ON SOME CONGRUENCES OF POWER ALGEBRAS

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Abstract. In a natural way we can "lift" any operation defined on a set A to an operation on the set of all non-empty subsets of A and obtain from any algebra (A, Ω) its *power algebra* of subsets. In this paper we investigate extended power algebras (power algebras of non-empty subsets with one additional semilattice operation) of modes (entropic and idempotent algebras). We describe some congruence relations on these algebras such that their quotients are idempotent. Such congruences determine some class of non-trivial subvarieties of the variety of all semilattice ordered modes (modals).

1. Introduction

Power structure of a structure $\mathcal{A}$ is an appropriate structure defined on the power set $\mathcal{P}A$. There are many papers on this topic, we refer the reader to C. Brink [3], I. Bošnjak, R. Madarász [2] which give an overview of the results and extensive reference lists.

For a given set A denote by $\mathcal{P}_{>0}A$ the family of all non-empty subsets of A. For any n-ary operation $\omega : A^n \to A$ we define *the complex* (*or power*) *operation* $\omega : (\mathcal{P}_{>0}A)^n \to \mathcal{P}_{>0}A$ in the following way:

$$\omega(A_1, \ldots, A_n) := \{\omega(a_1, \ldots, a_n) \mid a_i \in A_i\},$$

where $\emptyset \neq A_1, \ldots, A_n \subseteq A$. *The power* (*complex* or *global*) *algebra* of an algebra (A, Ω) is the algebra $(\mathcal{P}_{>0}A, \Omega)$.

We can also lift a relation from a set to its power set. For example, if θ is a binary relation on A then we can define a binary relation θ on $(\mathcal{P}A)^2$ as follows:

$$X \theta Y \quad \Leftrightarrow \quad (\forall x \in X)(\exists y \in Y)\; x\theta y \text{ and } (\forall y \in Y)(\exists x \in X)\; x\theta y. \tag{1.0.1}$$

Algebras considered in this paper are extended power algebras of modes, i.e. power algebras (of modes) with the additional operation of join of sets.

Definition 1.1. An algebra (M, Ω) is called a *mode* if it is *idempotent*, in the sense that each singleton is a subalgebra, and *entropic*, i.e. any two of its operations commute. Both properties may also be expressed by means of identities:

$$\omega(x, \ldots, x) \approx x, \qquad \text{(idempotent law)},$$
$$\omega(\phi(x_{11}, \ldots, x_{n1}), \ldots, \phi(x_{1m}, \ldots, x_{nm})) \approx$$
$$\phi(\omega(x_{11}, \ldots, x_{1m}), \ldots, \omega(x_{n1}, \ldots, x_{nm})), \qquad \text{(entropic law)},$$

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for every m-ary $\omega \in \Omega$ and n-ary $\phi \in \Omega$.

Modes were introduced and investigated in detail by A. Romanowska and J.D.H. Smith [19, 20] and for more information about modes we refer the reader to these monographs.

Closely related to power algebras of sets are complex algebras of subalgebras. Let $S(A)$ be the set of all (non-empty) subalgebras of (A, Ω). In general, the family $S(A)$ has not to be closed under complex operations. However if it does, $(S(A), \Omega)$ is a subalgebra of the algebra $(\mathcal{P}_{>0}A, \Omega)$ and is called *the algebra of subalgebras* of (A, Ω). For example, if an algebra (A, Ω) is a mode, then its algebra of subalgebras is always defined and moreover is again a mode. (See [16]-[18] and [19, Sections 1.4 and 3.1].) It is in contrast with power algebras of modes which are entropic but very rarely idempotent. In [12, 13] we described the congruence on a power algebra of a mode which gives as a factor the mode of submodes. In this paper we investigate other congruences on power algebras of modes which also give an idempotent factor. Briefly, such congruences we call *idempotent congruence relations*. By results of C. Brink [3] it is known that if θ is a congruence relation on a mode (M, Ω), then also the "lifted" relation θ defined as in (1.0.1) is a congruence on the power algebra. But the quotient is idempotent only in very special situations (see Section 2) so all congruences considered here are not obtained from congruences on (M, Ω).

The paper is organized as follows. In Section 2 we study the basic properties of power algebras and put a necessary and sufficient condition for the power algebra to be idempotent.

In [13] we introduced the concept of extended power algebras, i.e. power algebras with one additional union operation. Obviously, each congruence on the extended power algebra is a congruence on the power algebra. We also investigated the smallest congruence of an extended power algebra of a mode that gives an idempotent factor. In Section 3 other descriptions of such idempotent replica congruence are presented. Namely, we describe three idempotent congruence relations of extended power algebras of modes. We present some conditions when these congruences coincide and examples when they are different.

In Section 4 we introduce the concept of Γ-sinks of an algebra as a special kind of its subalgebras. It generalizes the concept of sinks introduced in [19]. Next we use Γ-sinks to define a family of congruences of extended power algebras of a mode such that the quotient is an idempotent algebra. Such congruences describe some non-trivial subvarieties of the variety of all semilattice ordered modes.

We conclude the paper with a list of open problems.

The set of all equivalence classes of a relation $\varrho \subseteq A \times A$ is denoted by A^{ϱ}. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0.

2. Idempotent power algebras

In this Section we will present some basic properties of idempotent power algebras and the relationship between complex operations and the set-theoretical union and intersection.

Let (A, Ω) be an algebra. The set $\mathcal{P}_{>0}A$ also carries a join semilattice structure under the set-theoretical union $\cup$. By adding the operation $\cup$ to the set of fundamental operations of the power algebra of (A, Ω) we obtain *the extended power algebra* $(\mathcal{P}_{>0}A, \Omega, \cup)$.

B. Jónsson and A. Tarski [6] proved that complex operations *distribute* over the union $\cup$, i.e. for each n-ary operation $\omega \in \Omega$ and non-empty subsets $A_1, \dots, A_i, \dots, A_n, B_i$ of A

$$\omega(A_1, \dots, A_i \cup B_i, \dots, A_n) = \omega(A_1, \dots, A_i, \dots, A_n) \cup \omega(A_1, \dots, B_i, \dots, A_n), \tag{2.0.1}$$

for any $1 \leq i \leq n$.

Power algebras have also the following two elementary properties for any non-empty subsets $A_i \subseteq B_i$ and A_{ij} of A for $1 \leq i \leq n$, $1 \leq j \leq r$:

$$\omega(A_1, \dots, A_n) \subseteq \omega(B_1, \dots, B_n), \tag{2.0.2}$$

$$\omega(A_{11}, \dots, A_{n1}) \cup \dots \cup \omega(A_{1r}, \dots, A_{nr}) \subseteq \omega(A_{11} \cup \dots \cup A_{1r}, \dots, A_{n1} \cup \dots \cup A_{nr}). \tag{2.0.3}$$

Note that the definition of a complex operation extends to each linear derived operation w:

$$w(A_1, \dots, A_n) = \{w(a_1, \dots, a_n) \mid a_i \in A_i\}.$$

Each non-linear term t can be obtained from a linear one t^* by identification of some variables. Let $t^*(x_{11}, \dots, x_{1k_1}, \dots, x_{m1}, \dots, x_{mk_m})$ be a linear term such that

$$t(x_1, \dots, x_m) = t^*(\underbrace{x_1, \dots, x_1}_{k_1-times}, \dots, \underbrace{x_m, \dots, x_m}_{k_m-times}).$$

Then for any subsets $A_1, \dots, A_m$ of A

$$\begin{aligned}&\{t(a_1, \dots, a_m) \mid a_i \in A_i\} \subseteq t(A_1, \dots, A_m) = \\ &\{t^*(a_{11}, \dots, a_{1k_1}, \dots, a_{m1}, \dots, a_{mk_m}) \mid a_{ij} \in A_i\} = \\ &t^*(\underbrace{A_1, \dots, A_1}_{k_1-times}, \dots, \underbrace{A_m, \dots, A_m}_{k_m-times}).\end{aligned}$$

It is easy to see that in general both (2.0.2) and (2.0.3) hold also for all derived operations. The proofs go by induction on the complexity of terms. In such a case, we also obtain the inclusion

$$t(A_1, \dots, A_i, \dots, A_n) \cup t(A_1, \dots, B_i, \dots, A_n) \subseteq t(A_1, \dots, A_i \cup B_i, \dots, A_n)$$

that generalizes the distributive law (2.0.1).

The idempotent law will play a special rôle in this paper. First note that if the power algebra $(\mathcal{P}_{>0}A, \Omega)$ of (A, Ω) is idempotent then the algebra (A, Ω) must be idempotent too. Next, if (A, Ω) is idempotent then for any non-empty subset B of A and $\omega \in \Omega$, we have $B \subseteq \omega(B, \dots, B)$. Moreover, as an easy consequence of results of A. Romanowska and J.D.H. Smith [17, Proposition 2.1] for an idempotent algebra (A, Ω), a non-empty subset $B \in \mathcal{P}_{>0}A$ is a subalgebra of (A, Ω) if and only if $\omega(B, \dots, B) = B$ for each $\omega \in \Omega$.

Theorem 2.1. *Let (A, Ω) be an idempotent algebra. The power algebra $(\mathcal{P}_{>0}A, \Omega)$ is idempotent if and only if for each n-ary basic operation $\omega \in \Omega$ and subsets $A_1, \dots, A_n \in \mathcal{P}_{>0}A$*

$$\omega(A_1, \dots, A_n) \subseteq A_1 \cup \dots \cup A_n. \tag{2.1.1}$$

Proof. Let $(\mathcal{P}_{>0}A, \Omega)$ be an idempotent algebra. By (2.0.2), for an n-ary basic operation $\omega \in \Omega$ and subsets $A_1, \ldots, A_n \in \mathcal{P}_{>0}A$ we have:

$$\omega(A_1, \ldots, A_n) \subseteq \omega(A_1 \cup \ldots \cup A_n, \ldots, A_1 \cup \ldots \cup A_n) = A_1 \cup \ldots \cup A_n.$$

The last equality follows by idempotency of $(\mathcal{P}_{>0}A, \Omega)$.

On the other hand, let $(\mathcal{P}_{>0}A, \Omega)$ be the power algebra of an idempotent algebra (A, Ω), which satisfies (2.1.1). Let B be a non-empty subset of A, then

$$B \subseteq \omega(B, \ldots, B) \overset{(2.1.1)}{\subseteq} B \cup \ldots \cup B = B.$$

This shows that $\omega(B, \ldots, B) = B$ and $(\mathcal{P}_{>0}A, \Omega)$ is idempotent. □

If an algebra (A, Ω) is not idempotent then (2.1.1) does not hold.

Example 2.2. Let $(G, \cdot, ^{-1}, 1)$ be a group, $1 \neq g \in G$, $G_1 = \{g\}$ and $G_2 = \{g^{-1}\}$. Then $G_1G_2 = \{g\}\{g^{-1}\} = \{1\}$ and G_1G_2 is not contained in the set $G_1 \cup G_2 = \{g, g^{-1}\}$.

Corollary 2.3. *The power algebra $(\mathcal{P}_{>0}A, \Omega)$ of an idempotent algebra (A, Ω) is idempotent if and only if each non-empty subset B of A is a subalgebra of (A, Ω). In such a case $(\mathcal{P}_{>0}A, \Omega) = (S(A), \Omega)$.*

Example 2.4. An algebra (A, Ω) such that $\omega(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$, for each n-ary $\omega \in \Omega$ and $a_1, \ldots, a_n \in A$, is called *conservative*. By Corollary (2.3), the power algebra of any conservative algebra is always idempotent. In particular, the power algebra of a chain, the power algebra of a left zero-semigroup [20], the power algebra of an equivalence algebra [4] or the power algebra of a tournament [5] are all idempotent.

Let θ be a congruence on an idempotent algebra (A, Ω). Obviously, $a\ \theta\ \omega(a, \ldots, a)$ for each $a \in A$ and $\omega \in \Omega$. On the other hand, it is not always true that $X\ \theta\ \omega(X, \ldots, X)$ for a subset X of A, if $(\mathcal{P}_{>0}A, \Omega)$ is not idempotent. It is enough to consider the equality relation on (A, Ω) in such case.

Consider now the family $\mathcal{P}A$ of all subsets of A. By the definition of power operations for arbitrary subsets

$$\omega(A_1, \ldots, A_{i-1}, \emptyset, A_{i+1}, \ldots, A_n) = \emptyset,$$

for all $1 \leq i \leq n$ and each (n-ary) operation $\omega \in \Omega$. Then the power algebra of all subsets of A can also be viewed as *the Boolean algebra* $(\mathcal{P}A, \cup, \cap, -, A, \emptyset, \Omega)$ *with operators* Ω. This concept was introduced and studied by B. Jónsson and A. Tarski [6, 7].

If $\omega : A^n \to A$ is an idempotent operation, then the complex operation ω and the intersection operation of $\mathcal{P}A$ are related as follows:

$$A_1 \cap \ldots \cap A_n \subseteq \omega(A_1, \ldots, A_n).$$

If for a subset $S \subseteq A$, each n-ary $\omega \in \Omega$ and any $1 \leq i \leq n$,

$$\omega(A, \ldots, \underbrace{S}_{i}, \ldots, A) \subseteq S,$$

then S is said to be a *sink* of (A, Ω). (See [19, Section 3.6].) Of course each sink is a subalgebra of (A, Ω).

Remark 2.5. *Let (A, Ω) be an idempotent algebra and $\omega \in \Omega$ an n-ary operation. By results of A. Romanowska and J.D.H. Smith* [19, Corollary 366], *if $A_1, \dots, A_n$ are sinks of (A, Ω), then $\omega(A_1, \dots, A_n) = A_1 \cap \dots \cap A_n$. On the other hand, if for all subsets $A_1, \dots, A_n$ of A, $\omega(A_1, \dots, A_n) = A_1 \cap \dots \cap A_n$, then the set $\mathcal{P}_{>0}A$ is the set of sinks of (A, Ω).*

3. Congruences α, ρ and β

Let (M, Ω) be a mode. Denote by $\mathcal{I}$ a variety of all idempotent τ-algebras of type $\tau : \Omega \uplus \{\cup\} \to \mathbb{N}$. Then $Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$ is the set of all congruence relations γ on $(\mathcal{P}_{>0}M, \Omega, \cup)$, such that the quotient $((\mathcal{P}_{>0}M)^{\gamma}, \Omega)$ is idempotent. This set is an algebraic lattice when ordered by inclusion. In [13], we defined two elements of $Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$:

$$X \,\rho\, Y \;\Leftrightarrow\; \text{there exist a } k\text{-ary term } t \text{ and an } m\text{-ary term } s \text{ both of type } \Omega \text{ such that } X \subseteq t(Y, Y, \dots, Y) \text{ and } Y \subseteq s(X, X, \dots, X),$$

$$X \,\alpha\, Y \;\Leftrightarrow\; \langle X \rangle = \langle Y \rangle,$$

where $\langle X \rangle$ denotes the subalgebra of (M, Ω) generated by X.

We also showed there that ρ is the least element of $(Con_{\mathcal{I}}(\mathcal{P}_{>0}M), \subseteq)$, i.e. it is so-called the *idempotent replica congruence.*

Let $\mathcal{P}^{<\omega}_{>0}M$ be the set of all finite non-empty subsets of a mode (M, Ω). The congruences α and ρ restricted to the subalgebra $\mathcal{P}^{<\omega}_{>0}M$ of $(\mathcal{P}_{>0}M, \Omega, \cup)$ coincide (see [13]).

In this section we describe one more idempotent congruence β and discuss with details when ρ, α and β are equal. First we recall some results obtained in [13]. For $1 \le i \le k$ and $k \ge 2$, let t_i be m_i-ary terms. By *the composition term* $t_1 \circ t_2 \circ \dots \circ t_k$ of the terms $t_1, t_2, \dots, t_k$ is meant an $m := m_1 \cdot \ldots \cdot m_k$ - ary term defined by the rule:

$$t_1 \circ t_2(\overline{x}_1, \dots, \overline{x}_{m_1}) := t_1(t_2(\overline{x}_1), \dots, t_2(\overline{x}_{m_1})),$$
$$t_1 \circ \dots \circ t_k(\overline{x}_1, \dots, \overline{x}_r) := t_1 \circ \dots \circ t_{k-1}(t_k(\overline{x}_1), \dots, t_k(\overline{x}_r)),$$

where $r = m_1 \cdot \ldots \cdot m_{k-1}$ and $\overline{x}_i = (x_{i1}, \dots, x_{im_k})$, for $i = 1, \dots, r$.

Note that for a mode (M, Ω) and a non-empty subset X of M

$$t_1 \circ \dots \circ t_k(X, \dots, X) = t_{\sigma(1)} \circ \dots \circ t_{\sigma(k)}(X, \dots, X),$$

for any permutation σ of the set $\{1, \dots, k\}$. Note also that for any derived operation t, we have $X \subseteq t(X, \dots, X)$. Hence, by (2.0.2) we can observe the following remark.

Remark 3.1. [13] *Let (M, Ω) be a mode. For $1 \le i \le k$, let t_i be m_i-ary terms and $\emptyset \ne X \subseteq M$. For the composition term $t = t_1 \circ t_2 \circ \dots \circ t_k$ we have*

$$t_i(\underbrace{X, X, \dots, X}_{m_i}) \subseteq t(\underbrace{X, X, \dots, X}_{m_1 \cdot \ldots \cdot m_k}),$$

for each $1 \le i \le k$.

Let (M, Ω) be a mode and let $\emptyset \ne X \subseteq M$ and $\Delta \subseteq \Omega$. For any $n \in \mathbb{N}$ let us define sets $X^{[n]}_{\Delta}$ in the following way:

$$X^{[0]}_{\Delta} := X,$$
$$X^{[n+1]}_{\Delta} := \bigcup_{\delta \in \Delta} \delta(X^{[n]}_{\Delta}, \dots, X^{[n]}_{\Delta}) = (X^{[n]}_{\Delta})^{[1]}_{\Delta}.$$

If $\Delta = \Omega$ we will use the abbreviated notation $X^{[n]}$ instead of $X^{[n]}_{\Omega}$. It is well known that

$$\langle X\rangle = \bigcup_{n\in\mathbb{N}} X^{[n]}.$$

Lemma 3.2. [13] *Let (M,Ω) be a mode, Δ be a finite subset of Ω and $\gamma \in Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$. Then $X\ \gamma\ X^{[n]}_{\Delta}$, for any $n \in \mathbb{N}$.*

It is clear that $X \subseteq X^{[n]} \subseteq X^{[n+1]}$ for any $n \in \mathbb{N}$. Moreover if $X^{[n]} \subseteq Y^{[m]}$, then $X^{[n+1]} \subseteq Y^{[m+1]}$, for any $n, m \in \mathbb{N}$. Note also that $\omega_1 \circ \ldots \circ \omega_n(X,\ldots,X) \in X^{[n]}$, for operations $\omega_1,\ldots,\omega_n \in \Omega$.

Lemma 3.3. *Let (M,Ω) be a mode and let $\omega \in \Omega$ be an n-ary operation. For any non-empty subsets $X_1,\ldots,X_n$ of M, and $m \in \mathbb{N}$ we have*

$$\omega(X_1^{[m]},\ldots,X_n^{[m]}) \subseteq \omega(X_1,\ldots,X_n)^{[mn]}. \tag{3.3.1}$$

Proof. Take $m = 1$. Let $x \in \omega(X_1^{[1]},\ldots,X_n^{[1]})$. Then there are basic operations $\omega_i \in \Omega$ such that $x \in \omega(\omega_1(X_1,\ldots,X_1),\ldots,\omega_n(X_n,\ldots,X_n))$. By Remark (3.1) for the composition term $t = \omega_1 \circ \ldots \circ \omega_n$ we have

$$\omega(\omega_1(X_1,\ldots,X_1),\ldots,\omega_n(X_n,\ldots,X_n)) \subseteq \omega(t(X_1,\ldots,X_1),\ldots,t(X_n,\ldots,X_n)) =$$
$$= t(\omega(X_1,\ldots,X_n),\ldots,\omega(X_1,\ldots,X_n)) \subseteq \omega(X_1,\ldots,X_n)^{[n]}$$

and (3.3.1) is true for $m = 1$.

Now assume that the hypothesis (3.3.1) is established for $m > 1$. Hence by (3.3.1) for $m = 1$ and the induction assumption we obtain

$$\omega(X_1^{[m+1]},\ldots,X_n^{[m+1]}) = \omega((X_1^{[m]})^{[1]},\ldots,(X_n^{[m]})^{[1]}) \subseteq \omega(X_1^{[m]},\ldots,X_n^{[m]})^{[n]} \subseteq$$
$$\subseteq (\omega(X_1,\ldots,X_n)^{[mn]})^{[n]} = \omega(X_1,\ldots,X_n)^{[(m+1)n]},$$

which completes the proof. □

Now we define a binary relation β on the set $\mathcal{P}_{>0}M$ in the following way:

$$X\ \beta\ Y \quad\Leftrightarrow\quad (\exists k, n \in \mathbb{N})\ \ X \subseteq Y^{[k]} \quad\text{and}\quad Y \subseteq X^{[n]}.$$

Theorem 3.4. *For a mode (M,Ω), the relation β belongs to the set $Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$.*

Proof. Obviously, β is reflexive and symmetric. To show that it is transitive let $X\ \beta\ Y$ and $Y\ \beta\ Z$ for some non-empty subsets X, Y, Z of M. This means that

$$(\exists k, n \in \mathbb{N})\ \ X \subseteq Y^{[k]} \quad\text{and}\quad Y \subseteq X^{[n]}$$

and

$$(\exists p, m \in \mathbb{N})\ \ Y \subseteq Z^{[p]} \quad\text{and}\quad Z \subseteq Y^{[m]}.$$

Hence $X \subseteq Y^{[k]} \subseteq Z^{[p+k]}$ and $Z \subseteq Y^{[m]} \subseteq X^{[m+n]}$. Consequently, $X\ \beta\ Z$ and β is an equivalence relation.

To show that β is a congruence on the extended power algebra $(\mathcal{P}_{>0}M,\Omega,\cup)$, let $\omega \in \Omega$ be an n-ary complex operation and let $X_1,\ldots,X_n,Y_1,\ldots,Y_n$ be non-empty subsets of M. Now let $X_i\ \beta\ Y_i$ for $1 \leq i \leq n$. This means that for each $1 \leq i \leq n$,

$$(\exists k_i, m_i \in \mathbb{N})\ \ X_i \subseteq Y_i^{[k_i]} \quad\text{and}\quad Y_i \subseteq X_i^{[m_i]}.$$

By (2.0.2),

$$\omega(X_1,\ldots,X_n) \subseteq \omega(Y_1^{[k_1]},\ldots,Y_n^{[k_n]}) \subseteq \omega(Y_1^{[\max(k_1,\ldots,k_n)]},\ldots,Y_n^{[\max(k_1,\ldots,k_n)]}).$$

Finally, by Lemma (3.3),

$$\omega(X_1,\dots,X_n) \subseteq \omega(Y_1,\dots,Y_n)^{[n\max(k_1,\dots,k_n)]}.$$

Similarly we can show that

$$\omega(Y_1,\dots,Y_n) \subseteq \omega(X_1,\dots,X_n)^{[n\max(m_1,\dots,m_n)]},$$

and consequently $\omega(X_1,\dots,X_n)\ \beta\ \omega(Y_1,\dots,Y_n)$.

Moreover,

$$(\exists k,p\in\mathbb{N})\ \ X_1\cup X_2 \subseteq Y_1^{[k]}\cup Y_2^{[p]} \subseteq Y_1^{[\max(k,p)]}\cup Y_2^{[\max(k,p)]} \subseteq (Y_1\cup Y_2)^{[\max(k,p)]}$$

and similarly

$$(\exists m,r\in\mathbb{N})\ \ Y_1\cup Y_2 \subseteq (X_1\cup X_2)^{[\max(m,r)]}.$$

Hence $(X_1\cup X_2)\ \beta\ (Y_1\cup Y_2)$ and β is also a congruence relation on $(\mathcal{P}_{>0}M,\Omega,\cup)$.

It is also clear that

$$X\subseteq\omega(X,\dots,X)\ \text{ and }\ \omega(X,\dots,X)\subseteq X^{[1]},$$

whence $X\ \beta\ \omega(X,\dots,X)$, and $((\mathcal{P}_{>0}M)^\beta,\Omega)$ is idempotent. □

We will show that in some cases the congruence β is the least element of the lattice $(Con_{\mathcal{I}}(\mathcal{P}_{>0}M),\subseteq)$. By Lemma (3.2) we immediately obtain the following.

Corollary 3.5. *Let (M,Ω) be a mode and Ω be a finite set of operations. The relation β is the least element of the lattice $(Con_{\mathcal{I}}(\mathcal{P}_{>0}M),\subseteq)$, i.e $\beta=\rho$.*

Theorem 3.6. *Let (M,Ω) be a mode. The congruences β and ρ restricted to the subalgebra $\mathcal{P}^{<\omega}_{>0}M$ of $(\mathcal{P}_{>0}M,\Omega,\cup)$ coincide:*

$$\beta_{\mathcal{P}^{<\omega}_{>0}M}=\rho_{\mathcal{P}^{<\omega}_{>0}M}.$$

Proof. Let $X,Y\in\mathcal{P}^{<\omega}_{>0}M$ and $X\ \beta\ Y$ and $\gamma\in Con_{\mathcal{I}}(\mathcal{P}^{<\omega}_{>0}M)$. Since X and Y are finite, there exist $k,m\in\mathbb{N}$ and finite subsets Δ_1,Δ_2 of Ω such that $X\subseteq Y^{[k]}_{\Delta_1}$ and $Y\subseteq X^{[m]}_{\Delta_2}$. By Lemma (3.2), $X\ \gamma\ X^{[m]}_{\Delta_2}$ and $Y\ \gamma\ Y^{[k]}_{\Delta_1}$. Hence

$$X\ \gamma\ X^{[m]}_{\Delta_2}=X^{[m]}_{\Delta_2}\cup Y\ \gamma\ X\cup Y\ \gamma\ X\cup Y^{[k]}_{\Delta_1}=Y^{[k]}_{\Delta_1}\ \gamma\ Y.$$

Thus, $\beta_{\mathcal{P}^{<\omega}_{>0}M}\subseteq\gamma$ and $\beta_{\mathcal{P}^{<\omega}_{>0}M}$ is the least element of the lattice $(Con_{\mathcal{I}}(\mathcal{P}^{<\omega}_{>0}M),\subseteq$). □

By results of [13] one obtains

Corollary 3.7. *Let (M,Ω) be a mode. The congruences α, β and ρ restricted to the subalgebra $\mathcal{P}^{<\omega}_{>0}M$ of $(\mathcal{P}_{>0}M,\Omega,\cup)$ coincide:*

$$\rho_{\mathcal{P}^{<\omega}_{>0}M}=\alpha_{\mathcal{P}^{<\omega}_{>0}M}=\beta_{\mathcal{P}^{<\omega}_{>0}M}.$$

The following example shows that in general, congruences α and β are different.

Example 3.8. A *differential groupoid* is a mode groupoid $(D,\cdot)$ satisfying the additional identity:

$$x(xy)=x.$$

Note also two other identities true in differential groupoids:

$$x(yz)=xy,\quad (xy)z=(xz)y.$$

Let $(D(x,y),\cdot)$ be the free differential groupoid on two generators x and y. As was shown in [15] each element of $(D(x,y),\cdot)$ may be expressed as $(\dots((x\underbrace{y)y)\dots)y}_{k-times} =: xy^k$ or yx^n for some $k,n\in\mathbb{N}$. Hence subalgebras of $(D(x,y),\cdot)$ are any finite or infinite subsets of the sets $\{x,xy,xy^2,xy^3,\dots\}$ or $\{y,yx,yx^2,yx^3,\dots\}$ or subalgebras generated by two elements xy^k and yx^n for some $k,n\in\mathbb{N}$, i.e.:

$$\langle\{xy^k,yx^n\}\rangle=\{xy^k,xy^{k+1},xy^{k+2},\dots\}\cup\{yx^n,yx^{n+1},yx^{n+2},\dots\}.$$

Let us consider congruences α and β defined on $(\mathcal{P}_{>0}(D(x,y)),\cdot,\cup)$. Note that each finite subset which contains x and y belongs to the same congruence class $\{x,y\}^{\beta}$. But none of the finite sets is in the relation β with a non-finite one, so the algebra $\langle\{x,y\}\rangle$ could not belong to the class $\{x,y\}^{\beta}$. Hence, $\alpha\neq\beta$.

Remark 3.9. *Let (M,Ω) be a mode with a finite set Ω of operations. Assume that (M,Ω) is not locally finite. Then congruences α and β of $(\mathcal{P}_{>0}M,\Omega,\cup)$ are different.*

Proof. Let (B,Ω) be an infinite subalgebra of (M,Ω), generated by a finite set X. Hence, of course $B\ \alpha\ X$. But there is no $k\in\mathbb{N}$ such that

$$B\subseteq\bigcup_{\nu\in\Omega}\nu(X^{[k-1]},\dots,X^{[k-1]})=X^{[k]},$$

whence $(B,X)\notin\beta$ and $\alpha\neq\beta$. □

Also the congruences β and ρ are different in a general case.

Example 3.10. Let $\mathbb{D}$ be the ring of *dyadic* rational numbers, i.e. rationals of the form $m\cdot 2^{-n}$ for integers m and n. Consider the mode $(\mathbb{D},\underline{\mathbb{D}})$, where $\underline{\mathbb{D}}=\{\underline{d}\mid d\in\mathbb{D}\}$ is the set of binary operations defined as follows:

$$\underline{d}(x,y):=(1-d)x+dy.$$

It is known that such algebra belongs to the variety of modes defined by the identities:

$$\underline{0}(x,y)=x=\underline{1}(y,x),$$
$$\underline{r}(\underline{p}(x,y),\underline{q}(x,y))=\underline{r(p,q)}(x,y)$$

and that this variety is equivalent to the variety $\underline{\underline{\mathbb{D}}}$ of affine $\mathbb{D}$-spaces. (See [20, Chapter 6.3].) In particular, each m-ary derived operation $t(x_1,\dots,x_m)$ of $(\mathbb{D},\underline{\mathbb{D}})$ can be expressed as

$$t(x_1,\dots,x_m)=d_1x_1+\dots+d_mx_m,$$

where $d_1,\dots,d_m\in\mathbb{D}$ and $\sum_{i=1}^{m}d_i=1$.

Consider two subsets $\mathbb{Z}$ and $\{0,1\}$ of the set $\mathbb{D}$. We can see that $\mathbb{Z}\ \beta\ \{0,1\}$ since $\mathbb{Z}\subset\{0,1\}^{[1]}=\mathbb{D}$ and $\{0,1\}\subset\mathbb{Z}^{[0]}=\mathbb{Z}$. Obviously, the sets $\mathbb{Z}$ and $\{0,1\}$ are α-related since $\langle\{0,1\}\rangle=\langle\mathbb{Z}\rangle=\mathbb{D}$. In fact, since the only subalgebras of $\mathbb{D}$ are singletons and $\mathbb{D}$ itself, both congruences are equal. Now for each m-ary (linear) derived operation t we have $t(\{0,1\},\{0,1\},\dots,\{0,1\})=\{t(x_1,\dots,x_m)\mid x_i\in\{0,1\}\}=\{d_1x_1+\dots+d_mx_m\mid x_i\in\{0,1\}\}$. Note that $\{d_1x_1+\dots+d_mx_m\mid x_i\in\{0,1\}\}$ is a finite subset of a commutative submonoid of $(\mathbb{D},+)$ generated by the set $\{d_1,\dots,d_m\}$. It follows that there is no such m-ary term t of type

$\underline{\underline{\mathbb{D}}}$ that $\mathbb{Z} \subseteq t(\{0,1\},\{0,1\},\ldots,\{0,1\})$. As a consequence $(\mathbb{Z},\{0,1\}) \notin \rho$, whence $\rho \subsetneq \beta = \alpha$.

The next example shows that for equivalent algebras their extended power algebras need not be equivalent.

Example 3.11. Recall that a quasigroup $(Q,\cdot,\backslash,/)$ is an algebra with three binary operations of multiplication $\cdot$, right division $\backslash$ and left division $/$ satisfying the identities:

$$(x\cdot y)/y = x = (x/y)\cdot y$$
$$y\backslash(y\cdot x) = x = y\cdot(y\backslash x).$$

A quasigroup is described as commutative if its multiplication is commutative. A quasigroup mode is an idempotent and entropic quasigroup. It is known (see e.g. [20, Chapter 6.6]) that the variety $\mathcal{Q}$ of commutative quasigroup modes is equivalent to the variety $\underline{\underline{\mathbb{D}}}$ of affine $\mathbb{D}$-spaces. The quasigroup operations of affine $\mathbb{D}$-spaces are defined by: $\cdot := \underline{2^{-1}}$, $\backslash := \underline{-1}$, and $/ := \underline{2}$.

In particular, $(\mathbb{D},\cdot,\backslash,/) = (\mathbb{D},\underline{2^{-1}},\underline{-1},\underline{2}) \in \mathcal{Q}$ and it is equivalent to the $\mathbb{D}$-affine space $(\mathbb{D},\underline{\underline{\mathbb{D}}})$. Let us consider congruences α, β and ρ of $(\mathcal{P}_{>0}\mathbb{D},\underline{2^{-1}},\underline{-1},\underline{2},\cup)$. By Corollary (3.5), the congruences β and ρ are equal. The quasigroup $(\mathbb{D},\underline{2^{-1}},\underline{-1},\underline{2})$ is infinite and is generated by the set $\{0,1\}$, then by Remark (3.9), $\alpha \neq \beta$. On the other hand, as we showed in Example (3.10), the congruences α, β and ρ of the power algebra of the affine $\mathbb{D}$-space $(\mathbb{D},\underline{\underline{\mathbb{D}}})$ satisfy $\alpha = \beta$ and $\beta \neq \rho$. Since equivalent algebras must have the same congruences (see [9, Section 4.12]), we obtain that $(\mathcal{P}_{>0}\mathbb{D},\underline{2^{-1}},\underline{-1},\underline{2},\cup)$ and $(\mathcal{P}_{>0}\mathbb{D},\underline{\underline{\mathbb{D}}},\cup)$ are not equivalent.

4. Γ-sinks

Let (M,Ω) be a mode. In this section we will investigate in full detail a subfamily of congruences in $Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$ which are closely connected to the concept of Γ-sinks. Such congruences determine some non-trivial subvarieties of the variety of all semilattice ordered modes.

Definition 4.1. Let $\Gamma \subseteq \Omega$. A subalgebra (S,Ω) of a mode (M,Ω) is said to be a Γ-*sink* of (M,Ω) if for each n-ary operation $\nu \in \Gamma$ and $i = 1,\ldots,n$,

$$\nu(M,\ldots,\underbrace{S}_{i},\ldots,M) \subseteq S.$$

In particular, an Ω-sink is a sink, as defined in Section 2. Obviously, for a Γ-sink S and an n-ary $\nu \in \Gamma$, $\nu(M,\ldots,\underbrace{S}_{i},\ldots,M) = S$ for each $i = 1,\ldots,n$. Note also that for any m-ary $\omega \in \Omega$ and $i = 1,\ldots,m$, a set $\omega(M,\ldots,\underbrace{S}_{i},\ldots,M)$ is a Γ-sink.

A. Romanowska and J.D.H. Smith showed in [19] that for a mode (M,Ω), the set $S(M)$ of all non-empty subalgebras of (M,Ω) has a mode structure under the Ω-complex operations.

Let $S_\Gamma(M)$ denote the set of all non-empty Γ-sinks of a mode (M,Ω). (Of course $S_\emptyset(M) = S(M)$.)

Lemma 4.2. *For a mode (M,Ω) and any subset $\Gamma \subseteq \Omega$, $(S_\Gamma(M),\Omega)$ is a submode of $(S(M),\Omega)$.*

Proof. For an n-ary operation $\omega \in \Omega$, consider the complex ω-product $\omega(S_1, \dots, S_n)$ of Γ-sinks $S_1, \dots, S_n$. Obviously it is a non-empty subalgebra of (M, Ω). By the entropic law, for each operation $\nu \in \Gamma$

$$\nu(M, \dots, \omega(S_1, \dots, S_n), \dots, M) = \omega(S_1, \dots, S_n).$$

□

Let (M, Ω) be a mode, $\emptyset \neq X \subseteq M$ and $\Gamma \subseteq \Omega$. For each operation $\nu \in \Gamma$ let n_ν denote its arity. Let us define sets $X^{[n]_\Gamma}$ by the following recursion:

$$X^{[0]_\Gamma} := X,$$

$$X^{[n+1]_\Gamma} := \bigcup_{\nu \in \Gamma} \bigcup_{1 \leq i \leq n_\nu} \nu(M, \dots, \underbrace{X^{[n]_\Gamma}}_{i}, \dots, M) \cup \bigcup_{\nu \in \Omega \setminus \Gamma} \nu(X^{[n]_\Gamma}, \dots, X^{[n]_\Gamma}).$$

It is clear that for any $n, m \in \mathbb{N}$, $X \subseteq X^{[n]_\Gamma} \subseteq X^{[n+1]_\Gamma}$. Moreover, if $X^{[n]_\Gamma} \subseteq Y^{[m]_\Gamma}$ then $X^{[n+1]_\Gamma} \subseteq Y^{[m+1]_\Gamma}$ and obviously $X^{[n+1]_\Gamma} = (X^{[n]_\Gamma})^{[1]_\Gamma}$. If $\Gamma = \emptyset$ we will use the abbreviated notation $X^{[n]}$ instead of $X^{[n]_\emptyset}$ (see Section 3).

Let us denote by $\langle X \rangle_\Gamma$ the Γ-sink generated by a non-empty set X, i.e. the intersection of all Γ-sinks that include X.

Theorem 4.3. *Let (M, Ω) be a mode, $\emptyset \neq X \subseteq M$ and $\Gamma \subseteq \Omega$. Then*

$$\langle X \rangle_\Gamma = \bigcup_{n \in \mathbb{N}} X^{[n]_\Gamma}.$$

Proof. Let $\omega \in \Omega$ be an l-ary basic operation and $X^{[m_i]_\Gamma}$ be a subset of $\bigcup_{n \in \mathbb{N}} X^{[n]_\Gamma}$ for $i = 1, \dots l$. Consider now a set $\omega(X^{[m_1]_\Gamma}, X^{[m_2]_\Gamma}, \dots, X^{[m_l]_\Gamma})$. We can easily see that the latter is a subset of $\omega(X^{[m]_\Gamma}, X^{[m]_\Gamma}, \dots, X^{[m]_\Gamma})$ for some large enough $m \in \mathbb{N}$. But then $\omega(X^{[m]_\Gamma}, \dots, X^{[m]_\Gamma}) \subseteq X^{[m+1]_\Gamma} \subseteq \bigcup_{n \in \mathbb{N}} X^{[n]_\Gamma}$ and $\bigcup_{n \in \mathbb{N}} X^{[n]_\Gamma}$ is a subalgebra of (M, Ω). Further, for any k-ary operation $\nu \in \Gamma$, $1 \leq i \leq k$ and for any $m \in \mathbb{N}$,

$$\nu(M, \dots, \underbrace{X^{[m]_\Gamma}}_{i}, \dots, M) \subseteq \bigcup_{n \in \mathbb{N}} X^{[n]_\Gamma},$$

which proves that $\bigcup_{n \in \mathbb{N}} X^{[n]_\Gamma}$ is a Γ-sink.

Now let S be a Γ-sink that includes X. It suffices to show that for every $n \in \mathbb{N}$, $X^{[n]_\Gamma} \subseteq S$. But this immediately follows by induction on n. □

As it was proved by A. Romanowska and J.D.H. Smith [19], for each n-ary complex operation $\omega \in \Omega$ and any non-empty subsets $X_1, \dots, X_n \subseteq M$

$$\langle \omega(X_1, \dots, X_n) \rangle = \omega(\langle X_1 \rangle, \dots, \langle X_n \rangle).$$

Similar results are also true for any Γ-sinks. First we notice an useful property which holds for power algebra of modes.

Remark 4.4. *Let X_{ij} be a nonempty subset of a mode (M, Ω) for $i = 1, \dots, n$ and $j = 1, \dots, m_i$. Let ω, ν_i be respectively n-ary and m_i-ary operations in Ω. Then*

$$\omega(\nu_1(X_{11}, X_{12}, \dots, X_{1m_1}), \dots, \nu_n(X_{n1}, X_{n2}, \dots, X_{nm_n})) \subseteq$$
$$\nu_{\delta(1)} \circ \nu_{\delta(2)} \circ \dots \circ \nu_{\delta(n)} \circ \omega(\{X_{11}, X_{12}, \dots, X_{1m_1}\} \times \dots \times \{X_{n1}, X_{n2}, \dots, X_{nm_n}\}),$$

for any permutation δ of the set $\{1, \dots, n\}$.

Proof. It goes by straightforward calculations with use of idempotency, entropicity and basic properties of power algebras of modes given in Section 2. □

Lemma 4.5. *For each n-ary complex operation $\omega \in \Omega$ and any non-empty subsets $X_1, \dots, X_n$ of M*

$$\langle \omega(X_1, \dots, X_n) \rangle_\Gamma = \omega(\langle X_1 \rangle_\Gamma, \dots, \langle X_n \rangle_\Gamma).$$

Proof. For any $1 \le i \le n$, $X_i \subseteq \langle X_i \rangle_\Gamma$. So by (2.0.2), $\omega(X_1, \dots, X_n) \subseteq \omega(\langle X_1 \rangle_\Gamma, \dots, \langle X_n \rangle_\Gamma)$. As we showed before, $\omega(\langle X_1 \rangle_\Gamma, \dots, \langle X_n \rangle_\Gamma)$ is a Γ-sink. Hence,

$$\langle \omega(X_1, \dots, X_n) \rangle_\Gamma \subseteq \omega(\langle X_1 \rangle_\Gamma, \dots, \langle X_n \rangle_\Gamma).$$

On the other side let $x \in \omega(\langle X_1 \rangle_\Gamma, \dots, \langle X_n \rangle_\Gamma)$. Then $x \in \omega(X_1^{[m]\Gamma}, \dots, X_n^{[m]\Gamma})$ for some large enough $m \in \mathbb{N}$. We will show that for every $k \in \mathbb{N}$, $\omega(X_1^{[k]\Gamma}, \dots, X_n^{[k]\Gamma}) \subseteq \langle \omega(X_1, \dots, X_n) \rangle_\Gamma$. We start with the set $\omega(X_1^{[1]\Gamma}, \dots, X_n^{[1]\Gamma})$. Note that $X_i^{[1]\Gamma} \subseteq \bigcup_{\nu \in \Omega} \bigcup_{1 \le j \le n_\nu} \nu(M, \dots, \underbrace{X_i}_{j}, \dots, M)$. Let $x \in \omega(X_1^{[1]\Gamma}, \dots, X_n^{[1]\Gamma})$, then there exist operations $\nu_i \in \Omega$ for $i = 1, \dots, n$ such that

$$\begin{aligned} x \in & \omega(\nu_1(M, \dots, \underbrace{X_1}_{j_1}, \dots, M), \dots, \nu_n(M, \dots, \underbrace{X_n}_{j_n}, \dots, M)) \subseteq \\ & \nu_{\delta(1)} \circ \dots \circ \nu_{\delta(k)} \circ \nu_{\delta(k+1)} \circ \dots \circ \nu_{\delta(n)}(M, \dots, \underbrace{\omega(X_1, \dots, X_n)}_{j}, \dots, M), \end{aligned}$$

where δ is such permutation that all operations $\nu_{\delta(1)}, \dots, \nu_{\delta(k)}$ are in $\Omega - \Gamma$ and $\nu_{\delta(k+1)}, \dots, \nu_{\delta(n)}$ in Γ. The latter set is a subset of

$$\begin{aligned} & \nu_{\delta(1)} \circ \dots \circ \nu_{\delta(k)}(\omega(X_1, \dots, X_n)^{[n-k]\Gamma}, \dots, \omega(X_1, \dots, X_n)^{[n-k]\Gamma}) \\ & \subseteq \omega(X_1, \dots, X_n)^{[n]\Gamma} \subseteq \langle \omega(X_1, \dots, X_n) \rangle_\Gamma. \end{aligned}$$

Now by induction on k we obtain

$$\begin{aligned} & \omega(X_1^{[k+1]\Gamma}, \dots, X_n^{[k+1]\Gamma}) = \omega((X_1^{[k]\Gamma})^{[1]\Gamma}, \dots, (X_n^{[k]\Gamma})^{[1]\Gamma}) \subseteq \\ & \omega(X_1^{[k]\Gamma}, \dots, X_n^{[k]\Gamma})^{[n]\Gamma} \subseteq \langle \omega(X_1, \dots, X_n) \rangle_\Gamma, \end{aligned}$$

which finishes the proof. □

Lemma (4.5) shows that if each subset $A_1, \dots, A_n \subseteq M$ is finite then the complex ω-product $\omega(\langle A_1 \rangle_\Gamma, \dots, \langle A_n \rangle_\Gamma)$ is finitely generated Γ-sink of (M, Ω).

Theorem 4.6. *Let (M, Ω) be a mode, $\emptyset \ne X \subseteq M$ and $\Gamma_1, \Gamma_2 \subseteq \Omega$. Then*

$$\langle X \rangle_{\Gamma_1 \cup \Gamma_2} = \langle \langle X \rangle_{\Gamma_1} \rangle_{\Gamma_2}.$$

Proof. It is easy to observe that $\langle X \rangle_{\Gamma_1} \subseteq \langle X \rangle_{\Gamma_1 \cup \Gamma_2}$, so

$$\langle \langle X \rangle_{\Gamma_1} \rangle_{\Gamma_2} \subseteq \langle \langle X \rangle_{\Gamma_1 \cup \Gamma_2} \rangle_{\Gamma_2} \subseteq \langle \langle X \rangle_{\Gamma_1 \cup \Gamma_2} \rangle_{\Gamma_1 \cup \Gamma_2} = \langle X \rangle_{\Gamma_1 \cup \Gamma_2}.$$

On the other hand, by induction on n one can show that $X^{[n]\Gamma_1 \cup \Gamma_2} \subseteq \langle \langle X \rangle_{\Gamma_1} \rangle_{\Gamma_2}$ for each $n \in \mathbb{N}$. Obviously, it holds for $n = 0$. Then $X^{[n+1]\Gamma_1 \cup \Gamma_2} = (X^{[n]\Gamma_1 \cup \Gamma_2})^{[1]\Gamma_1 \cup \Gamma_2} \subseteq (\langle \langle X \rangle_{\Gamma_1} \rangle_{\Gamma_2})^{[1]\Gamma_1 \cup \Gamma_2}$. Further, for ν in Γ_2, $\nu(M, \dots, \langle \langle X \rangle_{\Gamma_1} \rangle_{\Gamma_2}, \dots, M) = \langle \langle X \rangle_{\Gamma_1} \rangle_{\Gamma_2}$. For $\nu \in \Gamma_1$, by Lemma (4.5)

$$\begin{aligned} & \nu(M, \dots, \langle \langle X \rangle_{\Gamma_1} \rangle_{\Gamma_2}, \dots, M) = \nu(\langle M \rangle_{\Gamma_2}, \dots, \langle \langle X \rangle_{\Gamma_1} \rangle_{\Gamma_2}, \dots, \langle M \rangle_{\Gamma_2}) = \\ & \langle \nu(M, \dots, \langle X \rangle_{\Gamma_1}, \dots, M) \rangle_{\Gamma_2} = \langle \langle X \rangle_{\Gamma_1} \rangle_{\Gamma_2}. \end{aligned}$$

Finally, if $\omega \in \Omega \setminus (\Gamma_1 \cup \Gamma_2)$, then

$$\omega(\langle\langle X\rangle_{\Gamma_1}\rangle_{\Gamma_2}, \dots, \langle\langle X\rangle_{\Gamma_1}\rangle_{\Gamma_2}) = \langle\omega(\langle X\rangle_{\Gamma_1}, \dots, \langle X\rangle_{\Gamma_1})\rangle_{\Gamma_2} = \langle\langle X\rangle_{\Gamma_1}\rangle_{\Gamma_2}.$$

It shows that $(\langle\langle X\rangle_{\Gamma_1}\rangle_{\Gamma_2})^{[1]\Gamma_1\cup\Gamma_2} = \langle\langle X\rangle_{\Gamma_1}\rangle_{\Gamma_2}$, and implies

$$\langle X\rangle_{\Gamma_1\cup\Gamma_2} = \bigcup_{n\in\mathbb{N}} X^{[n]\Gamma_1\cup\Gamma_2} \subseteq \langle\langle X\rangle_{\Gamma_1}\rangle_{\Gamma_2}.$$

□

Corollary 4.7. *For a mode (M,Ω), $\emptyset \neq X \subseteq M$ and $\Gamma_1, \Gamma_2 \subseteq \Omega$ we have*

$$\langle\langle X\rangle_{\Gamma_1}\rangle_{\Gamma_2} = \langle\langle X\rangle_{\Gamma_2}\rangle_{\Gamma_1}.$$

A *modal* (*or semilattice ordered mode*) is an algebra $(M,\Omega,+)$ such that (M,Ω) is a mode, $(M,+)$ is a (join) semilattice (with semilattice order $\leq$, i.e. $x \leq y \Leftrightarrow x+y=y$) and the operations $\omega \in \Omega$ distribute over $+$. Similarly as in the case of the set of all non-empty subalgebras of (M,Ω), the set of all non-empty Γ-sinks also forms a semilattice $(S_\Gamma(M),+)$, where the operation $+$ is defined by setting

$$S_1 + S_2 := \langle S_1 \cup S_2\rangle_\Gamma.$$

Moreover, by Lemma (4.5) and (2.0.1), for any Γ-sinks $A_1,\dots,A_n,B,C$ of (M,Ω) and $\omega \in \Omega$

$$\omega(A_1,\dots,B+C,\dots,A_n) = \omega(A_1,\dots,\langle B\cup C\rangle_\Gamma,\dots,A_n) =$$
$$\langle\omega(A_1,\dots,B\cup C,\dots,A_n)\rangle_\Gamma = \langle\omega(A_1,\dots,B,\dots,A_n)\cup\omega(A_1,\dots,C,\dots,A_n)\rangle_\Gamma =$$
$$\omega(A_1,\dots,B,\dots,A_n)+\omega(A_1,\dots,C,\dots,A_n).$$

Hence, each operation $\omega \in \Omega$ distributes over $+$.

Corollary 4.8. *For any subset $\Gamma \subseteq \Omega$, $(S_\Gamma(M),\Omega,+)$ is a modal.*

Now we can define on the set $\mathcal{P}_{>0}M$, a family $\{\alpha_\Gamma \mid \Gamma \subseteq \Omega\}$ of the following relations:

$$X\alpha_\Gamma Y \quad\Leftrightarrow\quad \langle X\rangle_\Gamma = \langle Y\rangle_\Gamma.$$

It is obvious that each α_Γ is an equivalence relation. Note also that $\alpha_\emptyset$ is just relation α defined in Section 3.

Lemma 4.9. *For each subset $\Gamma \subseteq \Omega$, the relation α_Γ is in $Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$.*

Proof. To show that α_Γ is a congruence relation on the extended power algebra $(\mathcal{P}_{>0}M,\Omega,\cup)$, let $\omega \in \Omega$ be an n-ary complex operation and for non-empty subsets $X_1,\dots,X_n,Y_1,\dots,Y_n \subseteq M$ let $X_i\alpha_\Gamma Y_i$ for $1 \leq i \leq n$. This means that $\langle X_i\rangle_\Gamma = \langle Y_i\rangle_\Gamma$, for $1 \leq i \leq n$.
Hence by Lemma (4.5),

$$\langle\omega(X_1,\dots,X_n)\rangle_\Gamma = \omega(\langle X_1\rangle_\Gamma,\dots,\langle X_n\rangle_\Gamma) =$$
$$\omega(\langle Y_1\rangle_\Gamma,\dots,\langle Y_n\rangle_\Gamma) = \langle\omega(Y_1,\dots,Y_n)\rangle_\Gamma.$$

Moreover, it is also obvious that $X_1 \subseteq \langle X_1\rangle_\Gamma = \langle Y_1\rangle_\Gamma \subseteq \langle Y_1\cup Y_2\rangle_\Gamma$ and $X_2 \subseteq \langle X_2\rangle_\Gamma = \langle Y_2\rangle_\Gamma \subseteq \langle Y_1\cup Y_2\rangle_\Gamma$. Hence $\langle X_1\cup X_2\rangle_\Gamma \subseteq \langle Y_1\cup Y_2\rangle_\Gamma$. Similarly, we can show that $\langle Y_1\cup Y_2\rangle_\Gamma \subseteq \langle X_1\cup X_2\rangle_\Gamma$. Since for each $X \subseteq M$, the Γ-sink $\langle X\rangle_\Gamma$ is a subalgebra of (M,Ω), then

$$\langle\omega(X,\dots,X)\rangle_\Gamma = \omega(\langle X\rangle_\Gamma,\dots,\langle X\rangle_\Gamma) = \langle X\rangle_\Gamma.$$

This proves that $(\omega(X,\dots,X),X) \in \alpha_\Gamma$ and in fact $\alpha_\Gamma \in Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$. □

By First Isomorphism Theorem one immediately obtains the following generalization of Theorem 3.3. in [13].

Corollary 4.10. *Let* (M, Ω) *be a mode. For any* $\Gamma \subseteq \Omega$*, the quotient algebra* $((\mathcal{P}_{>0}M)^{\alpha_\Gamma}, \Omega, \cup)$ *is isomorphic to the modal* $(S_\Gamma(M), \Omega, +)$.

We say that an algebra is *plural* if all its operations have arities greater than one. Let $\emptyset \neq \Gamma \subseteq \Omega$. Each semilattice $(S, \cdot)$ may be considered as a plural Γ-algebra (S, Γ) on setting

$$\nu(x_1, \ldots, x_{n_\nu}) := x_1 \cdot \ldots \cdot x_{n_\nu},$$

for each n_ν-ary $\nu \in \Gamma$. Such an algebra is often referred to as a Γ-semilattice obtained from a semilattice $(S, \cdot)$ (see [19]). A (distributive) Γ-lattice is an algebra $(L, \Gamma, \vee)$, where $(L, \vee)$ is a semilattice, (L, Γ) is a Γ-semilattice obtained from a semilattice $(L, \wedge)$ and $(L, \wedge, \vee)$ is a (distributive) lattice.

Corollary 4.11. *Let* (M, Ω) *be a plural mode and* $\emptyset \neq \Gamma \subseteq \Omega$*. Then* $((\mathcal{P}_{>0}M)^{\alpha_\Gamma}, \Gamma, \cup)$ *is isomorphic to the distributive* Γ*-lattice* $(S_\Gamma(M), \Gamma, +)$ *obtained from the semilattice* $(S_\Gamma(M), \cap)$.

Proof. Similarly as it was shown for sinks in [19], for any operation $\nu \in \Gamma \neq \emptyset$ and Γ-sinks $S_1, \ldots, S_n$ in $S_\Gamma(M)$, one has $\nu(S_1, \ldots, S_n) \subseteq S_i$, so $\nu(S_1, \ldots, S_n) \subseteq S_1 \cap \ldots \cap S_n$. Conversely, for $s \in S_1 \cap \ldots \cap S_n$, $s = \nu(s, \ldots, s) \in \nu(S_1, \ldots, S_n)$. Thus $\nu(S_1, \ldots, S_n) = S_1 \cap \ldots \cap S_n$.
Moreover,

$$S_1 \cap (S_1 + S_2) = S_1 \cap \langle S_1 \cup S_2 \rangle_\Gamma = \nu(S_1, \ldots, S_1, \langle S_1 \cup S_2 \rangle_\Gamma) =$$
$$\langle \nu(S_1, \ldots, S_1, S_1 \cup S_2) \rangle_\Gamma = \langle \nu(S_1, \ldots, S_1) \cup \nu(S_1, \ldots, S_1, S_2) \rangle_\Gamma =$$
$$\langle S_1 \cup (S_1 \cap S_2) \rangle_\Gamma = \langle S_1 \rangle_\Gamma = S_1,$$

and

$$S_1 + (S_1 \cap S_2) = \langle S_1 \cup (S_1 \cap S_2) \rangle_\Gamma = \langle S_1 \rangle_\Gamma = S_1.$$

Hence, by Corollary (4.8), $(S_\Gamma(M), \cap, +)$ is a distributive lattice. □

By results of [19] we know that for Ω-sinks S_1 and S_2 of a mode (M, Ω), the union $S_1 \cup S_2$ is an Ω-sink, too. This implies the following corollary.

Corollary 4.12. [19] *Let* (M, Ω) *be a plural mode. The quotient algebra* $((\mathcal{P}_{>0}M)^{\alpha_\Omega}, \Omega, \cup)$ *is isomorphic to the distributive* Ω*-lattice* $(S_\Omega(M), \Omega, \cup)$ *obtained from the semilattice* $(S_\Omega(M), \cap)$.

Now, we consider the properties of the set $\{\alpha_\Gamma \mid \Gamma \subseteq \Omega\}$.

Lemma 4.13. *The set* $\{\alpha_\Gamma \mid \Gamma \subseteq \Omega\}$ *is partially ordered by a set inclusion with the greatest element* α_Ω *and the least element* $\alpha_\emptyset$*. In particular,*

$$\alpha_{\Gamma_1 \cap \Gamma_2} \subseteq \alpha_{\Gamma_1} \cap \alpha_{\Gamma_2}$$

and

$$\alpha_{\Gamma_1} \cup \alpha_{\Gamma_2} \subseteq \alpha_{\Gamma_1 \cup \Gamma_2}.$$

Proof. It is evident that, if $\Gamma_1 \subseteq \Gamma_2$ then $\langle X \rangle_{\Gamma_1} \subseteq \langle X \rangle_{\Gamma_2}$. Let $(X, Y) \in \alpha_{\Gamma_1}$. Then $X \subseteq \langle X \rangle_{\Gamma_1} = \langle Y \rangle_{\Gamma_1} \subseteq \langle Y \rangle_{\Gamma_2}$, so $\langle X \rangle_{\Gamma_2} \subseteq \langle Y \rangle_{\Gamma_2}$. Similarly $\langle Y \rangle_{\Gamma_2} \subseteq \langle X \rangle_{\Gamma_2}$, which implies $\alpha_{\Gamma_1} \subseteq \alpha_{\Gamma_2}$. □

Theorem 4.14. *The ordered set* $(\{\alpha_\Gamma \mid \Gamma \subseteq \Omega\}, \subseteq)$ *is a bounded join-semilattice* $(\{\alpha_\Gamma \mid \Gamma \subseteq \Omega\}, \vee)$, *where for any* $\Gamma_1, \Gamma_2 \subseteq \Omega$,

$$\alpha_{\Gamma_1} \vee \alpha_{\Gamma_2} = \alpha_{\Gamma_1 \cup \Gamma_2}.$$

Proof. Let $\emptyset \neq X, Y \subseteq M$. By Lemma (4.13), $\alpha_{\Gamma_1} \vee \alpha_{\Gamma_2} \subseteq \alpha_{\Gamma_1 \cup \Gamma_2}$. Assume now that $(X,Y) \in \alpha_{\Gamma_1 \cup \Gamma_2}$, so $\langle X\rangle_{\Gamma_1 \cup \Gamma_2} = \langle Y\rangle_{\Gamma_1 \cup \Gamma_2}$. Hence, by Theorem (4.6), $\langle\langle X\rangle_{\Gamma_1}\rangle_{\Gamma_2} = \langle\langle Y\rangle_{\Gamma_1}\rangle_{\Gamma_2}$. Consequently, $\langle X\rangle_{\Gamma_1} \alpha_{\Gamma_2} \langle Y\rangle_{\Gamma_1}$ and $X \alpha_{\Gamma_1} \langle X\rangle_{\Gamma_1} \alpha_{\Gamma_2} \langle Y\rangle_{\Gamma_1} \alpha_{\Gamma_1} Y$, which proves that $(X,Y) \in \alpha_{\Gamma_1} \vee \alpha_{\Gamma_2}$. □

Note that congruences $\alpha_{\Gamma_1} \cap \alpha_{\Gamma_2}$ and $\alpha_{\Gamma_1 \cap \Gamma_2}$ do not need to be equal and in general, the congruence $\alpha_{\Gamma_1} \cap \alpha_{\Gamma_2}$ does not even have to belong to the set $(\{\alpha_\Gamma \mid \Gamma \subseteq \Omega\}, \subseteq)$. Hence, such congruences give another examples of congruences with idempotent quotient. It is also possible that $\alpha_{\Gamma_1} \subseteq \alpha_{\Gamma_2}$ even in the case where Γ_1 and Γ_2 are incomparable.

Example 4.15. Let $(R, +, \cdot)$ be a commutative ring with unit 1 and let $I(R)$ be the set of all ideals of $(R, +, \cdot)$. For each $r \in R$ define binary operations

$$\cdot_r : I(R) \times I(R) \to I(R) \ \text{ by}$$

$$a \cdot_r b := \{s \in R \mid rs \in a\} \cap b.$$

(Note that $a \cdot_1 b = a \cap b$.) It was shown by K. Kearnes in [8] that $(I(R), \{\cdot_r\}_{r \in R})$ is a mode.

Let us consider the ring $(\mathbb{Z}_{30}, +_{30}, \cdot_{30})$, its ideals $a := \{0\}$, $b := \{0, 15\}$, $c := \{0, 10, 20\}$, $d := \{0, 6, 12, 18, 24\}$ and the set $\Omega := \{\cdot_2, \cdot_3, \cdot_5\}$ of operations. The algebra $(\{a,b,c,d\}, \Omega)$ is a subreduct of $(I(\mathbb{Z}_{30}), \{\cdot_r\}_{r \in \mathbb{Z}_{30}})$. So it is a 4-element mode with the three binary operations:

$\cdot_2$	a	b	c	d
a	a	b	a	a
b	a	b	a	a
c	a	b	c	a
d	a	b	a	d

$\cdot_3$	a	b	c	d
a	a	a	c	a
b	a	b	c	a
c	a	a	c	a
d	a	a	c	d

$\cdot_5$	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	a	a	a	d

Let $\Gamma_2 := \{\cdot_2\}$, $\Gamma_3 := \{\cdot_3\}$ and $\Gamma_5 := \{\cdot_5\}$. As a consequence of easy calculations we obtain the following.

All subsets $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a,b\}$, $\{a,c\}$, $\{a,d\}$ are subalgebras of $(\{a,b,c,d\}, \Omega)$ and $\langle\{b,c\}\rangle = \{a,b,c\}$, $\langle\{b,d\}\rangle = \{a,b,d\}$, $\langle\{c,d\}\rangle = \{a,c,d\}$, $\langle\{b,c,d\}\rangle = \{a,b,c,d\}$.

$\langle\{a\}\rangle_{\Gamma_2} = \langle\{b\}\rangle_{\Gamma_2} = \{a,b\}$, $\langle\{c\}\rangle_{\Gamma_2} = \langle\{b,c\}\rangle_{\Gamma_2} = \langle\{a,c\}\rangle_{\Gamma_2} = \{a,b,c\}$
$\langle\{d\}\rangle_{\Gamma_2} = \langle\{a,d\}\rangle_{\Gamma_2} = \langle\{b,d\}\rangle_{\Gamma_2} = \{a,b,d\}$
$\langle\{c,d\}\rangle_{\Gamma_2} = \langle\{b,c,d\}\rangle_{\Gamma_2} = \langle\{a,c,d\}\rangle_{\Gamma_2} = \{a,b,c,d\}$

$\langle\{a\}\rangle_{\Gamma_3} = \langle\{c\}\rangle_{\Gamma_3} = \{a,c\}$, $\langle\{b\}\rangle_{\Gamma_3} = \langle\{b,c\}\rangle_{\Gamma_3} = \langle\{a,b\}\rangle_{\Gamma_3} = \{a,b,c\}$
$\langle\{d\}\rangle_{\Gamma_3} = \langle\{a,d\}\rangle_{\Gamma_3} = \langle\{c,d\}\rangle_{\Gamma_3} = \{a,c,d\}$
$\langle\{b,d\}\rangle_{\Gamma_3} = \langle\{a,b,d\}\rangle_{\Gamma_3} = \langle\{b,c,d\}\rangle_{\Gamma_3} = \{a,b,c,d\}$

$\langle\{a\}\rangle_{\Gamma_5} = \langle\{d\}\rangle_{\Gamma_5} = \{a,d\}$, $\langle\{b\}\rangle_{\Gamma_5} = \langle\{a,b\}\rangle_{\Gamma_5} = \langle\{b,d\}\rangle_{\Gamma_5} = \{a,b,d\}$
$\langle\{c\}\rangle_{\Gamma_5} = \langle\{a,c\}\rangle_{\Gamma_5} = \langle\{c,d\}\rangle_{\Gamma_5} = \{a,c,d\}$
$\langle\{b,c\}\rangle_{\Gamma_5} = \langle\{a,b,c\}\rangle_{\Gamma_5} = \langle\{b,c,d\}\rangle_{\Gamma_5} = \{a,b,c,d\}$

$$\langle\{a\}\rangle_{\Gamma_2\cup\Gamma_3} = \langle\{b\}\rangle_{\Gamma_2\cup\Gamma_3} = \langle\{c\}\rangle_{\Gamma_2\cup\Gamma_3} = \{a,b,c\},\ \langle\{d\}\rangle_{\Gamma_2\cup\Gamma_3} = \{a,b,c,d\}$$

$$\langle\{a\}\rangle_{\Gamma_2\cup\Gamma_5} = \langle\{b\}\rangle_{\Gamma_2\cup\Gamma_5} = \langle\{d\}\rangle_{\Gamma_2\cup\Gamma_5} = \{a,b,d\},\ \langle\{c\}\rangle_{\Gamma_2\cup\Gamma_5} = \{a,b,c,d\}$$

$$\langle\{a\}\rangle_{\Gamma_3\cup\Gamma_5} = \langle\{c\}\rangle_{\Gamma_3\cup\Gamma_5} = \langle\{d\}\rangle_{\Gamma_3\cup\Gamma_5} = \{a,c,d\},\ \langle\{b\}\rangle_{\Gamma_3\cup\Gamma_5} = \{a,b,c,d\}$$

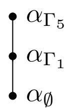
Finally, there is only one Ω-sink $\{a,b,c,d\}$.

This shows that the semilattice $(\{\alpha_\Gamma \mid \Gamma \subseteq \{\cdot_2,\cdot_3,\cdot_5\}\},\subseteq)$ is isomorphic to the lattice of all divisors of the number 30.

Moreover, in this case $(\{d\},\{a,d\}) \in \alpha_{\Gamma_2}\cap\alpha_{\Gamma_3}$, $(\{b\},\{a,b\}) \in \alpha_{\Gamma_2}\cap\alpha_{\Gamma_5}$ and $(\{c\},\{a,c\}) \in \alpha_{\Gamma_3}\cap\alpha_{\Gamma_5}$ hence $\alpha_{\Gamma_2}\cap\alpha_{\Gamma_3} \neq \alpha_{\Gamma_2\cap\Gamma_3} = \alpha_\emptyset$, $\alpha_{\Gamma_2}\cap\alpha_{\Gamma_5} \neq \alpha_{\Gamma_2\cap\Gamma_5} = \alpha_\emptyset$, $\alpha_{\Gamma_3}\cap\alpha_{\Gamma_5} \neq \alpha_{\Gamma_3\cap\Gamma_5} = \alpha_\emptyset$ and $\alpha_{\Gamma_2}\cap\alpha_{\Gamma_3}, \alpha_{\Gamma_2}\cap\alpha_{\Gamma_5}, \alpha_{\Gamma_3}\cap\alpha_{\Gamma_5} \notin \{\alpha_\Gamma \mid \Gamma \subseteq \{\cdot_2,\cdot_3,\cdot_5\}\}$.

Now let $\Gamma_1 := \{\cdot_1\}$ and consider the mode $(\{a,b,c,d\},\cdot_1,\cdot_5)$. It is also easy to check that
$\langle\{a\}\rangle_{\Gamma_1} = \{a\}$, $\langle\{b\}\rangle_{\Gamma_1} = \{a,b\}$, $\langle\{c\}\rangle_{\Gamma_1} = \{a,c\}$, $\langle\{d\}\rangle_{\Gamma_1} = \{a,d\}$,
$\langle\{b,c\}\rangle_{\Gamma_1} = \{a,b,c\}$, $\langle\{b,d\}\rangle_{\Gamma_1} = \{a,b,d\}$,
$\langle\{c,d\}\rangle_{\Gamma_1} = \{a,c,d\}$, $\langle\{b,c,d\}\rangle_{\Gamma_1} = \{a,b,c,d\}$.

Obviously, Γ_5-sinks are exactly the same as in the previous case and $\alpha_{\Gamma_1} \subseteq \alpha_{\Gamma_5} = \alpha_{\Gamma_1}\cup\alpha_{\Gamma_5}$. Hence the semilattice $(\{\alpha_\Gamma \mid \Gamma \subseteq \{\cdot_1,\cdot_5\}\},\subseteq)$ is the 3-element chain:



Now let X be a non-finite set, $\mathcal{V}$ be a variety of Ω-modes and $(F_\mathcal{V}(X),\Omega)$ be the free $\mathcal{V}$-algebra over the set X.

Theorem 4.16. *For any subset $\Gamma \subseteq \Omega$, the relation α_Γ is a fully invariant congruence on $(\mathcal{P}^{<\omega}_{>0}F_\mathcal{V}(X),\Omega,\cup)$.*

Proof. Let $Q,R \in \mathcal{P}^{<\omega}_{>0}F_\mathcal{V}(X)$. We have to prove that for every endomorphism ψ of $(\mathcal{P}^{<\omega}_{>0}F_\mathcal{V}(X),\Omega,\cup)$, if $\langle Q\rangle_\Gamma = \langle R\rangle_\Gamma$ then also $\langle\psi(Q)\rangle_\Gamma = \langle\psi(R)\rangle_\Gamma$.

Assume that the notation $t(x_1,\dots,x_n)$ means the term $t \in F_\mathcal{V}(X)$ contains no other variables than $x_1,\dots,x_n$ (but not necessarily all of them).

For each $x_i \in X$ let us choose a subset $P_i \in \mathcal{P}^{<\omega}_{>0}F_\mathcal{V}(X)$. The mapping $\psi : \mathcal{P}^{<\omega}_{>0}F_\mathcal{V}(X) \to \mathcal{P}^{<\omega}_{>0}F_\mathcal{V}(X)$

$$\psi(Q) = \psi(\{q_1(x_1,\dots,x_n),\dots,q_k(x_1,\dots,x_n)\}) := \bigcup_{q\in Q} q(P_1,\dots,P_n) \tag{4.16.1}$$

is a homomorphism of the algebra $(\mathcal{P}^{<\omega}_{>0}F_\mathcal{V}(X),\Omega,\cup)$.

Note that the algebra $(\mathcal{P}^{<\omega}_{>0}F_\mathcal{V}(X),\Omega,\cup)$ is generated by the set $\{\{x\} \mid x \in X\}$ and for any $\{x_i\}$, we have $\psi(\{x_i\}) = P_i$. Because each homomorphism is uniquely defined on generators of an algebra then we obtain that each endomorphism ψ of $(\mathcal{P}^{<\omega}_{>0}F_\mathcal{V}(X),\Omega,\cup)$ is of the form (4.16.1).

Now we will prove an auxiliary result.

Claim. Let $m \in \mathbb{N}$, and $R,P_1,\dots,P_n \in \mathcal{P}^{<\omega}_{>0}F_\mathcal{V}(X)$. For each $r \in R^{[m]_\Gamma}$ we have $r(P_1,\dots,P_k) \subseteq (\psi(R))^{[m]_\Gamma}$.

Proof of the Claim. For $m = 0$ the result is obvious. Now assume that the Claim is established for $m > 0$ and let $r \in R^{[m+1]_\Gamma}$. Then there exists a n_ν-ary operation $\nu \in \Gamma$ such that $r \in \nu(F_\mathcal{V}(X), \dots, R^{[m]_\Gamma}, \dots, F_\mathcal{V}(X))$ or there exists a n_ω-ary operation $\omega \in \Omega \setminus \Gamma$ such that $r \in \omega(R^{[m]_\Gamma}, \dots, R^{[m]_\Gamma})$. In the first case

$$r(x_1, \dots, x_n) = \nu(t_1(x_1, \dots, x_n), \dots, s(x_1, \dots, x_n), \dots, t_{n_\nu - 1}(x_1, \dots, x_n)),$$

for some terms $t_1, \dots, t_{n_\nu-1} \in F_\mathcal{V}(X)$ and $s \in R^{[m]_\Gamma}$. By the induction hypothesis we obtain

$$\begin{aligned} r(P_1, \dots, P_n) &= \nu(t_1(P_1, \dots, P_n), \dots, s(P_1, \dots, P_n), \dots, t_{n_\nu-1}(P_1, \dots, P_n)) \\ &\subseteq \nu(F_\mathcal{V}(X), \dots, (\psi(R))^{[m]_\Gamma}, \dots, F_\mathcal{V}(X)) \subseteq (\psi(R))^{[m+1]_\Gamma}. \end{aligned}$$

In the second case,

$$r(x_1, \dots, x_n) = \omega(r_1(x_1, \dots, x_n), \dots, r_{n_\omega}(x_1, \dots, x_n)),$$

for some terms $r_1, \dots, r_{n_\omega} \in R^{[m]_\Gamma}$. Hence, also by the induction hypothesis,

$$\begin{aligned} r(P_1, \dots, P_n) &= \omega(r_1(P_1, \dots, P_n), \dots, r_{n_\omega}(P_1, \dots, P_n)) \\ &\subseteq \omega((\psi(R))^{[m]_\Gamma}, \dots, (\psi(R))^{[m]_\Gamma}) \subseteq (\psi(R))^{[m+1]_\Gamma}, \end{aligned}$$

which completes the proof of the Claim.

Now assume that $\langle Q\rangle_\Gamma = \langle R\rangle_\Gamma$. For any $q \in Q \subseteq \langle Q\rangle_\Gamma = \langle R\rangle_\Gamma$ there exist $m \in \mathbb{N}$ and $r \in R^{[m]_\Gamma}$ such that $q = r$. Hence for any subsets $P_1, \dots, P_n \in \mathcal{P}^{<\omega}_{>0} F_\mathcal{V}(X)$, $q(P_1, \dots, P_n) \subseteq (\psi(R))^{[m]_\Gamma}$. This implies

$$\psi(Q) = \bigcup_{q \in Q} q(P_1, \dots, P_n) \subseteq (\psi(R))^{[m]_\Gamma} \subseteq \langle \psi(R)\rangle_\Gamma,$$

and consequently, $\langle\psi(Q)\rangle_\Gamma \subseteq \langle\psi(R)\rangle_\Gamma$. Analogously, we can show that $\langle\psi(R)\rangle_\Gamma \subseteq \langle\psi(Q)\rangle_\Gamma$ which finishes the proof. □

By Corollary (4.10) one immediately obtains that for any $\Gamma \subseteq \Omega$ the quotient algebra $((\mathcal{P}^{<\omega}_{>0} M)^{\alpha_\Gamma}, \Omega, \cup)$ is isomorphic to the modal of finitely generated Γ-sinks of (M, Ω).

Let $\mathcal{V}_\Omega$ be the variety of all Ω-modes. By results of A. Romanowska and J.D.H. Smith ([19]) the modal $((\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}_\Omega}(X))^{\alpha}, \Omega, \cup)$ is free over a set X in the variety $\mathcal{M}$ of all modals $(M, \Omega, +)$. As it was indicated in [14] very little is known about varieties of modals. Only the variety of dissemilattices (see [10]) and the variety of entropic modals (semilattice modes, see [8]) are well described.

By Theorem (4.16), for any subset $\Gamma \subseteq \Omega$, each algebra $((\mathcal{P}^{<\omega}_{>0} F_{\mathcal{V}_\Omega}(X))^{\alpha_\Gamma}, \Omega, \cup)$ determines a non-trivial modal subvariety $\mathcal{M}_\Gamma := \mathrm{HSP}((\mathrm{S}_\Gamma(\mathrm{F}_{\mathcal{V}_\Omega}(\mathrm{X})), \Omega, +))$ of the variety $\mathcal{M}$. Since the congruence α_Ω is the greatest element in $\{\alpha_\Gamma \mid \Gamma \subseteq \Omega\}$, then by Corollary (4.12), none of the subvarieties $\mathcal{M}_\Gamma$ is entropic (see Problem (5.4)).

5. Open problems

All congruences described in this paper are congruences on the extended power algebras of modes, and hence also on the power algebras of modes. But in the proof that the relation ρ is an idempotent replica congruence in [13] the assumption that ρ is a congruence with respect to the operation $\cup$ was essential.

Problem 5.1. *Is the relation ρ the idempotent replica congruence for the power algebra $(\mathcal{P}_{>0} M, \Omega)$ of a mode (M, Ω)?*

This problem is closely related to the following problem which was partially solved in [13].

Problem 5.2. [11, 14, 1] *Let $\mathcal{V}$ be a variety of modes. It is known that the variety generated by the class $\{(\mathcal{P}_{>0}M, \Omega) \mid (M, \Omega) \in \mathcal{V}\}$ satisfies the identities being a result of identification of variables from the linear identities true in $\mathcal{V}$. Is it true that the variety generated by the class $\{(S(M), \Omega) \mid (M, \Omega) \in \mathcal{V}\}$ satisfies the identities being consequences of the idempotence and the linear identities true in $\mathcal{V}$?*

Obviously, congruences considered here are not all idempotent congruences on the (extended) power algebras of modes. It follows by general observation that if γ is a congruence of the (extended) power algebra $(\mathcal{P}_{>0}M, \Omega, \cup)$ of a mode (M, Ω), then it is in $Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$ if and only if each coset is a subalgebra of $(\mathcal{P}_{>0}M, \Omega)$. In this way we know how to find elements of $Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$.

Problem 5.3. *Describe other congruences on the (extended) power algebras of modes which give an idempotent factor.*

Moreover, for each $\gamma \in Con_{\mathcal{I}}(\mathcal{P}_{>0}M)$ its quotient is a modal (similarly as in Corollary (4.10)). For detailed results on representation of modals (also by means of congruences) we refer the reader to [12]. In [8] semilattice modes (in our nomenclature entropic modals) were thoroughly investigated. It raises the following.

Problem 5.4. *Find the congruences on the extended power algebras of modes which give as a quotient an entropic modal.*

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