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Interval Bilattices and Some Other Simple Bilattices

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Abstract. In a number of papers M.Ginsberg introduced algebras called bilattices having two separate lattice structure and one additional basic unary operation. They originated as an algebraization of some non-classical logics that arise in artificial intelligence and knowledge-based logic programming. In this paper we introduce some new class of bilattices which originate from interval lattices and show that each of them is simple. A known simple lattices are used to give other examples of simple bilattices. We also describe simple bilattices satisfying some additional identities so called P-bilattices (or interlaced bilattices).

1 Introduction

In a number of papers M.Ginsberg [6], [7] introduced algebras called bilattices having two separate lattice structure and one additional basic unary operation acting on both lattices in a very regular way. Bilattices originated as an algebraization of some non-classical logics that arise in artificial intelligence and knowledge-based logic programming. The two lattice orderings of a bilattice may be viewed as the degrees of truth and knowledge of possible events. In general, both lattice structures of a bilattice are "almost" not connected to each other. However bilattices appearing in applications usually satisfy some additional conditions. Bilattices were also investigated by Fitting, Romanowska, Avron, Mobasher, Pigozzi, Slutzki, Voutsadakis, Pynko and others.

In [5] M.Fitting described a structure based on intervals of a lattice with one lattice and one semilattice orderings. The main purpose of this paper is to introduce a new class of bilattices which also originates from interval lattices and to show that such bilattices are simple.

In Sect. 1 we collect some useful facts about interval lattices, and bilattices. Interval bilattices, a new class of bilattices which originate from interval lattices, are described in Sect. 2. In Sect. 3 we show that all interval bilattices are simple. In Sect. 4 a known simple lattices are used to give other examples of simple bilattices. Finally, we describe simple P-bilattices (bilattices satisfying some additional identities which were introduced and investigated in [13]. P-bilattices were also investigated under the name of "interlaced" bilattices by other authors.).

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2 Preliminaries

Let $\mathbf{L} = (L, \vee, \wedge)$ be a lattice with the ordering relation $\leq$. The sublattice $[a,b] = \{x \in L | a \leq x \leq b\}$ of $\mathbf{L}$ is called *an interval* with the end-points a and b. The empty subset of L is also considered as an interval $[\ ,\ ]$. The family of all intervals in $\mathbf{L}$ is denoted by $IntL$.
$IntL$ is a lattice $\mathbf{IntL_k} = (IntL, \circ, +)$ under set inclusion $\leq_k$. We have that $[a,b] \leq_k [c,d]$ if and only if $a \geq c$ and $b \leq d$. The lattice operations are defined by the following formulas:

$$[a,b] \circ [c,d] := \begin{cases} [a \vee c, b \wedge d] & \text{if } a \vee c \leq b \wedge d \\ [\ ,\ ] & \text{otherwise} \end{cases} \tag{1}$$

$$[a,b] + [c,d] := [a \wedge c, b \vee d]\ . \tag{2}$$

The lattice $\mathbf{IntL_k}$ will be called *an interval lattice*. Note that the empty interval $[\ ,\ ]$ is the least element in this lattice. The interval lattices were studied by W.Duthie [2] and by V.Igoshin [8].

Definition 1. *A bilattice* $\mathbf{B} = (B, \wedge, \vee, \circ, +, \neg, 0_1, 1_1, 0_2, 1_2)$ *is an algebra such that*

1. $(B, \wedge, \vee, 0_1, 1_1)$ *and* $(B, \circ, +, 0_2, 1_2)$ *are bounded lattices.*
2. *A negation* $\neg : B \to B$ *is an unary operation on* B *satisfying for all* $x, y \in B$ *the identities:*

$$\neg\neg x = x, \tag{3}$$

$$\neg(x \vee y) = \neg x \wedge \neg y \tag{4}$$

$$\neg(x \wedge y) = \neg x \vee \neg y \tag{5}$$

$$\neg(x + y) = \neg x + \neg y \tag{6}$$

$$\neg(x \circ y) = \neg x \circ \neg y\ . \tag{7}$$

Definition 2. *A bilattice* $\mathbf{B} = (B, \wedge, \vee, \circ, +, \neg, 0_1, 1_1, 0_2, 1_2)$ *satisfying the following identities:*

$$((x \wedge y) \circ z) \wedge (y \circ z) = (x \wedge y) \circ z \tag{8}$$

$$((x \circ y) \wedge z) \circ (y \wedge z) = (x \circ y) \wedge z \tag{9}$$

$$((x \wedge y) + z) \wedge (y + z) = (x \wedge y) + z \tag{10}$$

$$((x + y) \wedge z) + (y \wedge z) = (x + y) \wedge z \tag{11}$$

will be called P-bilattice.

The following results were proved in [13]. (See also [1], [4] and [12].)

Theorem 1. *Let* $\mathbf{L} = (L, \wedge, \vee, 0, 1)$ *be a bounded lattice. On a set* $L \times L$ *we define an algebra* $\mathbf{B}(\mathbf{L}) = (L \times L, \wedge, \vee, \circ, +, \neg, 0_1, 1_1, 0_2, 1_2)$ *as follows:*
for all $(x, x'), (y, y')$ *in* $L \times L$,

$$(x, x') \wedge (y, y') := (x \wedge y, x' \vee y') \tag{12}$$

$$(x, x') \vee (y, y') := (x \vee y, x' \wedge y') \tag{13}$$

$$(x, x') \circ (y, y') := (x \wedge y, x' \wedge y') \tag{14}$$

$$(x, x') + (y, y') := (x \vee y, x' \vee y') \tag{15}$$

$$\neg(x, y) := (y, x) \tag{16}$$

$$0_1 := (0, 1), \quad 1_1 := (1, 0), \quad 0_2 := (0, 0), \quad 1_2 := (1, 1) \ . \tag{17}$$

The algebra $\mathbf{B}(\mathbf{L})$ *is a P-bilattice.*

The bilattice $\mathbf{B}(\mathbf{L})$ will be called *a product bilattice associated with the lattice* $\mathbf{L}$.

Theorem 2. *An algebra* $\mathbf{B} = (B, \wedge, \vee, \circ, +, \neg, 0_1, 1_1, 0_2, 1_2)$ *is a P-bilattice if and only if there is a bounded lattice* $\mathbf{L} = (L, \wedge, \vee, 0, 1)$, *such that* $\mathbf{B}$ *is isomorphic to the product bilattice* $\mathbf{B}(\mathbf{L})$ *associated with the lattice* $\mathbf{L}$.

3 A construction of interval bilattices

Let $\mathbf{L} = (L, \wedge, \vee, \neg, 0, 1)$ be a bounded lattice with a polarity $\neg : L \to L$ and ordering relation $\leq$. Let $Int_0 L$ be the family of all intervals of $\mathbf{L}$ excluding the empty interval. In $Int_0 L$ define two binary operations as follows:

$$[a, b] \wedge [c, d] := [a \wedge c, b \wedge d] \tag{18}$$

$$[a, b] \vee [c, d] := [a \vee c, b \vee d] \ . \tag{19}$$

Proposition 1. *The algebra* $\mathbf{Int_0 L_t} = (Int_0 L, \wedge, \vee, [0, 0], [1, 1])$ *is a bounded lattice.*

Proof. It is evident that if $a \leq b$ and $c \leq d$ then $a \wedge c \leq b \wedge d$ and $a \vee c \leq d \vee d$.
Therefore
$[a, b] \vee [a, b] = [a \vee a, b \vee, b] = [a, b]$,
$[a, b] \vee [c, d] = [a \vee c, b \vee d] = [c \vee a, d \vee b] = [c, d] \vee [a, b]$,
$([a, b] \vee [c, d]) \vee [e, f] = [a \vee c, b \vee d] \vee [e, f] = [a \vee c \vee e, b \vee d \vee f] =$
$= [a, b] \vee [c \vee e, d \vee f] = [a, b] \vee ([c, d] \vee [e, f])$,
$[a, b] \vee ([a, b] \wedge [c, d]) = [a \vee b] \vee [a \wedge c, b \wedge d] = [a \vee (a \wedge c), b \vee (b \wedge d)] = [a, b]$.
Simillarly, we can show that the second operation $\wedge$ is idempotent, commutative and associative.
Moreover, for all $[a, b]$ in $Int_0 L$:
$[a, b] \vee [1, 1] = [a \vee 1, b \vee 1] = [1, 1]$,
$[a, b] \wedge [0, 0] = [a \wedge 0, b \wedge 0] = [0, 0]$.
It follows that $\mathbf{Int_0 L_t}$ is a bounded lattice with the least element $[0, 0]$ and the greatest element $[1, 1]$. ∎

Corollary 1. *The ordering relation* $\leq_t$ *of a lattice* $\mathbf{Int_0L_t}$ *is defined by*

$$[a,b] \leq_t [c,d] \text{ if and only if } a \leq c \text{ and } b \leq d \ . \tag{20}$$

Now let $\mathbf{IntL_t}$ be the extension of the partially ordered set $(Int_0L, \leq_t)$ by the empty interval $[\ ,\ ]$. By definition $[\ ,\ ]$ covers $[0,0]$, is covered by $[1,1]$ and is not comparable with any other interval of $\mathbf{L}$.
In $IntL$ we define one unary operation $\neg : IntL \to IntL$ by

$$\neg[a,b] := [\neg b, \neg a] \tag{21}$$

$$\neg[\ ,\ ] := [\ ,\ ] \ . \tag{22}$$

Note that if $a \leq b$ then $\neg b \leq \neg a$, whence $[\neg b, \neg a]$ is an interval in $\mathbf{L}$.

Proposition 2. *The algebra* $\mathbf{BIntL} = (IntL, \wedge, \vee, \circ, +, \neg, [0,0], [1,1], [\ ,\ ], [0,1])$ *is a bilattice.*

Proof. Since the lattice $\mathbf{L} = (L, \wedge, \vee, \neg, 0, 1)$ is bounded, the lattice $\mathbf{IntL_k}$ has the greatest element $[0,1]$. Hence $\mathbf{IntL_t} = (IntL, \wedge, \vee, [0,0], [1,1])$ and $\mathbf{IntL_k} = (IntL, \circ, +, [\ ,\ ], [0,1])$ are bounded lattices.

Note that the unary operation $\neg : IntL \to IntL$ is an involution

$$\neg\neg[a,b] = \neg[\neg b, \neg a] = [a,b] \ . \tag{23}$$

It reverses the order $\leq_t$ and preserves the order $\leq_k$.
Indeed, if $[a,b] \leq_t [c,d]$ then $a \leq c$ and $b \leq d$ implying $\neg a \geq \neg c$ and $\neg b \geq \neg d$.
Hence $[\neg b, \neg a] \geq_t [\neg d, \neg c]$ and $\neg[a,b] \geq_t \neg[c,d]$.
If $[a,b] \leq_k [c,d]$ then $[a,b] + [c,d] = [a \wedge c, b \vee d] = [c,d]$ implying
$\neg[a \wedge c, b \vee d] = [\neg(b \vee d), \neg(a \wedge c)] = [\neg b \wedge \neg d, \neg a \vee \neg c] =$
$= [\neg b, \neg a] + [\neg d, \neg c] = \neg[a,b] + \neg[c,d] = \neg[c,d]$.
Hence, $\neg[a,b] \leq_k \neg[c,d]$.
It follows that $\mathbf{BIntL}$ is a bilattice. □

Definition 3. *The bilattice* $\mathbf{BIntL} = (IntL, \wedge, \vee, \circ, +, \neg, [0,0], [1,1], [\ ,\ ], [0,1])$ *will be called an interval bilattice.*

Example 1. Let $(\{0,1\}, \wedge, \vee, \neg, 0, 1)$ be two element chain. In this case $IntL = \{[\ ,\ ], [0,0], [1,1], [0,1]\}$ and the interval bilattice $\mathbf{BIntL}$ is isomorphic to four element distributive bilattice $\mathbf{B_4}$.

Lemma 1. *Let* $\mathbf{L} = (L, \wedge, \vee, \neg, 0, 1)$ *be an arbitrary bounded polarity lattice with more than two elements. Then the interval bilattice* $\mathbf{BIntL} = (IntL, \wedge, \vee, \circ, +, \neg, [0,0], [1,1], [\ ,\ ], [0,1])$ *is not a P-bilattice.*

Proof. Let $a \neq 0$ and $a \neq 1$ be an element in L. We have
$(([\ ,\ ] \wedge [0,a]) \circ [a,1]) \wedge ([0,a] \circ [a,1]) = ([0,0] \circ [a,1]) \wedge [0 \vee a, a \wedge 1] =$
$= [0 \vee a, 0 \wedge 1] \wedge [a,a] = [\ ,\ ] \wedge [a,a] = [0,0]$
and

$([\ ,\ ] \wedge [0, a]) \circ [a, 1] = [0, 0] \circ [a, 1] = [\ ,\ ]$.
It follows that identity

$$((x \wedge y) \circ z) \wedge (y \circ z) = (x \wedge y) \circ z \tag{24}$$

is not satisfied in the interval bilattice **BIntL**. Hence **BIntL** is not a P-bilattice. □

4 Congruence lattices of interval bilattices

In this section we give a characterization of the lattice of congruence relations of an interval bilattice. Let **BIntL** be an interval bilattice obtained from some bounded polarity lattice **L**$= (L, \wedge, \vee, \neg, 0, 1)$. The lattice of congruence relations of **BIntL** will be denoted by Con**BIntL**.

Theorem 3. *For any bounded polarity lattice* **L**$= (L, \wedge, \vee, \neg, 0, 1)$*, the congruence lattice* Con**BIntL** *is isomorphic to the 2-element chain.*

Proof. Let a, b, c, d be in L, $d \neq 0$, $d \neq 1$ and θ be a non-zero congruence relation of **BIntL**.
First note that if $[a, b] \leq_k [c, d]$ then $c \leq a \leq b \leq d$ and
$[a, b] \circ [d, d] = [a \vee d, b \wedge d] = [\ ,\ ]$
and
$[c, d] \circ [d, d] = [c \vee d, d \wedge d] = [d, d]$.
Therefore it is easy to see that if $[a, b] \leq_k [c, d]$ and $[a, b]$ and $[c, d]$ are in θ then $[d, d]$ and $[\ ,\ ]$ are in θ too.
It is well known that in a lattice $(L, \wedge, \vee)$, elements x and y are in some congruence relation φ if and only if $x \wedge y$ and $x \vee y$ are in φ.

Since by definition intervals $[d, d]$ and $[\ ,\ ]$ are not comparable with respect to $\leq_t$, it is clear that if $[d, d]$ and $[\ ,\ ]$ lie in θ, then θ also contains $[0, 0]$ and $[1, 1]$, and hence all other intervals.

Moreover, if [1,1] and $[\ ,\ ]$ (or [0,0] and $[\ ,\ ]$) are in θ, then $\neg[1, 1] = [0, 0]$ and $\neg[\ ,\ ] = [\ ,\ ]$ (or $\neg[0, 0] = [1, 1]$ and $[\ ,\ ]$) are in θ too.

It follows that θ is the largest trivial congruence relation. □

Corollary 2. *For any bounded polarity lattice* **L***, the interval bilattice* **BIntL** *is simple.*

5 Further examples of simple bilattices

In this section we give further examples of simple bilattices.

Example 2. Let n, i and j be natural numbers and let
$Q^n := \{(0, 0), (1, 1)\} \cup \{(2^{-i}, 0) | 0 \leq i \leq n - 1\} \cup \{(0, 2^{-i}) | 0 \leq i \leq n - 1\} \cup$
$\cup \{(2^{-i}, 2^{-j}) | 1 \leq i, j \leq n$ and $|i - j| \leq 1\}$.
Let $\mathbf{Q_k^n} = (Q^n, \leq_k)$ be the set Q^n with the order $\leq_k$ defined by

$$(a_1, b_1) \leq_k (a_2, b_2) \text{ iff } a_1 \leq a_2 \text{ and } b_1 \leq b_2 \ . \tag{25}$$

And let $\mathbf{Q^n_t} = (Q^n, \leq_t)$ be the set Q^n with the order $\leq_t$ defined by

$$(a_1, b_1) \leq_t (a_2, b_2) \text{ iff } a_1 \leq a_2 \text{ and } b_1 \geq b_2 \ . \tag{26}$$

Obviously, $\mathbf{Q^n_k}$ and $\mathbf{Q^n_t}$ are lattices.

The lattices $\mathbf{Q^n_k}$ and $\mathbf{Q^n_t}$ with the unary operation $\neg(a, b) := (b, a)$ form the bilattice $\mathbf{Q^n}$. First note that the unary operation $\neg : Q^n \to Q^n$ is an involution: $\neg\neg(a, b) = (a, b)$. Moreover, it reverses the order $\leq_t$ and preserves the order $\leq_k$, because
$(a_1, b_1) \leq_k (a_2, b_2) \Leftrightarrow a_1 \leq a_2$ and $b_1 \leq b_2 \Leftrightarrow$
$\Leftrightarrow \neg(a_1, b_1) = (b_1, a_1) \leq_k (b_2, a_2) = \neg(a_2, b_2)$
and
$(a_1, b_1) \leq_t (a_2, b_2) \Leftrightarrow a_1 \leq a_2$ and $b_1 \geq b_2 \Leftrightarrow$
$\Leftrightarrow \neg(a_1, b_1) = (b_1, a_1) \geq_t (b_2, a_2) = \neg(a_2, b_2)$.
By R.Wille [14] the lattice $\mathbf{Q^n_k}$ is simple. Because if a lattice reduct of a bilattice is simple then the bilattice in question is always simple, too, the bilattice $\mathbf{Q^n}$ is simple.

Example 3. Let $n \geq 2$ be a natural number and $Q_0^{2n} := Q^n \cup P^n - \{(0, 1), (1, 0)\}$, where Q^n is the set defined in Ex. 2 and $P^n := \{(1 - x, 1 - y) | (x, y) \in Q^n\}$.

In Q_0^{2n} define partial order $\leq_t$ as follows:

$$(a, b) \leq_t (c, d) \text{ iff } a \leq c \text{ and } b \leq d \ . \tag{27}$$

Now let $\mathbf{Q^{2n}_t}$ be the extension of the partially ordered set $(Q_0^{2n}, \leq_t)$ by a single element (0,1). By definition (0,1) covers $(2^{-n}, 2^{-n})$, is covered by $(1 - 2^{-n}, 1 - 2^{-n})$ and is not comparable with any other element of the set Q_0^{2n} different from (1,1) and (0,0). It is clear that Q_t^{2n} is a bounded lattice with (1,1) as the greatest element and (0,0) as the least element.

Moreover in $Q^{2n} := Q_0^{2n} \cup \{(0, 1)\}$ define two binary operations as follows:

$$(a, b) \circ (c, d) := \begin{cases} (max(a, c), max(b, d)) & \text{if } (a, b), (c, d) \in Q^n - \{(0, 1), (1, 1)\} \\ (min(a, c), min(b, d)) & \text{if } (a, b), (c, d) \in P^n - \{(0, 1), (0, 0)\} \\ (\frac{1}{2}, \frac{1}{2}) & \text{otherwise} \end{cases}$$

$$(a, b) + (c, d) := \begin{cases} (min(a, c), min(b, d)) & \text{if } (a, b), (c, d) \in Q^n - \{(0, 1), (1, 1)\} \\ (max(a, c), max(b, d)) & \text{if } (a, b), (c, d) \in P^n - \{(0, 1), (0, 0)\} \\ (0, 1) & \text{otherwise} \end{cases} \ .$$

It is easy to see that $\mathbf{Q^{2n}_k} = (Q^{2n}, \circ, +, (\frac{1}{2}, \frac{1}{2}), (0, 1))$ is a bounded lattice.

On the set Q^{2n} we define the unary operation

$$\neg(a, b) := (1 - b, 1 - a) \ . \tag{28}$$

It is easy to note that $\mathbf{Q^{2n}_k}$ and $\mathbf{Q^{2n}_t}$ with the above unary operation is a bilattice $\mathbf{Q^{2n}}$. As was shown by R. Wille [14], the lattice $\mathbf{Q^{2n}_t}$ is simple. Hence the bilattice $\mathbf{Q^{2n}}$ is simple too.

The following result was proved in [13].

Lemma 2. *Let* $\mathbf{B} = (B, \wedge, \vee, \circ, +, \neg, 0_1, 1_1, 0_2, 1_2)$ *be a P-bilattice isomorphic to the product bilattice* $\mathbf{B}(\mathbf{L})$ *associated with the lattice* $\mathbf{L} = (L, \wedge, \vee)$. *Then the lattice* $Con\mathbf{B}$ *is isomorphic to the lattice* $Con\mathbf{L}$.

As an easy consequence of this lemma we obtain the following corollary.

Corollary 3. *A P-bilattice* $\mathbf{B}(\mathbf{L})$ *is simple if and only if the lattice* $\mathbf{L}$ *is simple.*

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