine space $(E,\underline{R},P)$. Obviously, for an arbitrary T_R, the set $\Re(T_R)$ contains the full idempotent reduct of $(E,+,R)$. □

Note that modules over commutative rings are entropic, hence any two operations $(r_1,\ldots,r_n),(s_1,\ldots,s_m)$, not necessarily in $\Re(T_R)$, are entropic. In

particular $(E, \Re(T_R))$ is an entropic and Mal'cev algebra.
Note however that for $(r_1, r_2, \ldots, r_n)$ in $\Re(T_R)$

$$xx\ldots x(r_1, r_2, \ldots, r_n) = r_1x + r_2x + \ldots + r_nx = (r_1 + r_2 + \ldots + r_n)x = \sum_{i=1}^{n} r_i \cdot x.$$

Since $\sum_{i=1}^{n} r_i - 1$ is in T_R, and in general $T_R \neq \{0\}$, $\sum_{i=1}^{n} r_i \cdot x$ is not necessarily equal to x. This implies that $(E, \Re(T_R))$ is not always an idempotent algebra. We have the following lemma.

Lemma 1.3. Let $(E, +, R)$ be a faithful module over R. A T_R-enrichment of the affine R-space $(E, \underline{R}, P)$ is idempotent iff $T_R = \{0\}$.

Proof. Suppose that a T_R-enrichment of the affine R-space $(E, \underline{R}, P)$ is idempotent. This means that for each $(r_1, r_2, \ldots, r_n)$ in $\Re(T_R)$

$$xx\ldots x(r_1, r_2, \ldots, r_n) = r_1x + r_2x + \ldots + r_nx = \sum_{i=1}^{n} r_i \cdot x = x.$$

The faithfulness of E yields $\sum_{i=1}^{n} r_i = 1$ and $\sum_{i=1}^{n} r_i - 1 = 0$ for each $(r_1, r_2, \ldots, r_n)$ in $\Re(T_R)$. Thus $T_R = \{0\}$. Obviously the affine R-space $(E, \underline{R}, P)$ is an idempotent algebra. □

Proposition 1.4. Let U be a submodule of the module $(E, +, R)$ and let a be in E. If $T_Ra = \{ra \mid r \in T_R\} \subseteq U$, then the coset

$$a + U = \{a + u \mid u \in U\}$$

is a subalgebra of $(E, \Re(T_R))$. For each subalgebra $(A, \Re(T_R))$ of $(E, \Re(T_R))$ there is an element a in E and a uniquely defined submodule U of $(E, +, R)$ such that $A = a + U$ and $T_Ra \subseteq U$.

Proof. Let U be a submodule of $(E, +, R)$, let a be in E and $T_Ra \subseteq U$. For $x_1 = a + u_1, x_2 = a + u_2, \ldots, x_n = a + u_n$ in $a + U$, with $u_1, u_2, \ldots, u_n$ in U and for an n-ary operation $(r_1, r_2, \ldots, r_n)$ in $\Re(T_R)$

$$x_1x_2\ldots x_n(r_1,r_2,\ldots,r_n) = (a+u_1)(a+u_2)\ldots(a+u_n)(r_1,r_2,\ldots,r_n)$$

$$= r_1(a+u_1) + r_2(a+u_2) + \ldots + r_n(a+u_n)$$

$$= r_1a + r_2a + \ldots + r_na + r_1u_1 + r_2u_2 + \ldots + r_nu_n$$

$$= (r_1 + r_2 + \ldots + r_n)a + r_1u_1 + r_2u_2 + \ldots + r_nu_n$$

$$= a + (\sum_{i=1}^{n} r_i)a - a + \sum_{i=1}^{n} r_iu_i =$$

$$= a + (\sum_{i=1}^{n} r_i - 1)a + \sum_{i=1}^{n} r_iu_i.$$

By assumption, $\sum_{i=1}^{n} r_i - 1$ is in T_R and $T_Ra \subseteq U$. Hence $(\sum_{i=1}^{n} r_i - 1)a$ is in U and $x_1x_2\ldots x_n(r_1,r_2,\ldots,r_n)$ is in $a+U$. Consequently $a+U$ is a subalgebra of $(E, \Re(T_R))$.

Let $(A, \Re(T_R))$ be a subalgebra of $(E, \Re(T_R))$ containing 0. We observe that for each r in R and a, b in A

$$a0(r, 1-r) = ra + (1-r)0 = ra,$$

and

$$a0b(1, -1, 1) = a - 0 + b = a + b$$

are in A. So A is a submodule of $(E, +, R)$.

Now let $(A, \Re(T_R))$ be an arbitrary subalgebra of $(E, \Re(T_R))$ and let a be in A. For any x in A we have $x = a + x - a$. Then

$$A = \{a + (x-a) \mid x \in A\} = a + \{x - a \mid x \in A\}.$$

Note that $0 = a-a$ and hence 0 is in $\{x-a \mid x \in A\}$. Now for an n-ary operation $(r_1, r_2, \ldots, r_n)$ in $\Re(T_R)$ and $x_1 - a, x_2 - a, \ldots, x_n - a$ in $\{x - a \mid x \in A\}$ we have:

$$(x_1 - a)(x_2 - a) \dots (x_n - a)(r_1, r_2, \dots, r_n)$$

$$= r_1(x_1 - a) + r_2(x_2 - a) + \dots + r_n(x_n - a)$$

$$= r_1x_1 + r_2x_2 + \dots + r_nx_n - (\sum_{i=1}^{n} r_i)a$$

$$= r_1x_1 + r_2x_2 + \dots + r_nx_n - (\sum_{i=1}^{n} r_i)a + a - a$$

$$= x_1x_2 \dots x_n aa(r_1, r_2, \dots, r_n, -\sum_{i=1}^{n} r_i, 1) - a.$$

Of course, $(r_1, r_2, \dots, r_n, -\sum_{i=1}^{n} r_i, 1)$ is in $\Re(T_R)$, and

$$x_1x_2 \dots x_n aa(r_1, r_2, \dots, r_n, -\sum_{i=1}^{n} r_i, 1)$$

is in A. This implies that $(x_1 - a)(x_2 - a) \dots (x_n - a)(r_1, r_2, \dots, r_n)$ is in $\{x - a \mid x \in A\}$ and that $\{x - a \mid x \in A\}$ is a subalgebra of $(E, \Re(T_R))$ containing 0. It follows that $\{x - a \mid x \in A\}$ is a submodule of $(E, +, R)$, and A is a coset as required.

Now let b be an element in A. Let us consider the following set

$$V = \{x - b \mid x \in A\}.$$

As was shown before, V is a submodule of $(E, +, R)$. Let y be in V. Then there is x in A such that $y = x - b = x - b + a - a$. Because $(A, \Re(T_R))$ is a subalgebra of $(E, \Re(T_R))$ then $z = x - b + a = xba(1, -1, 1)$ is in A. This implies that $y = z - a$ is in $\{x - a \mid x \in A\}$ and $V \subseteq \{x - a \mid x \in A\}$. Similarly we can show that $\{x - a \mid x \in A\} \subseteq V$, and consequently $\{x - b \mid x \in A\} = \{x - a \mid x \in A\}$ for any a and b in A. This shows that the submodule U of $(E, +, R)$ such that $A = a + U$, is defined uniquely.

Finally note that for each t in T_R, the operation $(t, 1)$ is in $\Re(T_R)$. Then for each a in A, $aa(t, 1) = ta + a$ is in A. Let $u = ta + a$. Then, $ta = u - a$ for u in A. It follows that $T_Ra \subseteq \{u - a \mid u \in A\}$ and this completes the proof. □

By Proposition 1.4. the cosets $a + U$ of submodules U of $(E, +, R)$ with $T_Ra \subseteq U$, are exactly the subalgebras of the algebra $(E, \Re(T_R))$. It is clear that for $T_R = \{0\}$ subalgebras of affine R-spaces are exactly the cosets of submodules of the module $(E, +, R)$.

Note that if T_Ra is not included in U, the coset $a + U$ of the submodule U of $(E, +, R)$ does not necessarily have to be a subalgebra of $(E, \Re(T_R))$. Consider the following example.

Example 1.5. Consider the Z-module $(Z, +, Z)$ and let $T_R\colon = Z$. Let U be the submodule $(2Z, +, Z)$ of $(Z, +, Z)$ and $a = 1$. Of course $Z \cdot 1$ is not included in $2Z$. Consider the coset

$$a + U = 1 + 2Z = \{1 + 2z \mid z \in Z\}$$

of the submodule $(2Z, +, Z)$. Note that the operation $(2, 2)$ is in $\Re(Z)$ and for any $x = 1 + 2z_1$ and $y = 1 + 2z_2$ in the coset $1 + 2Z$, the element $xy(2, 2) = 4z_1 + 4z_2 + 4$ is not in the set $\{1 + 2z \mid z \in Z\}$. Thus the coset $1 + 2Z$ is not a subalgebra of $(Z, \Re(Z))$. □

2. Algebras of subalgebras of T_R-enrichments of affine R-spaces

Let $(E, \underline{R}, P)$ be an affine space, $T_R \subseteq R$ and $(E, \Re(T_R))$ be the T_R -enrichment of $(E, \underline{R}, P)$. Consider the set ESA of non-empty subalgebras of $(E, \Re(T_R))$. The set ESA forms an algebra under the complex $(r_1, r_2, \ldots, r_n)$-products

$$(r_1, r_2, \ldots, r_n)\colon ESA^n \to ESA;$$

$$(A_1, A_2, \ldots, A_n) \mapsto \{a_1 a_2 \ldots a_n(r_1, r_2, \ldots, r_n) \mid a_i \in A\}.$$

By results of [7] and [3] if an algebra (A, Ω) is entropic, then (AS, Ω) is again entropic and satisfies each linear identity true in (A, Ω). It turns out that the algebra $(ESA, \Re(T_R))$ is an entropic algebra satisfying each linear identity satisfied by $(E, \Re(T_R))$.

To describe the structure of $(ESA, \Re(T_R))$ we will need the notion of Ponka sum of algebras ([6], [7]). Let us recall the definition. Let (Ω) denote the category of Ω-algebras and homomorphisms between them. Consider the semilattice $(H, +)$ as a small category (H) with a set H of objects and with unique morphism $h \to k$ precisely when $h + k = k$, i.e. $h \leq k$.
Let $F : (H) \to (\Omega)$ be a functor. Then the *Ponka sum* of the Ω-algebras (hF, Ω), for h in H, over the semilattice $(H, +)$ by the functor F, is the disjoint union $HF = \bigcup(hF \mid h \in H)$ of the underlying sets hF, equipped with the Ω-algebra structure, given for each n-ary operation ω in Ω and $h_1, h_2, \ldots, h_n, h = h_1 + h_2 + \ldots + h_n$ in H, by

$$\omega : h_1F \times \ldots \times h_nF \to hF;\ \ (x_1, \ldots, x_n) \mapsto x_1(h_1 \to h)F \ldots x_n(h_n \to h)F\omega.$$

The *canonical projection* of the Ponka sum HF is the homomorphism $\pi : (HF, \Omega) \to (H, \Omega)$ with restriction $\pi : hF \to \{h\}$. The subalgebras $(hF, \Omega) = (h\pi^{-1}, \Omega)$ of (HF, Ω) are the *Ponka fibres*.

Recall that for Ω-algebras in an idempotent irregular variety $\mathbf{V}$, the identities satisfied by their Ponka sums are precisely the regular identities satisfied in the fibres. On the other hand, the regularisation $\tilde{\mathbf{V}}$ of $\mathbf{V}$, the variety defined by all the regular identities true in $\mathbf{V}$, consists precisely of Ponka sums of $\mathbf{V}$-algebras (Cf. [6], [7]).

Let ESM be the set of submodules of the R-module $(E, +, R)$. The internal structure of certain reducts of $(ESA, \Re(T_R))$ in the case R is an arbitrary commutative ring with unity and $T_R = \{0\}$ was described in [5] using the concept of Ponka sums. In that paper, one considered a certain subset Ω_R of R. It was shown that the algebra (ESA, Ω_R) is a Ponka sum over the Ω_R-semilattice (ESM, Ω_R) of submodules of the R-module $(E, +, R)$ corresponding to the projective geometry. In particular, one obtained an invariant algebraic passage from affine to projective geometry. The corresponding Ponka fibres were shown to be quotient spaces of the affine R-space $(E, \underline{R}, P)$.

In the more general case considered in this paper, the rôle of Ω_R is taken by the subset $\underline{J}^0_R := \{(r_1, r_2, \ldots, r_n) \in \Re(T_R) \mid n \geq 1$ and for each $1 \leq i \leq n,\ r_i$ is a unit of $R\}$ of $\Re(T_R)$. The motivation for choosing the set $\underline{J}^0_R$ is to obtain an adequate description of the structure of $(ESA, \Re(T_R))$. Moreover in the case $T_R = \{0\}$ we want to have an invariant passage from affine to projective geometry as in [10]. We will prove that for each commutative ring R with unity, the algebra $(ESA, \underline{J}^0_R)$ is a Ponka sum of cosets $a + U$ with $T_R a \subseteq U$, over the $\underline{J}^0_R$-semilattice $(ESM, \underline{J}^0_R)$. Projecive space is considered here as the set ESM of submodules of the R-module $(E, +, R)$, together with the semilattice operation $+$, where for submodules U and V of $(E, +, R)$,

$$U + V = \{u + v \mid u \in U, v \in V\}$$

is the sum of U and V. The inclusion structure is recovered from $(ESM, +)$ via $U \leq V$ iff $U + V = V$.

Let us recall that for a submodule U of $(E, +, R)$ and r in R, rU is a submodule of U.

Lemma 2.1. Let U be a submodule of the module $(E, +, R)$. Then U is a subalgebra of $(E, \Re(T_R))$.

Proof. For any submodule U of the module $(E, +, R)$ we have $U = 0 + U = \{0 + u \mid u \in U\}$. Moreover for an arbitrary subset T_R of R, $T_R \cdot 0 = \{0\} \subseteq U$. Hence by Proposition 1.4, U is a subalgebra of $(E, \Re(T_R))$. □

Corollary 2.2. The set ESM is a subalgebra of $(ESA, \Re(T_R))$.

Proof. For $(r_1, r_2, \ldots, r_n)$ in $\Re(T_R)$ and for submodules $U_1, U_2, \ldots, U_n$ of $(E, +, R)$

$$U_1 U_2 \ldots U_n(r_1, r_2, \ldots, r_n) = \{u_1 u_2 \ldots u_n(r_1, r_2, \ldots, r_n) \mid u_i \in U_i\}$$

$$= \{r_1u_1 + r_2u_2 + \ldots + r_nu_n \mid u_i \in U_i\} = r_1U_1 + r_2U_2 + \ldots + r_nU_n$$

is a submodule of the module $U_1 + U_2 + \ldots + U_n$. Hence ESM is a subalgebra of $(ESA, \Re(T_R))$. □

Lemma 2.3. Let $U_1, U_2, \ldots, U_n$ be submodules of the module $(E, +, R)$ and $(r_1, r_2, \ldots, r_n)$ be in $\Re(T_R)$. Then one has:

$$(x_1 + U_1)(x_2 + U_2) \ldots (x_n + U_n)(r_1, r_2, \ldots, r_n)$$

$$= x_1x_2 \ldots x_n(r_1, r_2, \ldots, r_n) + U_1U_2 \ldots U_n(r_1, r_2, \ldots, r_n).$$

Proof. For $x_1, x_2, \ldots, x_n$ in E

$$(x_1 + U_1)(x_2 + U_2) \ldots (x_n + U_n)(r_1, r_2, \ldots, r_n)$$

$$= \{(x_1 + u_1)(x_2 + u_2) \ldots (x_n + u_n)(r_1, r_2, \ldots, r_n) \mid u_i \in U_i\}$$

$$= \{r_1x_1 + r_2x_2 + \ldots + r_nx_n + r_1u_1 + r_2u_2 + \ldots + r_nu_n \mid u_i \in U_i\}$$

$$= x_1x_2 \ldots x_n(r_1, r_2, \ldots, r_n) + \{r_1u_1 + r_2u_2 + \ldots + r_nu_n \mid u_i \in U_i\}$$

$$= x_1x_2 \ldots x_n(r_1, r_2, \ldots, r_n) + U_1U_2 \ldots U_n(r_1, r_2, \ldots, r_n).$$

□

Lemma 2.4. Let $U_1, U_2, \ldots, U_n$ be submodules of the module $(E, +, R)$. Then for each n-ary $\Re(T_R)$-word w

$$(x_1 + U_1)(x_2 + U_2) \ldots (x_n + U_n)w = x_1x_2 \ldots x_nw + U_1U_2 \ldots U_nw.$$

Proof. The proof goes by induction on the length of w. First note that by Lemma 2.3 and for (r) in $\Re(T_R)$

$$(x_1 + U_1)(r) = x_1(r) + U_1(r).$$

Hence the result holds for words of the length 1.
Now for $x_1, x_2, \ldots, x_n$ in E and $(r_1, r_2, \ldots, r_p)$ in $\Re(T_R)$, let

$$(x_1 + U)(x_2 + U) \ldots (x_n + U)w$$

$$= (x_1 + U_1) \ldots (x_k + U_k)((x_{k+1} + U_{k+1}) \ldots$$

$$(x_{k+p} + U_{k+p})(r_1, \ldots, r_p))(x_{k+p+1} + U_{k+p+1}) \ldots (x_n + U_n)w.$$

Then by the induction hypothesis, we have

$$(x_1 + U)(x_2 + U) \dots (x_n + U)w$$

$$= (x_1 + U_1) \dots (x_k + U_k)(x_{k+1} \dots x_{k+p}(r_1, \dots, r_p)+$$

$$U_{k+1} \dots U_{k+p}(r_1, \dots, r_p))(x_{k+p+1} + U_{k+p+1}) \dots (x_n + U_n)w$$

$$= x_1 \dots x_k x_{k+1} \dots x_{k+p}(r_1, \dots, r_p)x_{k+p+1} \dots x_n w+$$

$$U_1 \dots U_k U_{k+1} \dots U_{k+p}(r_1, \dots, r_p)U_{k+p+1} \dots U_n w$$

$$= x_1 x_2 \dots x_n w + U_1 U_2 \dots U_n w.$$

□

Note that for a submodule U of $(E, +, R)$ and $(r_1, r_2, \dots, r_n)$ in $\Re(\{0\})$

$$UU \dots U(r_1, r_2, \dots, r_n) = r_1U + r_2U + \dots + r_nU \subseteq U$$

$$= \sum_{i=1}^{n} r_i \cdot U = \{\sum_{i=1}^{n} r_i \cdot u \mid u \in U\}$$

$$= \{r_1u + r_2u + \dots + r_nu \mid u \in U\} \subseteq r_1U + r_2U + \dots + r_nU.$$

Hence $UU \dots U(r_1, r_2, \dots, r_n) = U$. This gives the following corollary.

Corollary 2.5. Let U be a submodule of the module $(E, +, R)$. Then for each n-ary $\Re(\{0\})$-word w

$$(x_1 + U)(x_2 + U) \dots (x_n + U)w = x_1x_2 \dots x_nw + U.$$

□

Lemma 2.6. For submodules $U_1, U_2, \dots, U_n$ of the module $(E, +, R)$ and $(r_1, r_2, \dots, r_n)$ in , $\underline{J}_R^0$, one has

$$U_1U_2 \dots U_n(r_1, r_2, \dots, r_n) = U_1 + U_2 + \dots + U_n.$$

Proof. If r is a unit of R, then $U = \{r \cdot r^{-1}u \mid u \in U\} \subseteq rU$. Since obviously, $rU \subseteq U$, it follows that $U = rU$.
Hence

$$U_1U_2 \dots U_n(r_1, r_2, \dots, r_n) = r_1U_1 + r_2U_2 + \dots + r_nU_n = U_1 + U_2 + \dots + U_n.$$

□

Note that the semilattice $(ESM, +)$ of submodules of $(E, +, R)$ may be considered as an $\Re(T_R)$-semilattice on defining

$$U_1U_2 \dots U_n(r_1, r_2, \dots, r_n) := U_1 + U_2 + \dots + U_n$$

for U_i in ESM. By the previous lemma, the $\underline{J}^0_R$-algebra $(ESM, \underline{J}^0_R)$ is in fact the $\underline{J}^0_R$- semilattice of submodules of $(E, +, R)$.

Note that if $(r_1, r_2, \dots, r_n)$ is not in $\underline{J}^0_R$, the subset $U_1U_2 \dots U_n(r_1, r_2, \dots, r_n)$ is not necessarily equal to $U_1 + U_2 + \dots + U_n$. First observe that rU may be different from U. Indeed, in Z_6, $\{0, 3\}$ and $\{0, 2, 4\}$ are submodules of Z_6 and $4\{0, 3\} = \{0\} \neq \{0, 3\}$. Similarly, $3\{0, 2, 4\} = \{0\} \neq \{0, 2, 4\}$. Now in Z_6SA,

$$\{0, 3\}\{0, 2, 4\}(4, 3) = 4\{0, 3\}+3\{0, 2, 4\} = \{0\}+\{0\} = \{0\} \neq Z_6 = \{0, 3\}+\{0, 2, 4\}.$$

The operation $(4, 3)$ is not in $\underline{J}^0_{Z_6}$. In fact, for each $(Z_{2n}, \Re(T_{Z_{2n}}))$, the subset $\{(r, 1-r) \mid r \in R\}$ of $\underline{J}^0_{Z_{2n}}$ is empty. Indeed, if for k, l in Z_{2n}, $kl = 1$, then both k and l are odd. But in this case, $1-k$ and $1-l$ are even, and hence non-invertible.

Lemma 2.7. For a submodule U of $(E, +, R)$, the set $\{x+U \mid x \in E, T_Rx \subseteq U\}$ is a subalgebra of $(ESA, \underline{J}^0_R)$.

Proof. First note that by Lemma 2.3 and Lemma 2.6, for $(r_1, r_2, \dots, r_n)$ in $\underline{J}^0_R$ and for $(x_1 + U), (x_2 + U), \dots, (x_n + U)$ in $\{x + U \mid x \in E, T_Rx \subseteq U\}$ we have

$$(x_1 + U)(x_2 + U) \dots (x_n + U)(r_1, r_2, \dots, r_n)$$

$$= x_1x_2 \dots x_n(r_1, r_2, \dots, r_n) + UU \dots U(r_1, r_2, \dots, r_n)$$

$$= x_1x_2 \dots x_n(r_1, r_2, \dots, r_n) + r_1U + r_2U + \dots + r_nU$$

$$= x_1x_2 \dots x_n(r_1, r_2, \dots, r_n) + U.$$

Moreover for each x_i in E, $1 \leq i \leq n$, and for each t in T_R, there are some u_i in U such that $tx_i = u_i$, and also $r_itx_i = r_iu_i$. It follows, that for each t in T_R,

$$t(x_1x_2 \dots x_n(r_1, r_2, \dots, r_n)) = r_1tx_1 + r_2tx_2 + \dots + r_ntx_n$$

$$= r_1u_1 + r_2u_2 + \dots + r_nu_n = u_1u_2 \dots u_n(r_1, r_2, \dots, r_n)$$

is in U. Then $T_R(x_1x_2 \dots x_n(r_1, r_2, \dots, r_n)) \subseteq U$ and $\{x+U \mid x \in E, T_Rx \subseteq U\}$ is a subalgebra of $(ESA, \underline{J}^0_R)$. □

Lemma 2.8. For each $(r_1, r_2, \dots, r_n)$ in $\Re(T_R)$, the mapping

$$\pi : ESA \to ESM; \quad x + U \mapsto U$$

is a homomorphism from the algebra $(ESA, (r_1, r_2, \dots, r_n))$ into the $(r_1, r_2, \dots, r_n)$-algebra of submodules of the module $(E, +, R)$.

Proof. For submodules $U_1, U_2, \ldots, U_n$ and $(r_1, r_2, \ldots, r_n)$ in $\Re(T_R)$, we have

$$(x_1 + U_1)(x_2 + U_2)\ldots(x_n + U_n)(r_1, r_2, \ldots, r_n)\pi$$

$$= (x_1x_2\ldots x_n(r_1, r_2, \ldots, r_n) + U_1U_2\ldots U_n(r_1, r_2, \ldots, r_n))\pi$$

$$= U_1U_2\ldots U_n(r_1, r_2, \ldots, r_n)$$

$$= (x_1 + U_1)\pi(x_2 + U_2)\pi\ldots(x_n + U_n)\pi(r_1, r_2, \ldots, r_n).$$

It follows that the mapping π is an $(r_1, r_2, \ldots, r_n)$-homomorphism. □

Proposition 2.9. For each $(r_1, r_2, \ldots, r_n)$ in $\underline{J}_R^0$ and for the functor $F : (ESM) \to ((r_1, r_2, \ldots, r_n))$ with $UF = U\pi^{-1}$ and $(U \to V)F : U\pi^{-1} \to V\pi^{-1}$; $x + U \mapsto x + V$, the algebra $(ESA, (r_1, r_2, \ldots, r_n))$ is the Ponka sum of $(r_1, r_2, \ldots, r_n)$-reducts of $(\{x+U \mid x \in E, T_Rx \subseteq U\}, \Re(T_R))$ over the projective space $(ESM, +)$ by the functor F.

Proof. By previous lemmas, the mapping π is an $(r_1, r_2, \ldots, r_n)$-homomorphism onto the semilattice $(ESM, +)$. The fibres

$$(U\pi^{-1}, (r_1, r_2, \ldots, r_n)) = (\{x + U \mid x \in E, T_Rx \subseteq U\}, (r_1, r_2, \ldots, r_n))$$

are $(r_1, r_2, \ldots, r_n)$-reducts of the algebra $(\{x + U \mid x \in E, T_Rx \subseteq U\}, \Re(T_R))$.

Further, for $(r_1, r_2, \ldots, r_n)$ in $\underline{J}_R^0$,

$$(x_1 + U_1)(x_2 + U_2)\ldots(x_n + U_n)(r_1, r_2, \ldots, r_n)$$

$$= x_1x_2\ldots x_n(r_1, r_2, \ldots, r_n) + U_1 + U_2 + \ldots + U_n$$

$$= (x_1 + (U_1 + U_2 + \ldots + U_n))(x_2 + (U_1 + U_2 + \ldots + U_n))\ldots$$

$$(x_n + (U_1 + U_2 + \ldots + U_n))(r_1, r_2, \ldots, r_n)$$

$$= (x_1 + U_1)(U_1 \to U_1 + U_2 + \ldots + U_n)F$$

$$(x_2 + U_2)(U_2 \to U_1 + U_2 + \ldots + U_n)F\ldots$$

$$(x_n + U_n)(U_n \to U_1 + U_2 + \ldots + U_n)F(r_1, r_2, \ldots, r_n).$$

Thus $(ESA, (r_1, r_2, \ldots, r_n))$ is a Ponka sum as claimed. □

Theorem 2.10. For an algebra $(E, \Re(T_R))$, each algebra (ESA, Ω) of subalgebras of $(E, \Re(T_R))$, where $\Omega \subseteq \underline{J}_R^0$, is the Ponka sum of Ω-reducts of

$(\{x+U \mid x \in E, T_R x \subseteq U\}, \Re(T_R))$ over the projective space $(ESM, +)$ by the functor $F : (ESM) \to (\Omega)$ with $UF = \{x+U \mid x \in E, T_R x \subseteq U\}$ and $(U \to V)F : UF \to VF;\ x+U \mapsto x+V$.

Proof. The fact that (ESA, Ω) is a Ponka sum over $(ESM, +)$ by the functor F follows directly by Proposition 2.9. □

As noted in Example 1.2, the $\{0\}$-enrichment $(E, \Re(\{0\}))$ of an affine R-space $(E, \underline{R}, P)$ is equivalent to the affine R-space $(E, \underline{R}, P)$.
Let $J_R^2 := \{r \in R \mid r \text{ and } 1-r \text{ are units of } R\}$ and $\underline{J}_R^2 := \{\underline{r} \mid r \in J_R^2\}$. Theorem 2.10 has the following corollary.

Corollary 2.11. [5] For an affine space $(E, \underline{R}, P)$ in $\underline{\underline{R}}$, each algebra (ESA, Ω) of affine subspaces of $(E, \underline{R}, P)$, where $\Omega \subseteq \underline{J}_R^2$, is a Ponka sum of Ω-reducts of affine R-spaces $(\{x+U \mid x \in E\}, \underline{R}, P)$ over the projective space $(ESM, +)$ by the functor $F : (ESM) \to (\Omega)$ with $UF = \{x+U \mid x \in E\}$ and $(U \to V)F : UF \to VF;\ x+U \mapsto x+V$. □

3. A sum of algebras over a mode.

In Section 2 the structure of the algebra $(ESA, \underline{J}_R^0)$ was described using the concept of a Ponka sum. In this Section, we use a more general construction of generalized coherent Lallement sum to describe a wider class of reducts of algebra $(ESA, \Re(T_R))$.

At first we recall some results presented in [11] concerning generalized coherent Lallement sums.

Definition 3.1. [11] Let (I, Ω) be an algebra of type $\tau : \Omega \to N$. The least reflexive relation $\prec$ on I containing the set

$$\{(i,j) \mid \text{ there exist } x_1 \dots x_n t \in X\Omega \text{ and } i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_n \in I \text{ such that } j = i_1 \dots i_{k-1} i i_{k+1} \dots i_n t\}$$

is called the *algebraic quasi-order* of the algebra (I, Ω). □

Definition 3.2. [11] Let (I, Ω) be an algebra of type $\tau : \Omega \to N$, let $\prec$ be its algebraic quasi-order, and $a_1, \dots, a_{\omega\tau}, b_1, \dots, b_{\omega\tau}$ its elements. An algebra (I, Ω) satisfying the condition

$$\forall \omega \in \Omega\ \ \forall i = 1, \dots, \omega\tau \text{ if } a_i \prec b_i \text{ then } a_1 \dots a_{\omega\tau}\omega \prec b_1 \dots b_{\omega\tau}\omega \tag{3.3}$$

is called *naturally quasi-ordered.* □

Let (B, Ω) be any algebra and S a subset of B. Then S is a *sink* of B if for each (n-ary) ω in Ω and $b_1, \dots, b_n$ in B, if for some $i = 1, \dots, n$, $b_i \in S$ then $b_1, \dots, b_n\omega \in S$. A congruence θ on (B, Ω) *preserves* the sink (S, Ω) if the restriction of the natural projection $B \to B^\theta$, $b \mapsto b^\theta$ to the subalgebra (S, Ω) injects. The algebra (B, Ω) is said to be an *envelope* of a subalgebra (S, Ω), if (S, Ω) is a sink of (B, Ω) such that equality is the only congruence on (B, Ω) preserving (S, Ω). (See [7].)

Definition 3.4. [11] Let (I, Ω) be an idempotent Ω-algebra with the algebraic quasi- ordering $\prec$. For each i in I, let an Ω-algebra (A_i, Ω) and its extension (E_i, Ω) be given, i.e. (A_i, Ω) is a subalgebra of (E_i, Ω). For $i \prec j$ in $(I, \prec)$, let $\phi_{i,j} : (A_i, \Omega) \to (E_j, \Omega)$ be an Ω-homomorphism such that:

a) $\phi_{i,j}$ is an embedding of A_i into E_i;

b) for each (n-ary) ω in Ω and for $i_i, \dots, i_n$ in I with $i_1 \dots i_n\omega = i$,

$$(A_{i_1}\phi_{i_1,i}) \dots (A_{i_n}\phi_{i_n,i})\omega \subseteq A_i;$$

c) for each $i_1 \dots i_n\omega = i \prec j$ in $(I, \prec)$

$$a_{i_1}\phi_{i_1,i} \dots a_{i_n}\phi_{i_n,i}\omega\phi_{i,j} = a_{i_1}\phi_{i_1,j} \dots a_{i_n}\phi_{i_n,j}\omega,$$

where for $k = 1, \dots, n$, $a_{i_k} \in A_{i_k}$;

d) for each j in I, the extension E_j of A_j is its envelope and $E_j = \{a_i\phi_{i,j} \mid i \prec j\}$.

Then the disjoint union $A = \bigcup(A_i \mid i \in I)$ equipped with the operations

$$\omega : A_{i_1} \times \dots \times A_{i_n} \to A_i; \quad (a_{i_1}, \dots, a_{i_n}) \mapsto a_{i_1}\phi_{i_1,i} \dots a_{i_n}\phi_{i_n,i}\omega$$

for all ω in Ω, all $(i_1, \dots, i_n) \in I^n$ and $i = i_1 \dots i_n\omega$, is called a *generalized coherent Lallement sum* or briefly *gcL-sum.*

A gcL-sum is said to be *strict* if $E_j = A_j$ for each j in I. □

Using the construction of gcL-sum one can represent each algebra having a homomorphism onto an idempotent naturally quasi-ordered algebra.

Theorem 3.5. [11] Let (A, Ω) be an Ω-algebra having a homomorphism onto an idempotent naturally quasi-ordered algebra (I, Ω), with corresponding fibres (A_i, Ω) for i in I. Then (A, Ω) is a gcL-sum of (A_i, Ω). □

Let

$\underline{J}_R^{-1} := \{ \quad (r_1, r_2, \dots, r_n) \in \Re(T_R) \mid n \geq 1$ and there exists at least one
$1 \leq i \leq n$ such that r_i is a unit of $R\}$

and

$$\underline{J}_R^\infty := \{ \quad (r_1, r_2, \ldots, r_n) \in \underline{J}_R^{-1} \mid n \geq 1 \text{ and there exists at least one } 1 \leq i \leq n \text{ such that } r_i = 0\}.$$

Now we will show that certain reducts of $(EMS, \Re(T_R))$ are idempotent naturally quasi-ordered algebras.

Lemma 3.6. For each $\Omega \subseteq \Re(T_R)$ such that $\Omega \cap \underline{J}_R^\infty \neq \emptyset$, the algebra (ESM, Ω) is naturally quasi-ordered.

Proof. By Corollary 2.2, the set ESM is a subalgebra of (ESA, Ω). By Definition 3.1, the quasi-order $\prec$ of the algebra (ESM, Ω) is the set

$$S = \{ \quad (U, V) \mid \exists x_1 \ldots x_n t \in X\Omega,\ U_1, \ldots, U_{k-1}, U_{k+1}, \ldots, U_n \in ESM \quad V = U_1 \ldots U_{k-1} U U_{k+1} \ldots U_n t\}.$$

Note that for submodules U and V in ESM and $(r_1, \ldots, r_i, \ldots, 0, \ldots, r_n)$ in the set $\Omega \cap \underline{J}_R^\infty$, we have

$$V \ldots V \ldots U \ldots V(r_1, \ldots, r_i, \ldots, 0, \ldots, r_n)$$

$$= r_1 V + \ldots + r_i V + \ldots + 0U + \ldots + r_n V =$$

$$= r_1 V + \ldots + V + \ldots + \{0\} + \ldots + r_n V = V.$$

Hence, for all submodules U and V in ESM, $(U, V) \in S$ and the algebraic quasi-order $\prec$ of (ESM, Ω) is full. Consequently, the set ESM is naturally quasi-ordered. □

Corollary 3.7. The algebra $(ESM, \Re(T_R))$ of submodules of the R-module $(E, +, R)$ is naturally quasi-ordered. □

Lemma 3.8. For each $\Omega \subseteq \underline{J}_R^{-1}$, the algebra (ESM, Ω) is a mode.

Proof. By Corollary 2.2, the set ESM is a subalgebra of (ESA, Ω), so it is an entropic algebra. Moreover, let us recall that for each r in R and U in ESM we have

$$U + U = U,$$

$$rU = \{ru \mid u \in U\},$$

$$U + rU = U.$$

If r is a unit of R, then $rU = U$. Hence for each n-ary operation $(r_1, \ldots, r_n)$ in

$\Omega \subseteq \underline{J}_R^{-1}$ one has

$$U \dots U \dots U(r_1, \dots, r_i, \dots, r_n)$$

$$= r_1U + \dots + r_iU + \dots + r_nU = r_1U + \dots + U + \dots + r_nU = U.$$

This implies that (ESM, Ω) is an idempotent algebra. Hence, the algebra (ESM, Ω) is a mode. □

Finally, we can use the construction of gcL-sum to describe certain reducts of algebras of subalgebras of $(E, \Re(T_R))$. We have the following theorem.

Theorem 3.9. For an algebra $(E, \Re(T_R))$, the algebra (ESA, Ω) of subalgebras of $(E, \Re(T_R))$, where $\Omega \subseteq \underline{J}_R^{\infty}$, is the gcL-sum of Ω-reducts of $(\{x + U \mid x \in E, T_Rx \subseteq U\}, \Re(T_R))$ over an Ω-mode (ESM, Ω).

Proof. As was shown in Proposition 1.4. non-empty subalgebras of $(E, \Re(T_R))$ are cosets $a+U$ of submodules of the module $(E, +, R)$ such that $T_Ra \subseteq U$. By Lemma 2.8, for each $(r_1, \dots, r_n)$ in $\Re(T_R)$, the mapping

$$\pi : ESA \to ESM; \quad x + U \mapsto U$$

is a homomorphism from the algebra $(ESA, (r_1, \dots, r_n))$ to the $(r_1, \dots, r_n)$-algebra of submodules of the module $(E, +, R)$. Then, by Lemmas 3.6 and 3.8, the algebra (ESA, Ω) has a homomorphism onto an idempotent naturally quasi-ordered algebra (ESM, Ω) with the corresponding fibers $(M_U, \Omega) := (\{x + U \mid x \in E, T_Rx \subseteq U\}, \Omega)$, for U in ESM. Hence, by Theorem 3.5, (ESA, Ω) is a gcL-sum of (M_U, Ω) over (ESM, Ω). Moreover, for each pair $(U, V) \in ESM \times ESM$ the mappings

$$\phi_{U,V} : M_U \to M_V; \quad x + U \mapsto x + V$$

are Ω-homomorphisms. Indeed, by Lemmas 2.3 and 3.8, for each $(r_1, \dots, r_n)$ in Ω we have

$$(x_1 + U) \dots (x_n + U)(r_1, \dots, r_n)\phi_{U,V}$$

$$= (x_1 \dots x_n(r_1, \dots, r_n) + U \dots U(r_1, \dots, r_n))\phi_{U,V}$$

$$= (x_1 \dots x_n(r_1, \dots, r_n) + U)\phi_{U,V} = x_1 \dots x_n(r_1, \dots, r_n) + V$$

$$= x_1 \dots x_n(r_1, \dots, r_n) + V \dots V(r_1, \dots, r_n)$$

$$= (x_1 + V) \dots (x_n + V)(r_1, \dots, r_n)$$

$$= (x_1 + U)\phi_{U,V} \dots (x_n + U)\phi_{U,V}(r_1, \dots, r_n).$$

So the extension E_V of M_V is equal to

$$\{(x+U)\phi_{U,V} \mid U \in ESM, x \in E, T_R x \subseteq U\} = \{x+V \mid x \in E\}.$$

□

As a consequence of Theorem 3.9. one has the following.

Corollary 3.10. For an affine space $(E, \underline{R}, P)$ in $\underline{\underline{R}}$, each algebra (ESA, Ω) of affine subspaces of $(E, \underline{R}, P)$, where $\Omega \subseteq \{\underline{r} \mid r$ or $1-r$ are units of $R\}$ and $\Omega \cap \{\underline{0}, \underline{1}\} \neq \emptyset$, is the strict gcL-sum of Ω-reducts of affine R-spaces $(\{x+U \mid x \in E\}, \underline{R}, P)$ over an Ω-mode (ESM, Ω).

Proof. As noted in Example 1.2, the $\{0\}$-enrichment $(E, R(\{0\}))$ of an affine R-space $(E, \underline{R}, P)$ is equivalent to the affine R-space $(E, \underline{R}, P)$. In particular, subalgebras of $\{0\}$-enrichments of affine R-spaces are exactly the cosets of submodules of the module $(E, +, R)$. By Theorem 3.9, each algebra (ESA, Ω), where $\Omega \subseteq \{\underline{r} \mid r$ or $1-r$ are units of $R\}$ and $\Omega \cap \{\underline{0}, \underline{1}\} \neq \emptyset$, is the gcL-sum of Ω-reducts $(M_U, \Omega) := (\{x+U \mid x \in E\}, \Omega)$ of affine R-spaces, over an Ω-mode (ESM, Ω), with for submodules U, V in ESM, Ω-homomorphisms

$$\phi_{U,V} : M_U \to M_V; \quad x+U \mapsto x+V.$$

Moreover, the extension E_V of M_V is equal to

$$\{(x+U)\phi_{U,V} \mid U \in ESM, x \in E\} = \{x+V \mid x \in E\} = M_V.$$

So in fact this gcL-sum is strict. □

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