THE LATTICE OF SUBVARIETIES OF THE VARIETY OF SOME TERNARY MODES

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Abstract. We describe the semiring associated with the variety $\mathbf{V}(R)$ generated by some ternary algebras of submodules of R-modules. We show that this semiring is isomorphic to the semiring of finitely generated ideals of the ring R. We also describe the lattice of all subvarieties of the variety $\mathbf{V}(Z_n)$.

Algebras considered in this paper are semilattice modes. A *mode* is an idempotent algebra, in the sense that each singleton is a subalgebra, and an entropic algebra, i.e. each operation as a mapping from a direct power of the algebra into the algebra is a homomorphism. A mode is called a *semilattice mode* if some binary term interprets as a semilattice operation.

Let R be a commutative ring with unity 1. We can describe affine R-spaces as certain ternary reducts $(E, \overline{R})$ of R-modules, and consider algebras $(ESA, \overline{R})$ of subalgebras of algebras $(E, \overline{R})$. For each R-module, its submodules form a subalgebra $(ESM, \overline{R})$ of $(ESA, \overline{R})$. Since among the operations $\overline{R}$ is a semilattice operation $x+y := xyy\overline{1}$, the algebras $(ESM, \overline{R})$ are semilattice modes, and the variety $\mathbf{V}(R)$ they generate forms a semilattice mode variety.

General semilattice modes were investigated by K. Kearnes in [1]. He has shown there that to each variety $\mathbf{V}$ of semilattice modes one can associate a certain commutative semiring $S(\mathbf{V})$. The semiring of a semilattice mode variety plays a similar rôle as the ring of a variety of affine spaces. Similarly as in the case of affine R-spaces, the lattice of subvarieties of a variety of semilattice modes is determined by the congruences of the associated semiring.

Section 1 is devoted to the variety $\mathbf{V}(R)$ generated by the class $MS(R)$ of all algebras $(ESM, \overline{R})$ of submodules. We describe a standard form of words in the free $\mathbf{V}(R)$-algebra on two generators. We use this to describe the semiring associated with the variety $\mathbf{V}(R)$ in Section 2. We show that this semiring is isomorphic to the semiring of finitely generated ideals of the ring R. In Section 3 we give some properties of congruence relations of the semiring of finitely generated ideals of the ring R. Finally, in Section 4 we describe the lattice of all subvarieties of the variety $\mathbf{V}(Z_n)$. We will show that this lattice is isomorphic to the lattice of divisors of n.

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The notation and terminology of the paper is basically as in the book [5]. We refer the reader to the book for all undefined notions and results. We use "Polish" notation for words (terms) and operations, e.g. instead of $w(x_1, \dots, x_n)$ we write $x_1 \dots x_n w$.

1. Ternary semilattice modes of submodules of modules

Let R be a commutative ring with unity and let $(E, +, R)$ be a module over R. For each element r of R, define a ternary operation $\overline{r}$ by

$$\overline{r} : E \times E \times E \to E; \quad (x, y, z) \mapsto xyz\overline{r} := x - ry + rz,$$

and consider the algebra $(E, \overline{R})$ with the set $\overline{R} = \{\overline{r} : r \in R\}$ of operations. We will call the algebra $(E, \overline{R})$ the *ternary affine R-space.*

Note that the ternary affine R-space is term equivalent to the affine R-space. The cosets of submodules of $(E, +, R)$ are exactly the nonempty subalgebras of the algebra $(E, \overline{R})$. Consider the set ESA of non-empty subalgebras of $(E, \overline{R})$. The set ESA forms the algebra $(ESA, \overline{R})$ under the following complex product operations:

$$\overline{r} : ESA \times ESA \times ESA \to ESA;$$

$$(X, Y, Z) \mapsto \{xyz\overline{r} |\ x \in X, y \in Y, z \in Z\}.$$

Let ESM be the set of submodules of the R-module $(E, +, R)$. Note that for each r in R and U, V in ESM the sets

$$rU = \{ru | u \in U\}, \quad U + V = \{u + v | u \in U, v \in V\}$$

are submodules of the R-module $(E, +, R)$. Then

$$UVW\overline{r} = U - rV + rW = U + rV + rW$$

is a submodule of $(E, +, R)$, too. Hence $(ESM, \overline{R})$ is a subalgebra of the algebra $(ESA, \overline{R})$.

The sum of submodules

$$U + V = U + V + V = UVV\overline{1}$$

is the semilattice operation. The inclusion structure is recovered from $(ESM, +)$ via $U \leq V$ iff $U + V = V$. It turns out that the algebra $(ESM, \overline{R})$ is a semilattice mode.

Let $MS(R)$ denote the class of algebras $(ESM, \overline{R})$ of submodules of all modules $(E, +, R)$ over the ring R. Let $\mathbf{V}(R)$ be the variety generated by the class $MS(R)$.

We have the following lemma.

Lemma 1.1. *In the variety* $\mathbf{V}(R)$ *the following identities are satisfied for any* $r, r_1, r_2, \ldots, r_n, s_1, s_2, \ldots, s_m, t_1, t_2, \ldots, t_p$ *in* R*:*

i) $xyy\overline{r_1}yy\overline{r_2}\ldots yy\overline{r_n}xyy\overline{s_1}yy\overline{s_2}\ldots yy\overline{s_m}xyy\overline{t_1}yy\overline{t_2}\ldots yy\overline{t_p}\,\overline{r}$
$= xyy\overline{r_1}yy\overline{r_2}\ldots yy\overline{r_n}yy\overline{rs_1}yy\overline{rs_2}\ldots yy\overline{rs_m}yy\overline{rt_1}yy\overline{rt_2}\ldots yy\overline{rt_p}$,

ii) $xyy\overline{r_1}yy\overline{r_2}\ldots yy\overline{r_n}yxx\overline{s_1}xx\overline{s_2}\ldots xx\overline{s_m}yxx\overline{t_1}xx\overline{t_2}\ldots xx\overline{t_p}\,\overline{r}$
$= xyy\overline{r_1}yy\overline{r_2}\ldots yy\overline{r_n}yy\overline{r}$,

iii) $xyy\overline{r_1}yy\overline{r_2}\ldots yy\overline{r_n}xyy\overline{s_1}yy\overline{s_2}\ldots yy\overline{s_m}yxx\overline{t_1}xx\overline{t_2}\ldots xx\overline{t_p}\,\overline{r}$
$= xyy\overline{r_1}yy\overline{r_2}\ldots yy\overline{r_n}yy\overline{rs_1}yy\overline{rs_2}\ldots yy\overline{rs_m}yy\overline{r}$,

,iv) $xyy\overline{r_1}yy\overline{r_2}\ldots yy\overline{r_n} = xyy\overline{r_{1\sigma}}yy\overline{r_{2\sigma}}\ldots yy\overline{r_{n\sigma}}$
for each permutation σ *of the set* $\{1, 2, \ldots, n\}$.

Lemma 1.1. gives the following theorem.

Theorem 1.2. *In the free* $\mathbf{V}(R)$*-algebra* $F_{V(R)}(x, y)$ *on two generators* x *and* y*, each further element may be expressed in the standard form*

$$x_1x_2x_2\overline{r_1}x_2x_2\overline{r_2}\ldots x_2x_2\overline{r_n},$$

where x_1 *and* x_2 *are in* $\{x, y\}$, $x_1 \neq x_2$, $r_1, r_2, \ldots, r_n$ *are in* R *and* $r_i \neq r_j$ *for* $i \neq j$.

2. The semiring of the variety $\mathbf{V}(R)$.

In [1] K. Kearnes has shown that to each variety of semilattice modes one can associate a certain semiring. Such a semiring determines many properities of the variety.

Let $\mathbf{V}$ be a variety of semilattice modes with the semilattice operation $+$ and $F_V(x, y)$ denote the free $\mathbf{V}$-algebra on two generators x and y. Let $S(\mathbf{V})$ be the subset of $F_V(x, y)$ consisting of all t in $F_V(x, y)$ such that $t + y = t$. It is easy to check that $S(\mathbf{V})$ is a subalgebra of $F_V(x, y)$. For each t in $F_V(x, y)$ define the endomorphism

$$e_t : F_V(x, y) \to F_V(x, y)$$

determined by $x \mapsto t$ and $y \mapsto y$. For t and s in $F_V(x, y)$ let

$$s \bullet t := se_t.$$

Theorem 2.1 (1)**.** *Let* $\mathbf{V}$ *be a variety of semilattice modes with the semilattice operation* $+$*. The algebra* $(S(\mathbf{V}), +, \bullet, 1, 0)$ *with two binary operations* $+$ *and* $\bullet$ *defined as above and two constants* $1 := x + y$ *and* $0 := y$*, is a commutative semiring, satisfying* $1 + s = 1$ *for each* s *in* $S(\mathbf{V})$.

We will call the semiring $(S(\mathbf{V}), +, \bullet, 1, 0)$ the *semiring associated with the variety* $\mathbf{V}$.

As was noticed before the algebras $(ESM, \overline{R})$ are semilattice modes and the variety $\mathbf{V}(R)$ they generate is a variety of semilattice modes. The semilattice operation is defined by

$$x + y := xyy\overline{1} = yxx\overline{1}.$$

In this section we describe the semiring associated with the variety $\mathbf{V}(R)$ explicitly.

Lemma 2.2. *The subset $S(\mathbf{V}(R))$ of $F_{V(R)}(x, y)$ consisting of all t in $F_{V(R)}(x, y)$ such that $t + y = t$, is equal to the set*

$$\{yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n} \mid r_1, r_2, \dots, r_n \in R\}.$$

Proof. By Theorem 1.2. each element of $F_{V(R)}(x, y)$ can be represented by $xyy\overline{r_1}yy\overline{r_2}\dots yy\overline{r_n}$ or $yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}$ for $r_1, r_2, \dots, r_n$ in R. If for $r_1, r_2, \dots, r_n$ in R, $t = yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}$ then by Lemma 1.1.

$$\begin{aligned} t + y &= yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n} + yxx\overline{0} \\ &= yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}yxx\overline{0}yxx\overline{0}\;\overline{1} \\ &= yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}xx\overline{0}xx\overline{0} \\ &= yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n} = t. \end{aligned}$$

On the other hand, if $t = xyy\overline{r_1}yy\overline{r_2}\dots yy\overline{r_n}$ then

$$t + y = xyy\overline{r_1}yy\overline{r_2}\dots yy\overline{r_n}yy\overline{1} = xyy\overline{1} = yxx\overline{1}.$$

In this case, $t + y = t$ implies $t = yxx\overline{1}$, what completes the proof. □

Let $(r_1, r_2, \dots, r_n)$ denote the ideal of the ring R generated by the set $\{r_1, r_2, \dots, r_n\} \subseteq R$. Let $I_f(R)$ be the set of all finitely generated ideals of R. For $r_1, \dots, r_n, s_1, \dots, s_m$ in R

$$(r_1, \dots, r_n) + (s_1, \dots, s_m) = (r_1, \dots, r_n, s_1, \dots, s_m),$$

$$(r_1, \dots, r_n) \cdot (s_1, \dots, s_m) = (r_1s_1, \dots, r_1s_m, \dots, r_ns_1, \dots, r_ns_m).$$

Lemma 2.3. *For a commutative ring R with unity 1, the algebra $(I_f(R), +, \cdot, (1), (0))$, where $+$ denote the ideal sum and $\cdot$ denote the ideal multiplication, is a commutative semiring satisfying $1 + t = 1$.*

Theorem 2.4. *Let R be a commutative ring with unity. Let $\mathbf{V}(R)$ be the variety of semilattice modes generated by the algebras $(ESM, \overline{R})$ of submodules of all modules $(E, +, R)$. The semiring $S(\mathbf{V}(R))$ associated with $\mathbf{V}(R)$ is isomorphic to the semiring $(I_f(R), +, \cdot, (1), (0))$ of finitely generated ideals of the ring R.*

Proof. To show that $(S(\mathbf{V}(R)), +, \bullet, x + y, y)$ is isomorphic to the semiring $(I_f(R), +, \cdot, (1), (0))$ let us define the following mapping

$$h : S(\mathbf{V}(R)) \to I_f(R); \quad yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n} \mapsto (r_1, r_2, \dots, r_n).$$

First note that the mapping h is well-defined. Let $yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}$ and $yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m}$ be equal in $S(\mathbf{V}(R))$. If $x = R$ and $y = \{0\}$ then

$$\begin{aligned}yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n} &= \{0\}RR\overline{r_1}RR\overline{r_2}\dots RR\overline{r_n}\\ &= \{0\} + (r_1)R + (r_2)R + \dots + (r_n)R\\ &= (r_1, r_2, \dots, r_n)R = (r_1, r_2, \dots, r_n).\end{aligned}$$

On the other hand

$$\begin{aligned}yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m} &= \{0\}RR\overline{s_1}RR\overline{s_2}\dots RR\overline{s_m}\\ &= \{0\} + (s_1)R + (s_2)R + \dots + (s_m)R\\ &= (s_1, s_2, \dots, s_m)R = (s_1, s_2, \dots, s_m).\end{aligned}$$

This shows that $(r_1, r_2, \dots, r_n) = (s_1, s_2, \dots, s_m)$ and consequently the mapping h is well defined.
Now we want to show that h is a semiring homomorphism.
Let $yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}$ and $yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m}$ be in $S(\mathbf{V}(R))$. Then by Lemma 1.1.

$$\begin{aligned}&(yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n} + yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m})h\\ &\quad= (yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m}yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m}\overline{1})h\\ &\quad= yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}xx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m}h\\ &\quad= (r_1, r_2, \dots, r_n, s_1, s_2, \dots, s_m)\\ &\quad= (r_1, r_2, \dots, r_n) + (s_1, s_2, \dots, s_m)\\ &\quad= yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}h + yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m}h.\end{aligned}$$

Similarly

$$\begin{aligned}&(yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n} \bullet yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m})h\\ &\quad= (yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}e_{yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m}})h\\ &\quad= (r_1s_1, r_1s_2, \dots, r_1s_m, r_2s_1, r_2s_2, \dots, r_2s_m, \dots, r_ns_1, r_ns_2, \dots, r_ns_m)\\ &\quad= (r_1, r_2, \dots, r_n) \cdot (s_1, s_2, \dots, s_m)\\ &\quad= yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n}h \cdot yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m}h.\end{aligned}$$

Moreover $(x + y)h = yxx\overline{1}h = (1)$ and $yh = yxx\overline{0}h = (0)$.

So the mapping h is a semiring homomorphism.
To show that h is an isomorphism, consider two ideals of the ring R generated by $\{r_1, r_2, \dots, r_n\}$ and by $\{s_1, s_2, \dots, s_m\}$. Assume that $(r_1, r_2, \dots, r_n) = (s_1, s_2, \dots, s_m)$. It follows that for any submodules U and V in any ESM,

$$\begin{aligned}r_1U + r_2U + \dots + r_nU &= (r_1, r_2, \dots, r_n)U\\ &= (s_1, s_2, \dots, s_m)U = s_1U + s_2U + \dots + s_mU\end{aligned}$$

and consequently

$$\begin{aligned} VUU\overline{r_1}UU\overline{r_2}\dots UU\overline{r_n} &= V + r_1U + r_2U + \dots + r_nU \\ &= V + s_1U + s_2U + \dots + s_mU = VUU\overline{s_1}UU\overline{s_2}\dots UU\overline{s_m}. \end{aligned}$$

Thus the identity

$$yxx\overline{r_1}xx\overline{r_2}\dots xx\overline{r_n} = yxx\overline{s_1}xx\overline{s_2}\dots xx\overline{s_m}$$

holds in $\mathbf{V}(R)$ and h is an embedding. Moreover it is clear that h is onto what completes the proof of Theorem. □

Note that for invertible r in R the following identity

$$xyy\overline{r}yy\overline{r_1}yy\overline{r_2}\dots yy\overline{r_n} = xyy\overline{1}$$

also holds in $\mathbf{V}(R)$. This gives the following Corollary.

Corollary 2.5. *Let F be a field. Then the semiring associated with the variety $\mathbf{V}(F)$ generated by algebras $(ESV, \overline{F})$ of subspaces of vector spaces $(E, +, F)$ over the field F, is two element semiring $(\{(0), (1)\}, +, \cdot, (1), (0))$ isomorphic to the semiring associated with the variety* $\mathbf{Sl}$ *of semilattices.*

3. Congruence relations on $I_f(Z_n)$.

Lemma 3.1. *Let R be a commutative ring with unity, n be natural number and $r_{11}, \dots, r_{1k_1}, \dots, r_{n1}, \dots, r_{nk_n}$ be in R. For any congruence relation Θ on $I_f(R)$*

$$(r_{i1}, \dots, r_{ik_i})\Theta(0), \quad for \;\; i = 1, \dots, n \;\; and \;\; k_i \in N$$

if and only if

$$(r_{11}, \dots, r_{1k_1}, \dots, r_{n1}, \dots, r_{nk_n})\Theta(0).$$

Now let us consider the ring Z_n. This ring is a principal ideal ring and for r, s in Z_n, $(r) + (s) = (GCD(r, s))$.

Note that the semilattice $(I_f(Z_n), +)$ of ideals of Z_n is dually isomorphic to the semilattice $(\downarrow n, GCD)$ of divisors of n with the greatest common divisor as a semilattice operation.

Lemma 3.2. *Let p be in Z_n and $i \neq j \geq 0$ be natural numbers. Then for any congruence relation Θ on $I_f(Z_n)$ if*

$$(p^i)\Theta(p^j)$$

then

$$(p^{min(i,j)})\Theta(p^k),$$

for each $k \geq min(i, j)$.

Lemma 3.3. *Suppose that n factorizes as $abcde$ and a, b, c, d and e are pairwise relatively prime. Let $r := a'b'c'$ and $s := a''b'd'$, where a' and a'' divide a, b' divides b, c' divides c and d' divides d. Moreover let $t := GCD(a', a'')be$. Then for any congruence relation Θ on $I_f(Z_n)$*

$$(r)\Theta(s)$$

if and only if

$$(t)\Theta(0).$$

Proof. Let

$$a = p_{11}^{n_{11}} \dots p_{1k_1}^{n_{1k_1}}, \quad a' = p_{11}^{i_{11}} \dots p_{1k_1}^{i_{1k_1}}, \quad a'' = p_{11}^{j_{11}} \dots p_{1k_1}^{j_{1k_1}},$$

$$b = p_{21}^{n_{21}} \dots p_{2k_2}^{n_{2k_2}}, \quad b' = p_{21}^{i_{21}} \dots p_{2k_2}^{i_{2k_2}},$$

$$c = p_{31}^{n_{31}} \dots p_{3k_3}^{n_{3k_3}}, \quad c' = p_{31}^{i_{31}} \dots p_{3k_3}^{i_{3k_3}},$$

$$d = p_{41}^{n_{41}} \dots p_{4k_4}^{n_{4k_4}}, \quad d' = p_{41}^{i_{41}} \dots p_{4k_4}^{i_{4k_4}},$$

where for $u = 1, 2, 3, 4$ and $w = 1, \dots, k_u$, p_{uw} are prime numbers, $n_{uw} \geq 0$ are natural numbers and $0 \leq i_{uw} \leq n_{uw}$, $0 \leq j_{1w} \leq n_{1w}$.

Let $(r)\Theta(s)$. Then

$$(GCD(r, p_{1s_1}^{max(i_{1s_1}, j_{1s_1})}))\Theta(GCD(s, p_{1s_1}^{max(i_{1s_1}, j_{1s_1})})).$$

So

$$(p_{1s_1}^{i_{1s_1}})\Theta(p_{1s_1}^{j_{1s_1}}), \text{ for } s_1 = 1, \dots, k_1.$$

Hence by Lemma 3.2.

$$(p_{1s_1}^{i_{1s_1}})\Theta(p_{1s_1}^{j_{1s_1}})\Theta(p_{1s_1}^{n_{1s_1}}), \text{ for } s_1 = 1, \dots, k_1,$$

and

$$(a)\Theta(a')\Theta(a'').$$

Similarly

$$(GCD(r, p_{3s_3}^{i_{3s_3}})) = (p_{3s_3}^{i_{3s_3}})\Theta(1) = (GCD(s, p_{3s_3}^{i_{3s_3}})),$$

for $s_3 = 1, \dots, k_3$, and

$$(GCD(r, p_{4s_4}^{i_{4s_4}})) = (1)\Theta(p_{4s_4}^{i_{4s_4}}) = (GCD(s, p_{4s_4}^{i_{4s_4}})),$$

for $s_4 = 1, \dots, k_4$.
Hence by Lemma 3.2.

$$(p_{3s_3}^{n_{3s_3}})\Theta(1)\Theta(p_{4s_4}^{n_{4s_4}}),$$

for $s_3 = 1, \dots, k_3$ and $s_4 = 1, \dots, k_4$.
Consequently

$$(c')\Theta(c)\Theta(1)\Theta(d')\Theta(d).$$

Therefore we obtain that $(a'c')\Theta(ac)$ and $(a''d')\Theta(ad)$.
This implies the following:

$$(a'c'be)\Theta(abcde) = (0) \text{ and } (a''d'be)\Theta(abcde) = (0).$$

Finally, we get

$$(t) = (GCD(a', a'')be) = (GCD(a'bc'e, a''bd'e))\Theta(0).$$

Now assume that $(t) = (GCD(a', a'')be)\Theta(0)$. We have that $(0) \subseteq (a'be) \subseteq (GCD(a', a'')be)$ and $(0) \subseteq (a''be) \subseteq (GCD(a', a'')be)$. So

$$(a'be)\Theta(0)\Theta(a''be).$$

Hence

$$(a'bc'e)\Theta(0)\Theta(a''bd'e),$$

and

$$(r) = (a'b'c') = GCD(ab'cd', a'bc'e)\Theta(GCD(ab'cd', a''bd'e) = (a''b'd') = (s).$$

what complets the proof. □

4. The lattice of subvarieties of $\mathbf{V}(Z_n)$.

K. Kearnes has shown that the lattice of subvarieties of a variety of semilattice modes is determined by the congruences of the associated semiring. He has proved the following theorems.

Theorem 4.1 (1). *If $\mathbf{V}$ is a variety of semilattice modes, then any subvariety $\mathbf{U} \subseteq \mathbf{V}$ is axiomatized by the set of all equations $xys = xyt$ satisfied by $\mathbf{U}$, where $xys \geq y$ and $xyt \geq y$.*

In particular, every variety of semilattice modes is axiomatized by entropic laws and binary equations.

Theorem 4.2 (1). *If $\mathbf{V}$ is a variety of semilattice modes, then the lattice of subvarieties of $\mathbf{V}$ is dually isomorphic to $ConS(\mathbf{V})$.*

Let us denote by V_r the subvariety of $\mathbf{V}(Z_n)$ that satisfies one additional identity $yxx\overline{r} = y$.

By Lemma 3.1. we have the following corollaries.

Corollary 4.3. *Let R be a commutative ring with unity, n and $k_1, \dots, k_n$ be natural numbers and $r_{i1}, \dots, r_{ik_i}$ be in R for $i = 1, \dots, n$. In the variety $\mathbf{V}(R)$ the set of identities*

$$yxx\overline{r_{i1}} \dots xx\overline{r_{ik_i}} = y,$$

is equivalent to the identity

$$yxx\overline{r_{11}} \dots xx\overline{r_{1k_1}} \dots xx\overline{r_{n1}} \dots xx\overline{r_{nk_n}} = y.$$

Corollary 4.4. *Let* $r_1, \ldots, r_m$ *be in* Z_n. *Then*

$$V_{r_1} \cap \ldots \cap V_{r_m} = V_{GCD(r_1,\ldots,r_m)}.$$

By Lemma 3.3. the following Theorem holds.

Theorem 4.5. *If* $\mathbf{V}$ *is the subvariety of the variety* $\mathbf{V}(Z_n)$ *then* $\mathbf{V}$=$\mathbf{V}(Z_n)$ *or* $\mathbf{V}$ *is trivial or* $\mathbf{V}$=V_r *for some* r *in* Z_n, *such that* $GCD(r, n) \neq 1$.

Lemma 4.6. *Let* r, s *be in* Z_n. $V_r \subseteq V_s$ *if and only if* $r \mid s$.

Proof. Assume that $r \mid s$. Then $(0) \subseteq (s) \subseteq (r)$.
So for any congruence relation Θ, if $(r)\Theta(0)$, then $(s)\Theta(0)$. This implies that $V_r \subseteq V_s$.
On the other hand, if $V_r \subseteq V_s$ then by Corollary 4.4.

$$V_r = V_r \cap V_s = V_{GCD(r,s)}$$

and consequently $r = GCD(r, s)$ and $r \mid s$. □

Let $(\downarrow n, GCD, LCM)$ be the lattice of divisors of n with the meet of two numbers i and j being their greatest common divisor $GCD(i, j)$ and join of i and j being their least common multiple $LCM(i, j)$.

Finally we have the following Theorem.

Theorem 4.7. *The lattice of all subvarieties of the variety* $\mathbf{V}(Z_n)$ *is isomorphic to the lattice* $(n \downarrow, GCD, LCM)$. *The isomorphism is given by the mapping* h, *where the image of the trivial variety is* 1, *the image of the variety* $\mathbf{V}(Z_n)$ *is* n *and for* r *in* Z_n *with* $GCD(n, r) \neq 1$, $h(V_r) = r$.

Proof. It is evident that the mapping h is onto. We show that it is one-to-one. Indeed, if $V_r = V_s$ then by Lemma 4.6. $r = s$.
By Corollary 4.4.

$$h(V_r \cap V_s) = h(V_{GCD(r,s)}) = GCD(r, s) = GCD(h(V_r), h(V_s)).$$

It follows that h is a meet-homomorphism.
Now we show that

$$V_r \vee V_s = V_{LCM(r,s)}.$$

By Lemma 4.6. it follows that

$$V_r \vee V_s \subseteq V_{LCM(r,s)}.$$

Now if $V_r \subseteq V$ and $V_s \subseteq V$, then by Theorem 4.5. there is w in Z_n such that $V = V_w$ and $r \mid w$ and $s \mid w$, whence $LCM(r, s) \mid w$.
It follows that $V_{LCM(r,s)} \subseteq V_w$ and consequently

$$V_r \vee V_s = V_{LCM(r,s)}.$$

This implies that h is a join-homomorphism. Indeed,

$$h(V_r \vee V_s) = h(V_{LCM(r,s)}) = LCM(r, s) = LCM(h(V_r), h(V_s)).$$

It complets the proof. □

Note that, if Z_n is a field (n is a prime number) the variety $\mathbf{V}(Z_n)$ has only trivial subvarieties.

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