ON THE STRUCTURE OF SOME BILATTICES

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ABSTRACT. In a number of papers, M.L.Ginsberg introduced algebras called bilattices having two lattice structures and one additional unary operation acting on both lattices in a very regular way. Balattices originated as an algebraization of some non-classical logics that appeared recently in investigations on artificial inteligence. Bilattices that appear in application usually satisfy some additional conditions. In this paper, we investigate the structure of bilattices satisfying some additional identities and show that each of them may be obtained from some products of lattices.

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INTRODUCTION.

In a number of papers [G1], [G2], [G3], M.L.Ginsberg introduced algebras called bilattices having two lattice structures, and one additional basic unary operation acting on both lattices in very regular way. Bilattices originated as an algebraization of some non-classical logics that appeared recently in investigation on artificial inteligence.

In general, the both lattice structures of a bilattice are "almost" not connected to each other. However bilattices appearing in applications usually satisfy some additional (sometimes quite strong) conditions. In this paper we investigate the structure of bilattices satisfying some additional identities that were pointed out by

Padmanabhan [Pa] by investigating regular identities in lattices.

In Section 1 we collect all basic definitions we need then in this paper, as for example these of 2-lattice, bilattice, quasilattice and Płonka sum. In Section 2, a construction of a bilattice by means of some products of two lattices is described. Section 3 gives the main representation theorem (3.1) of bilattices satisfying Padmanabhan identities, using the construction described in Section 2. In Section 4 the representation theorem is specified for distributive bilattices satisfying all distributive laws between all basic binary operations and other basic algebraic properties of such bilattices are given. In Section 5 we discuss a possibility of "good" graphical illustration for bilattices.

1. PRELIMINARIES

An algebra $\underline{L}=(L,o_{11},o_{12},o_{21},o_{22})$, where for $i=1,2$, $\underline{L}_i=(L,o_{i1},o_{i2})$ are lattices is called a <u>2-lattices</u>. In the case the lattices $\underline{L}_i$ are bounded by $0_i,1_i$ respectively, we write $(L,o_{11},o_{12},0_1,1_1,o_{21},o_{22},0_2,1_2)$.
The symbol o_{i1} (staying on the second place in $\underline{L}_i$) always denotes the operation of the meet, the symbol o_{i2} (staying on the third place in $\underline{L}_i$) always denotes the operation of the join. In the case $\underline{L}_i$ is bounded, the symbol o_{i2} is always followed by the symbol of the least element of $\underline{L}_i$.
The lattices $\underline{L}_1$ and $\underline{L}_2$ are called the first and the second <u>main reducts</u> of $\underline{L}$.
An interval in the lattice $\underline{L}_i$ bounded by elements a and b is denoted by $(a,b)_i$.
Usually, we use the symbols $\wedge$ or $\circ$ for o_{i1} and $\vee$ or $+$ for o_{i2}, but there are some exceptions. For example $(L,\wedge,\vee,0,1,\wedge,\vee,0,1)$ is a 2-lattice with the equal main reducts, and $(L,\wedge,\vee,0,1,\vee,\wedge,1,0)$ is a 2-lattice with the dual main reducts. In the last example, the ordering

relation $\leq_2$ of $\underline{L}_2$ is the reverse of the ordering relation $\leq_1$ of $\underline{L}_1$. In $\underline{L}_2$ the operation $\vee$ plays a rôle of the meet and the operation $\wedge$ plays a rôle of the join.

A <u>bilattice</u> is an algebra $\underline{B}=(B,\wedge,\vee,0_1,1_1,\circ,+,0_2,1_2,')$ such that

(B1) $\underline{B}_1=(B,\wedge,\vee,0_1,1_1)$ and $\underline{B}_2=(B,\circ,+,0_2,1_2)$ are bounded lattices,

(B2) $':B\longrightarrow B$ is a unary operation satisfying the following identities

(i) $x''=x$,

(ii) $(x\vee y)'=x'\wedge y'$,

$(x\wedge y)'=x'\vee y'$,

(iii) $(x+y)'=x'+y'$,

$(x\circ y)'=x'\circ y'$.

Note that

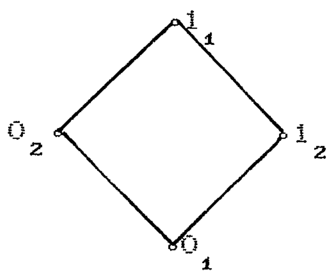
$$x\leq_1 y \quad\Leftrightarrow\quad y'\leq_1 x'$$

and

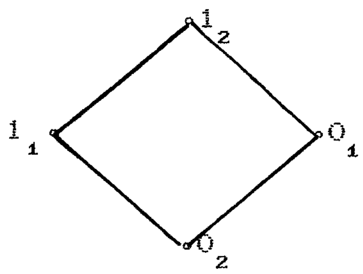
$$x\leq_2 y \quad\Leftrightarrow\quad x'\leq_2 y'.$$

Moreover $':(B,\wedge,\vee)\longrightarrow(B,\vee,\wedge)$ and $':(B,\circ,+)\longrightarrow(B,\circ,+)$ are lattice isomorphisms.

The bound s of $\underline{B}_1$ are incomparable in $\underline{B}_2$, i.e. $0_1\|_2 1_1$ and the bounds of $\underline{B}_2$ are incomparable in $\underline{B}_1$, i.e. $0_2\|_1 1_2$. The reduct $\underline{B}'_1=(B,\wedge,\vee,0_1,1_1,')$ is a polarity lattice as defined in [S] and in the case $\underline{B}_1$ is distributive, it is a de Morgan algebra [BD]. A finite bilattice $\underline{B}$ may be illustrated graphically by means of two Hasse diagrams, the diagrams of main reducts $\underline{B}_1$ and $\underline{B}_2$. In particular, the smallest non-trivial bilattice $\underline{B}_4$ has the following diagrams.



$(B_4,\wedge,\vee)$



$(B_4,\circ,+)$

A lattice $\underline{L}$ is called <u>locally finite</u> if each bounded chain in $\underline{L}$ is finite. A graph $G(\underline{L})$ of a locally finite lattice $\underline{L}$ is an (undirected) graph whose vertices are elements of L and whose edges are those pairs (a,b) of $L \times L$ such that a covers b or b cover a. Given graphs $\underline{G}_1$ and $\underline{G}_2$ whose set of vertices are G_1 and G_2, respectively, a bijection $f: G_1 \longrightarrow G_2$ is to be an <u>isomorphism of $\underline{G}_1$ and $\underline{G}_2$</u> if for each x, y in G_1, (x,y) is an edge in $\underline{G}_1$ if and only if (xf, yf) is an edge in $\underline{G}_2$. We say that a <u>bilattice $\underline{B}$ have isomorphic graphs</u> if the graphs $G(\underline{B}_1)$ and $G(\underline{B}_2)$ are isomorphic.

In the proof of a structure theorem for bilattices satisfying Padmanabhan identities, in Section 3, we use the notion of a quasilattice introduced and investigated in [P2] and [Pa]. It is an algebra $(Q, \circ, +)$ such that both reducts $(Q, \circ)$ and $(Q, +)$ are semilattices and the following identities are satisfied

(Q1) $(x+y) \circ z + (y \circ z) = (x+y) \circ z$,

(Q2) $((x \circ y)+z) \circ (y+z) = (x \circ y)+z$.

As was shown in [Pa] each quasilattice is a Płonka sum of lattices. On the other hand, by Płonka's results in [P2] and [P3], it is known that a Płonka sum of lattices is a quasilattice. Let us recall the definition of a Płonka sum of lattices. Let $(I, \circ)$ be a semilattice. For each i in I, let a lattice $(L_i, \circ, +)$ be given. For each pair (i,j) in I^2 with $i \geq j$ let $\varphi_{i,j}: (L_i, \circ, +) \longrightarrow (L_j, \circ, +)$ be a homomorphism satisfying:

(i) $\varphi_{i,i}$ is the identity mapping on L_i ,

(ii) $\varphi_{i,j}\varphi_{j,k} = \varphi_{i,k}$ for $i \geq j \geq k$.

Then the <u>Płonka sum of lattices $(L_i, \circ, +)$</u> (over the semilattice $(I, \circ)$ by the homomorpisms $\varphi_{i,j}$) is the algebra $(\bigcup (L_i \mid i \in I), \circ, +)$ defined on the disjoint sum $S = \bigcup (L_i \mid i \in I)$ by

$$x + y = x\varphi_{i,ij} + y\varphi_{j,ij} ,$$
$$x \circ y = x\varphi_{i,ij} \circ y\varphi_{j,ij} ,$$

where x is in L_i and y is in L_j.

The decomposition of a quasilattice $(Q,\circ,+)$ into a Płonka sum of lattices $(L_i,\circ,+)$, i in I, is given by means of a <u>partition function</u> $o: L\times L \longrightarrow L$, $(x,y) \longmapsto x\circ y = x+xy$. In this decomposition x,y are in L_i if and only if $x\circ y=x$ and $y\circ x=y$. For x in L_i, and $i \geq j$, $x\varphi_{i,j} := x\circ y$, where y is an arbitrary element of L_j.
The lattices $(L_i,\circ,+)$ are referred to as the <u>Płonka fibres</u>.
The mapping $p: (Q,\circ,+) \longrightarrow (I,\circ,o)$, $x \longmapsto i$ for x in L_i is a homomorphism referred to as the <u>Płonka homomorphism</u>.
For more information concerning Płonka sums and quasilattices we refer the reader to [P2], [P3], [Pa] and [RS].

For undefined concepts of universal algebra and lattice theory, and terminology we refer the reader to [Gr], [RS] and [BD].

2. A CONSTRUCTION OF BILATTICES

Let $\underline{L}=(L,\wedge,\vee,0,1)$ be a bounded lattice. Let $\tilde{\underline{B}}(L)$ be the direct product $\underline{L}_1\times\underline{L}_2$ of two 2-lattices $\underline{L}_1=(L,\wedge,\vee,0,1,\wedge,\vee,0,1)$ and $\underline{L}_2=(L,\vee,\wedge,1,0,\wedge,\vee,0,1)$. The carrier $B(L)$ of $\tilde{\underline{B}}(L)$ is $L\times L$.
Denote the lattice operations of the first main reduct of $\tilde{\underline{B}}(L)$ by $\wedge$ and $\vee$, and of the second main reduct of $\tilde{\underline{B}}(L)$ by $\circ$ and $+$.
Note that

$$(B,\wedge,\vee) = (L,\wedge,\vee)\times(L,\wedge,\vee)^d ,$$

where $(L,\wedge,\vee)^d$ is the dual of $(L,\wedge,\vee)$,
and

$$(B,\circ,+) = (L,\wedge,\vee)\times(L,\wedge,\vee) .$$

In $B(L) = L\times L$ define one unary and four nullary operations as follows

$$(2.1) \qquad \begin{array}{ll} (a,b)' := (b,a) , & \\ 0_1 := (0,1), & 0_2 := (0,0) , \\ 1_1 := (1,0), & 1_2 := (1,1) . \end{array}$$

2.2. PROPOSITION. The algebra $\underline{B}(L)=(B(L),\wedge,\vee,0_1,1_1,\circ,+,0_2,1_2,')$ is a bilattice. Moreover $\underline{B}(L)$ satisfies the folliwing identities:

(2.3) $((x \wedge y)\circ z) \wedge (y \circ z) = (x \wedge y) \circ z$,

(2.4) $((x \circ y)\wedge z) \circ (y \wedge z) = (x \circ y)\wedge z$,

(2.5) $((x \wedge y)+ z) \wedge (y + z) = (x \wedge y)+ z$,

(2.6) $((x + y)\wedge z) + (y \wedge z) = (x + y)\wedge z$.

Remark. Note that ~~the~~ identities (2.3)-(2.6) imply the following ones

(2.7) $((x \vee y)\circ z) \vee (y \circ z) = (x \vee y)\circ z$,

(2.8) $((x \circ y)\vee z) \circ (y \vee z) = (x \circ y)\vee z$,

(2.9) $((x \vee y)+ z) \vee (y + z) = (x \vee y)+ z$,

(2.10) $((x + y)\vee z) + (y \vee z) = (x +y)\vee z$.

It follows that each reduct $(B(L),o_1,o_2)$, where o_1, $o_2 \in \{\wedge,\vee,\circ,+\}$ is a quasilattice.
A bilattice satisfying the identities (2.3)-(2.6) will be called Padmanabhan bilattice or briefly P-bilattice.

Proof. It is evident that $(a,b)''=(a,b)$ and the four constants defined by (2.1) play the role of the least and the greatest elements of the main reducts of $\underline{B}(L)$. We have to prove that the mappings $':(B,\wedge,\vee)\longrightarrow(B,\vee,\wedge)$ and $':(B,\circ,+)\longrightarrow(B,\circ,+)$ are homomorphisms. Let a,b,c,d be in L. Note that

$$(a,b) \wedge (c,d) = (a \wedge c,d \vee d) ,$$

$$(a,b) \vee (c,d) = (a \vee c,b \wedge d) .$$

Then $((a,b) \wedge (c,d))' = (a \wedge c,b \vee d)' = (b \vee d, a \wedge c) = (b,a) \vee (d,c) = (a,b)' \vee (c,d)'$,

and $((a,b) \vee (c,d))' = (a \vee c,b \wedge d)' = (b \wedge d, a \vee c) = (b,a) \wedge (d,c) = (a,b)' \wedge (c,d)'$.

The proof of the second statement is similar. It follows that $\underline{B}(L)$ is a bilattice.

To prove the second part of the proposition let us note that $(B(L),\wedge,+)$ $((B(L),\vee,\circ)$, $(B(L),\vee,+)$, $(B(L),\circ,\wedge)$

respectively) is the Płonka sum of lattices isomorphic to $(L, \wedge, \vee)$ (to $(L, \wedge, \vee)$, $(L, \vee, \wedge)$, $(L, \vee, \wedge)$ respectively). The Płonka homomorphisms are just isomorphisms. By results of Padmanabhan [Pa], it follows that ~~the~~ identities (2.3)–(2.6) are satisfied in $\underline{B}(L)$. □

In the sequel, the bilattice $\underline{B}(L)$ constructed in this section will be called a <u>product bilattice associated with the lattice L</u>.

3. A REPRESENTATION OF PADMANABHAN BILATTICES

In this section we show that each P-bilattice, i.e. each bilattice satisfying ~~the~~ identities (2.3)–(2.6) is isomorphic to a bilattice constructed in the way described in Section 2.

3.1. THEOREM. An algebra $\underline{B}=(B, \wedge, \vee, 0_1, 1_1, \circ, +, 0_2, 1_2, ')$ of type (2,2,0,0,2,2,0,0,1) is a P-bilattice if and only if there is a bounded lattice $\underline{L}=(L, \wedge, \vee, 0, 1)$ such that $\underline{B}$ is isomorphic to the product bilattice $\underline{B}(L)$ associated with the lattice $\underline{L}$.

<u>Proof</u>. ($\Leftarrow$) Proposition 2.2.

($\Rightarrow$) The proof is devided in several lemmas.

LEMMA A. The elements $0_1, 1_1, 0_2, 1_2$ of B form a subbilattice of $\underline{B}$ isomorphic to $\underline{B}_4$.

<u>Proof</u>. Using ~~the~~ identities (2.4) and (2.8) one gets the following

$$(0_1 \circ 1_1) \wedge 0_2 = ((0_1 \circ 1_1) \wedge 0_2) \circ (1_1 \wedge 0_2)$$
$$= ((0_1 \circ 1_1) \wedge 0_2) \circ 0_2) = 0_2 ,$$

whence $0_2 \leq_1 0_1 \circ 1_1$, and

$$(0_1 \circ 1_1) \vee 0_2 = ((1_1 \circ 0_1) \vee 0_2) \circ (0_1 \vee 0_2)$$
$$= ((1_1 \circ 0_1) \vee 0_2) \circ 0_2 = 0_2 ,$$

whence $0_1 \circ 1_1 \leq_1 0_2$. It follows that $0_1 \circ 1_1 = 0_2$.

Similarly, using ~~the~~ identities (2.6) and (2.10) we show that $0_1 + 1_1 = 1_2$, using ~~the~~ identities (2.3) and (2.5) we show that $0_2 \wedge 1_2 = 0_1$, and using ~~the~~ identities (2.7) and (2.9) we show that $0_2 \vee 1_2 = 1_1$ □

Now, let us recall that each of the reducts $(B,\wedge,+)$, $(B,\wedge,\circ)$, $(B,\vee,+)$, $(B,\vee,\circ)$ is a quasilattice, whence by results of Padmanabhan [Pa], it is represented as a Płonka sum of lattices. In particular, let $(B,\wedge,+)$ be the Płonka sum of lattices $(B_i,\wedge,+)$ over a semilattice $(I,\wedge)$.

LEMMA B. The Płonka fibre B_2 containing 0_2 coincides with the interval $(0_2,1_1)_1$.

Proof. Note that $x+(x \wedge y)$ is a partition function for $(B,\wedge,+)$. Since

$$1_1+(1_1 \wedge 0_2) = 1_1 ,$$
$$0_2+(1_1 \wedge 0_2) = 0_2 ,$$

it follows that 1_1 is in B_2, as well and hence the inerval $(0_2,1_1)_1$, is contained in B_2.

Now let a be in B_2 and suppose a is not in $(0_2,1_1)_1$. Then $b := a \wedge 0_2$ is not in $(0_2,1_1)_1$ either and

$$0_2 + (b \wedge 0_2) = 0_2 + b = b .$$

On the other hand, b is in B_2, whence

$$b + (b \wedge 0_2) = b ,$$
$$0_2 + (b \wedge 0_2) = 0_2 .$$

Consequently $b=0_2$, a contradiction. It follows that B_2 is contained in $(0_2,1_1)_1$, whence $B_2=(0_2,1_1)_1$. □

LEMMA C. If a is in B_i for some i in I, then $a\wedge 0_2$ and $a+1_1$ are in B_i as well.

Proof. Let a be in B_i. Then since the identity

$$x + (x \circ y) = (x + y) \circ x$$

is satisfied in each quasilattice (see [Pa]), it follows that

$$a + (a \wedge (a \wedge 0_2)) = a + (a \wedge 0_2) = (a + 0_2) \wedge a = a ,$$
$$(a \wedge 0_2) + (a \wedge (a \wedge 0_2)) = a \wedge 0_2 ,$$

whence $a \wedge 0_2$ is in B_i. □

Similarly,

$$(a+1_1)+((a+1_1)\wedge a) = (a+1_1)+(a+(1_1+a)) = (a+1_1)+a = a+1_1 ,$$

whence $a+1_1$ is in B_i.

LEMMA D. For every i in I, the set $B_i \cap (0_1,0_2)_1$ has exactly one element, $|B_i \cap (0_1,0_2)_1| = 1$.

Proof. First note that the following four statements are equivalent.

$$0_2 \leq_1 a \leq_1 b \leq_1 1_1 ,$$
$$0_2 \leq_2 a \leq_2 b \leq_2 1_1 ,$$
$$0_1 \leq_1 b' \leq_1 a' \leq_1 0_2 ,$$
$$0_2 \leq_2 a' \leq_2 b' \leq_2 0_2 .$$

Consequently

$$(0_1 \leq_1 a \leq_1 b \leq_1 0_1) \Leftrightarrow (0_2 \leq_2 b \leq_2 a \leq_2 0_1) .$$

It follows that if a,b are in $(0_1,0_2)_1$ and a is in B_i, b is in B_j, then $i \neq j$. By Lemma C, each B_i contains an element in $(0_1,0_2)_1$. Hence the lemma holds. □

LEMMA E. For every i in I, the set $B_i \cap (1_2,1_1)_1$ has exactly one element, $|B_i \cap (1_2,1_1)_1| = 1$.

Proof: It is similar to the proof of lemma D. □

LEMMA F. For every i in I, the Płonka fibre $\underline{B}_i$ is the bounded lattice $(B_i,\wedge,+,0_i,1_i) = (B_i,\wedge,\vee,0_i,1_i) = (B_i,\circ,+,0_i,1_i)$, where $0_i = a \wedge 0_2$, $1_i = a + 1_1$ for any a in B_i.

Proof: Lemmas B, C, D. □

LEMMA G. $(I,\wedge) \cong ((0_1,0_2)_1,\wedge) \cong ((0_2,0_1)_2,+)$.

Proof: It follows directly by Lemma F. □

LEMMA H. For i,j in I with $i \geq j$, the Płonka homomorphism $\varphi_{i,j}: B_i \longrightarrow B_j$ is defined by $a \longmapsto a+0_j = a \wedge 1_j$.

Proof: Let a be in B_i. Since $a\varphi_{i,j} = a + (a \wedge b)$ for any b in B_i, it follows that in particular for $b=0_j$, $a\varphi_{i,j} = a + (a \wedge 0_j) = a + 0_j$. On the other hand for $b=1_j$, $a\varphi_{i,j} = a + (a \wedge 1_j) = a \wedge (a + 1_j) = a \wedge 1_j$. □

PROPOSITION I. The quasilattice $(B,\wedge,\vee)$ is the Płonka sum of lattices $B_i=(0_i,1_i)_1,\wedge,+) = ((0_i,1_i)_1,\wedge,\vee)$

$= ((0_i, 1_i)_1, \circ, +)$ over the semilattice $\underline{I}=((0_1,0_2)_1,\wedge) \cong ((1_2,1_1)_1,\wedge)$ by the homomorphisms $\varphi_{i,j}:\underline{B}_i \longrightarrow \underline{B}_j$, $a \longmapsto a+0_j = a \wedge 1_j$.

<u>Proof:</u> Previous lemmas.

□

In a similar way we describe the structure of the quasilattice $(B,\wedge,\circ)$.

We denote elements of $(0_1,1_2)_1$ by 0_j where j are in a set J. Since the cardinalities of $(0_1,1_2)_1$ and $(0_2,1_1)_1$ are equal (similarly as cardinalities of $(0_1,0_2)_1$ and $(1_2,1_1)_1$) we can denote the elements of $(0_2,1_1)_1$ by 1_j where j are in J.

PROPOSITION J. The quasilattice $(B,\wedge,\circ)$ is the Płonka sum of lattices $\underline{A}_j = ((0_j,1_j)_1,\wedge,\circ) = ((0_j,1_j)_1,\wedge,\vee) = ((0_j,1_j)_1,+,\circ)$, (where for a in A_j, $0_j=1_2\wedge a$ and $1_j=1_1\circ a$) over the semilattice $\underline{J}=((0_1,1_2)_1,\wedge) \cong ((0_2,1_1)_1,\wedge)$ by the homomorphisms $\varphi_{i,j}:\underline{A}_i \longrightarrow \underline{A}_j$, $a \longmapsto a \wedge 1_j = a \circ 0_j$.

<u>Proof:</u> It is analogous to the proof of Proposition I.

□

LEMMA K. For every i in I, j in J, $|B_i \cap A_j| = 1$.

<u>Proof:</u> First note that the set $B_i\cap A_j$ is not empty. Indeed, since 1_i is in $B_i\cap A_1$ and 1_j is in $B_2\cap A_j$ it follows that $1_i \wedge 1_j$ is in $B_i\cap A_j$. Now suppose there are a and b in $B_i\cap A_j$ with $a \neq b$. Without loss of generality we can assume that $a \leq_1 b$. Since a,b are in B_i, it follows that $a \leq_2 b$. Since a,b are in A_j, it follows that $b \leq_2 a$. Consequently, a=b and $B_i\cap A_j = \{1_i\wedge 1_j\}$.

□

Let $\Theta(I)$ be the congruence relation of the 2-lattice $\tilde{\underline{B}}=(B,\wedge,\vee,0_1,1_1,\circ,+,0_2,1_2)$ induced by the decomposition of $\tilde{\underline{B}}$ into lattices $\underline{B}_i$ and let $\Theta(J)$ be the congruence relation of $\tilde{\underline{B}}$ induced by the decomposition of $\tilde{\underline{B}}$ into lattices $\underline{A}_j$.

LEMMA L. (i) The congruence $\Theta(I) \wedge \Theta(J)$ is the equality relation.

(ii) The congruence $\Theta(I) \vee \Theta(J)$ is the trivial congruence collapsing all elements of B.

(iii) The congruences $\Theta(I)$ and $\Theta(J)$ permute.

<u>Proof</u>: The first part of the lemma follows by Lemma K. For to prove the second one, let a be in $A_j \cap B_i$ and b be in $A_{j'} \cap B_{i'}$, where i, i' are in I, and j, j' are in J. Then by Lemma K, $a\ \Theta(J)\ (1_j \wedge 1_i)\ \Theta(I)\ b$ and $a\ \Theta(I)\ (1_i \wedge 1_j)\ \Theta(J)\ b$. Consequently $a\ (\Theta(J) \vee \Theta(I))\ b$ and $\Theta(I)\Theta(J) = \Theta(J)\Theta(I)$. □

<u>The proof of</u> (⇒). By Theorem 3 p.120 in [G] and Lemma L, the 2-lattice $\tilde{\underline{B}}$ is isomorphic to $\tilde{\underline{B}}/\Theta(I) \times \tilde{\underline{B}}/\Theta(J)$. Let $\underline{L}=(L,\wedge,\vee,0,1)$ be a bounded lattice isomorphic to $((0_2,1_1)_1,\wedge,\vee,0_2,1_1)$. Since by Proposition I and J, $\tilde{\underline{B}}/\Theta(I)$ is isomorphic to $((0_1,0_2)_1,\wedge,\vee,0_1,0_2)$ and $\tilde{\underline{B}}/\Theta(J)$ is isomorphic to $((0_2,1_1)_1,\wedge,\vee,0_2,1_1)$ which is dual to $((0_1,0_2)_1,\wedge,\vee,0_1,0_2)$, it follows that $\tilde{\underline{B}}$ is isomorphic to $(L,\wedge,\vee,0,1,\wedge,\vee,0,1)\times(L,\vee,\wedge,1,0,\wedge,\vee,0,1)$.

By Lemma K, each element b of B is the unique element of $B_i \cap A_j$ for some i in I and j in J. Let $i:\tilde{\underline{B}} \longrightarrow \tilde{\underline{B}}/\Theta(I) \times \tilde{\underline{B}}/\Theta(J)$ be the isomorphism meantioned at the beginning of this proof. The image bi of b may be then identify with the pair $(0_i,1_j)$, where 0_i is in $(0_1,0_2)_1$, and 1_j is in $(0_2,1_1)_1$. We will show that $(0_i,1_j)'=(1_j,0_i)$. At first note that since b is in B_i, $0_i \leq_1 b \leq_1 1_i$, $0_i \leq_2 b \leq_2 1_i$, whence

$$1'_i \leq_1 b' \leq_1 0'_i \;,$$
$$0'_i \leq_2 b' \leq_2 1'_i \;.$$

Note that $0'_i$ is in $(0_2,1_1)_1$.

Similarly since b is in A_j, $0_j \leq_1 b \leq_1 1_j$ implying

$$1'_j \leq_1 b' \leq_1 0'_j \;,$$

and $1_j \leq_2 b \leq_2 0_j$ implying

$$0'_j \leq_2 b' \leq_2 1'_j \;.$$

Note that $1'_j$ is in $(0_1,0_2)_1$.

By Lemma K, it follows that b' is the unique element of

$B_k \cap A_l$, where k is such that $0_k = 1'_j$ and l is such that $1_l = 0'_i$. Now since $((0_1, 0_2)_1, \wedge, \vee)$ is the dual of $((0_2, 1_1)_1, \wedge, \vee)$, the element $1'_j$ may be identified with 1_j and the element $0'_i$ with 0_i. It follows that $(0_i, 1_j)' = (1_j, 0_i)$.

The last condition (2.1) of the definition of a product bilattice is satisfied in an obvious way.

□

4. DISTRIBUTIVE BILATTICES

A 2-lattice $(B, \wedge, \vee, \circ, +)$ is called distributive, if each of the basic binary operations of $(B, \wedge, \vee, \circ, +)$ distributes over each other. A bilattice $\underline{B}$ is called distributive if its 2-lattice reduct is distributive. In fact by results of [JK1] it follows that for a bilattice to be distributive it is enough to require only two of distributive laws.

4.1. THEOREM [JK1]. (i) Let $(B, \wedge, \vee, \circ, +)$ be a 2-lattice. If the operation $\circ$ distributes over both the operations $\wedge$ and $\vee$, then $(B, \wedge, \vee, \circ, +)$ is a distributive 2-lattice.

(ii) A 2-lattice $(B, \wedge, \vee, \circ, +)$ is distributive if and only if there are distributive lattices $(A, \wedge, \vee)$ and $(C, \wedge, \vee)$ such that $(B, \wedge, \vee) \cong (A, \wedge, \vee) \times (C, \wedge, \vee)$ and $(B, \circ, +) \cong (A, \wedge, \vee) \times (C, \wedge, \vee)^d$.

□

The main structure Theorem 3.1 may be formulated in the case of distributive bilattices as follows.

4.2. THEOREM [T]. An algebra $\underline{B} = (B, \wedge, \vee, 0_1, 1_1, \circ, +, 0_2, 1_2, ')$ of type (2,2,0,0,2,2,0,0,1) is a distributive bilattice if and only if there is a bounded distributive lattice $\underline{L} = (L, \wedge, \vee, 0, 1)$ such that $\underline{B}$ is isomorphic to the product bilattice $\underline{B}(L)$ associated with the lattice $\underline{L}$.

Proof: It is an easy corollary from Theorem 3.1

since distributive laws imply Padmanabhan's identities (2.3)-(2.6). □

Note that Theorem I in [P1] gives in particular a structure theorem for distributive 2-lattices as well. However this structure theorem is not very suitable for bilattices.

Combining the last theorem with the well known representation of distributive lattices as rings of sets one obtains a set theoretical representation for distributive bilattices.

Let $\underline{L}=(L,\wedge,\vee)$ be a distributive lattice. Let $(P(\underline{L}),\cap,\cup)$ be the ring of subsets of the set $P(\underline{L})$ of all prime filters of $\underline{L}$, isomorphic to $\underline{L}$ via $a \longmapsto \{F\in P(\underline{L}) \mid a\in F\}$.

4.3. THEOREM. Each distributive bilattice $\underline{B}$ is isomorphic to a product bilattice $\underline{B}(L)$ associated with some ring of sets $\underline{L}$, namely $\underline{L}=P((0_1,0_2)_1,\wedge,\vee)$. □

4.4. COROLLARY [G1]. Each distributive bilattice $\underline{B}$ is a subbilattice of a product bilattice $\underline{B}(L)$ associated with some ring of all subsets of a set . □

Note that the bilattice $\underline{B}(L)$ of Corollary 4.4 is called "world based" bilattice in [G1], [G2] and [G3].

If $\underline{B}$ is a distributive bilattice, then the reduct $(B,\wedge,\vee,0_1,1_1,')$ is a de Morgan algebra. This fact makes it easy to find all subdirectly irreducible distributive bilattices and hence to find subdirect product representation for these algebras. We will show that there exists exactly one non-trivial subdirectly irreducible distributive bilattice. This implies that the variety of distributive bilattices does not contain any non-trivial variety. We first need some lemmas.

4.5. LEMMA. Let $\underline{B}$ be a P-bilattice. A relation $\Theta \subseteq B\times B$ is a congruence relation of the bilattice $\underline{B}$ if and only if it is a congruence relation of the reduct $(B,\wedge,\vee,')$.

Proof: ($\Rightarrow$) If Θ is a congruence relation of the bilattice $\underline{B}$, then obviously it is a congruence relation of the reduct $(B,\wedge,\vee,')$.

($\Leftarrow$) Let Θ be a congruence relation of the reduct $(B,\wedge,\vee,')$. Obviously, Θ is a congruence relation of the lattice $(B,\wedge,\vee)$. By Theorem 3.1, $(B,\wedge,\vee)$ is the direct product $(L,\wedge,\vee)\times(L,\wedge,\vee)^{d}$. By results of [FH], $\Theta = \Theta_1 \times \Theta_2$, where Θ_1 and Θ_2 are congruence relations of the lattice $(L,\wedge,\vee)$. On the other hand, $(B,\circ,+)$ is the direct product $(L,\wedge,\vee)\times(L,\wedge,\vee)$. It follows that $\Theta = \Theta_1 \times \Theta_2$ is a congruence relation of the lattice $(B,\circ,+)$ as well. □

4.6. COROLLARY. A P-bilattice $\underline{B}$ is subdirectly irreducible if and only if the reduct $(B,\wedge,\vee,')$ is subdirectly irreducible. □

4.7. THEOREM [T]. Let $\underline{B}$ be a non-trivial distributive bilattice. Then $\underline{B}$ is subdirectly irreducible if and only if it is isomorphic to the bilattice $\underline{B}_4$.

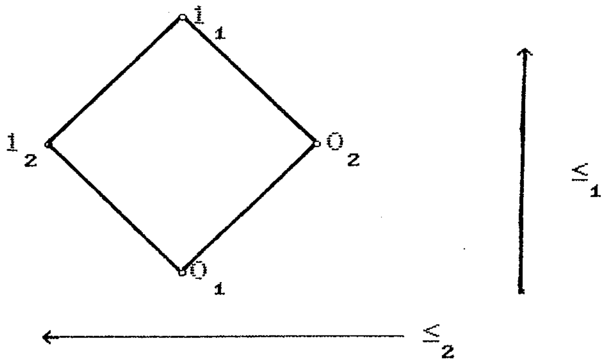
Proof: By Lemma 4.4, a non-trivial bilattice $\underline{B}$ is subdirectly irreducible if and only if **the** de Morgan algebra $(B,\wedge,\vee,',0,1)$ is subdirectly irreducible. By results of [K] and since the least non-trivial bilattice $\underline{B}$ has four elements, it follows that $\underline{B}$ is isomorphic to $\underline{B}_4$. □

5. SOME REMARKS ON DIAGRAMS OF BILATTICES

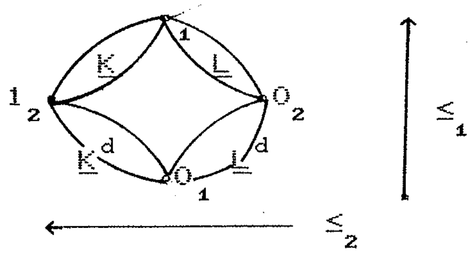
Let $\underline{B}=(B,\wedge,\vee,0_1,1_1,\circ,+,0_2,1_2)$ be a locally finite P-bilattice.

From Theorem 3.1 and results of [JK2,D⇒F,p2] it follows that the main reducts $\underline{B}_1=(B,\wedge,\vee)$ and $\underline{B}_2=(B,\circ,+)$ of $\underline{B}$ have isomorphic graphs. Moreover, it is easy to see that each of the Hasse diagrams of $\underline{B}_1$ and $\underline{B}_2$ may be drawn in such a way that $\underline{B}_i$, where i=1,2, can be obtained from the other one by rotating it through an angle of $\frac{\pi}{2}$.

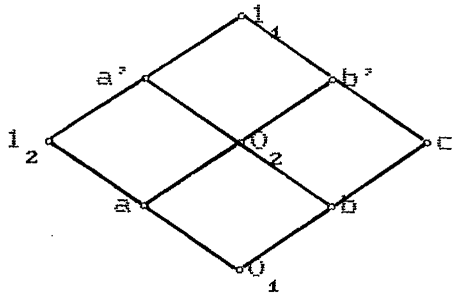
In this situation one can follow a convention proposed by Ginsberg [G1], [G2], [G3] to illustrate graphically a bilattice by means of only one Hasse diagram as for example



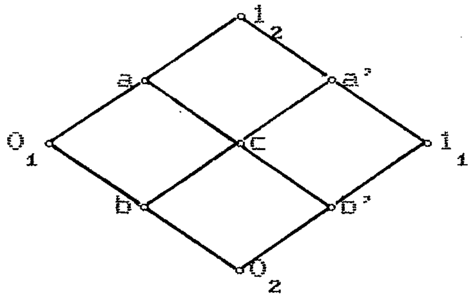
and interpret both ordering relation $\leq_1$ and $\leq_2$ as on the picture above. P-bilattices are not the only bilattices that can be presented by means of one Hasse diagram. A typical example of such a bilattice that is not necessarily P-bilattice is given on the next picture



On this picture, $\underline{L}$ and $\underline{K}$ are arbitrary bounded lattices ($\underline{L}^d$ and $\underline{K}^d$ are the duals of $\underline{L}$ and $\underline{K}$ respectively, with respect to the order $\leq_1$). Note however that not all bilattices have isomorphic graphs of their main reducts. The next picture shows an example of a bilattice $\underline{B}$ given by E.Gedeonova having distributive main reducts but not isomorphic graphs.

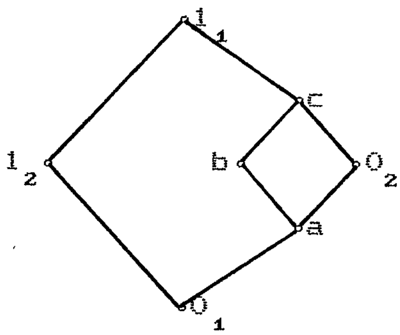


$\underline{B}_1$

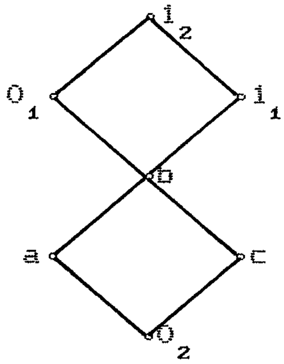


$\underline{B}_2$

Another example is provided by the bilattice $\underline{D}$ of so called "default logic" of Ginsberg



$\underline{D}_1$



$\underline{D}_2$

This shows that it is not always justified to use only one Hasse diagram to illustrate graphically a bilattice.

The examples above show that one should define precisely what is meant by "illustrate a bilattice using one diagram only" and arise the following problems.

1. Characterise bilattice that can be illustrate "using only one Hasse diagrams".

2. Characterise lattices that can serve as a "unique Hasse diagram" of a bilattice.

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