VARIETIES OF DIFFERENTIAL MODES EMBEDDABLE INTO SEMIMODULES

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Abstract. Differential modes provide examples of modes that do not embed as subreducts into semimodules over commutative semirings. The current paper studies differential modes, so-called Szendrei differential modes, which actually do embed into semimodules. These algebras form a variety. The main result states that the lattice of non-trivial subvarieties is dually isomorphic to the (non-modular) lattice of congruences of the free commutative monoid on two generators. Consequently, all varieties of Szendrei differential modes are finitely based.

1. Introduction

As a consequence of results of Ježek-Kepka [4], each idempotent and entropic groupoid, i.e. each groupoid mode as defined e.g. in [11], embeds into a semimodule over a commutative semiring. As shown by M. Stronkowski [13, 14] and D. Stanovský [12], this is no longer true for modes with operations of larger arity. In [5], A. Kravchenko, A. Pilitowska, A. Romanowska and D. Stanovský presented a broad class of modes that are not embeddable into such semimodules. These algebras generalize an example of Stanovský [12], and belong to the class $\mathcal{D}_3$ of so-called (ternary) differential modes. Recall that a *differential mode* is an idempotent algebra (A, f) with one ternary operation $f(x, y, z) =: (xyz)$ satisfying the identities

$$((xy_1y_2)z_1z_2) = ((xz_1z_2)y_1y_2) \qquad (\textit{left normal law}) \tag{1.1}$$

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and

$$(x(y_1z_1z_2)(y_2t_1t_2)) = (xy_1y_2) \qquad (\textit{left reductive law}). \tag{1.2}$$

These identities imply entropicity. Consequently, differential modes are indeed modes, i.e. they are idempotent and entropic. Note one further identity

$$(x(xy_1z_1)(xy_2z_2)) = x \tag{1.3}$$

satisfied by differential modes.

The paper [5] explains the relationship between differential modes and differential groupoids. It contains some structural investigations and analysis of the lattice of varieties. It also provides a survey of some other classes of modes that are known to be embeddable into semimodules over a commutative semiring (see also [7]), and shows that the class of differential modes splits into two subclasses: the variety of those that do embed, and the class of those that do not. The former class is characterized as the variety $Sz(\mathcal{D}_3)$ of differential modes satisfying two (equivalent) *Szendrei identities*:

$$((x_{11}X_{12}x_{13})X_{21}x_{31}) = ((x_{11}X_{21}x_{13})X_{12}x_{31}), \tag{1.4}$$

$$((x_{11}x_{12}X_{13})x_{21}X_{31}) = ((x_{11}x_{12}X_{31})x_{21}X_{13}). \tag{1.5}$$

The present paper studies the variety of Szendrei differential modes, differential modes that are embeddable into semimodules. The main goal is to describe the lattice of their varieties. We start with an alternative characterization of free Szendrei differential modes in Section 2, and then use it in Section 3 to show that subvarieties of $Sz(\mathcal{D}_3)$ can be based by identities in two variables. The main result of Section 3 is that non-trivial subvarieties of the variety $Sz(\mathcal{D}_3)$ of Szendrei differential modes form a lattice dually isomorphic to the lattice of congruences of the square of the additive monoid of natural numbers. As a corollary, we show that all subvarieties are finitely based, and that the lattice is not modular. The final sections contain a more detailed analysis of the varieties relatively based by one binary identity, and a discussion concerning the size of relative bases and of the covers of the unique atom of the lattice of subvarieties of $Sz(\mathcal{D}_3)$. This unique atom is the variety of left-zero algebras defined by the identity $(xyz) = x$.

The present paper is a continuation of [5], and we refer readers to that paper for all unexplained details. The notation and terminology follows that paper. Note also that, as in [5], the results of this paper may be generalized to the case of n-ary differential modes in an obvious way. For further information concerning the theory of modes, we refer readers to the two monographs [9] and [11]; for universal algebra, one may also consult standard books on the subject.

2. Free Szendrei differential modes revisited

Free Szendrei modes were characterized by Theorem 4.4 of [5]. Here we will provide an alternative characterization, better suited to the needs of the current paper. First let us recall a useful notation: the expression xR^k_{yz} is an abbreviation for $(\ldots((xyz)yz)\ldots yz)$, where yz is repeated k times, and R_{ij}, with natural numbers i, j, refers to $R_{x_ix_j}$.

By Lemma 2.5 of [5], the Szendrei identities are equivalent in the variety $\mathcal{D}_3$ of ternary differential modes to the unique identity

$$(xyz) = ((xyx)xz). \tag{2.1}$$

Then, by the left normal law and (2.1), the following equation is satisfied in all Szendrei modes for each natural number k:

$$xR^k_{yz} = xR^k_{yx}R^k_{xz}. \tag{2.2}$$

By Theorem 4.2 of [5], each word $w = w(x_1, \ldots, x_n)$ with the leftmost variable x_1 is equivalent in $\mathcal{D}_3$ to the standard word

$$x_1R^{k_{12}}_{12}\ldots R^{k_{1n}}_{1n}\,R^{k_{21}}_{21}\ldots R^{k_{2n}}_{2n}\ldots R^{k_{n1}}_{n1}\ldots R^{k_{nn}}_{nn}, \tag{2.3}$$

where the indices ij run over the set $\mathbf{n}\times\mathbf{n}$ and are ordered lexicographically. Now using (2.2) and the left normal law, we can reduce (2.3) to

$$x_1R^{l_{12}}_{12}R^{l_{21}}_{21}R^{l_{13}}_{13}R^{l_{31}}_{31}\ldots R^{l_{1n}}_{1n}R^{l_{n1}}_{n1}. \tag{2.4}$$

Note that some of the l_{ij} may be equal to 0. Now recall that the *orbit* of an element a in a differential mode M is the smallest set containing a and closed under all the mappings R_{bc}, for $b, c \in M$. The alternative description of free Szendrei differential modes follows immediately.

Theorem 2.1. *Let $w = w(x_1, \ldots, x_n)$ be an element of the free $Sz(\mathcal{D}_3)$-algebra $F_{Sz}(X)$ over a set X, where the set $\{x_1, \ldots, x_n\}$ is precisely the set of variables in w, with x_i as its leftmost variable. Then w may be expressed uniquely in the standard form*

$$x_iR^{l_{i1}}_{i1}R^{l_{1i}}_{1i}\ldots R^{l_{in}}_{in}R^{l_{ni}}_{ni} \tag{2.5}$$

for some $l_{ij}, l_{ji} \in \mathbb{N}$ with $l_{ii} = 0$. The elements (2.5) form the orbit of x_i, and a left-zero subalgebra. The algebra $F_{Sz}(X)$ is the $LZ \circ LZ$-sum of the orbits of its generators in X.

Recall that the orbits of generators define a (unique) congruence of $F_{Sz}(X)$ with left-zero classes and left-zero quotient. The $LZ \circ LZ$-sum construction allows one to reconstruct the whole algebra from these classes and the quotient.

Each orbit of a free generator also carries the structure of a commutative monoid. The monoid operation $\circ$ in the orbit of x_i is defined by

$$\begin{aligned}(2.6)\qquad & x_i R_{i1}^{k_{i1}} R_{1i}^{k_{1i}} \dots R_{in}^{k_{in}} R_{ni}^{k_{ni}} \circ x_i R_{i1}^{l_{i1}} R_{1i}^{l_{1i}} \dots R_{in}^{l_{in}} R_{ni}^{l_{ni}} := \\ & x_i R_{i1}^{k_{i1}+l_{i1}} R_{1i}^{k_{1i}+l_{1i}} \dots R_{in}^{k_{in}+l_{in}} R_{ni}^{k_{ni}+l_{ni}},\end{aligned}$$

while the monoid identity is the generator x_i. We denote by $M(x_i)$ the monoid defined in this way on the orbit of x_i.

Corollary 2.2. *Let $F_{Sz}(n)$ be the free Szendrei differential mode on the set $X = \{x_1, \dots, x_n\}$ of n free generators. Then the mapping*

$$\iota : M(x_i) \to (\mathbb{N} \times \mathbb{N})^{n-1};$$

$$x_i R_{i1}^{l_{i1}} R_{1i}^{l_{1i}} \dots R_{in}^{l_{in}} R_{ni}^{l_{ni}} \mapsto (l_{i1}, l_{1i}, \dots, l_{i(i-1)}, l_{(i-1)i}, l_{i(i+1)}, l_{(i+1)i}, \dots, l_{in}, l_{ni})$$

is an isomorphism of the monoids $M(x_i)$ and $((\mathbb{N} \times \mathbb{N})^{n-1}, +, \underline{0})$.

In particular, the orbit of x in the free Szendrei differential mode on two generators x and y consists of elements of the form $xR_{xy}^i R_{yx}^j$ with (i, j) as the image under ι. Each of the monoids $M(x)$ and $M(y)$ is isomorphic to the monoid $\mathbb{N} \times \mathbb{N}$.

3. The lattice of varieties

The aim of this section is to describe the lattice of varieties of Szendrei differential modes. First recall that the lattice has a unique atom, the variety of left-zero algebras. This follows immediately from the fact that each differential mode decomposes as an $LZ \circ LZ$-sum. In particular, both sides of an identity satisfied by a non-trivial differential mode must have the same leftmost variable.

Next we show that to describe the lattice of varieties of Szendrei differential modes, we only need the free algebra on two free generators.

Proposition 3.1. *Each subvariety of the variety $Sz(\mathcal{D}_3)$ of Szendrei differential modes possesses a relative basis consisting of identities in two variables.*

Proof. Let $s = t$ be an identity in variables $x_1, \dots, x_n$, where x_1 is the leftmost variable in both s and t. By Theorem 2.1,

$$s = x_1 R_{12}^{k_{12}} R_{21}^{k_{21}} \dots R_{1n}^{k_{1n}} R_{n1}^{k_{n1}}$$

and

$$t = x_1 R_{12}^{l_{12}} R_{21}^{l_{21}} \dots R_{1n}^{l_{1n}} R_{n1}^{l_{n1}}.$$

We show that the identity $s = t$ is equivalent to the following set E of identities in two variables:

$$(e_2) \qquad x_1 R_{12}^{k_{12}} R_{21}^{k_{21}} = x_1 R_{12}^{l_{12}} R_{21}^{l_{21}},$$

$$(e_3) \qquad x_1 R_{13}^{k_{13}} R_{31}^{k_{31}} = x_1 R_{13}^{l_{13}} R_{31}^{l_{31}},$$

$$\dots$$

$$(e_n) \qquad x_1 R_{1n}^{k_{1n}} R_{n1}^{k_{n1}} = x_1 R_{1n}^{l_{1n}} R_{n1}^{l_{n1}}.$$

(Obviously, we can replace all the variables x_i in E, for $i = 2, \dots, n$, by the same variable, say y.)

First note that by substituting x_1 for x_j in $s = t$, for each $j \neq i$, one obtains the identity (e_i) . On the other hand, an easy induction shows that the set E implies the identity $s = t$. Indeed, if the identities (e_2) - (e_i) imply the identity

$$(E_i) \qquad x_1 R_{12}^{k_{12}} R_{21}^{k_{21}} \dots R_{1i}^{k_{1i}} R_{i1}^{k_{i1}} = x_1 R_{12}^{l_{12}} R_{21}^{l_{21}} \dots R_{1i}^{l_{1i}} R_{i1}^{l_{i1}},$$

then by replacing x_1 in (E_i) by the two sides of the identity (e_{i+1}), and then using the left normal law, one readily obtains (E_{i+1}). □

Now it is easy to see that the proper fully invariant congruences of the free Szendrei mode $F_{Sz}(2)$ are determined by their restriction to the orbit of any free generator. Note that the monoids of any two free generators in each free Szendrei differential mode are isomorphic. Then the invariance under substitution corresponds to the preservation of the monoid operation on each of the orbits. This remark justifies the following proposition.

Proposition 3.2. *The following conditions hold for the free Szendrei differential mode $F_{Sz}(2)$ on two generators:*

(a) *The restriction of a fully invariant congruence of $F_{Sz}(2)$ to the orbit of a free generator is a congruence of its monoid.*
(b) *A proper congruence of the monoid of a free generator extends uniquely to a fully invariant congruence of $F_{Sz}(2)$.*

Note that the improper congruence of the monoid of a generator extends to two congruences of $F_{Sz}(2)$: to its largest congruence, and to the congruence whose blocks are precisely the two orbits. In terms of the corresponding varieties, the former corresponds to the trivial variety defined by $x = y$, and the latter to the variety of left-zero algebras defined by $(xyx) = (xxy) = x$.

Corollary 3.3. *Proper fully invariant congruences of* $F_{Sz}(2)$ *are uniquely determined by the congruences of the monoid of any free generator.*

As a corollary to Propositions 3.1 and 3.2, and Corollary 2.2, one obtains the following theorem.

Theorem 3.4. *The lattice of non-trivial subvarieties of the variety* $Sz(\mathcal{D}_3)$ *of Szendrei differential modes is dually isomorphic to the lattice* $Cg(\mathbb{N} \times \mathbb{N})$ *of congruences of the monoid* $(\mathbb{N} \times \mathbb{N}, +, \underline{0})$.

Theorem 3.4 has several interesting consequences.

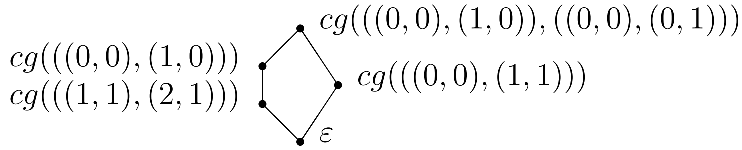
Corollary 3.5. *Each variety of Szendrei differential modes has a finite basis for its identities.*

Proof. By Rédei's Theorem (see e.g. [1]), each finitely generated commutative semigroup is finitely presented. In particular, this means that every congruence of a free commutative monoid on two generators is generated by a finite number of pairs of natural numbers. They correspond to the finite number of equations in two variables. □

It is known that the lattice of varieties of differential groupoids is distributive. This is no longer true in the case of (ternary) differential modes.

Corollary 3.6. *The lattice of varieties of Szendrei differential modes is not modular.*

Proof. The congruence lattice of $(\mathbb{N} \times \mathbb{N}, +, \underline{0})$ contains the following lattice isomorphic to the "pentagon" lattice N_5:



In the picture, the symbol $cg((j,k),(m,n))$ denotes the principal congruence generated by the pairs (j,k) and (m,n), while ε denotes the least congruence of $\mathbb{N} \times \mathbb{N}$ corresponding to the equality relation. □

Congruences of free commutative monoids have been extensively studied, see e.g. [2, Ch. 9]. They are completely determined by means of certain abelian groups, the so-called Rédei groups, and certain mappings from these groups into ideals of the monoids. However, in the case considered in this paper, when we deal only with the monoid $(\mathbb{N} \times \mathbb{N}, +, \underline{0})$, there is a more direct description of the congruences that we present in the next section. Our description provides an easy comparison of varieties, and an easy way of relating some of them to corresponding varieties of differential groupoids.

4. Varieties defined by a single binary identity

In this section we will describe the poset of subvarieties of $Sz(\mathcal{D}_3)$ relatively based by one identity in two variables.

Note that by [5, §5], each of the derived binary operations of differential modes is in fact a differential groupoid operation. (In particular, a derived operation in two variables x and y has the form $xR^i_{xy}R^j_{yx}$.) The defining identities of the subvarieties of $\mathcal{D}_2$, when applied to the derived operation $x \circ y := xR^i_{xy}R^j_{yx}$, also define subvarieties of the variety $\mathcal{D}_3$, and provide a sublattice of the lattice $\mathcal{L}(\mathcal{D}_3)$ of varieties of differential modes.

Recall (see [8]) that each proper non-trivial subvariety of the variety $\mathcal{D}_2$ of differential groupoids is relatively based by a unique identity of the form $xy^k = xy^{k+l}$ for some natural number k and positive integer l. Denote such a variety by $\mathcal{D}_2^{k,k+l}$. These subvarieties form the lattice $\mathcal{L}(\mathcal{D}_2)^- \cong \mathbb{N} \times \mathbb{Z}^+$, the first factor with the usual linear ordering and the second ordered by the divisibility relation. Note also that the lattice of non-trivial subvarieties is dually isomorphic to the congruence lattice $Cg(\mathbb{N})$ of the monoid $(\mathbb{N}, +, \underline{0})$ of natural numbers. (This latter is described e.g. in [3, Ch. I.7].)

By Theorem 3.4, to describe varieties relatively based by a unique binary identity, it suffices to consider principal congruences of the monoid $\mathbb{N} \times \mathbb{N}$, and then to deduce the relation between the corresponding subvarieties.

For $(m, n), (m', n') \in \mathbb{N} \times \mathbb{N}$, let $cg((m, n), (m', n'))$ be the principal congruence generated by the pair $((m, n), (m', n'))$. Note that if there is (k, l) in $\mathbb{N} \times \mathbb{N}$ such that

$$((p, q), (r, s)) = ((m + k, n + l), (m' + k, n' + l)),$$

then

$$((p, q), (r, s)) \in cg((m, n), (m', n')).$$

In what follows, a congruence $cg((m, n), (m', n'))$ corresponds to the subvariety $\mathcal{V}^{m',n'}_{m,n}$ of $Sz(\mathcal{D}_3)$ defined by the identity $xR^m_{xy}R^n_{yx} = xR^{m'}_{xy}R^{n'}_{yx}$. Note also that $\mathcal{V}^{m',n'}_{m,n} \leq \mathcal{V}^{m'+k,n'+l}_{m+k,n+l}$.

Consider elements of $\mathbb{N} \times \mathbb{N}$ located as the corresponding points of the real plane $\mathbb{R} \times \mathbb{R}$ with the usual coordinate axes. Note that any two points (m, n) and (m', n') of $\mathbb{N} \times \mathbb{N}$ uniquely define a straight line L in $\mathbb{R} \times \mathbb{R}$ containing these two points and described by the equation

$$y = \frac{n' - n}{m' - m}(x - m) + n,$$

or by

$$y = \pm\frac{j}{i}(x - m) + n, \tag{4.1}$$

where j and i are relatively prime positive integers, or else by $y = c$ or $x = c$ for some non-negative integer c. The complete set of integral solutions of (4.1) consists of points $(m + ki, n + kj)$ if the coefficient of x in (4.1) is positive or $(m - ki, n + kj)$ if it is negative, where k runs over the set $\mathbb{Z}$.

Now there are two basic cases (Case 1 and Case 2 below) to consider for the principal congruences $cg((m, n), (m', n'))$ of the monoid $\mathbb{N} \times \mathbb{N}$. Without loss of generality, assume further that $n \leq n'$.

Case 1: *Congruences of the form* $\theta = cg((m, n), (m', n'))$, *where* $(m, n) < (m', n')$.

In this case the line L is determined by the point (m, n) and the non-negative coefficient j/i of x. Let

$$L(i, j, m, n) := \{(m + ki, n + kj) \mid k \in \mathbb{N}\}.$$

Let

$$L_{ij} := L(i, j, 0, 0),$$

the intersection of $\mathbb{N} \times \mathbb{N}$ with the line L^0 parallel to L, given by the equation $y = \frac{j}{i}x$. Note that it consists of the points (ki, kj) for $k \in \mathbb{N}$.

We will first consider the principal congruences θ of the form

$$cg((ki, kj), ((k + l)i, (k + l)j)) \tag{4.2}$$

for a positive number l, i.e. congruences generated by two points of L_{ij}. Note that each finite θ-class is a singleton, and that any two points of $\mathbb{N} \times \mathbb{N}$ related by θ belong to a line parallel to L^0. Observe also that each such congruence restricts to a congruence of the submonoid L_{ij} of $\mathbb{N} \times \mathbb{N}$. On the other hand, since for all $(p, r) > (0, 0)$, we have

$$((p+ki, r+kj), (p+(k+l)i, r+(k+l)j)) \in cg((ki, kj), ((k+l)i, (k+l)j)),$$

each such restriction extends uniquely to the original congruence θ of $\mathbb{N} \times \mathbb{N}$. Now the mapping

$$\theta \mapsto \theta | L_{ij} \mapsto cg(k, k+l)$$

defines a one-to-one correspondence between the congruences θ of the form (4.2) from $Cg(\mathbb{N} \times \mathbb{N})$ and the congruences from $Cg(\mathbb{N})$. As all non-trivial congruences of $(\mathbb{N}, +, \underline{0})$ are principal [3, Ch. I.7], it follows that for fixed i and j as above, the mapping actually defines a lattice isomorphism. Note that similar observations hold for points of lines parallel to the y-axis. This proves the following lemma.

Lemma 4.1. *For each pair (i, j) of relatively prime natural numbers i and j, the varieties $\mathcal{V}_{ki,kj}^{(k+l)i,(k+l)j}$, with $k \in \mathbb{N}$ and $l \in \mathbb{Z}^+$, form a lattice isomorphic to the dual of the lattice $Cg(\mathbb{N})$ with the top element removed, and hence isomorphic to the lattice $\mathcal{L}(\mathcal{D}_2)^-$. Each of these varieties contains the variety $\mathcal{V}_{0,0}^{i,j}$.*

It is clear that for any two distinct coprime pairs (i, j) and (i', j'), the corresponding varieties $\mathcal{V}_{0,0}^{i,j}$ and $\mathcal{V}_{0,0}^{i',j'}$ are incomparable, and hence also any two varieties $\mathcal{V}_{ki,kj}^{(k+l)i,(k+l)j}$ and $\mathcal{V}_{ri',rj'}^{(r+s)i',(r+s)j'}$.

Note that for fixed i and j as above, the varieties $\mathcal{V}_{0,0}^{li,lj}$, where l runs over $\mathbb{Z}^+$, form a lattice isomorphic to the lattice $\mathbb{Z}^+$ with the divisibility relation, and are in one-to-one correspondence with the varieties $\mathcal{D}_2^{0,l}$ of differential groupoids. It is easy to see that for any point $(m, n) \in \mathbb{N} \times \mathbb{N}$ the mapping

$$\mathcal{V}_{0,0}^{li,lj} \mapsto \mathcal{V}_{m,n}^{m+li,n+lj}$$

also provides a lattice isomorphism. This justifies the following proposition.

Proposition 4.2. *Let (i, j) be a pair of relatively prime natural numbers. Then the varieties $\mathcal{V}_{m,n}^{m+li,n+lj}$, with $(m, n) \in \mathbb{N} \times \mathbb{N}$ and $l \in \mathbb{Z}^+$, form a lattice isomorphic to the lattice $\mathbb{N} \times \mathbb{N} \times \mathbb{Z}^+$ and hence to $\mathbb{N} \times \mathcal{L}(\mathcal{D}_2)^-$.*

In particular, for fixed i and j, the varieties $\mathcal{V}_{ki,kj}^{(k+l)i,(k+l)j}$ and the varieties $\mathcal{D}_2^{k,k+l}$ form isomorphic lattices.

Case 2: *Congruences of the form $\theta = cg((m, n), (m', n'))$, where (m, n) and (m', n') are incomparable*

We may assume now that $n' > n$ and $m' < m$. In the case under consideration, the line L is determined by the point (m, n) and the negative coefficient $-\frac{j}{i}$ of x. As in Case 1, set

$$L(i, j, m, n) := \mathbb{N} \times \mathbb{N} \cap \{(m - ki, n + kj) \mid k \in \mathbb{Z}\},$$

the intersection of the line L with the set $\mathbb{N} \times \mathbb{N}$. Also set

$$L_{ij} := L(i, j, i, 0),$$

the intersection of $\mathbb{N} \times \mathbb{N}$ with the line L^0 parallel to L, given by the equation $y = -\frac{j}{i}(x - i)$. It consists of the points $(i, 0)$ and $(0, j)$. Note that the sets $L(i, j, m, n)$ are finite.

First consider the principal congruences θ of the form $cg((i, 0), (0, j))$ for positive integers i and j. Note that in this case all θ-classes are finite, and any two points of $\mathbb{N} \times \mathbb{N}$ related by θ belong to a line parallel to L^0. In particular, for any $(p, r) \in \mathbb{N} \times \mathbb{N} \setminus \{(0, 0)\}$,

$$((p + i, r), (p, r + j)) \in cg((i, 0), (0, j)).$$

Let $[(k, l)) := \{(s, t) \in \mathbb{N} \times \mathbb{N} \mid (s, t) \geq (k, l)\}$. Note that the elements of $\mathbb{N} \times \mathbb{N} \smallsetminus [(i, 0)) \cup [(0, j))$ form one-element θ-classes. And if at least one of i and j is bigger than 1, then the elements of $\{(p, r) \mid i \leq p < 2i,\ 0 \leq r < j\} \cup \{(p, r) \mid 0 \leq p < i,\ j \leq r < 2j\}$ form two element θ-classes. (Each point represents a two element θ-class consisting precisely of one point in the first summand and one in the second). Each of the remaining classes has more than two elements. The same remarks hold in the case when $i = j$ and $i > 1$.

Now, for fixed i and j, consider the congruences of the form

$$cg((p + i, r), (p, r + j))$$

with $(p, r) > (0, 0)$, collapsing two points of a line parallel to L_0. Clearly $cg((p+i, r), (p, r+j)) < cg((i, 0), (0, j))$. For (p, r) different from (p', r') the congruences $cg((p + i, r), (p, r + j))$ and $cg((p' + i, r'), (p', r' + j))$ are distinct, since they have different numbers of one-element classes. It follows that for fixed (i, j) and (p, r) running over $\mathbb{N} \times \mathbb{N} \smallsetminus \{(0, 0)\}$, one obtains distinct congruences. Moreover, if $(p, r) < (p', r')$ then $cg((p' + i, r'), (p', r' + j)) < cg((p + i, r), (p, r + j))$. This implies the following proposition.

Proposition 4.3. *For each pair (i, j) of coprime natural numbers i and j different from $(0, 0)$, the varieties $\mathcal{V}^{p,r+j}_{p+i,r}$, where $(p, r) \in \mathbb{N} \times \mathbb{N}$, form a lattice isomorphic to the lattice $\mathbb{N} \times \mathbb{N}$ with the usual ordering relation. They all contain the variety $\mathcal{V}^{0,j}_{i,0}$.*

A similar proposition holds in the case $i = j$ and $i > 1$. Note also that $\mathcal{V}^{0,j}_{i,0} < \mathcal{V}^{0,jk}_{ik,0}$.

5. The size of relative bases

As shown already in [5], a variety of differential modes defined by a finite set of identities has a relative basis consisting of a single identity.

In the case of Szendrei varieties, the single identity may be obtained as follows. First write each identity of the basis in the standard form as in the proof of Proposition 3.1. Then replace it by the equivalent set of identities in two variables. Finally, replace the union of these sets by a unique identity, as was done in the last part of the proof of Proposition 3.1. Note however that the number of variables in this identity may be quite large.

Although varieties of Szendrei differential modes may be relatively based by identities in two variables, the following examples show that a single such identity may not be sufficient.

Example 5.1. The variety $\mathcal{LZ}_3$ of left-zero algebras is relatively based by two identities $(xxy) = x$ and $(xyx) = x$ in two variables. But there is no single identity in two variables that would imply both of them.

Example 5.2. It is clear that if any two pairs of points of $\mathbb{N} \times \mathbb{N}$ are related by the same principal congruence then they belong to two parallel lines. If two pairs of points belong to two non-parallel lines, they determine a subvariety that is defined by two (but no fewer) binary identities.

There are varieties of differential modes defined by more than two, say n, binary identities, but not by $k < n$ binary identities.

Example 5.3. Consider the join θ of the principal congruences $\theta_i := cg((n-i, i), (n-i+1, i))$ of the (additive) monoid $\mathbb{N}\times\mathbb{N}$, for $i = 0, \ldots, n$. It is easy to see that the congruence θ is not generated by a smaller number of principal congruences. It follows that the variety determined by the congruence θ has a relative basis consisting of n binary identities, but not by a smaller number of such identities.

6. Covers of the variety of left-zero differential modes

First note that the free algebra on two generators in a variety covering $\mathcal{LZ}_3$ must have precisely one proper non-trivial fully invariant congruence, with the quotient being a left-zero algebra. When restricted to the orbit of any free generator, this congruence must be the improper congruence of the monoid of the orbit, and its only non-trivial congruence. The only commutative monoids with this property are the simple monoids. Such finite simple commutative monoids are isomorphic to $(\mathbb{Z}_p, +, 0)$ for a prime p, or to the 2-element semilattice $\underline{\mathbf{2}}$. To obtain $(\mathbb{Z}_p, +, 0)$ or $\underline{\mathbf{2}}$ as a quotient of the monoid $\mathbb{N} \times \mathbb{N}$, one should apply an appropriate maximal congruence. Now $(\mathbb{Z}_p, +, 0)$ may be obtained by first applying a congruence $cg((0, 0), (p, 0))$ or $((0, 0), (0, p))$, and then a congruence that will reduce lines parallel to the axes to points. This procedure provides two types of congruences:

$$cg(((0,0),(p,0)),((a,0),(0,1))) \quad \text{and} \quad cg((0,0),(0,p)),(0,a),(1,0))),$$

where $a = 0, 1, \ldots, p-1$. For given p and $a \neq 0$, two such congruences actually coincide. The congruences correspond to varieties defined by two identities in two variables:

$$x = xR_{xy}^p \quad \text{and} \quad xR_{xy}^a = xR_{yx}$$

in the case $a = 0, 1, \ldots, p-1$, or

$$x = xR_{yx}^p \quad \text{and} \quad x = xR_{xy}.$$

The following maximal congruences of $\mathbb{N} \times \mathbb{N}$ provide the 2-element semilattice as a quotient:

$$cg(((1,0),(2,0)),((0,0),(0,1))),\ cg(((0,1),(0,2)),((0,0),(1,0))),$$
$$cg(((1,0),(2,0)),((0,1),(1,0))) = cg(((0,1),(0,2)),((1,0),(0,1))).$$

The corresponding varieties are defined by the following pairs of identities:

$$xR_{xy} = xR_{xy}^2 \quad \text{and} \quad x = xR_{yx},$$
$$xR_{yx} = xR_{yx}^2 \quad \text{and} \quad x = xR_{xy},$$
$$xR_{xy} = xR_{xy}^2 \quad \text{and} \quad xR_{yx} = xR_{xy}.$$

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