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IDENTITIES FOR CLASSES OF ALGEBRAS CLOSED UNDER THE COMPLEX STRUCTURES

Agata Pilitowska

Warsaw Technical University, Department of Mathematics
00-661 Warsaw, Poland
e-mail: irbis@pol.pl

Abstract

The aim of this paper is to study identities satisfied in complex algebras of subalgebras of a given algebra. In particular, we show that many axioms defining affine spaces are consequences of linear identities, and thus are satisfied by their algebras of subalgebras.

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The *complex* (*power* or *global*) *algebra* (AUS, Ω) *of sets* of an algebra (A, Ω) of type $\tau : \Omega \to N$ is the family of non-void subsets of A with operations of complex ω-products given by

$$\omega : AUS^{\omega\tau} \to AUS; \qquad X_1 \ldots X_{\omega\tau}\omega = \{x_1 \ldots x_{\omega\tau}\omega \mid x_i \in X_i\},$$

for each operation $\omega : A^{\omega\tau} \to A$ in the set Ω (in the notation of [10]).

According to [1] the concept of complex algebras of subsets originated with Frobenius in the context of group theory. Any subset of a group is refereed to as a "complex" and the complex products yield a "calculus of complexes". There are numerous examples of complex operations. In group theory, for instance, any coset xN is the complex product of a singleton $\{x\}$ and a normal subgroup N. In lattice theory, the set $I(L)$ of ideals of any lattice $(L, \wedge, \vee)$ again forms a lattice under the set inclusion. If $(L, \wedge, \vee)$ is distributive, then joins and meets in $I(L)$ are precisely the complex operations obtained from joins and meets in $(L, \wedge, \vee)$ (see [1]). For distributive lattice $(L, \wedge, \vee)$ its lattice of ideals is a subalgebra of the complex algebra of $(L, \wedge, \vee)$.

Complex algebras of subalgebras are closely related to complex algebras of sets. Given an algebra (A, Ω) of type $\tau : \Omega \to N$, one may form the set $(A, \Omega)S$ or AS of non-empty subalgebras of (A, Ω). This set AS may carry an Ω-algebra structure under the ω-complex products

$$\text{(CP)} \qquad \omega : AS^{\omega\tau} \to AS; \qquad A_1 \dots A_{\omega\tau}\omega = \{a_1 \dots a_{\omega\tau}\omega \mid a_i \in A_i\},$$

for each ω in Ω. But in general, the family AS does not need to be closed under complex operations. However if it is, AS is a subalgebra of the algebra (AUS, Ω). In this case, the algebra (AS, Ω) is called the *complex algebra of subalgebras* of the algebra (A, Ω), or briefly the *algebra of subalgebras.*

In general, a class of algebras need not to be closed under power algebras. The complex algebra of a group, for example, is not again a group [3]. This is due to the fact that although a complex operation may preserve some of the properties of (A, Ω) it will in general not retain them all. In particular, not all identities true in (A, Ω) will be satisfied in (AUS, Ω) or in (AS, Ω).

In this paper we study identities satisfied in complex algebras of subalgebras of a given algebra. Recall that a *term* p *is linear* if no variable symbol occurs more than once in p. An *identity* $p = q$ *is linear* if both terms p and q are linear.

As was proved by G. Grätzer and H. Lakser, for any variety $\boldsymbol{V}$ of Ω-algebras (A, Ω) and algebras (AUS, Ω) of non-empty subsets of A, the identities satisfied by the variety generated by $\{(AUS, \Omega)|\ (A, \Omega) \text{ in } \boldsymbol{V}\}$ are precisely the consequences of linear identities true in $\boldsymbol{V}$. The aim of this paper is to find a similar characterisation for varieties generated by algebras (AS, Ω) of non-empty subalgebras of (A, Ω). We are looking for an answer to the following question. Is it true that identities satisfied by the variety generated by $\{(AS, \Omega)|\ (A, \Omega) \text{ in } \boldsymbol{V}\}$ are precisely the consequences of linear and idempotent identities true in $\boldsymbol{V}$? We present many examples which confirm our conjecture and we do not find a counterexample. However the full answer to this question is still unknown.

In Section 1 we recall a characterization of complex algebras of sets given by G. Grätzer and H. Lakser. Section 2 is devoted to complex algebras of subalgebras. We give a sufficient condition, called complex condition, for the set of non-empty subalgebras of an algebra to be closed under the complex products. We present examples of algebras satisfying this condition.

We show that the complex condition is not necessary to make the set AS of non-empty subalgebras of an algebra (A,Ω) closed under complex products. In Section 3 the role of the idempotent law in complex algebras of subalgebras is studied. An example of idempotent algebras is mode i.e. idempotent and entropic algebra. As was shown by A. Romanowska and J.D.H. Smith in [10], if (A,Ω) is a mode, then (AS,Ω) is again a mode satisfying each linear identity satisfied by (A,Ω). Many examples of modes and their algebras of submodes are demonstrated using this result. Moreover we show that for a variety $\boldsymbol{V}$ the class $\{(AS,\Omega)|(A,\Omega) \text{ in } \boldsymbol{V}\}$ is not necessarily a variety. In the last, main Section 4 we present some (not necessarily linear) identities satisfied by the complex algebras of subalgebras in the case varieties in question are idempotent. Some consequences of these results are then applied to describe identities satisfied by algebras of subalgebras of affine spaces. We show that many of the axioms defining affine spaces are consequences of linear identities, and thus are satisfied by their algebras of subalgebras.

The notation and terminology of the paper is similar to that in the book [10]. We use "Polish notation" for words (terms) and operations, e.g. instead of $w(x_1,...,x_n)$ we write $x_1...x_nw$. Moreover, the symbol $x_1...x_nw$ means that $x_1,...,x_n$ are exactly variables appearing in the word w. The traditional notation is used in the case of groupoid words. For such words we frequently use non-brackets notation. The cardinality of a set A is denoted by $|A|$. We refer the reader to the book [10] for all undefined notions and results.

1. Complex algebras of subsets

At first we recall some known results concerning complex algebras of sets. It is convenient to use the following formalism. Let $1 \leq m \leq n$ and let $\varphi : \{1,\ldots,n\} \to \{1,\ldots,m\}$. Then from the term $x_1\ldots x_np$, with $x_1,\ldots,x_n$ distinct variables, we get the term $x_{1\varphi}\ldots x_{n\varphi}p$ with the variables $x_1,\ldots,x_m$ by substituting $x_{i\varphi}$ for x_i.

Lemma 1.1 (see [3]). *Given two terms p and q with variables in the set $\{x_1,\ldots,x_m\}$, there are an integer $n \geq m$, a mapping $\varphi : \{1,\ldots,n\} \to \{1,\ldots,m\}$, and linear terms p^*, q^* with variables in the set $\{x_1,\ldots,x_n\}$ such that p is obtained from $p*$ and q is obtained from q^* by substituting $x_{i\varphi}$ for x_i.* ∎

If $x_1 \dots x_n p$ is a linear term, (A, Ω) an algebra and $A_1, \dots, A_n$ subsets of A, then the definition of complex ω-product extends to

$$A_1 \dots A_n\, p = \{a_1 \dots a_n\, p \mid a_i \in A_i\}\,.$$

As was shown in [3], the linearity of p is essential. For example, let $(A, \cdot)$ be a groupoid which is not a left or right zero band and let $xp := xx$. Then, for a subset A_1 of A, with more than one elements, the subset $A_1 p$ is equal to $A_1 A_1 = \{ab \mid a, b \in A_1\}$ and not necessarily equal to $\{ap \mid a \in A_1\} = \{aa \mid a \in A_1\}$. (See [3]).

For any variety $\boldsymbol{V}$ of Ω-algebras we denote by $\boldsymbol{VUS}$ the variety generated by $\{(AUS, \Omega) \mid (A, \Omega) \in \boldsymbol{V}\}$.

Proposition 1.2 (see [3]). *Let $\boldsymbol{V}$ be a variety of algebras. Then the identities satisfied by $\boldsymbol{VUS}$ are precisely those identities resulting through identification of variables from the linear identities true in $\boldsymbol{V}$.* ■

Corollary 1.3 (see [3]). *Let $\boldsymbol{V}$ be a variety of algebras. Then $\boldsymbol{VUS} = \boldsymbol{V}$ if and only if $\boldsymbol{V}$ is defined by a set of linear identities.* ■

Let us call a term p^* a *generalization* of a term p if p is obtained from p^* by identification of some variables. If p^* is linear, we call it a *linearisation* of p. This leads to the following. If for $1 \leq m \leq n$, $\varphi : \{1, \dots, n\} \to \{1, \dots, m\}$ is a mapping and $x_1 \dots x_n\, p^*$ is any linearisation of the term $x_{1\varphi} \dots x_{n\varphi}\, p$, then for subsets $A_1, \dots, A_m$ of an Ω- algebra (A, Ω),

$$A_1 \dots A_m\, p = \{a_1 \dots a_n\, p^* \mid a_i \in A_{i\varphi}\}\,.$$

For instance, if $(A, \cdot)$ is a groupoid, $xyp = xxy$, then for subsets A_1 and A_2 of A

$$A_1 A_2\, p = \{a_{11} a_{12} a_2 \mid a_{11}, a_{12} \in A_1,\ a_2 \in A_2\}\,.$$

Using the linearisation $x_1 x_2 y\, p^* = x_1 x_2 y$ of p, we see that

$$A_1 A_2\, p = \{a_{11} a_{12} a_2\, p^* \mid a_{11}, a_{12} \in A_1,\ a_2 \in A_2\} = A_1 A_1 A_2\, p^*.$$

2. COMPLEX ALGEBRAS OF SUBALGEBRAS

If an algebra (A, Ω) is *entropic*, i.e. each operation, as a mapping from a direct power of the algebra into the algebra, is actually a homomorphism, then for each ω in Ω, the complex ω-product $A_1 \dots A_{\omega\tau} \omega$ is a subalgebra of

(A,Ω). Consequently the set AS of non-empty subalgebras of (A,Ω) carries an Ω-algebra structure under the complex products, and (AS,Ω) is a well defined algebra (see [10]). But note that $A_1 \dots A_{\omega\tau}\omega$ is not a subalgebra of (A,Ω) for an arbitrary algebra (A,Ω).

Example 2.1. Let $(S_3,\circ)$ be the permutation group of the set $\{1,2,3\}$. It is easy to verify that

$$\begin{aligned} S_3S \;=\; & \{\{id\}, \{id,(23)\}, \{id,(13)\}, \{id,(12)\}, \{id,(123),(132)\}, \\ & \{id,(123),(132),(23),(13),(12)\}\}. \end{aligned}$$

But $\{id,(23)\} \circ \{id,(13)\} = \{id,(13),(23),(123)\}$ is not a subalgebra of $(S_3,\circ)$. ■

Let ω be in Ω and let (A_1,Ω),...,$(A_{\omega\tau},\Omega)$ be subalgebras of an algebra (A,Ω). It is easy to see that $A_1 \dots A_{\omega\tau}\,\omega$ is a subalgebra of (A,Ω) if and only if for each υ in Ω and elements a_{ij} in A_i with $1 \le i \le \omega\tau$ and $1 \le j \le \upsilon\tau$, there are elements a_1 in $A_1,\dots,a_{\omega\tau}$ in $A_{\omega\tau}$ such that $a_{11}\dots a_{\omega\tau 1}\,\omega \dots\dots a_{1\upsilon\tau}\dots a_{\omega\tau\upsilon\tau}\,\omega\upsilon = a_1 \dots a_{\omega\tau}\,\omega$. This suggests the following definition.

Definition 2.2. An *algebra* (A,Ω) *satisfies the complex condition* if for each pair (ω,υ) in $\Omega\times\Omega$, there exist terms $t_1,\dots,t_{\omega\tau}$ such that the following identities hold in (A,Ω)

$$\text{(CC)} \qquad \begin{aligned} & x_{11}\dots x_{\omega\tau 1}\,\omega \dots\dots x_{1\upsilon\tau}\dots x_{\omega\tau\upsilon\tau}\,\omega\upsilon \\ & = y_{11}\dots y_{1\,k_1}t_1 \dots\dots y_{\omega\tau 1}\dots y_{\omega\tau k_{\omega\tau}}t_{\omega\tau}\,\omega, \end{aligned}$$

with $\{y_{i1},\dots,y_{ik_i}\}$ a subset of $\{x_{i1},\dots,x_{i\,\upsilon\tau}\}$ for each $i=1,\dots,\omega\tau$.

A *variety* $\boldsymbol{V}$ of Ω-algebras (A,Ω) *satisfies the complex condition* if for each pair (ω,υ) in $\Omega\times\Omega$, there exist Ω-words $t_1,\dots,t_{\omega\tau}$ such that the identities (CC) hold in each algebra (A,Ω) in the variety $\boldsymbol{V}$. ■

Lemma 2.3. *The complex condition (CC) is sufficient to make the set AS of non-empty subalgebras of an algebra (A,Ω) closed under the complex ω-products for all ω in Ω.*

Proof. Let an algebra (A,Ω) satisfy the condition (CC). For an operation ω in Ω, consider the complex ω-product $A_1\dots A_{\omega\tau}\,\omega$ of elements $A_1,\dots,A_{\omega\tau}$ of AS. Clearly $A_1\dots A_{\omega\tau}\,\omega$ is non-empty. It is a subalgebra of (A,Ω), since for each operation υ in Ω, there exist terms $t_1,\dots,t_{\omega\tau}$ such that for elements $a_{i\,j}$ of A_i with $1 \le i \le \omega\tau, 1 \le j \le \upsilon\tau$ and corresponding subsets $\{b_{i1},\dots,b_{ik_i}\}$ of $\{a_{i1},\dots,a_{i\,\upsilon\tau}\}$, $a_{11}\dots a_{\omega\tau 1}\,\omega\dots\dots a_{1\upsilon\tau}\dots a_{\omega\tau\upsilon\tau}\,\omega\upsilon = b_{11}\dots b_{1\,k_1}\,t_1\dots\dots b_{\omega\tau 1}\dots b_{\omega\tau k_{\omega\tau}}\,t_{\omega\tau}\,\omega$, is in $A_1\dots A_{\omega\tau}\,\omega$. ∎

Thus, if an algebra (A,Ω) satisfies (CC), then (AS,Ω) is a subalgebra of (AUS,Ω).

Example 2.4. Let $(G,\cdot)$ be a groupoid which satisfies the entropic identity

$$xy\cdot x'y' = xx'\cdot yy'.$$

Then, the complex condition is satisfied in $(G,\cdot)$ with terms $t_1 = xx'$ and $t_2 = yy'$. ∎

Example 2.4 may be generalised as follows.

Example 2.5. According to [10] the property of *entropicity* of an algebra (A,Ω) may be expressed algebraically by means of the linear identities

$$x_{11}\dots x_{\omega\tau 1}\,\omega\dots\dots x_{1\upsilon\tau}\dots x_{\omega\tau\upsilon\tau}\,\omega\upsilon$$

$$= x_{11}\dots x_{1\upsilon\tau}\,\upsilon\dots\dots x_{\omega\tau 1}\dots x_{\omega\tau\upsilon\tau}\,\upsilon\omega,$$

that are satisfied for all operations ω and υ in Ω. It is easy to see that entropic law is a special case of the complex condition, where each term t_i is equal $x_{i1}\dots x_{i\upsilon\tau}\upsilon$. ∎

Example 2.6. Let $(G,\cdot)$ be a groupoid, which satisfies any one of the following identities:

(i) $xy\cdot x'y' = x\cdot yy'$;
(ii) $xy\cdot x'y' = x'x'\cdot(yy\cdot yy)$;
(iii) $xy\cdot x'y' = x^kx\cdot yy^p$.

Then $(G,\cdot)$ satisfies the condition (CC). ∎

Example 2.7 (see [5] and [10]. Let $(S,\cdot)$ be a semigroup which satisfies any one of the following identities:

(i) $xyx'y' = xx'yy'$;
(ii) $xyx'y' = xx'y$;
(iii) $xyx'y' = xxxxy$.

Clearly $(S,\cdot)$ satisfies the complex condition. ■

Example 2.8 (see [8]). Let (A,p) be an algebra with one n-ary operation which satisfies the diagonal identity:

$$x_{11}\dots x_{1n}\,p\dots\dots x_{n1}\dots x_{nn}\,p\,p = x_{11}\dots x_{nn}\,p.$$

Obviously, the algebra (A,p) satisfies the complex condition, where each term t_i is a variable.

Note, that also $x_{11}\dots x_{n1}\,p\dots\dots x_{1n}\dots x_{nn}\,p\,p = x_{11}\dots x_{nn}\,p$.
So the algebra (A,p) satisfies the entropic law. ■

Example 2.9. There is a well known necessary and sufficient condition to make the set of non-empty subgroups of a group closed under the complex products. It was shown, e.g., in [15], that if $(G_1,\cdot,^{-1})$ and $(G_2,\cdot,^{-1})$ are subgroups of a group $(G,\cdot,^{-1})$, then G_1G_2 is a subgroup of $(G,\cdot,^{-1})$ if and only if $G_1G_2 = G_2G_1$. In particular, abelian groups are closed under the complex products. Obviously, every abelian group $(G,\cdot,^{-1})$ satisfies the condition (CC), because it satisfies the identities:

$$xy\cdot x'y' = xx'\cdot yy',$$

$$(xy)^{-1} = x^{-1}y^{-1}.$$

But, by Example 2.1. this is not longer true for non-abelian groups. ■

Now we will show that the complex condition (CC) is not necessary to make the set AS of non-empty subalgebras of an algebra (A,Ω) closed under the complex products.

Example 2.10. Let $(G,\cdot)$ be a four element non-idempotent groupoid, with the following multiplication table:

$\cdot$	a	b	c	d
a	c	c	c	d
b	c	c	c	d
c	c	c	c	c
d	b	a	c	d

It is easy to check that $\{c\}$, $\{d\}$, $\{c,d\}$, $\{a,c\}$, $\{b,c\}$, $\{a,b,c\}$ and $\{a,b,c,d\}$ are all subgroupoids of $(G,\cdot)$ and by the following table

$\cdot$	$\{c\}$	$\{d\}$	$\{c,d\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$	$\{a,b,c,d\}$
$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$
$\{d\}$	$\{c\}$	$\{d\}$	$\{c,d\}$	$\{b,c\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,b,c,d\}$
$\{c,d\}$	$\{c\}$	$\{c,d\}$	$\{c,d\}$	$\{b,c\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,b,c,d\}$
$\{a,c\}$	$\{c\}$	$\{c,d\}$	$\{c,d\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c,d\}$
$\{b,c\}$	$\{c\}$	$\{c,d\}$	$\{c,d\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c,d\}$
$\{a,b,c\}$	$\{c\}$	$\{c,d\}$	$\{c,d\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{c,d\}$
$\{a,b,c,d\}$	$\{c\}$	$\{c,d\}$	$\{c,d\}$	$\{b,c\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,b,c,d\}$

the set GS of subalgebras of $(G,\cdot)$ is closed under the complex $\cdot$-product.

However this algebra does not satisfy the condition (CC). Suppose that there exist terms t_1 and t_2 such that the following identity holds in $(G,\cdot)$

$$x_{11}x_{21}\cdot x_{12}x_{22} = y_{11}y_{12}\,t_1 \cdot y_{21}y_{22}\,t_2, \tag{2.11}$$

with $\{y_{i1}, y_{i2}\}$ a subset of $\{x_{i1}, x_{i2}\}$ for each i=1,2.

A simple case analysis establishes that the free algebra on two generators x and y in the variety generated by the groupoid $(G,\cdot)$ consists of 15 elements: $x, y, xy, yx, xx, yy, x\cdot xy, x\cdot yx, y\cdot yx, y\cdot xy, yy\cdot x, xx\cdot y, xx\cdot xy, yy\cdot yx, xy\cdot xx$.

Let xyt and xys denote terms different from x and y in the free algebra on two generators x and y in the variety generated by the groupoid $(G,\cdot)$. It is easy to see that for $x_{11}=a$, $x_{21}=d$, $x_{12}=b$ and $x_{22}=d$,

$$x_{11}x_{21}\cdot x_{12}x_{22} = ad\cdot bd \neq abt\cdot d = c = x_{11}x_{12}t\cdot x_{21},$$

$$x_{11}x_{21}\cdot x_{12}x_{22} = ad\cdot bd \neq abt\cdot d = c = x_{11}x_{12}t\cdot x_{22},$$

$$x_{11}x_{21}\cdot x_{12}x_{22} = ad\cdot bd \neq abt\cdot dds = c = x_{11}x_{12}\,t\cdot x_{21}x_{22}\,s.$$

Additionally, for $x_{11}=x_{21}=x_{22}=d$ and $x_{12}=c$

$$x_{11}x_{21}\cdot x_{12}x_{22} = dd\cdot cd = c \neq d\cdot dds = d = x_{11}\ \cdot x_{21}x_{22}\,s,$$

and for $x_{11}=c$ and $x_{21}=x_{12}=x_{22}=d$

$$x_{11}x_{21}\cdot x_{12}x_{22} = cd\cdot dd = c \neq d\cdot dds = d = x_{12}\ \cdot x_{21}x_{22}s.$$

Finally, substitutions as below give the following

$$\begin{aligned}
x_{11}x_{21}\cdot x_{12}x_{22} &= dd\cdot cd = c \neq dd = d = x_{11}x_{21},\\
x_{11}x_{21}\cdot x_{12}x_{22} &= dd\cdot cd = c \neq dd = d = x_{11}x_{22},\\
x_{11}x_{21}\cdot x_{12}x_{22} &= cd\cdot dd = c \neq dd = d = x_{12}x_{21},\\
x_{11}x_{21}\cdot x_{12}x_{22} &= cd\cdot dd = c \neq dd = d = x_{12}x_{22}.
\end{aligned}$$

So there exist no terms t_1 and t_2 such that the identity (2.11) holds in $(G,\cdot)$. ■

Example 2.12 (see [11]). Let $(L,\cdot)$ be a left zero semigroup and let $(L',\cdot)$ denote the semigroup obtained from $(L,\cdot)$ by adjoining an identity element 1 such that $1\cdot 1=1$ and for each l in $L, 1\cdot l = l\cdot 1 = l$.

Now note that although $(L,\cdot)$ itself is entropic, $(L',\cdot)$ is not if L has at least two distinct elements l and m. Indeed, $(1\cdot l)\cdot(m\cdot 1) = l \neq m = (1\cdot m)\cdot(l\cdot l)$.

Note moreover that subsemigroups of $(L',\cdot)$ are exactly the subsets of L'. Hence the set of subsemigroups of $(L',\cdot)$ is closed under the complex product. But, similar arguments as in example 2.10 show that the algebra $(L',\cdot)$ does not satisfy the complex condition. ■

3. Idempotent law and modes of submodes

An algebra (A,Ω) is idempotent if each singleton is a subalgebra, i.e. the identity $x\dots x\,\omega = x$ is satisfied in (A,Ω) for each operation ω in Ω. The idempotent law plays a special role in complex algebras of subalgebras. It was shown in [10] that if (A,Ω) is idempotent and entropic, then (AS,Ω) satisfies idempotent law, too. In fact, if (A,Ω) is idempotent and AS is a subalgebra of (AUS,Ω), then (AS,Ω) is idempotent, too. Indeed, for a non-empty subalgebra (S,Ω) of (A,Ω) and ω in Ω, $S\dots S\omega = \{s_1\dots s_n\omega \mid s_i\in S\} \subseteq S$. Conversely, since (A,Ω) is idempotent, $S=\{s\mid s\in S\} = \{s\dots s\omega \mid s\in S\}\subseteq S\dots S\omega$, whence $S\dots S\omega = S$ and we have the idempotence of (AS,Ω).

Let $\boldsymbol{V}$ be a variety of idempotent Ω-algebras (A,Ω) such that AS is a subalgebra of (AUS,Ω). Let $\boldsymbol{VS}$ be the variety generated by the class $\{(AS,\Omega)\mid (A,\Omega)\in\boldsymbol{V}\}$. Since algebras of one-element subalgebras of algebras in $\boldsymbol{V}$ satisfy exactly the identities true in $\boldsymbol{V}$, it follows that $\boldsymbol{V}\subseteq\boldsymbol{VS}\subseteq\boldsymbol{VUS}$. The following example shows that the class $\{(AS,\Omega)\mid (A,\Omega)\in\boldsymbol{V}\}$ is not necessarily a variety.

Example 3.1. Let $\boldsymbol{Lz}$ be the variety of *left zero bands*, i.e. the variety of semigroups defined by the identity $xy = x$. We will show that the class $\{(AS, \cdot) \mid (A, \cdot) \in \boldsymbol{Lz}\}$ is closed neither under subalgebras, nor under products, nor under homomorphic images.

Let $(A, \cdot) = (\{a_1, a_2\}, \cdot)$ and $(B, \cdot) = (\{b_1, b_2\}, \cdot)$ be two element left zero bands. Then $AS = \{\{a_1\}, \{a_2\}, \{a_1, a_2\}\}$, $BS = \{\{b_1\}, \{b_2\}, \{b_1, b_2\}\}$ and $|AS \times BS| = 9$.

Note that $(\{\{a_1\}, \{a_2\}\}, \cdot)$ is a subalgebra of $(AS, \cdot)$. However $(\{\{a_1\}, \{a_2\}\}, \cdot)$ cannot be a complex algebra of subalgebras of some left zero band $(C, \cdot)$ because such an algebra must contain as its element the whole algebra $(C, \cdot)$ as well.

Note that for each left zero band $(C, \cdot)$ with $|C| = n$, the set CS is equal to the set of all non-empty subsets of C, and hence $|CS| = 2^n - 1$. So there is no algebra $(C, \cdot)$ in $\boldsymbol{Lz}$, such that $|AS \times BS| = |CS|$. Consequently, $(AS \times BS, \cdot)$ cannot be a complex algebra of subalgebras of some left zero band. Finally, let us define a mapping

$$h : AS \to BS; \{a_1\} \mapsto \{b_1\},\ \{a_2\} \mapsto \{b_2\},\ \{a_1, a_2\} \mapsto \{b_1\}.$$

It is easy to see, that h is an $\boldsymbol{Lz}$-homomorphism and $ASh = \{\{b_1\}, \{b_2\}\}$. But, as was observed above the algebra $(\{\{b_1\}, \{b_2\}\}, \cdot)$ cannot be a complex algebra of subalgebras of some left- zero band. ■

One of the more important examples of idempotent algebras satisfying complex condition is given by modes (i.e. idempotent and entropic algebras). The following property of complex algebras of modes was given in [10].

Proposition 3.2 (see [10]). *If (A, Ω) is a mode, then (AS, Ω) is again a mode satisfying each linear identity satisfied by (A, Ω).* ■

Example 3.3. Semigroup modes are *normal bands* (see [5], [10]). Since the variety of normal bands is specified by idempotent and linear identities, it follows that for each normal band $(A, \cdot)$, the algebra $(AS, \cdot)$ also lies in this variety. The same is true for any subvariety of this variety, in particular for varieties of semilattices, left and right-zero bands, rectangular bands, and left and right normal bands. ■

Example 3.4. Let $(L, \cdot)$ be a *differential* or $\boldsymbol{LIR}$-groupoid (see, e.g., [9], [12]), i.e. a groupoid mode satisfying the following reduction law

$$x \cdot yz = xy.$$

By Proposition 3.2. the algebra $(LS, \cdot)$ is a $\boldsymbol{LIR}$-groupoid, too. ∎

Example 3.5. *A commutative groupoid mode* (see [6], [13]) is a mode $(C, \cdot)$ satisfying the following commutative law

$$xy = yx.$$

Because the above identity is linear, the algebra $(CS, \cdot)$ is also a commutative groupoid mode. ∎

Let E be a vector space over a field F. Let $\underline{F}$ denote the set $\{\underline{r} \mid r \in F\}$ of binary operations $\underline{r}$ on E given by $\underline{r} : E \times E \to E; \qquad (x, y) \mapsto (1-r)x + ry$. The algebra $(E, \underline{F})$ is a mode ([10]).

Example 3.6. For a vector space E over the field $\mathbb{R}$ of real numbers, the subalgebras of $(E, \underline{\mathbb{R}})$ are the affine subspaces of E and the algebra $(ES, \underline{\mathbb{R}})$ is the mode of affine subspaces of the space E (see [10]). A more general case will be considered in Example 4.9. ∎

Note that it is not always the case that non-empty subalgebras of $(E, \underline{F})$, for an arbitrary field F, are affine subspaces of E. For example, if $F = GF(2)$, the Galois field of order 2, then the binary operations r are just the projections, and every subset of E is a subalgebra of $(E, \underline{F})$.

Example 3.7. Let I^0 be the open unit interval in the set $\mathbb{R}$ of real numbers (the interior of the closed unit interval [0,1]). One of the most important varieties of modes is the variety of barycentric algebras $(A, \underline{I}^0)$, the smallest variety containing all convex subsets of real affine spaces. The variety of barycentric algebras is defined by the
idempotence

$$xx\underline{p} = x,$$

skew commutativity

$$xy\underline{p} = yx\underline{(1-p)},$$

and skew associativity

$$xy\underline{p}z\underline{q} = x\,yz\underline{(q/(1-(1-p)(1-q)))}\,\underline{(1-(1-p)(1-q))},$$

for all p and q in I^0.

The skew commutativity and skew associativity are linear identities, so if $(A, \underline{I}^0)$ is a barycentric algebra, then so is $(AS, \underline{I}^0)$ (see [10], p. 215). ■

Finally, observe that the algebra (AS, Ω) may be a mode even if (A, Ω) is not.

Example 3.8. Let $(G, \cdot, ^{-1})$ be an abelian group defined as an inverse semigroup satisfying the following identities:

$$xy \cdot z = x \cdot yz,$$

$$xy = yx,$$
$$y^{-1}yx = x = xyy^{-1},$$
$$xx^{-1} = yy^{-1}.$$

By Example 2.9, the abelian group $(G, \cdot, ^{-1})$ satisfies the condition (CC). By Lemma 2.3, the set GS is closed under complex multiplication and inversion. Let $(G_1, \cdot, ^{-1})$, $(G_2, \cdot, ^{-1})$, $(G_3, \cdot, ^{-1})$ and $(G_4, \cdot, ^{-1})$ be arbitrary subgroups of $(G, \cdot, ^{-1})$. It is easy to see that

$$\begin{aligned} G_1^{-1} &= \left\{g^{-1} \mid g \in G_1\right\} \subseteq G_1 = \{g \mid g \in G_1\} \\ &= \left\{(g^{-1})^{-1} \mid g \in G_1\right\} \subseteq \left\{h^{-1} \mid h \in G_1\right\} = G_1^{-1}, \end{aligned}$$

and

$$\begin{aligned} G_1G_1 &= \{g\tilde{g} \mid g, \tilde{g} \in G_1\} \subseteq G_1 \\ &= \{g \mid g \in G_1\} = \left\{g\,(yy^{-1}) \mid g, y \in G_1\right\} \subseteq G_1G_1. \end{aligned}$$

Hence we have

$$G_1^{-1} = G_1,$$

and

$$G_1G_1 = G_1.$$

Consequently, complex multiplication and inversion are idempotent operations. Moreover

$$\begin{aligned} (G_1G_2)^{-1} &= \left\{(g_1g_2)^{-1} \mid g_1 \in G_1, g_2 \in G_2\right\} \\ &= \left\{g_1^{-1}g_2^{-1} \mid g_1 \in G_1,\ g_2 \in G_2\right\} = G_1^{-1}G_2^{-1} \end{aligned}$$

and since commutativity and associativity are linear identities, it follows that

$$G_1G_2G_3G_4 = G_1G_3G_2G_4.$$

Hence $(GS, \cdot, ^{-1})$ is an entropic algebra and consequently is a mode. ■

Let (A, Ω) be an Ω-algebra. Note, that for a non-empty subalgebra (S, Ω) of (A, Ω) and for any ω in Ω

$$S \dots S\omega = \{s_1 \dots s_n\omega \mid s_i \in S\} \subseteq S.$$

The inverse inclusion $S \subseteq S \dots S\omega$ is satisfied if for each s in S and for each ω in Ω there exist $s_1, \dots, s_n$ in S such that $s = s_1 \dots s_n\omega$. In particular this is true for idempotent algebras.

Example 3.9. Let $(G, f, \mathbf{1})$ be an algebra with one n-ary operation f and one nullary operation $\mathbf{1}$, satisfying at least one of the following identities:

$$x\mathbf{1} \dots \mathbf{1}f = x,\ 1x1 \dots 1f = x, \dots, 111 \dots 1xf = x.$$

Then for a subalgebra $(S, f, 1)$ of $(G, f, 1)$ and for each element s in S, $s = 1...s...1f$ is in $S...Sf$. It means that $S \subseteq S...Sf$ and consequently (GS, f) is an idempotent algebra. ■

Example 3.10. A groupoid $(Q, \cdot)$ is said to be a *quasigroup* if, in the equation $xy = z$, knowledge of any two of x, y, z specifies the third uniquely. It means that for each a in Q the *left multiplication* $L_a : Q \to Q;\ b \mapsto a \cdot b$ and the *right multiplication* $R_a : Q \to Q;\ b \mapsto b \cdot a$ are bijections.

A quasigroup may also be defined as an algebra $(Q, \cdot, /, \backslash)$ with three binary operations satisfying the following identities:

$$(xy)/y = x = (x/y)y,$$

$$x\backslash(xy) = y = x(x\backslash y).$$

Each non-empty subalgebra $(S, \cdot, /, \backslash)$ of $(Q, \cdot, /, \backslash)$ is a quasigroup and for each s in s there exist a and b in s such that $s = L_a(b) = a \cdot b$. Hence s is in $S \cdot S$. It follows that the groupoid $(QS, \cdot)$ is an idempotent algebra. ■

4. Identities in complex algebras of subalgebras

Let us assume that each variety $\boldsymbol{V}$ considered in this section is a variety of Ω-algebras (A, Ω) such that AS is a subalgebra of (AUS, Ω). For example $\boldsymbol{V}$ may be a variety of Ω-algebras which satisfies the complex condition.

It is easy to see that if a variety $\boldsymbol{V}$ is defined by a set of linear identities, then $\boldsymbol{VS} \subseteq \boldsymbol{V}$. As was noticed in Section 3, that for a variety $\boldsymbol{V}$ of idempotent Ω-algebras we have $\boldsymbol{V} \subseteq \boldsymbol{VS}$. Hence by Proposition 3.2, for the variety $\boldsymbol{M}$ of all modes of a given type, $\boldsymbol{MS} = \boldsymbol{M}$. The variety of commutative binary modes, the variety of $\boldsymbol{LIR}$-groupoids, all varieties of normal semigroups also have this property.

The following examples show that there exist varieties $\boldsymbol{V}$ such that the inclusion $\boldsymbol{VS} \subseteq \boldsymbol{V}$ does not hold.

Example 4.1 (cf. 3.10). Let $\boldsymbol{QM}$ be the variety of quasigroup modes and $(Q, \cdot, /, \backslash)$ be in $\boldsymbol{QM}$. Then for each q in Q, Q and $\{q\}$ are subalgebras of $(Q, \cdot, /, \backslash)$. Because for each q in Q the right multiplication $R_q Q \longrightarrow Q$; $a \mapsto a \cdot q$ is a bijection, the set $\{aq \,|\, a \in Q\}$ is equal to Q. On the other hand, for each q in Q, $Q \cdot \{q\} = \{a\,q \mid a \in Q\} = Q$. Hence $(QS, \cdot, /, \backslash)$ cannot be a quasigroup, since the left multiplication $L_Q : QS \longrightarrow QS$; $P \mapsto QP$ is not a bijection. ■

Example 4.2. Let $\boldsymbol{S}$ be the variety of symmetric groupoid modes (see [14]), i.e. the variety of binary modes defined by the symmetric identity

$$(S) \qquad xy^2 = x.$$

Note that for an $\boldsymbol{S}$-groupoid $(G, \cdot)$, the groupoid $(GS, \cdot)$ does not necessarily satisfy the symmetric identity (S). Indeed, consider the groupoid $(Z_4, \cdot) = (Z_4, \underline{2})$ (see Example 3.5). In $(Z_4 S, \cdot)$ we have $(\{0\} \cdot \{0,1,2,3\}) \cdot \{0,1,2,3\} = \{0,2\} \cdot \{0,1,2,3\} = \{0,2\} \neq \{0\}$. ■

As was shown by G. Grätzer and H. Lakser [3], linear identities true in $\boldsymbol{V}$ are satisfied by $\boldsymbol{VUS}$. Since, the variety $\boldsymbol{VS}$ is a subvariety of $\boldsymbol{VUS}$, linear identities true in $\boldsymbol{V}$ are also satisfied by $\boldsymbol{VS}$.

Example 4.3. Let $(L', \cdot)$ be the algebra defined in Example 2.12. Because each subset of L' is a subalgebra of $(L', \cdot)$, then by Proposition 1.2 each identity satisfied by $(L'S, \cdot)$ is a consequence of a linear identity true in $(L', \cdot)$. ■

Let $\boldsymbol{V}$ be a variety of idempotent Ω-algebras. Now we will describe certain non-linear identities satisfied by $\boldsymbol{VS}$-algebras which are consequences of non-linear identities true in $\boldsymbol{V}$.

Let $x_{11}\dots x_{1k_1}\dots x_{l1}\dots x_{lk_l}x_{(l+1)1}\dots x_{(l+1)k_{l+1}}\dots x_{n1}\dots x_{nk_n}p^*$ and $x_1\dots x_l x_{n+1}\dots x_{n+r}\,q^*$ be linear Ω-terms. Let for each $1\le i\le l$ and $n+1\le i\le n+r$, $x_{i1}\dots x_{ik_i}\,q_i$ be an Ω-term. By substituting $x_{i1}\dots x_{ik_i}\,q_i$ for x_i in the term q^* we obtain the following Ω-term q:

$$\begin{aligned}&x_{11}\dots x_{1k_1}\dots x_{l1}\dots x_{lk_l}x_{(n+1)1}\dots x_{(n+1)k_{n+1}}\dots x_{(n+r)1}\dots x_{(n+r)k_{n+r}}q :=\\&= x_{11}\dots x_{1k_1}q_1\dots x_{l1}\dots x_{lk_l}q_l x_{(n+1)1}\dots x_{(n+1)k_{n+1}}q_{n+1}\cdots\\&\qquad\qquad\cdots x_{(n+r)1}\dots x_{(n+r)k_{n+r}}q_{n+r}\,q^*.\end{aligned}$$

Let us consider the following equation

$$\begin{aligned}x_{11}\dots x_{1k_1}\dots x_{l1}\dots x_{lk_l}x_{(l+1)1}\dots x_{(l+1)k_{l+1}}\dots x_{n1}\dots x_{nk_n}p^*\\ = x_{11}\dots x_{1k_1}\dots x_{l1}\dots x_{lk_l}x_{(n+1)1}\dots x_{(n+1)k_{n+1}}\dots x_{(n+r)1}\dots x_{(n+r)k_{n+r}}q.\end{aligned}\tag{4.4}$$

Note that if for at least one i with $1\ \le i\le l,\ n+1\le i\le n+r,\ q_i$ is a non-linear term then (4.4.) is not linear, either.

Let p be the following Ω-term:

$$x_1\dots x_l x_{l+1}\dots x_n p := \underbrace{x_1\dots x_1}_{k_1-times}\dots\underbrace{x_l\dots x_l}_{k_l-times}\underbrace{x_{l+1}\dots x_{l+1}}_{k_{l+1}-times}\dots\underbrace{x_n\dots x_n}_{k_n-times}p^*.$$

By substituting x_i for $x_{i1},\dots,x_{ik_i}$ in (4.4.) and then by idempotency we obtain the equation

$$\begin{aligned}x_1\dots x_l x_{l+1}\dots x_n p &= \underbrace{x_1\dots x_1}_{k_1-times}\dots\underbrace{x_l\dots x_l}_{k_l-times}\underbrace{x_{l+1}\dots x_{l+1}}_{k_{l+1}-times}\dots\underbrace{x_n\dots x_n}_{k_n-times}p^*\\&= \underbrace{x_1\dots x_1}_{k_1-times}q_1\dots\underbrace{x_l\dots x_l}_{k_l-times}q_l\underbrace{x_{n+1}\dots x_{n+1}}_{k_{n+1}-times}q_{n+1}\dots\underbrace{x_{n+r}\dots x_{n+r}}_{k_{n+r}-times}q_{n+r}q^*\\&= x_1\dots x_l x_{n+1}\dots x_{n+r}q^*.\end{aligned}\tag{4.5}$$

If there is at least one i with $1\le i\le n$ and $k_i\ge 2$, then (4.5) is a non-linear equation.

Proposition 4.6. *Let* (A,Ω) *be a* $\boldsymbol{V}$*-algebra satisfying the identity* (4.4.) *for some linear* Ω*-term* $x_{11}\dots x_{1k_1}\dots x_{l1}\dots x_{lk_l}x_{(l+1)1}\dots x_{(l+1)k_{l+1}}\dots \dots x_{n1}\dots x_{nk_n}p^*$, *with* $k_i\geq 2$ *for* $1\leq i\leq n$, *and* Ω*-term*

$$x_{11}\dots x_{1k_1}\dots x_{l1}\dots x_{lk_l}x_{(n+1)1}\dots x_{(n+1)k_{n+1}}\dots x_{(n+r)1}\dots x_{(n+r)k_{n+r}}q:=$$
$$=x_{11}\dots x_{1k_1}q_1\dots x_{l1}\dots x_{lk_l}q_l x_{(n+1)1}\dots x_{(n+1)k_{n+1}}q_{n+1}\dots$$
$$\dots x_{(n+r)1}\dots x_{(n+r)k_{n+r}}q_{n+r}\,q^*,$$

where $x_1\dots x_l x_{n+1}\dots x_{n+r}\,q^*$ *is a linear term and for 1* $\leq i\leq l$ *and* $n+1\leq i\leq n+r$, $x_{i1}\dots x_{ik_i}\,q_i$ *are* Ω*-terms such that at least one of them is non-linear. Then the algebra* (AS,Ω) *satisfies the identity* (4.5).

Note that (4.5) is a non-linear consequence of a non-linear identity.

Proof. For $1\leq i\leq n+r$, let (A_i,Ω) be subalgebras of (A,Ω). Then

$$A_1\dots A_l A_{l+1}\dots A_n p=\underbrace{A_1\dots A_1}_{k_1-times}\dots\underbrace{A_l\dots A_l}_{k_l-times}\underbrace{A_{l+1}\dots A_{l+1}}_{k_{l+1}-times}\dots\underbrace{A_n\dots A_n}_{k_n-times}\,p^*$$
$$=\Big\{a_{11}\dots a_{1k_1}\dots a_{l1}\dots a_{lk_l}a_{(l+1)1}\dots a_{(l+1)k_{l+1}}\dots a_{n1}\dots a_{nk_n}\,p^*\mid a_{ij}\in A_i\Big\}$$
$$=\Big\{a_{11}\dots a_{1k_1}q_1\dots a_{l1}\dots a_{lk_1}q_l a_{n+11}\dots a_{(n+1)k_{n+1}}q_{n+1}\dots$$
$$\dots a_{(n+r)1}\dots a_{(n+r)k_{n+r}}q^*\mid a_{ij}\in A_i\Big\}\subseteq A_1\dots A_l A_{n+1}\dots A_{n+r}\,q^*,$$

because for each $1\leq i\leq l$ and $n+1\leq i\leq n+r$, $a_{i\,1}\dots a_{i\,k_i}\,q_i$ is in A_i.

On the other hand, since for $\boldsymbol{V}$-algebras (4.4) implies (4.5), one has

$$A_1\dots A_l A_{n+1}\dots A_{n+r}\,q^*=\{a_1\dots a_l a_{n+1}\dots a_{n+r}\,q^*\mid a_i\in A_i\}$$
$$=\{\,a_1\dots a_l a_{l+1}\dots a_n\,p\mid a_i\in A_i\}\subseteq A_1\dots A_l A_{l+1}\dots A_n p.$$

Hence $A_1\dots A_l A_{l+1}\dots A_n\,p=A_1\dots A_l A_{n+1}\dots A_{n+r}\,q^*$ and (4.5) is satisfied in (AS,Ω). ■

The next results are based on the following Proposition.

Proposition 4.7. *Let* $\boldsymbol{V}$ *be a variety of idempotent* Ω*-algebras. Let* $x_1\dots x_l x_{l+1}\dots x_n p$ *and* $x_1\dots x_l x_{n+1}\dots x_{n+r}q$ *be* Ω*-terms and let* $p{=}q$ *hold in* $\boldsymbol{VS}$. *Then there are* Ω*-terms* $\tilde{p}$ *and* q^* *such that:*

(i) $\tilde{p}$ *is a generalisation of* p;

(ii) q^* *is a linearisation of* q;

(iii) $\tilde{p} = q^*$ *holds in* $\boldsymbol{V}$;

(iv) $p = q$ *is a consequence of* $\tilde{p} = q^*$.

Proof. Let X be a countable infinite set of variables and $F_V(X)$ be the free $\boldsymbol{V}$-algebra over X. Let $x_{11} \dots x_{1k_1} \dots x_{l1} \dots x_{lk_l} x_{(l+1)1} \dots x_{(l+1)k_{l+1}} \dots \dots x_{n1} \dots\ x_{nk_n} p^*$ be a linearisation of the term p and $x_{11} \dots x_{1k_1} \dots x_{l1} \dots \dots x_{lk_l}\, x_{(n+1)1} \dots x_{(n+1)k_{n+1}} \dots x_{(n+r)1} \dots x_{(n+r)k_{n+r}} q^*$ be a linearisation of the term q, such that p is obtain from p^* and q is obtain from q^* by substituting x_i for $x_{i1}, \dots, x_{ik_i}$, with $1 \leq i \leq n + r$. Such terms p^* and q^* exist by Lemma 1.1.

Let (F_i, Ω) be the subalgebra of $F_V(X)$ generated by the set $\{x_{ij} \in X \mid j = 1, \dots, k_i\}$. Note that all these subalgebras are non empty and pairwise disjoint.

Obviously the element

$$b = x_{11} \dots x_{1\,k_1} \dots x_{l1} \dots x_{l\,k_l}\, x_{(n+1)1} \dots x_{(n+1)k_{n+1}} \dots x_{(n+r)1} \dots x_{(n+r)k_{n+r}}\, q^*$$

belongs to the set

$$\begin{aligned}
&\underbrace{F_1 \dots F_1}_{k_1-times} \dots \underbrace{F_l \dots F_l}_{k_1-times} \dots \underbrace{F_{n+1} \dots F_{n+1}}_{k_{n+1}-times} \dots \underbrace{F_{n+r} \dots F_{n+r}}_{k_{n+r}-times} q^* \\
&= F_l \dots F_1 F_{n+1} \dots F_{n+r} q F_1 \dots F_l F_{l+1} \dots F_n p = \\
&\underbrace{F_1 \dots F_1}_{k_1-times} \dots \underbrace{F_l \dots F_l}_{k_1-times} \dots \underbrace{F_{l+1} \dots F_{l+1}}_{k_{l+1}-times} \dots \underbrace{F_n \dots F_n}_{k_n-times}\, p^* \\
&= \Big\{ b_{11} \dots b_{1\,k_1} \dots b_{l1} \dots b_{lk_l} b_{(l+1)1} \dots b_{(l+1)k_{l+1}} \dots b_{n1} \dots b_{n\,k_n}\, p^* \big| \\
&b_{ij} \in F_i,\, i = 1, \dots, n,\ j = 1, \dots, k_i \big\}\,.
\end{aligned}$$

So the element b is in the subalgebra $F_1 \dots F_l F_{l+1} \dots F_n\, p$ and $b = b_{11} \dots b_{1\,k_1} \dots b_{l1} \dots b_{lk_l} b_{(l+1)1} \dots b_{(l+1)k_{l+1}} \dots b_{n1} \dots b_{n\,k_n}\, p^*$, where $b_{ij} = y_{i1} \dots y_{i\,m_{ij}}\, t_{ij}$ for some terms t_{ij} with $y_{i1}, \dots, y_{i\,m_{ij}}$ a subset of $x_{i1}, \dots, x_{i\,k_i}$ for each $i = 1, \dots, n$ *and* $j = 1, \dots, k_i$. It follows that the identity

$$y_{11} \dots y_{1m_{11}} t_{11} y_{11} \dots y_{1m_{12}} t_{12} \dots y_{n1} \dots y_{nm_{nk_n}} t_{nk_n}\, p^*$$

$$= x_{11} \dots x_{1\,k_1} \dots x_{l1} \dots x_{lk_l}\, x_{(n+1)1} \dots x_{(n+1)\,k_{n+1}} \dots x_{(n+r)1} \dots x_{(n+r)\,k_{n+r}}\, q^*$$

holds in $\boldsymbol{V}$.

Let $\tilde{p}$ be the term $y_{11} \dots y_{1m_{11}} t_{11} y_{11} \dots y_{1m_{12}} t_{12} \dots y_{n1} \dots y_{nm_{nk_n}}\, t_{nk_n}\, p^*$. Observe, that $\tilde{p}$ is a generalization of p, because by substituting x_i for $x_{i\,1}, x_{i\,2}, \dots, x_{i\,k_i}$ for each $1 \leq i \leq n$ in $\tilde{p}$, we obtain

$$\underbrace{x_1 \dots x_1}_{m_{11}-times} t_{11} \underbrace{x_1 \dots x_1}_{m_{12}-times} t_{12} \dots \underbrace{x_n \dots x_n}_{m_{nk_n}-times} t_{nk_n} p^*$$

$$= \underbrace{x_1 \dots x_1}_{k_1-times} \dots \underbrace{x_n \dots x_n}_{k_n-times} p^* = x_1 \dots x_n p.$$

It is clear that the identity $p = q$ is a consequence of $\tilde{p} = q^*$. This completes the proof of the Proposition. ∎

Note that the term $\tilde{p}$ in Proposition 4.7. may not be linear because Ω-terms t_{ij} do not need to be linear. But, if t_{ij} are linear for each $i = 1, \dots, n$ and $j = 1, \dots,\ k_i$, then the identity

$$y_{11} \dots y_{1m_{11}} t_{11} y_{11} \dots y_{1m_{12}} t_{12} \dots y_{n1} \dots y_{nm_{nk_n}} t_{nk_n} p^*$$

$$= x_{11} \dots x_{1\,k_1} \dots x_{l1} \dots x_{lk_l} x_{(n+1)1} \dots x_{(n+1)\,k_{n+1}} \dots x_{(n+r)1} \dots x_{(n+r)\,k_{n+r}}\, q^*$$

is linear. Hence in this case the identity $p = q$ is a consequence of a linear identity satisfied in $\boldsymbol{V}$.

We denote by $\operatorname{var}(q)$ the set of all variables occurring in q. As a corollary of Proposition 4.7. we get the following.

Corollary 4.8. *Let* $\boldsymbol{V}$ *be a variety of idempotent* Ω*-algebras. Let* $x_1 \dots x_l x_{l+1} \dots x_n p$ *be an* Ω*-term and* $x_1 \dots x_l x_{n+1} \dots x_{n+r} q^*$ *be a linear* Ω*-term. Let* $p = q^*$ *be an identity true in* $\boldsymbol{VS}$*. Then there are* Ω*-terms* $p^*, q_1, \dots, q_l, q_{n+1}, \dots, q_{n+r}$ *such that:*

(i) p^* *is a linearisation of* p*;*

(ii) $\operatorname{var}(q_i) \cap \operatorname{var}(q_j) = \emptyset$ *for each* $1 \leq i, j \leq l$ *and* $n + 1 \leq i, j \leq n + r$ *,* $i \neq j$*;*

(iii) *if* q *is obtained from* q^* *by substituting* q_i *for* x_i *where* $1 \leq i \leq l$ *and* $n + 1 \leq i \leq n + r$*, then* $p^* = q$ *holds in* $\boldsymbol{V}$*;*

(iv) $p = q^*$ *is a consequence of* $p^* = q$*.*

Proof. Let X be a countable infinite set of variables and let $F_V(X)$ be the free $\boldsymbol{V}$-algebra over X. Let $x_{11} \dots x_{1k_1} \dots x_{l1} \dots x_{lk_l} x_{(l+1)1} \dots x_{(l+1)k_{l+1}} \dots$ $\dots x_{n1} \dots x_{nk_n} p^*$ be a linearisation of p, such that p is obtained from p^* by substituting x_i for $x_{i1}, \dots, x_{ik_i}$ with $1 \leq i \leq n$. Let (F_i, Ω) be the subalgebra of $F_V(X)$ generated by the set $\{x_{ij} \in X \mid j = 1, \dots, k_i\}$, as in Proposition 4.7. Then

$$F_1 \dots F_l F_{n+1} \dots F_{n+r} q^* = F_1 \dots F_l F_{l+1} \dots F_n p$$
$$= \Big\{ a_{11} \dots a_{1k_1} \dots a_{l1} \dots a_{lk_l} a_{(l+1)1} \dots a_{(l+1)k_{l+1}} \dots a_{n1} \dots a_{n+k_n} p^* |$$
$$a_{ij} \in F_i, i = 1, ..., n, \quad j = 1, ..., k_i \Big\}.$$

Moreover the element $x_{11} \dots x_{1k_1} \dots x_{l1} \dots x_{lk_l} x_{(l+1)1} \dots x_{(l+1)k_{l+1}} \dots \dots x_{n1} \dots x_{nk_n} p^*$ belongs to the set $F_1 \dots F_l F_{n+1} \dots F_{n+r} q^*$. Thus

$$x_{11} \dots x_{1k_1} \dots x_{l1} \dots x_{lk_l} x_{(l+1)1} \dots x_{(l+1)k_{l+1}} \dots x_{n1} \dots x_{nk_n} p^*$$
$$= b_1 \dots b_l b_{n+1} \dots b_{n+r} q^*,$$

where for each $1 \leq i \leq l$ and $n+1 \leq i \leq n+r$, $b_i = y_{i1} \dots y_{im_i} q_i$ for some terms q_i with $\{y_{i1}, \dots, y_{im_i}\}$ a subset of $\{x_{i1}, \dots, x_{ik_i}\}$. So for each $1 \leq i \leq l$ and $n+1 \leq i \leq n+r$, b_i is in (F_i, Ω). Because the sets F_i are pairwise disjoint, $\mathrm{var}(q_i) \cap \mathrm{var}(q_j) = \emptyset$, for $1 \leq i \leq l$ and $n+1 \leq i \leq n+r$. ∎

Note, that if for each $1 \leq i \leq l$ and $n+1 \leq i \leq n+r$, terms q_i are linear then the identity $p^* = q$ is also linear.

Proposition 4.7 and Corollary 4.8 do not give an effective way to decide whether an identity $p = q$ true in $\boldsymbol{VS}$ is a consequence of linear and idempotent identities satisfied in $\boldsymbol{V}$. However they may provide a good tool to find an identity satisfied in $\boldsymbol{V}$ such that the identity $p = q$ is its consequence.

Example 4.9. Let R be a commutative ring with unity and let $(E, +, R)$ be a module over R. For each element r of R, define a binary operation by

$$\underline{r} : E \times E \to E; \quad (x, y) \mapsto (1 - r)x + ry,$$

and the Mal'cev operation P by

$$P : E \times E \times E \mapsto E; \quad (x, y, z) \mapsto x - y + z.$$

Then the algebra $(E, \underline{R}, P)$, with the ternary operation P and the set $\underline{R} = \{\underline{r} \,|\, r \in R\}$ of binary operations, has as its derived operations (those obtained from successive compositions of the basic operations P and $\underline{r}$ for r in R) precisely the affine combinations $r_1x_1 + r_2x_2 + \dots + r_nx_n$ with $r_1 + r_2 + \dots + r_n = 1$ of elements $x_1, x_2, \dots, x_n$ of E. The set

E together with all the idempotent term operations (considered as fundamental) is called the full idempotent reduct of the R-module $(E,+,R)$ ([17]). It follows that the algebra $(E,\underline{R},P)$ is equivalent to the full idempotent reduct $(E,\{r_1x_1+r_2x_2+\ldots+r_nx_n \mid r_1,r_2,\ldots,r_n\in R,\ \sum_{i=1}^{n} r_i=1\})$ of the module $(E,+,R)$. Note that the algebra (E,$\underline{\text{R}}$,P) has the affine group as its group of automorphisms, and may thus be identified with the affine geometry ([7]). Carrying out this identification, we will refer to the algebra $(E,\underline{R},P)$ as an *affine space over* R or an *affine* R*-space*. (Note, however, that such algebras have also been called *affine modules*).

It is well known that the class of affine spaces over a commutative ring r with unity is a variety. According to [10], this variety is equivalent to the variety $\underline{\underline{R}}$ of *Mal'cev modes* $(E,\underline{R},P)$, algebras with the ternary Mal'cev operation P and one binary operation $\underline{r}$ for each r in R, satisfying the identities defining modes and for all p, q, r in r the identities:

(A1) $xyx\,P = yx\,\underline{2}\,,$

(A2) $xy\underline{p}\,xy\underline{q}\,\underline{r} = xy\underline{pqr},$

(A3) $xy\underline{p}\,xy\underline{q}\,xy\underline{r}\,P = xy\underline{pqr\,P},$

(A4) $xy\,\underline{0} = yx\,\underline{1} = x.$ ■

The algebra $(ES,\underline{R},P)$ is again a mode satisfying each linear identity satisfied by $(E,\underline{R},P)$. In general the algebra $(ES,\underline{R},P)$ is not an affine space.

Note that the identity (A4) is linear. Hence it is clearly satisfied in the variety $\underline{\underline{R}}\boldsymbol{S}$. We will show that for some $p,q,$ r in R the identities (A1) – (A3) are also satisfied in the variety $\underline{\underline{R}}\boldsymbol{S}$. First we will prove the following lemma.

Lemma 4.10. *Suppose that for some* p, q, r *in* R*, the identity* (A2) *is satisfied in* $\underline{\underline{R}}\boldsymbol{S}$*. Then the identity* (A2) *is a consequence of a linear identity satisfied in* $\underline{\underline{R}}$*.*

Proof. To prove the lemma we will use a similar method as in the proof of Proposition 4.7. Let $Z=\{x_1,x_2,y_1,y_2\}$ be a set of variables and $F_{\underline{\underline{R}}}(Z)$ be the free $\underline{\underline{R}}$-algebra over Z. Note that the term

$$x_1x_2y_1y_2t^* \ := x_1y_1\underline{p}x_2y_2\underline{q}\underline{r}$$

is a linearisation of the term $xyt = xy\underline{p}\,xy\underline{q}\,\underline{r}$, where t is obtained from t^* by substituting x for x_1 and x_2 and y for y_1 and y_2.

Let $(A_1, \underline{\underline{R}}, P)$ be the subalgebra of $F_{\underline{\underline{R}}}(Z)$ generated by the set $\{x_1, x_2\}$ and $(A_2, \underline{\underline{R}}, P)$ be the subalgebra of $F_{\underline{\underline{R}}}(Z)$ generated by the set $\{y_1,\, y_2\}$.

If for some $p,\ q,\ r$ in R, (A2) is satisfied in $\underline{\underline{R}}\boldsymbol{S}$, then sets $A_1A_2t = A_1A_1A_2A_2t^*$ and $A_1A_2\underline{\underline{pqr}}$ are equal. Then obviously the element $b = x_1x_2y_1y_2t^*$ belongs to the set

$$A_1A_2\underline{\underline{pqr}} = \left\{a_1a_2\underline{\underline{pqr}} \mid a_1 \in A_1,\ a_2 \in A_2\right\}.$$

Hence $b = a_1a_2\underline{\underline{pqr}}$, where $a_1 = x_1x_2t_1$ and $a_2 = y_1y_2t_2$ for some term t_1 in $(A_1, \underline{\underline{R}}, P)$ and term t_2 in $(A_2, \underline{\underline{R}}, P)$. Recall that all binary term operations of $(E, \underline{\underline{R}}, P)$ have the form $wx_1 + (1-w)x_2$, for some w in R. This implies that there are w and v in r such that the following identity

$$x_1x_2\underline{w}y_1y_2\underline{v}\underline{\underline{pqr}} = x_1x_2y_1y_2t^* = x_1y_1\underline{p}x_2y_2\underline{qr} \tag{4.11}$$

holds in $\underline{\underline{R}}$. Consequently, if for some $p,\ q,\ r$ in R, (A2) is satisfied in $\underline{\underline{R}}\boldsymbol{S}$, then it is a consequence of the linear identity (4.11) satisfied in $\underline{\underline{R}}$. ■

Corollary 4.12. *For $p,\ q,\ r$ in R such that $((1-r)p + rq)$ and $(1-(1-r)p - rq)$ are invertible, the identity* (A2) *is satisfied in $\underline{\underline{R}}\boldsymbol{S}$.*

Proof. Note that for any $\underline{\underline{R}}$-algebra $(A, \underline{\underline{R}}, P)$ and x_1, x_2, y_1, y_2 in a $x_1x_2\underline{w}y_1y_2\underline{v}\underline{\underline{pqr}} = (1 - p(1-r) - qr)(1-w)x_1 + (1 - p(1-r) - qr)wx_2$ $+(p(1-r) + qr)(1-v)y_1 + (p(1-r) + qr)vy_2$ and

$$x_1y_1\underline{p}x_2y_2\underline{qr} = (1-r)(1-p)x_1 + r(1-q)x_2 + (1-r)py_1 + rqy_2.$$

If $(1-r)p + rq$ and $1 - ((1-r)p + rq)$ are invertible in R, then for $w = (1-q)r(1 - p(1-r) - qr)^{-1}$ and $v = qr(p(1-r) + qr)^{-1}$ the identity (4.11) is satisfied in $\underline{\underline{R}}$. It follows that for invertible $((1-r)p + rq)$ and $(1 - ((1-r) + rq))$, (A2) is a consequence of the linear identity (4.11) and consequently, it is satisfied in $\underline{\underline{R}}\boldsymbol{S}$. ■

Lemma 4.13. *Suppose that for some $p,\ q,\ r$ in R, the identity* (A3) *is satisfied in $\underline{\underline{R}}\boldsymbol{S}$. Then the identity* (A3) *is a consequence of a non-linear identity of the type* (4.4) *satisfied in $\underline{\underline{R}}$.*

Proof. Let $Z = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ be a set of variables and $F_{\underline{\underline{R}}}(Z)$ be the free $\underline{\underline{R}}$-algebra over Z. Note that the term

$$x_1x_2x_3y_1y_2y_3t^* := x_1y_1\underline{p}x_2y_2\underline{q}x_3y_3\underline{r}P$$

is a linearisation of the term $xyt := xy\underline{p}\,xy\underline{q}\,xy\underline{r}P$, where t is obtained from t^* by substituting x for x_1, x_2 and x_3 and y for y_1, y_2 and y_3.

Let $(A_1, \underline{R}, P)$ be the subalgebra of $F_{\underline{\underline{R}}}(Z)$ generated by the set $\{x_1,\ x_2,\ x_3\}$ and $(A_2, \underline{R}, P)$ be the subalgebra of $F_{\underline{\underline{R}}}(Z)$ generated by the set $\{y_1,\ y_2,\ y_3\}$.

If for some p, q, r in R, (A3) is satisfied in $\underline{\underline{R}}\boldsymbol{S}$, then sets $A_1A_2t = A_1A_1A_1A_2A_2A_2t^*$ and $A_1A_2\underline{pqrP}$ are equal. Then obviously the element $b = x_1x_2x_3y_1y_2y_3t^*$ belongs to the set

$$A_1A_2\underline{pqrP} = \Big\{a_1a_2\underline{pqrP} \,|\, a_1 \in A_1,\ a_2 \in A_2\Big\}.$$

Hence $b = a_1a_2\underline{pqrP}$, where $a_1 = x_1x_2x_3t_1$ and $a_2 = y_1y_2y_3t_2$ for some term t_1 in $(A_1, \underline{R}, P)$ and term t_2 in $(A_2, \underline{R}, P)$. Recall that all ternary term operations of $(E, \underline{R}, P)$ have the form, for some p_1 and p_2 in R. This implies that there are r_1, r_2, s_1, s_2 in r such that the following identity

$$x_3x_1\underline{r_1}x_3x_3x_2\underline{r_2}Py_3y_1\underline{s_1}y_3y_3y_2\underline{s_2}P\underline{pqrP} = x_1y_1\underline{p}x_2y_2\underline{q}x_3y_3\underline{r}P$$

holds in $\underline{\underline{R}}$.
Let

$$x_1x_2x_3y_1y_2y_3p^* := x_1y_1\underline{p}x_2y_2\underline{q}x_3y_3\underline{r}P,$$
$$x_1x_2x_3q_1 := x_3x_1\underline{r_1}x_3x_3x_2\underline{r_2}P,$$
$$y_1y_2y_3q_2 := y_3y_1\underline{s_1}y_3y_3y_2\underline{s_2}P$$

and

$$xyq^* := xy\underline{pqrP}.$$

Then, if for some p, q, r in R, (A3) is satisfied in $\underline{\underline{R}}\boldsymbol{S}$, then it is a consequence of the non- linear identity

$$x_1x_2x_3y_1y_2y_3p^* = x_1x_2x_3q_1y_1y_2y_3q_2q^* \tag{4.14}$$

satisfied in $\underline{\underline{R}}$. ∎

Corollary 4.15. *For p, q, r in R such that $(p-q+r)$ and $(1-p+q-r)$ are invertible, the identity* (A3) *is satisfied in* $\underline{\underline{R}}\boldsymbol{S}$.

Proof. Now note that for any $\underline{\underline{R}}$-algebra (A, R, P) and x_1, x_2, x_3, y_1, y_2, y_3 in A

$$x_1x_2x_3y_1y_2y_3p^* = x_1y_1\underline{p}x_2y_2\underline{q}x_3y_3\underline{r}P$$
$$= (1-p)x_1 - (1-q)x_2 + (1-r)x_3 + py_1 - qy_2 + ry_3$$

and

$$x_1x_2x_3q_1y_1y_2y_3q_2q^* = x_3x_1\underline{r_1}x_3x_3x_2\underline{r_2}Py_3y_1\underline{s_1}y_3y_3y_2\underline{s_2}P\underline{pqrP}$$

$$= (1-p+q-r)r_1x_1 + (1-p+q-r)r_2x_2 + (1-p+q-r)(1-r_1-r_2)x_3+$$

$$+(p-q+r)s_1y_1 + (p-q+r)s_2y_2 + (p-q+r)(1-s_1-s_2)y_3.$$

If $(p-q+r)$ and $(1-p+q-r)$ are invertible in R, then for $r_1 = (1-p)(1-p+q-r)^{-1}$, $r_2 = -(1-q)(1-p+q-r)^{-1}$, $s_1 = p(p-q+r)^{-1}$ and $s_2 = -q(p-q+r)^{-1}$ the identity (4.14.) is satisfied in $\underline{\underline{R}}$.

By substituting x for x_1, x_2 and x_3 and y for y_1, y_2 and y_3 in (4.14) we obtain the identity (A3). Hence for invertible $(p-q+r)$ and $(1-p+q-r)$, (A3) is a consequence of the non-linear identity (4.14) of the type (4.4) true in $\underline{\underline{R}}$. By Proposition 4.6. the identity (A3) is also satisfied in $\underline{\underline{R}}\boldsymbol{S}$. ■

Now we are left with the following question. Is the identity (4.14) a consequence of some linear identity true in $\underline{\underline{R}}$?

Lemma 4.16. *Suppose that the identity* (A1) *is satisfied in* $\underline{\underline{R}}\boldsymbol{S}$. *Then the identity* (A1) *is a consequence of a linear identity satisfied in* $\underline{\underline{R}}$.

Proof. Let $Z = \{x_1, x_2, y\}$ be a set of variables and $F_{\underline{\underline{R}}}(Z)$ be the free $\underline{\underline{R}}$-algebra over Z. Note that the term

$$x_1x_2yt^* := x_1yx_2P$$

is a linearisation of the term $xyt := xyxP$, where t is obtained from t^* by substituting x for x_1 and x_2.

Let $(A, \underline{R}, P)$ be the subalgebra of $F_{\underline{\underline{R}}}(Z)$ generated by the set $\{x_1,\, x_2\}$ and $(y, \underline{R}, P)$ be one element subalgebra of $F_{\underline{\underline{R}}}(Z)$.

If the identity (A1) is satisfied in $\underline{\underline{R}}\boldsymbol{S}$, then sets $Ayt = AAyt^*$ and $yA\underline{2}$ are equal. Then obviously the element $b = x_1x_2yt^*$ belongs to the set

$$\{y\}A\underline{2} = \{ya\underline{2} \mid a \in A\}.$$

Hence $b = ya\underline{2}$, where $a = x_1x_2s$ for some term s in $(A, \underline{R}, P)$. This implies that there is r in R such that the following identity

$$x_1yx_2P = yx_1x_2\underline{r2} \tag{4.17}$$

holds in $\underline{\underline{R}}$. Consequently, if (A1) is satisfied in $\underline{\underline{R}}\boldsymbol{S}$, then it is a consequence of the linear identity (4.17) satisfied in $\underline{\underline{R}}$. ■

Corollary 4.18. *If the element* 2 *of a ring* r *is invertible, the identity* (A1) *is satisfied in* $\underline{\underline{R}}\boldsymbol{S}$.

Proof. It is easy to see, that for any $\underline{\underline{R}}$-algebra (A, R, P) and x_1, x_2, y in A

$$x_1 y x_2 P = x_1 - y + x_2$$

and

$$y x_1 x_2 \underline{r2} = 2(1-r)x_1 + 2r x_2 - y.$$

If 2 is invertible in R, then for $r = 2^{-1}$ the identity (4.17) is satisfied in $\underline{\underline{R}}$. It follows that for invertible 2, (A1) is a consequence of the linear identity (4.17) and consequently it is satisfied in $\underline{\underline{R}}\boldsymbol{S}$. ∎

As was shown in [10], (see p. 256) if the element 2 of a ring R is invertible, the identities (A1) - (A4) for $\underline{\underline{R}}$ may be reduced to (A2) and (A4). Consequently, for p, q, r and 2 in R such that $((1-r)p + rq)$, $(1-(1-r)p-rq)$ and 2 are invertible, identities (A2) and (A4) are satisfied in $\underline{\underline{R}}\boldsymbol{S}$.

The following open question arised during preparation of this paper. Let $\boldsymbol{V}$ be a variety of idempotent algebras. Is it true that $\boldsymbol{V} = \boldsymbol{VS}$ if and only if $\boldsymbol{V}$ is defined by a set of linear and idempotent identities?

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