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DUALITY FOR QUASIPOLYTOPES

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Abstract

In an earlier paper, Romanowska, Ślusarski and Smith described a duality between the category of polytopes (finitely generated real convex sets considered as barycentric algebras) and a certain category of intersections of hypercubes, considered as barycentric algebras with additional constant operations. The present paper provides an extension of this duality to a much more general class of so-called quasipolytopes, that is, convex sets with polytopes as closures. The dual spaces of quasipolytopes are Płonka sums of open polytopes, which are considered as barycentric algebras with some additional operations. In constructing this duality, we use several known and new dualities: the Hofmann–Mislove–Stralka duality for semilattices; the Romanowska–Ślusarski–Smith duality for polytopes; a new duality for open polytopes; and a new duality for injective Płonka sums of polytopes.

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1. Introduction

The main motivation for this work is the search for a duality theory of barycentric algebras and, in particular, of convex sets. There is a well-known self-duality for the category of finite-dimensional (real) vector spaces which can be adapted to the category of corresponding affine spaces. However, this duality cannot be restricted to provide a duality for the category of all convex subsets of (real) affine spaces. The previously known dualities only cover restricted classes of convex sets, which include finite-dimensional real simplices, as finitely generated free barycentric algebras [19], and the class of quadrilaterals in [20]. A more general duality was provided by Romanowska *et al.* [21] for the category of polytopes (finitely generated real convex sets considered as barycentric algebras). Pontryagin duality for semilattices [11] may also be considered as a duality for a limited class of barycentric algebras.

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In this paper, we extend the earlier results and present a duality for the class of quasipolytopes, convex subsets of affine spaces over a subfield R of the field $\mathbb{R}$ of real numbers, having polytopes as closures, and as before considered previously to be (cancellative) barycentric algebras. As in the earlier cases, this duality is again of 'classical' type, and is given by an infinite object, the open unit interval of R. However, this time it is extended by the addition of a new element playing the role of 'zero'. Although this object is not a quasipolytope, it plays the role of a schizophrenic object. In the new duality, the class of representation spaces is a class of certain barycentric algebras which are Płonka sums of open polytopes. They are considered to be barycentric algebras with an additional left-zero operation and with new constant operations (and with the corresponding homomorphisms preserving the new operations). The duality does not involve any topology or additional relations.

In constructing this duality, we use several known and new dualities: the Hofmann–Mislove–Stralka duality for semilattices; the Romanowska–Ślusarski–Smith duality for convex polytopes; a new duality for open polytopes; and a new duality between the class of Płonka sums of polytopes with injective Płonka homomorphisms and a certain class of Płonka sums of polytopes considered as barycentic algebras with some additional operations.

The paper is organised as follows. Section 2 gives a brief introduction to real affine spaces, convex sets and barycentric algebras. Płonka sums of barycentric algebras and their basic properties are recalled in Section 3, where we also provide an alternative description of barycentric algebras and a characterisation of barycentric algebras in the smallest quasivariety containing all convex sets and all semilattices. Algebras in this class are Płonka sums of convex sets with injective Płonka homomorphisms. For the convenience of the reader, and to make the paper relatively self-contained, Sections 4 and 5 provide the background necessary for understanding the dualities considered in this paper. In particular, duality for semilattices is described in Section 5. Section 6 recalls those basic facts concerning duality for polytopes which we need to describe a duality for open polytopes in Section 7. An essential role is played by Section 8, where a duality for injective Płonka sums of polytopes is provided. The role of a schizophrenic object is played by the extended closed unit interval of R. The representation spaces are Płonka sums of first duals of summands of given injective Płonka sums, and are considered as barycentric algebras with one new binary operation and three new constant operations. In the proof, we use the duality for semilattices, and the duality for polytopes. The next two sections deal with the first and second duals of the duality for quasipolytopes. The duality is given by the extended open unit interval of the field R, which plays the role of a schizophrenic object. In a certain sense, it is constructed alongside the duality for injective Płonka sums of polytopes. The representation spaces are Płonka sums of the first duals of open polytopes, and are special subalgebras of Płonka sums of certain polytopes. They are considered to be barycentric algebras with additional operations. The main result (Corollary 10.2) establishes the full duality between the category of quasipolytopes and the category of corresponding representation spaces.

We use notation and conventions similar to those of [26, 29] and the earlier papers. For details and further information on affine spaces and barycentric algebras, we refer the reader to those papers and to the monographs [22, 26]. For convex polytopes, see [2, 10].

2. Modes, affine spaces and barycentric algebras

In the sense of [22, 26], *modes* are algebras in which each element forms a singleton subalgebra, and for which each operation is a homomorphism. For algebras (A, Ω) of a given type $\tau : \Omega \to \mathbb{N}$, these two properties are equivalent to satisfaction of the identity

$$x \ldots x\psi = x$$

of *idempotence* for each operator ψ in Ω, and the identity

$$(x_{1,1} \ldots x_{1,\psi\tau}\psi) \ldots (x_{\phi\tau,1} \ldots x_{\phi\tau,\psi\tau}\psi)\phi = (x_{1,1} \ldots x_{\phi\tau,1}\phi) \ldots (x_{1,\psi\tau} \ldots x_{\phi\tau,\psi\tau}\phi)\psi$$

of *entropicity* for each pair ψ, ϕ of operators in Ω.

One of the main families of examples of modes is given by affine spaces over a commutative unital ring R (*affine R-spaces*), or, more generally, by subreducts (subalgebras of reducts) of affine spaces. Affine spaces are considered here as Mal'tsev modes, as explained in the monographs [22, 26]. In particular, if 2 is invertible in R, an affine R-space can be considered as the reduct $(A, \underline{R})$ of an R-module $(A, +, R)$, where $\underline{R}$ is the family of binary operations

$$\underline{r} : A^2 \to A; \quad (x_1, x_2) \mapsto x_1x_2\underline{r} = x_1(1 - r) + x_2r$$

for each $r \in R$. The class of all affine R-spaces is a variety (see [4]) and is denoted by $\underline{\underline{R}}$.

An important class of subreducts of affine spaces is given by convex sets, defined as subreducts of affine $\mathbb{R}$-spaces, where $\mathbb{R}$ is the ring of real numbers. Convex sets are characterised as subsets of a real affine space closed under the operations $\underline{r}$ of weighted means coming from the open real unit interval $I^o = (0, 1)$. Thus a convex set contains, along with any two of its points, the line segment joining them. The class C of convex sets, considered as such algebras $(C, \underline{I}^o)$, generates the variety $\mathcal{B}$ of barycentric algebras, and forms a subquasivariety of $\mathcal{B}$ (see [15]). The definition of real convex sets and barycentric algebras is easily generalised to the case of subfields R of the field $\mathbb{R}$ (see, for instance, [26, Chs 5, 7]).

In this paper we assume that all affine spaces are over a subfield R of the field $\mathbb{R}$ of real numbers, and convex sets are convex subsets of affine R-spaces. In what follows, the adjective 'real' will be used only for affine spaces over $\mathbb{R}$ and their convex subsets. Both the variety $\underline{\underline{R}}$ of affine R-spaces and the variety $\mathcal{B}$ of barycentric algebras over R (as well as its subquasivariety C of convex sets) will also be considered as categories with algebras as objects and the corresponding homomorphisms as morphisms.

The following theorem, providing a characterisation of entropicity, will play an essential role in this paper. Let $\underline{\underline{\tau}}$ be the variety of all τ-algebras (algebras of a given type τ).

Theorem 2.1 [22, 26]. *A τ-algebra A is entropic if and only if, for each τ-algebra X, the morphism set $\underline{\tau}(X, A)$ is a subalgebra of the power τ-algebra A^X.*

Corollary 2.2. *If $\mathcal{K}$ is a prevariety of entropic algebras, then, for each pair A, B of $\mathcal{K}$-algebras, the morphism set $\mathcal{K}(B, A)$ is again a $\mathcal{K}$-algebra.*

Recall that the power algebra on the set A^B is defined by

$$\omega : (A^B)^m \to A^B; \quad (f_1 \dots f_m) \mapsto f_1 \dots f_m\omega = (f : B \to A),$$

where

$$f : B \to A; \quad x \mapsto xf = x(f_1 \dots f_m\omega) = (xf_1 \dots xf_m)\omega$$

for each (m-ary) ω in Ω.

The free barycentric algebra over a finite set $X = \{x_0, \dots, x_k\}$ is characterised as the $\underline{I}^o$-subreduct of the free affine R-space R^k over the same set X. The $\underline{I}^o$-subreduct is

$$\Big\{x_0r_0 + \cdots + x_kr_k \mid r_i \in I, \sum_{i=0}^{k} r_i = 1\Big\},$$

where I is the closed unit interval. Its elements are known as *convex combinations*. They form the k-dimensional simplex.

We say that a convex set A is *k-dimensional* if k is the smallest positive integer such that A embeds as a subreduct into the affine R-space R^k. In such a situation, the convex set A generates the affine space R^k. Each convex polytope A is finite-dimensional. The minimal set of generators of A is uniquely determined: it is the set of vertices of A. If a k-dimensional polytope A has n vertices, then the number of vertices is at least $k + 1$. Such a polytope A is closed when considered as a subset of the topological space R^k with the usual topology.

3. Płonka sums of barycentric algebras

Recall that the variety of barycentric algebras over a subfield R of the field $\mathbb{R}$ may be defined by the identity

$$xx\underline{p} = x$$

of *idempotence* for each p in I^o, the identities

$$xy\underline{p} = yx\underline{1-p}$$

of *skew-commutativity* for each p in I^o and the identities

$$xy\,\underline{p}\,z\,\underline{q} = x\,yz\,\underline{q/(p \circ q)}\,\underline{p \circ q}$$

of *skew-associativity* for each p, q in I^o. Here $p \circ q = p + q - pq$. Setting $p' := 1 - p$, one obtains $p \circ q = (p'q')'$ (see [23, 26, Section 5.8]).

The main models are provided by convex subsets of affine R-spaces and $\underline{I}^o$-semilattices, algebras equivalent to semilattices, where the operations in $\underline{I}^o$ are associative and any two of them are equal. In what follows, $\underline{I}^o$-semilattices will simply be referred to as semilattices. Convex subsets of affine R-spaces are described as $\underline{I}^o$-subreducts of affine spaces $(A, \underline{R})$. Among barycentric algebras, convex sets are characterised by cancellativity: that is, they form the subquasivariety C of $\mathcal{B}$ defined by the cancellation laws

$$(xy\underline{p} = xz\underline{p}) \longrightarrow (y = z),$$

which hold for all $p \in I^o$. The quasivariety of convex sets and the variety of $\underline{I}^\circ$-semilattices are the only minimal quasivarieties of barycentric algebras. The variety of barycentric algebras may, equivalently, be defined as the class of homomorphic images of convex sets in C.

Recall, also, that each barycentric algebra A is a semilattice sum $\bigcup_{s\in S} A_s$ of open convex sets A_s over its semilattice replica S: that is, it has a homomorphism onto the largest semilattice image S, with fibres A_s being open convex sets (each A_s is open in the affine space it generates) (see [26, Sections 3.3 and 7.5]). In fact, A is a subalgebra of a Płonka sum $\sum_{s\in S} E_s$ of convex extensions E_s of A_s over the semilattices replica S. For the convenience of the reader, we recall, here, the definition of a Płonka sum of barycentric algebras E_s. We consider the semilattice S as a small category. Let $F : S \to \mathcal{B}$ be a functor from the category S to the category $\mathcal{B}$ of barycentric algebras, assigning to each $s \in S$ an algebra E_s and to each arrow $s \to t$ in S a barycentric algebra homomorphism $\varphi_{s,t} : E_s \to E_t$. (The mappings $\varphi_{s,t}$ are called *Płonka homomorphisms*.) Then the *Płonka sum* $E = \sum_{s\in S} E_s$ of algebras E_s (over the semilattice S) is the disjoint sum of the sets E_s with the operations $\underline{r}$, for all $r \in I^o$, defined by

$$a_s b_t \underline{r} = a_s\varphi_{s,s\cdot t} b_t \varphi_{s,s\cdot t}\underline{r},$$

for $a_s \in E_s, b_t \in E_t$. Recall, also, that a *wall* of a barycentric algebra A is a subalgebra W such that, for $a, b \in A$ and $r \in I^o$,

$$ab\underline{r} \in W \quad \text{if and only if } a, b \in W.$$

The smallest wall $[a]$ containing an element a is called a *principal wall* generated by a. The semilattice replica S of A is the semilattice of its principal walls (see [26, Theorem 7.5.10]).

An essential role is played by the algebra A^∞, obtained from a barycentric algebra A, as the Płonka sum of $A_1 = A$ and the singleton algebra $A_0 = \{\infty\}$ over the two-element (meet) semilattice $\underset{\sim}{2} = \{0 < 1\}$. The element ∞ plays the role of a zero added to the algebra A.

We call a (quasi)variety of algebras *irregular* if it does not contain the variety $\mathcal{S}$ of semilattices (recall that the variety of semilattices may be considered as the variety of an arbitrary type with at least one symbol of at least binary operation and without nullary operations). A (quasi)variety is *strongly irregular* if it satisfies an equation $x \star y = x$ for some binary term $x \star y$. The *regularisation* $\widetilde{\mathcal{V}}$ of a strongly irregular variety $\mathcal{V}$ is the smallest variety containg both $\mathcal{V}$ and $\mathcal{S}$. It is known that

$\widetilde{\mathcal{V}}$ consists precisely of Płonka sums of $\mathcal{V}$-algebras, (see, for instance, [26, Ch. 4]). The *quasiregularisation* $\widetilde{Q}$ of an irregular quasivariety Q is the smallest quasivariety containing both Q and $\mathcal{S}$. If Q is a variety, its quasiregularisation does not necessary coincide with its regularisation (see [1]).

The quasivarieties of barycentric algebras were described by Ignatov [12] (see also [26, Section 7.6]). In particular, the quasiregularisation $\widetilde{C}$ of the quasivariety C of convex sets is the subquasivariety of $\mathcal{B}$ defined by the quasi-identities

$$(xz\underline{p} = yz\underline{p}) \to (xy\underline{r} = xy\underline{s}) \tag{3.1}$$

for any $p, r, s \in I^o$ with $r \neq s$.

The irregular subquasivarieties of $\mathcal{B}$ form a countably infinite chain with the quasivariety $\mathcal{B}_\omega$ defined by the quasi-identities

$$(xy\underline{p} = y) \to (x = y)$$

for any $p \in I^o$ as its upper bound. Its quasiregularisation $\widetilde{\mathcal{B}_\omega}$ is defined by

$$(xz\underline{p} = z = yz\underline{p}) \to (xy\underline{r} = xy\underline{s})$$

for any $p, r, s \in I^o$ with $r \neq s$.

Note that each irregular quasivariety of barycentric algebras is covered by its quasiregularisation. The structure of algebras in the quasiregularisation $\widetilde{\mathcal{B}_\omega}$ may be described in a similar way as in the case of the quasiregularisation of a strongly irregular variety provided in [1]. However, this requires a slightly different approach to barycentric algebras.

We will first define two special congruences on a Płonka sum A of algebras A_s, all in a strongly irregular quasivariety Q, over its semilattice replica S. The first congruence σ is the kernel of the semilattice replica homomorphism from A onto S. It is called the *semilattice replica congruence*. The second congruence δ is called the *thread congruence*, and is defined as follows: for $a_s \in A_s$ and $b_t \in A_t$, we have $(a_s, b_t) \in \delta$, if there is an index $u \in S$ such that $a_s\varphi_{s,u} = b_t\varphi_{t,u}$.

The following fact generalises [1, Proposition 3.3].

Proposition 3.1. *Let Q be any strongly irregular quasivariety. Let $\mathcal{S}$ be the (quasi)variety of semilattices (considered as algebras of the same type as Q). Let A be an algebra in the variety $\mathbf{V}(Q \cup \mathcal{S})$ generated by $Q \cup \mathcal{S}$. Then the following conditions are equivalent.*

(a) *A is a subalgebra of a product of algebras in $Q \cup \mathcal{S}$: that is $A \in \mathbf{SP}(Q \cup \mathcal{S})$.*
(b) *A is a Płonka sum of Q-algebras A_s over its semilattice replica S with injective Płonka homomorphisms $\varphi_{s,t}$.*
(c) *A is a Płonka sum of Q-algebras A_s over its semilattice replica S, and the intersection of the thread congruence δ and the semilattice replica congruence σ is the identity relation on A.*

PROOF. (a) $\Rightarrow$ (b). We may assume that A is a subalgebra of a product $B \times T$, where $B \in \mathcal{Q}$ and $T \in \mathcal{S}$. The algebra $B \times T$ is a Płonka sum of subalgebras $B_t = B \times \{t\}$, for $t \in T$, over the semilattice T with isomorphisms $\varphi_{s,t}$, for $s \geq t$, as Płonka homomorphisms. Since $A \cap B_t$ is a subalgebra of A for each $t \in T$, it follows that, for $b \in B_s \cap A$ and $a \in B_t \cap A$, one has $b\varphi_{s,t} = b\varphi_{s,t} \star a \in B_t \cap A$. The Płonka homomorphisms on A are the restrictions of those on $B \times T$. Hence they are injective.

(b) $\Rightarrow$ (c). Assume that $(a, b) \in \delta \wedge \sigma$. Then there are $s, t \in S$ with $s \geq t$ such that $a, b \in A_s$ and $a\varphi_{s,t} = b\varphi_{s,t}$. Since $\varphi_{s,t}$ are injective, we get $a = b$.

(c) $\Rightarrow$ (a). Since $\delta \wedge \sigma$ is the identity relation on A, there is a subdirect embedding of A into $A/\delta \times A/\sigma$. Since $A/\delta \in \mathcal{Q}$ and $A/\sigma \in \mathcal{S}$, it follows that $A \in \mathbf{SP}(\mathcal{Q} \cup \mathcal{S})$. □

Now note that barycentric algebras may also be defined equivalently, as algebras with the set $\underline{I}$ of binary operations coming from the closed unit interval $I = [0, 1]$ of R. The operations $\underline{0}$ and $\underline{1}$ are defined by

$$xy\underline{0} = x = yx\underline{1}.$$

In fact, it is sufficient to include the operation $\underline{0}$ in the type. However, in such a case, although the identities of idempotence and skew-commutativity are still satisfied for all the operations of $\underline{I}$, there are some problems with skew-associativity. They can be avoided by a certain small change in the axiomatisation of barycentric algebras [27]: skew-associativity for $(A, \underline{I})$ may be written as

$$xy\underline{p}\,z\underline{q} = x\,yz\underline{((p \circ q) \to_\pi q)}\;\underline{p \circ q},$$

where

$$p \circ q = q1\underline{p},$$

and

$$(p \circ q) \to_\pi q = \mathit{if}\;\; p \circ q \leq q \;\mathit{then}\; 1 \;\mathit{else}\; q/(p \circ q).$$

In this way, all the algebras in $\mathcal{B}$ obtain an additional binary strongly irregular operation $\star = \underline{0}$. Note, however, that while adding the operation $\star$ to algebras in the quasivariety $\mathcal{B}_\omega$ does not change its irregularity, adding this operation to algebras in its quasiregularisation will make it irregular. Since we will use the operation $\star$ in $\mathcal{B}_\omega$-algebras to describe the structure of algebras in the quasiregularisation $\widetilde{\mathcal{B}_\omega}$, in this section, we will only consider irregular quasivarieties of barycentric algebras as classes of algebras of the extended type containing $\star = \underline{0}$. In such a situation, we will add a superscript $\star$ to the symbols of these classes. Note, however, that the quasivarieties $\mathcal{B}_\omega$ and $\mathcal{B}^\star_\omega$ are equivalent. As a binary operation, the operation $\star$ in semilattices must coincide with the multiplication and, in $\widetilde{\mathcal{B}^\star_\omega}$-algebras, the operation $\star$ becomes a left-normal band operation, (see [17] and [26, Section 4]).

Consider the quasi-identity

$$(x \star y = x \,\&\, y \star x = y \,\&\, x \star z = z \star x = z \,\&\, y \star z = z \star y = z) \to (x = y). \tag{3.2}$$

LEMMA 3.2. *Let $\mathcal{Q}$ be an irregular quasivariety of barycentric algebras. Then the following conditions hold.*

(a) *$\mathcal{Q}^\star$ satisfies the quasi-identity (3.2).*
(b) *$\mathcal{S}$ satisfies the quasi-identity (3.2).*
(c) *For each nontrivial A in $\mathcal{Q}^\star$, A^∞ does not satisfy (3.2).*

We omit the simple proof, which is similar to the proof of [1, Lemma 3.2].

Theorem 3.3. *Let A be an algebra in the variety $\mathbf{V}(\mathcal{B}^\star_\omega \cup \mathcal{S})$ generated by $\mathcal{B}^\star_\omega \cup \mathcal{S}$. Then the following conditions are equivalent.*

(a) *A satisfies the quasi-identity (3.2).*
(b) *A satisfies the equivalent conditions of Proposition 3.1, in particular,*

$$A \in \mathbf{SP}(\mathcal{B}^\star_\omega \cup \mathcal{S}).$$

(c) *A belongs to the quasiregularisation $\widetilde{\mathcal{B}^\star_\omega}$ of $\mathcal{B}^\star_\omega$.*

Proof. (a) $\Rightarrow$ (b). First note that if A satisfies the quasi-identity (3.2) and, for $s \geq t$, one has $a_s\varphi_{s,t} = b_s\varphi_{s,t}$, then, substituting $a_s, b_s, a_s\varphi_{s,t}$ for x, y, z in (3.2), one concludes that $a_s = b_s$. Then use Proposition 3.1.

(b) $\Rightarrow$ (a). By Lemma 3.2, every member of $\mathcal{B}^\star_\omega \cup \mathcal{S}$ satisfies (3.2). Since the satisfaction of quasi-identities is inherited by both subalgebras and products, $\mathbf{SP}(\mathcal{B}^\star_\omega \cup \mathcal{S})$ also satisfies it.

(a) $\Leftrightarrow$ (c). Let $\mathcal{Q}$ be the subquasivariety of $\mathbf{V}(\mathcal{B}^\star_\omega \cup \mathcal{S})$ defined by the quasi-identity (3.2). Then Lemma 3.2 and the previous equivalence imply the inclusions

$$\widetilde{\mathcal{B}^\star_\omega} \subseteq \mathcal{Q} = \mathbf{SP}(\mathcal{B}^\star_\omega \cup \mathcal{S}) \subseteq \widetilde{\mathcal{B}^\star_\omega}. \qquad \square$$

Corollary 3.4. *Let $\mathcal{Q}$ be a subquasivariety of $\mathbf{V}(\mathcal{B}^\star_\omega \cup \mathcal{S})$. Then the following conditions are equivalent.*

(a) *$\mathcal{Q} = \widetilde{C^\star}$.*
(b) *$\mathcal{Q}$ is defined by the quasi-identities (3.1) and (3.2).*
(c) *$\mathcal{Q}$ consists of Płonka sums of convex sets with injective Płonka homomorphisms.*
(d) *$\mathcal{Q} = \mathbf{SP}(C^\star \cup \mathcal{S})$.*

Płonka sums of convex sets with injective Płonka homomorphisms will be called *injective Płonka sums*.

4. Duality

Let $\mathcal{A}$ and $\mathcal{X}$ be categories. The category $\mathcal{A}$ is usually a class of algebras, for instance a quasivariety of algebras, considered as a category with homomorphisms as arrows, while $\mathcal{X}$ is a concrete category of representation spaces for $\mathcal{A}$-algebras. We say that there is a *dual equivalence*, or simply *duality*, between $\mathcal{A}$ and $\mathcal{X}$ if there are contravariant functors

$$D : \mathcal{A} \to \mathcal{X} \quad \text{and} \quad E : \mathcal{X} \to \mathcal{A} \tag{4.1}$$

such that both $DE = E \circ D$ and $ED = D \circ E$ are naturally isomorphic with the corresponding identity functors on $\mathcal{A}$ and $\mathcal{X}$, respectively (see, for example, [13] and [14, page 91]).

In many cases, the functors of the duality (4.1) are represented by a *schizophrenic object* [6], [13, Section VI.4.1]. The schizophrenic object T appears simultaneously as an object $\underline{T}$ of $\mathcal{A}$ and as an object $\underset{\sim}{T}$ in $\mathcal{X}$. The underlying sets of $\underline{T}$ and $\underset{\sim}{T}$ coincide (with T).

The functors D and E are defined on objects and morphisms by

$$\begin{array}{cccc} A & & \mathcal{A}(A,\underline{T}) & fx : A \to B \to \underline{T} \\ \downarrow f & \overset{D}{\mapsto} & \uparrow f^D & \uparrow \\ B & & \mathcal{A}(B,\underline{T}) & x : B \to \underline{T}. \end{array}$$

$$\begin{array}{cccc} X & & \mathcal{X}(X,\underset{\sim}{T}) & \varphi\alpha : X \to Y \to \underset{\sim}{T} \\ \downarrow \varphi & \overset{E}{\mapsto} & \uparrow \varphi^E & \uparrow \\ Y & & \mathcal{X}(Y,\underset{\sim}{T}) & \alpha : Y \to \underset{\sim}{T}. \end{array}$$

In left-handed notation,

$$D(A) = A^D = \mathcal{A}(A,\underline{T}) \quad \text{and} \quad D(f) = f^D.$$

The natural isomorphisms e of DE with the identity functor and ε of ED with the identity functor are given by the evaluations

$$\begin{aligned} e_A : A \to A^{DE}; &\quad a \mapsto (ae_A : x \mapsto ax), \\ \varepsilon_X : X \to X^{ED}; &\quad x \mapsto (x\varepsilon_X : \alpha \mapsto x\alpha). \end{aligned}$$

In left-handed notation,

$$e_A(a)(x) = x(a) \quad \text{and} \quad \varepsilon_X(x)(\alpha) = \alpha(x).$$

Note that the 'natural dualities' considered in [3, 6, 8, 9, 18] are of a similar type. However, they require finite schizophrenic objects and satisfaction of certain additional conditions that are not necessarily obtained for the cases considered in this paper.

5. Duality for semilattices

One example of the duality (4.1) represented by a finite schizophrenic object is fundamental. This is the duality for semilattices given by

$$C : \mathcal{S} \to \mathcal{Z} \quad \text{and} \quad F : \mathcal{Z} \to \mathcal{S}. \tag{5.1}$$

Here $\mathcal{S}$ is the variety of semilattices, while $\mathcal{Z}$ is the category of all bounded compact topological semilattices carrying a Boolean topology. The $\mathcal{Z}$-morphisms are continuous homomorphisms of bounded semilattices. The two-element meet-semilattice $\underline{T} = (\{0 \leq 1\}, \wedge)$ is an object of $\mathcal{S}$, and $\underset{\sim}{T} = (\{0 \leq 1\}, \wedge, 0, 1, \Theta)$, where Θ

is the discrete topology and is an object of $\mathcal{Z}$. The variety $\mathcal{S}$ is the closure $\mathsf{ISP}(\underline{\mathrm{T}})$ of the singleton class $\{\underline{\mathrm{T}}\}$ under the closure operations P of power, S of subalgebra and I of isomorphic copy. The category $\mathcal{Z}$ is the closure $\mathsf{ISP}(\underset{\sim}{\mathrm{T}})$ under the closure operations P of power, S of closed substructure and I of isomorphic (homeomorphic) copy. This duality is a modification of the very well-known Hofmann–Mislove–Stralka duality (called also Pontryagin duality for semilattices) between the category $\mathcal{S}_0$ of semilattices with zero and the category $\mathcal{Z}_0$ of compact topological semilattices with zero carrying a Boolean topology (see [8, 11] and [13, Section VI.3.6] and also the brief description of the duality (4.1) provided in [25]).

Note also, that for a semilattice H, the set HC may be described in terms of walls. Let HW denote the set of walls of H. There is a natural $\mathcal{Z}$-isomorphism

$$HC \to HW; \quad \chi \mapsto \chi^{-1}(1).$$

All walls of a finite semilattice are principal.

Recall that each $\mathcal{Z}$-space is, in fact, an algebraic lattice [11]. For a $\mathcal{Z}$-space G, the set GK of compact elements forms a join-semilattice. Then there is a natural isomorphism

$$GF \to GK; \quad \theta \mapsto \inf \theta^{-1}(1).$$

For a semilattice H, an element $\Theta \in HW$ is compact if and only if it is principal. Then there is a natural isomorphism

$$H \to HWK; \quad h \mapsto [h].$$

For a $\mathcal{Z}$-space G, there is a natural isomorphism

$$G \to GKW; \quad g \mapsto GK \cap \downarrow g.$$

For more details see [25].

6. Duality for convex polytopes

Convex polytopes are convex sets with a finite nonempty set of vertices. Considered as barycentric algebras, they are (finitely) generated by their sets of vertices. The set $G(C)$ of vertices of a convex polytope C forms a minimal set of its generators. The paper [21, Theorem 6.5] establishes a duality between the category $\mathcal{P}$ of real convex polytopes considered as (cancellative) barycentric algebras and the category $\widehat{\mathcal{P}}$ of certain polytopes with constants (see also [28]). The category $\widehat{\mathcal{P}}$ is the class of intersections of some polytopes isomorphic to hypercubes, considered as barycentric algebras with additional constant operations and with the corresponding homomorphisms preserving these constants.

The duality is represented as

$$D : \mathcal{P} \to \widehat{\mathcal{P}} \quad \text{and} \quad E : \widehat{\mathcal{P}} \to \mathcal{P}, \tag{6.1}$$

and is given by an infinite schizophrenic object T, where T is the unit real interval I. In particular, for a polytope C, one has $CD = \mathcal{P}(C, \underline{T})$ and for $C \in \widehat{\mathcal{P}}$ one has $CE = \widehat{\mathcal{P}}(C, \underset{\sim}{T})$.

Each k-dimensional polytope C is the union $\bigoplus_{i \in I} C_i$ of k-dimensional simplices C_i, each generated by a $(k+1)$-element subset of vertices of C. Each $C_i D = D(C_i)$ is isomorphic to the $(k+1)$-dimensional hypercube I^{k+1}. Moreover,

$$CD = D\Big(\bigoplus_{i \in I} C_i\Big) \cong \bigcap_{i \in I} D(C_i).$$

The duality makes use of a strong connection between three structures: namely, a k-dimensional polytope, the k-dimensional simplex and the k-dimensional affine space. In a sense, three dualities are carried out in parallel fashion: for simplices, for polytopes and for barycentric algebra reducts of finitely generated real affine spaces. The first dual of an affine space $\mathbb{R}^k$ is isomorphic to $\mathbb{R}^{k+1}$, and the first dual CD of a k-dimensional polytope C embeds as a subreduct into $\mathbb{R}^{k+1}$. The set CD contains the points $\bar{0} = (0, \ldots, 0)$, $\bar{1} = (1, \ldots, 1)$ and hence all $\bar{r} = (r, \ldots, r)$ for $r \in I$. They are then considered as the constant operations of CD. The three dualities do not involve any topology or additional relations. The results obtained in [21] carry over, *mutatis mutandis*, to convex polytopes over subfields R of the field $\mathbb{R}$.

Finally, note that the duality (6.1) for the class of convex polytopes over R would not work for a broader class of convex sets.

Example 6.1. Consider the closed interval $(I, \underline{I}^o)$. Note that the homomorphisms from $(I, \underline{I}^o)$ to $(I, \underline{I}^o)$ are in one-to-one correspondence with pairs of elements of I, the pair $(0h, 1h)$ for each homomorphism $h : I \to I$. Hence, it is easy to see that the first dual ID of I is isomorphic to the square $(I^2, \underline{I}^o)$. However, we will get the same square when we replace the closed interval I by the open interval I^o and form the first dual $I^o D$ of I^o.

In his doctoral dissertation [28], Ślusarski established dualities for certain classes of real convex sets by considering different convex sets as schizophrenic objects. However these dualities are rather complicated and do not provide a uniform approach to a duality for more general classes of real convex sets. In this paper, we take a different approach.

7. Duality for open polytopes

In what follows, we fix a given subfield R of $\mathbb{R}$, and consider convex subsets of affine R-spaces (and, in particular, convex polytopes) as barycentric algebras $(A, \underline{I}^o)$ with barycentric algebra homomorphisms as morphisms.

As observed in [21, Section 3], each polytope A embeds as a subreduct into the uniquely defined canonical affine R-space $R(A)$ that it generates.

Definition 7.1. A convex set A is called a *quasipolytope* if its closure $\overline{A}$ in the canonical affine space extension $R(A)$ is a polytope. If, additionally, A coincides with the interior of $\overline{A}$ in $R(A)$, then we say that A is an *open polytope*.

A duality for open polytopes may be developed alongside the duality for (closed) polytopes, as follows. Instead of the schizophrenic object $T = I$, we will take as T the open unit interval I^o of R. The category of open polytopes is denoted by $\mathcal{P}^o$, and the category of representation spaces is the category $\widehat{\mathcal{P}}^o$ of interiors of members of $\widehat{\mathcal{P}}$ in their canonical affine space extensions. Algebras of the latter category are considered as barycentric algebras with constants $\bar{r}$ for all $r \in I^o$.

We will show that the duality is represented as

$$D^o : \mathcal{P}^o \to \widehat{\mathcal{P}}^o \quad \text{and} \quad E^o : \widehat{\mathcal{P}}^o \to \mathcal{P}^o, \tag{7.1}$$

and is given by the schizophrenic object $T = I^o$. In particular, for an open polytope C, one has $CD^o = \mathcal{P}^o(C, \underline{T})$ and for $C \in \widehat{\mathcal{P}}^o$ one has $CE^o = \widehat{\mathcal{P}}^o(C, \underset{\sim}{T})$.

First, let us note that there is a one-to-one correspondence between open and closed polytopes. The closure $\overline{A}$ of an open polytope A is a polytope, and the interior B^o in $R(B)$ of a polytope B is an open polytope. Next we will show, in a purely algebraic way, that there is also a one-to-one correspondence between surjective homomorphisms connecting two closed polytopes and surjective homomorphisms connecting two open polytopes.

Closures of convex sets in finite-dimensional affine R-spaces were characterised in algebraic fashion in [5]. In the case when R is a field, they can be described as follows. For a convex set C with elements a and b, the pair (a, b) is *eligible* (denoted by $(a, b) \in E(C)$) if, for any x in the interval generated by a and b with $x \neq a, b$, one has $xb\underline{s^{-1}} \in C$ for an arbitrarily chosen $s \in I^o$. It was shown in [5] that the definition does not depend on the choice of s. So, in the case of a field, one can choose, for instance, $s = 1/2$. Then the closure $\overline{C}$ of C is described as the set

$$\{ab\underline{2} \mid (a, b) \in E(C)\}. \tag{7.2}$$

Lemma 7.2. *Let A and B be two open polytopes, with closures $\overline{A}$ and $\overline{B}$. Then each barycentric algebra homomorphism $h : A \to B$ from A onto B extends uniquely to a barycentric algebra homomorphism $\overline{h} : \overline{A} \to \overline{B}$ from $\overline{A}$ onto $\overline{B}$.*

Proof. Describe the closures $\overline{A}$ and $\overline{B}$ as in (7.2) above. Note that, for each eligible pair (a, b) of A, the pair (ah, bh) is eligible in B. Then extend the homomorphism h to $\overline{h}$ by defining its values on elements of the form $ab\underline{2}$, which do not belong to A, in the only possible way as

$$(ab\underline{2})\overline{h} = ah\, bh\underline{2}.$$

Then, clearly, $\overline{h}$ is a homomorphism from $\overline{A}$ onto $\overline{B}$. □

To show that, for each surjective homomorphism between (closed) polytopes, there is a uniquely defined surjective homomorphism between their interiors, we first note the following.

Lemma 7.3. *Let A and B be nontrivial polytopes. Let $h : A \to B$ be a homomorphism from A onto B. Then, for each $a \in A^o$, one has $ah \in B^o$.*

PROOF. Let $G(A) = \{f_1, \ldots, f_m\}$ be the set of vertices of A, and let $G(B) = \{g_1, \ldots, g_n\}$ be the set of vertices of B. Note that, since h is surjective, it follows that each vertex g_i of B is the image $f_j h$ of a generator f_j of A.

Then note that if an element $a \in A$ has the form $a = t(f_{i_1}, \ldots, f_{i_k})$ for some convex combination t and generators $f_{i_1}, \ldots, f_{i_k}$ of A, then $ah = t(f_{i_1}h, \ldots, f_{i_k}h)$. In particular, for each element $b \in A^o$, there is a convex combination s such that $b = s(f_1, \ldots, f_m)$ (see [26, Section 5.8]). Then $bh = s(f_1 h, \ldots, f_m h)$ and $\{f_1 h, \ldots, f_m h\} \supseteq G(B)$. Hence, there is a convex combination s' such that $bh = s(f_1 h, \ldots, f_m h) = s'(g_1, \ldots, g_n)$. It is clear that $b \in B^o$. □

COROLLARY 7.4. *Let A and B be nontrivial polytopes. Then each homomorphism $h : A \to B$ from A onto B restricts to a homomorphism $h^o : A^o \to B^o$ from A^o onto B^o.*

PROOF. By Lemma 7.3, we already know that $h(A^o) \subseteq B^o$. By Lemma 7.2, the restriction $h : A^o \to h(A^o)$ extends uniquely to $\overline{h} : A = \overline{A^o} \to \overline{h(A^o)} \subseteq B$. Note that the images under h of all the vertices of A belong to $\overline{h(A^o)}$, and hence the set $G(B)$ of vertices of B is also contained in $\overline{h(A^o)}$. It follows that $\overline{h(A^o)} = B$, and hence $h(A^o) = B^o$. □

Let us also mention some further properties of walls of polytopes. First, recall that each wall of a polytope A is again a polytope, and that it is a principal wall. However, it is also generated (as a wall) by a (finite) subset of the set $G(A)$ of the vertices. In particular, the wall generated by an element $a \in A^o$ is the whole polytope A (see [26, Section 5.8]).

LEMMA 7.5. *Let A and B be nontrivial polytopes. Let $h : A \to B$ be a homomorphism from A onto B. Then, for each wall W of B, the preimage $h^{-1}(W)$ is a wall of A.*

PROOF. We keep the notation from the proof of Lemma 7.3. For each g_i in $G(B)$, let G_i be the set $\{f_{i_1}, \ldots, f_{i_{k(i)}}\}$ of vertices in $G(A)$ with $f_{i_p} h = g_i$ for $p = 1, \ldots, k(i)$. Then it is easy to see that, for the wall $[G_i]$ generated by G_i, one has $h([G_i]) = \{g_i\}$. Hence, the preimage of a principal wall of B consisting of one vertex is a wall. Now consider a wall $H = [g_{j_1}, \ldots, g_{j_l}]$ of B generated by vertices $g_{j_1}, \ldots, g_{j_l}$ and the wall $W = [\bigcup(G_{j_p} \mid p = 1, \ldots, l)]$ of A generated by the union of all G_{j_p}. An obvious calculation shows that $h(W) = H$. Consequently, the preimage of the wall H is a wall. □

Let us note that, for a wall W of A, the image $h(W)$ does not need to be a wall.

PROPOSITION 7.6. *Let A be a polytope. The elements $h : A \to I$ of AD which take some vertices of A to zero or one form the boundary $\partial(AD)$ of AD.*

PROOF. Assume that A is a k-dimensional polytope with $n + 1 \geq k$ vertices $g_0, g_1, \ldots, g_k, \ldots, g_n$. Recall that A is a union of k-dimensional simplices A_i, each generated by a $(k + 1)$-element subset of vertices of A. The dual $A_i D$ of A_i is isomorphic to the $(k + 1)$-dimensional hypercube I^{k+1}. The dual AD is the intersection of the duals $A_i D$. If the dimension of AD is equal to, say, m, then the boundary of AD

consists of walls of dimension $m-1$, and is determined by the vertices of AD. The vertices of AD are given by the homomorphisms of AD which take some generators g_i of the polytope A to zero or one. □

Theorem 7.7. *There is a duality between the categories $\mathcal{P}^o$ and $\widehat{\mathcal{P}}^o$ given by the schizophrenic object I^o.*

Proof. Let A be an open polytope. Consider the sets $AD^o = \mathcal{P}^o(A, I^o)$ and $\bar{A}D = \mathcal{P}(\overline{A}, I)$. It follows, by Lemma 7.2, that any element $h : A \to I^o$ of AD^o extends uniquely to the homomorphism $\overline{h} : \overline{A} \to \overline{I^o} = I$ of $\overline{A}D$. Let

$$f : AD^o \to \overline{A}D; \quad (h : A \to I^o) \mapsto (\overline{h} : \bar{A} \to I).$$

Note that f is an embedding of barycentric algebras. Moreover,

$$f(AD^o) = \{\overline{h} : \overline{A} \to I \mid g_i\overline{h} \neq 0, 1\},$$

for all generators $g_0, \ldots, g_n$ of $\overline{A}$. By Proposition 7.6, it follows that

$$f(AD^o) \cong \overline{A}D \setminus \partial(\overline{A}D).$$

This allows us to identify AD^o with the subalgebra $f(AD^o)$ of $\overline{A}D$, which is the interior of $\overline{A}D$.

Now the second dual of $\overline{A}$ is given by the algebra $\overline{A}DE = \widehat{\mathcal{P}}(\overline{A}D, I)$ of homomorphisms from $\overline{A}D$ to I preserving constants, and is isomorphic to $\overline{A}$. Using an argument similar to that in the first part of the proof, we may easily observe that the restriction of $\overline{A}DE$ to the interior $f(AD^o)$ of $\overline{A}D$ consists of homomorphisms with images contained in I^o. This gives

$$\overline{A}DE|_{f(AD^o)} \cong (\overline{A})^o \cong A$$

and, consequently,

$$AD^oE^o \cong A.$$

To complete the proof, one uses the duality for convex polytopes described in Section 6 and a similar argument to that used above. □

8. Duality for injective Płonka sums of polytopes

Let us recall that injective Płonka sums of polytopes are Płonka sums with injective Płonka homomorphisms, and that they are barycentric algebras. In particular, we will be interested in such Płonka sums over finite semilattices, and we will consider them as members of the category $\widetilde{C^\star}$. The class of injective Płonka sums of polytopes over finite semilattices will be denoted by $\mathcal{PP}$. A duality for such barycentric algebras may be described using two dualities: the duality for polytopes described in Section 6 and the duality for semilattices described in Section 5. The role of a schizophrenic object will be taken by the extended unit interval $T = I^\infty$. Note, however, that I^∞ is not an injective Płonka sum. Note also that the methods developed earlier concerning dualities for Płonka sums (see [7, 24, 25]) cannot be directly applied in our case, since the categories considered in this section do not satisfy all the assumptions of those works. The category of corresponding representation spaces will be denoted by $\widehat{\mathcal{PP}}$. The duality we look for should have the form

$$\bar{D} : \mathcal{PP} \to \widehat{\mathcal{PP}} \quad \text{and} \quad \bar{E} : \widehat{\mathcal{PP}} \to \mathcal{PP}.$$

For a Płonka sum A, its dual $A\bar{D}$ will consist of homomorphisms from A to $\underline{T}$. By Corollary 2.2, the algebra $A\bar{D}$ is again a barycentric algebra. It is equipped with certain constant operations, to be described later. The category $\widehat{\mathcal{PP}}$ will consist of barycentric algebras $A\bar{D}$ belonging to $\widetilde{C^\star}$ with constants and with homomorphisms preserving these constants. For $X \in \widehat{\mathcal{PP}}$, its dual $X\bar{E}$ will consist of homomorphisms from X to $\underset{\sim}{T}$ that preserve the constant operations.

As polytopes are considered, in this section, as members of the category $C^\star$, each homomorphism from A to $\underline{T}$ belonging to $A\bar{D}$ (and, similarly, each homomorphism from $A\bar{D}$ to $\underset{\sim}{T}$ belonging to $A\bar{D}\bar{E}$) maps convex subsets to convex subsets.

Lemma 8.1. *Let $A = \sum_{s\in S} A_s$ be a member of $\mathcal{PP}$. Then the following conditions hold.*

(a) *Each wall of A has the form $W_s = \sum_{t\geq s} A_t$ for some $s \in S$.*
(b) *For each $h \in A\bar{D}$, the elements of A mapped to I form a wall of A.*
(c) *There is a one-to-one correspondence between the walls of A and the homomorphisms from A to I^∞.*

Proof. We omit the obvious proofs of (a) and (b). To show that (c) holds, it is sufficient to note that a mapping $h : A \to I^\infty$ such that $h|_{W_s}$ is a homomorphism, $h(W_s) \subseteq I$ and $h(A \setminus W_s) = \{\infty\}$ is a barycentric algebra homomorphism. □

For a given $s \in S$, the homomorphism h corresponding to the wall W_s described in the proof of Lemma 8.1 is said to be *determined by* s, and the set of all such homomorphisms is denoted by H_s.

Lemma 8.2. *Let $A = \sum_{s\in S} A_s$ be a member of $\mathcal{PP}$. For $t \geq s$, let $\varphi_{t,s} : A_t \to A_s$ be the Płonka homomorphism. For each $h \in H_s$ and $x_t \in A_t$,*

$$x_t h = x_t \varphi_{t,s} h.$$

We omit the proof, which follows from a general description of homomorphisms between Płonka sums (see [7, 25]).

Corollary 8.3. *The set H_s of homomorphisms determined by an element $s \in S$ forms an algebra isomorphic to the algebra $A_s D$ of all homomorphisms from A_s to the unit interval I. It is a subalgebra of the first dual $A\bar{D}$ of A.*

Proof. Let $W_s = \sum_{t\geq s} A_t$ be a wall of A. Let

$$H_s = \{h : A \to I^\infty \mid h(W_s) \subseteq I \text{ and } h(A \setminus W_s) = \{\infty\}\}$$

be the set of homomorphisms determined by s. Note that, for each $h \in H_s$, the restriction $h|_{A_s} : A_s \to I$ belongs to $A_s D$. Then use Lemma 8.2 to show that each homomorphism of $A_s D$ extends uniquely to a homomorphism $h : A \to I^\infty$. It follows that the mapping

$$\alpha : H_s \to A_s D : h \mapsto h|_{A_s}$$

is a bijection. We omit a standard proof showing that α is a barycentric algebra homomorphism. □

The following lemma is a consequence of the fact that, for each s in S, the algebra $A_s D$ has a trivial semilattice replica. See, also, the more general results of [25] concerning so-called semilattice representations.

Lemma 8.4. *Let $A = \sum_{s \in S} A_s$ be a member of $\mathcal{PP}$. The semilattice replica R of $A\bar{D}$ is isomorphic to the first dual SC of the semilattice replica S of A under the functor C of the semilattice duality.*

Note that, since S is finite, its dual SC is a lattice isomorphic to the lattice obtained from S by adjoining a new greatest element corresponding to the improper wall S.

Lemmas 8.1–8.4 and Corollary 8.3 provide the proof of the following theorem.

Theorem 8.5. *Let $A = \sum_{s \in S} A_s$ be a member of $\mathcal{PP}$. Then the barycentric algebra $A\bar{D}$ is isomorphic to the Płonka sum $\sum A_s D$ of the first duals $A_s D$ of the polytopes A_s over the first dual SC of S.*

Recall that the duals $A_s D$ of polytopes A_s are polytopes with constants $\bar{0}$ and $\bar{1}$ (and hence with all $\bar{r}$ for $r \in I$). It follows that $A\bar{D}$ is, in fact, a Płonka sum with constants (see [16, 17]). The constants are chosen from the summand $A_S D$, determined by the largest element S of $R = SC$. (Note that the constants of other summands are obtained as homomorphic images (under Płonka homomorphisms) of the constants in the fibre $A_S D$.)

To describe the second dual, we will need one more basic constant operation $\bar{\infty}$ in the Płonka sum $A\bar{D}$, corresponding to the homomorphism $h : A \to \{\infty\}$. This is connected with the fact that the first dual of the semilattice replica S of A is a bounded semilattice. (See also similar phenomena in the other cases of dualisations of Płonka sums considered earlier, in [7, 9, 25].) In this way $A\bar{D}$ becomes the barycentric algebra with three constants $\bar{0}$, $\bar{1}$ and $\bar{\infty}$. The schizophrenic object I^∞ will also be considered as a barycentric algebra with three constants 0, 1 and ∞.

As the category $\widehat{\mathcal{PP}}$, we take the class of barycentric algebras $A\bar{D}$ with the three constant operations described above and homomorphisms respecting these constants.

A reasoning, similar to the one leading to Theorem 8.5 and the dualities for polytopes and for semilattices, provide the second dual of the category $\mathcal{PP}$.

Theorem 8.6. *Let $A = \sum_{s \in S} A_s$ be a member of $\mathcal{PP}$. Then the barycentric algebra $A\bar{D}\bar{E}$ is isomorphic to the Płonka sum $\sum A_s DE$ of the second duals $A_s DE$ of the polytopes A_s over the second dual SCF of S, and hence is isomorphic to $A = \sum_{s \in S} A_s$.*

The main result of this section follows by Theorem 8.6 and [21, Theorem 6.5].

Corollary 8.7. *There is a duality between the categories $\mathcal{PP}$ and $\widehat{\mathcal{PP}}$.*

9. Duality for quasipolytopes – first dual

In this section, we consider quasipolytopes as barycentric algebras from the class C with the set $\underline{I}^o$ of basic operations.

Our aim is to find a duality for the class $\mathcal{QP}$ of quasipolytopes. We look for a duality similar to (6.1) given by a (possibly extended) schizophrenic object T. We will discuss later which barycentric algebra T will be suitable for our purpose. The class of appropriate representation spaces will be denoted by $\widehat{\mathcal{QP}}$. We expect the duality to have the form

$$\widetilde{D} : \mathcal{QP} \to \widehat{\mathcal{QP}} \quad \text{and} \quad \widetilde{E} : \widehat{\mathcal{QP}} \to \mathcal{QP},$$

with $A\widetilde{D}$ as the set of barycentric algebra homomorphisms from a quasipolytope A to $\underline{\mathrm{T}}$, and $X\widetilde{E}$ as the set of morphisms from the representation space X to $\underset{\sim}{\mathrm{T}}$.

By Corollary 2.2, the algebra $A\widetilde{D}$ is again a barycentric algebra, although not necessarily a convex set. In describing our new duality, we will use the dualities described earlier: that is, the duality (6.1) for polytopes given in Section 6, the duality (7.1) for open polytopes of Section 7 and the duality (5.1) for semilattices considered in Section 5, as well as the duality for Płonka sum of polytopes (considered as algebras in the class $C^\star$) described in Section 8. We could already observe that neither the duality for polytopes nor the duality for open polytopes could be directly extended to the class containing both closed and open polytopes. (None of them 'recognises' both closed and open sets.) In particular, neither I nor I^o could play the role of a schizophrenic object T in the case in which we are interested, in this section.

The structure of quasipolytopes is crucial for our further investigations. As recalled in Section 3, each barycentric algebra A is a semilattice sum $\bigcup_{s\in S} A_s$ of open convex sets A_s over its semilattice replica S. The semilattice replica of A is the semilattice of its principal walls. (Note that walls of a finite semilattice are all principal.) Moreover, A is a subalgebra of the Płonka sum $\sum_{s\in S} E_s$ of certain convex extensions E_s (so-called envelopes of A_s (see [26, Section 7])) over its semilattice replica S. We will use a variant of this theorem in the current section. First, recall that if A is a convex set, then the envelopes E_s are subalgebras of A. In the case where A is a quasipolytope, the Płonka sum of the envelopes E_s has an even more transparent description.

Proposition 9.1. *Let A be a quasipolytope and a semilattice sum $\bigcup_{s\in S} A_s$ of open polytopes A_s over its semilattice replica S. Then the following conditions hold.*

(a) *All walls of A have the form $W_s = \bigcup(A_t \mid t \geq s)$ for $s \in S$. Moreover, each W_s is the smallest wall containing A_s.*
(b) *For each $s \in S$, the envelope E_s of A_s coincides with the wall W_s.*
(c) *The quasipolytope A embeds into the Płonka sum $\sum_{s\in S} W_s$ of the envelopes W_s of A_s, with embedding of walls as injective Płonka homomorphisms.*

Note that a wall W_s of a quasipolytope A is again a quasipolytope, and the closure $\bar{W}_s$ of W_s is a polytope, coinciding with the closure $\bar{A}_s$ of A_s.

We will also use another Płonka sum containing A as a subalgebra. First, note the following.

Proposition 9.2. *Let a quasipolytope A be a semilattice sum $\bigcup_{s\in S} A_s$ of open polytopes A_s over its semilattice replica S and a subalgebra of the Płonka sum $\sum_{s\in S} W_s$ of its walls. Then the following conditions hold.*

(a) *The Płonka sum $\sum_{s\in S} W_s$ extends to the Płonka sum $\sum_{s\in S} \bar{W}_s = \sum_{s\in S} \bar{A}_s$.*
(b) *The quasipolytope A may be reconstructed from the sum $\sum_{s\in S} \bar{A}_s$ by removing the boundaries of the closures $\bar{A}_s$.*

Proof. The first statement follows directly by use of Proposition 9.1 and Lemma 7.3. The second is obvious. □

Let us note that the decomposition of a quasipolytope A into the (disjoint) sum of open polytopes A_s is determined by its walls, and thus by the elements of the semilattice replica S. There is a one-to-one correspondence between any two of the three structures: the semilattice sums $\bigcup_{s\in S} A_s$ of open polytopes A_s, the Płonka sum $\sum_{s\in S} W_s$ of walls of A and the Płonka sum $\sum_{s\in S} \bar{A}_s$ of (closed) polytopes $\bar{A}_s$. In each of the three cases, the semilattice S is the semilattice replica of the sum. Moreover, the lattices of walls of $A = \bigcup_{s\in S} A_s$ and of $\sum_{s\in S} \bar{A}_s$ are isomorphic.

As the Płonka sum $\sum_{s\in S} \bar{A}_s$ has injective Płonka homomorphisms, the closures $\bar{A}_s$ are polytopes and S is a finite semilattice, it follows that Corollary 8.7 may be applied to this Płonka sum to, in particular, provide a dual space for this sum.

Before we describe our duality, let us consider the following example.

Example 9.3. As quasipolytopes with operations $\underline{I}^o$, consider the intervals I, I^o and $I^{\triangleleft} := [0, 1)$. We will try to describe the first and second duals of these algebras by taking $T = I^\infty$ as an extended schizophrenic object.

We start with the interval I. Note that the semilattice replica of I is the (meet) semilattice $S = \{a, b, 0\}$ with incomparable a and b and smallest element zero. Then I decomposes as the semilattice sum of $I_a = \{0\}$, $I_b = \{1\}$ and $I_0 = I^o$, and embeds into the Płonka sum $\sum_{s\in S} \bar{I}_s$ of the closures $\bar{I}_a = I_a$, $\bar{I}_b = I_b$ and $\bar{I}_0 = I$ over S. The homomorphisms h from $A = I$ to I^∞ consist of the four subalgebras:

- $A_1 = \{h : I \to I\}$;
- $A_l = \{h \mid 0 \mapsto I \text{ and } x \mapsto \infty \text{ for } x \in (0, 1]\}$;
- $A_r = \{h \mid 1 \mapsto I \text{ and } x \mapsto \infty \text{ for } x \in [0, 1)\}$; and
- $A_\infty = \{h : I \mapsto \infty\}$.

Then A_1 is isomorphic to the dual $AD = \mathcal{P}(A, I) = \mathcal{P}(I, I)$, and hence to I^2 (see [21]). The subalgebra A_l (and, similarly, the subalgebra A_r) can be considered as the dual $\mathbf{1}D$ of one-element polytope $\mathbf{1}$, and hence is isomorphic to I. The set $\{1, l, r, \infty\}$ forms the (meet) semilattice replica R of $I\widetilde{D}$ with smallest element ∞, largest element one and two incomparable atoms l and r. The semilattice R is isomorphic to the dual SC of the semilatice replica S of I. Finally, $I\widetilde{D}$ can be considered as the Płonka sum of A_1, A_l, A_r and A_∞ with Płonka homomorphisms given by

$$\varphi_{1,l} : (x, y) \mapsto (x, \infty), \quad \varphi_{1,r} : (x, y) \mapsto (\infty, y),$$
$$\varphi_{l,\infty} : (x, \infty) \mapsto (\infty, \infty), \quad \varphi_{r,\infty} : (\infty, y) \mapsto (\infty, \infty).$$

(Note that A_∞ can be considered as the dual of the empty algebra.) The constants of this Płonka sum are given by $\bar{0} = (0, 0)$ and $\bar{1} = (1, 1)$. (They generate other constants of the form $\bar{r} = (r, r)$ for $r \in I^o$.) Note, also, that $I\widetilde{D}$ is isomorphic to $I^\infty \times I^\infty$.

A similar analysis can be made for I^o and $I^\triangleleft$. As the first duals, one obtains subalgebras of the dual $I\widetilde{D}$. The algebra $I^o\widetilde{D}$ is the Płonka sum of A_1 and A_∞, and the algebra $I^\triangleleft\widetilde{D}$ is the Płonka sum of A_1, A_l and A_∞, all considered as barycentric algebras with constants. Note that the three duals are pairwise distinct.

The three algebras we obtained as the first duals are Płonka sums of polytopes with three constant operations $\bar{0}, \bar{1}, \bar{\infty}$. By applying the results of Section 8, we can see that the duals of these duals are given, respectively, by the Płonka sum $\sum_{s\in S} \bar{I}_s$, the Płonka sum $\sum_{s\in\{1,\infty\}} \bar{I}_s$ and the Płonka sum $\sum_{s\in\{1,l,\infty\}} \bar{I}_s$. In particular, none of them is isomorphic with the original algebra being dualised.

Example 9.3 shows that either I^∞ is not an appropriate candidate for a schizophrenic object, or the structure of the representation space should be chosen differently.

Now consider the extended open interval $(I^o)^\infty$ as a candidate for an extended schizophrenic object T. First, we describe the structure of the barycentric algebra $\mathcal{B}(A, (I^o)^\infty)$ of homomorphisms from a quasipolytope A to $(I^o)^\infty$. Note that open convex sets have no homomorphisms onto a nontrivial semilattice. This has two important consequences. For each $h \in \mathcal{B}(A, (I^o)^\infty)$ and $s \in S$, the image $h(A_s)$ is either contained in I^o or consists of one element ∞. Moreover, each homomorphism h determines a homomorphism from the semilattice replica S into the two-element semilattice $\underset{\sim}{\mathbf{2}}$, the semilattice replica of $(I^o)^\infty$. More precisely, in analogy with properties of walls of the Płonka sum $\sum_{s\in S} \bar{A}_s$, the following properties hold.

Lemma 9.4. *Let A be a quasipolytope. Then the following conditions hold.*

(a) *A mapping $h : A \to (I^o)^\infty$ such that $h|_{W_s}$ is a homomorphism, $h(W_s) \subseteq I^o$ and $h(A \setminus W_s) = \{\infty\}$, is a barycentric algebra homomorphism.*
(b) *For each barycentric algebra homomorphism $h : A \to (I^o)^\infty$, the elements mapped to I^o form a wall of A.*

In particular, there is a one-to-one correspondence between walls of A and homomorphisms of $A\widetilde{D}$.

We omit the obvious proof. Similarly, as in the case of Płonka sums of polytopes, for a given $s \in S$, homomorphisms described in Lemma 9.4(a) are said to be *determined by s*, and the set of all such homomorphisms is denoted by H_s.

Lemma 9.5. *Let A be a quasipolytope as above. Let*

$$W_s = \bigcup (A_t \mid t \geq s)$$

be a wall.

(a) *The set H_s of homomorphisms determined by s is a subalgebra of $A\widetilde{D}$.*
(b) *For each homomorphism $h \in H_s$, the restriction $h_s = h|_{W_s}$ extends uniquely to a homomorphism $\bar{h}_s : \bar{W}_s \to I$.*
(c) *For $t \geq s$, let $\varphi_{t,s} : W_t \to W_s$ be the Płonka homomorphism in the Płonka sum $\sum_{s\in S} W_s$, and let $\bar{\varphi}_{t,s} : \bar{W}_t \to \bar{W}_s$ be the Płonka homomorphism in the Płonka sum $\sum_{s\in S} \bar{W}_s$. For each $h \in H_s$ and $x_t \in \bar{W}_t$,*

$$x_t\bar{h}_s = x_t\varphi_{t,s}\bar{h}_s.$$

(d) *The algebra H_s is isomorphic to the algebra $H_s^w = \{W_s \to I^o\}$ of all homomorphisms from the wall W_s to the open unit interval I^o.*
(e) *The algebra H_s embeds into the algebra $\bar{H}_s = \{\bar{W}_s \to I\}$ of all homomorphisms from the closure $\bar{W}_s$ of W_s to the unit interval I, the closure of I^o.*

Again, we omit the easy proof.

Example 9.6. Consider again the three intervals of Example 9.3. But this time take as an object T the extended unit open interval $(I^o)^\infty$. Then the first dual of I is calculated similarly as in Example 9.3, and gives as $I\widetilde{D}$ the Płonka sum of the open polytopes:

- $A_1^o = \{h : I \to I^o\}$;
- $A_l^o = \{h \mid 0 \mapsto I^o$ and $x \mapsto \infty$ for $x \in (0, 1]\}$;
- $A_r^o = \{h \mid 1 \mapsto I^o$ and $x \mapsto \infty$ for $x \in [0, 1)\}$; and
- $A_\infty^o = \{h : I^o \mapsto \infty\}$,

with constant operations $\bar{r}$, for $r \in I^o$ and $\bar{\infty}$, and with Płonka homomorphisms being restrictions of Płonka homomorphisms of the previous example. Note that $I\widetilde{D}$ is isomorphic to $(I^o)^\infty \times (I^o)^\infty$.

In a similar way, one obtains the first duals of the intervals I^o and $I^{\triangleleft} := [0, 1)$ as subalgebras of $I\widetilde{D}$, the Płonka sums of the corresponding open polytopes, with the constants $\bar{r}$ and $\bar{\infty}$.

Corollary 9.7. *Let W_s be a wall of a quasipolytope $A = \bigcup_{s\in S} A_s$. The algebra H_s of all homomorphisms determined by s forms a barycentric subalgebra of $A\widetilde{D}$ isomorphic to the first dual A_sD^o of A_s under the functor D^o of the duality for open polytopes.*

Proof. We omit the obvious proof showing that H_s is a subalgebra of $A\widetilde{D}$. Then it is sufficient to note that, for each $s \in S$, the algebra A_sD^o is the interior of the first dual $\bar{A}_sD$ of the closure $\bar{A}_s$ of A_s under the functor D of the duality for convex polytopes, and to use the duality results for open polytopes of Section 7, Corollary 8.3 and Lemma 9.5(e). □

The following lemma may be proved in a similar fashion to Lemma 8.4.

Lemma 9.8. *Let A be a quasipolytope. The semilattice replica of $A\widetilde{D}$ is isomorphic to the first dual SC of the semilattice replica S of A under the functor C of the semilattice duality.*

The first main result describing the dual $A\widetilde{D}$ of A follows directly from the preceding results.

Theorem 9.9. *Let A be a quasipolytope and a semilattice sum $\bigcup_{s\in S} A_s$ of open polytopes A_s over its semilattice replica S. Then the barycentric algebra $A\widetilde{D}$ of homomorphisms from A to $(I^o)^\infty$ is isomorphic to the Płonka sum $\sum(A_sD^o)$ of the first duals A_sD^o of the open polytopes A_s over the first dual SC of S.*

In general, the barycentric algebra $A\widetilde{D}$ is not a quasipolytope. Observe, also, that the algebra $A\widetilde{D}$ is a subalgebra of the barycentric algebra $(\sum_{s\in S} \bar{A}_s)\bar{D} = \sum \bar{A}_sD$ of homomorphisms from the Płonka sum $\sum_{s\in S} \bar{A}_s$ into I^∞, with the Płonka fibres of $A\widetilde{D}$ being the interiors of the summands $\bar{A}_sD$.

10. Duality for quasipolytopes – second dual

The second dual of a quasipolytope A may be obtained using the second dual of the barycentric algebra $(\sum_{s\in S} \bar{A}_s)\bar{D} = \sum \bar{A}_sD$ and the duality for open polytopes of Section 7, by similar methods as in the case of the first dual. This provides the following theorem.

Theorem 10.1. *Let A be a quasipolytope and a semilattice sum $\bigcup_{s\in S} A_s$ of open polytopes A_s over its semilattice replica S. Then the barycentric algebra $A\widetilde{D}\widetilde{E}$ of homomorphisms from $A\widetilde{D}$ to $(I^o)^\infty$ is isomorphic to the semilattice sum $\bigcup(A_sD^oE^o)$ of the second duals $A_sD^oE^o$ of the open polytopes A_s over the second dual SCF of S.*

As a final consequence of the previous theorems, one obtains the required duality between the category of quasipolytopes and the corresponding category of representation spaces.

Corollary 10.2. *There is a duality between the categories $\mathcal{QP}$ and $\widehat{\mathcal{QP}}$.*

Example 10.3. We consider, again, the three intervals of Examples 9.3 and 9.6, and the duality given by the (extended) schizophrenic object $(I^o)^\infty$. The first dual $I\widetilde{D}$ was calculated in 9.6 as the Płonka sum of four open polytopes A_1^o, A_l^o, A_r^o and A_∞^o.

The algebra $I\widetilde{D}$ has three proper walls:

- $W_1 = A_1^o$;
- $W_r = A_r^o \cup A_1^o$; and
- $W_l = A_l^o \cup A_1^o$,

which determine, respectively, the sets of homomorphisms H_1, H_r and H_l from $I\widetilde{D}$ to $(I^o)^\infty$ and preserves the three constants $\bar{0}$, $\bar{1}$ and $\bar{\infty}$. Then the algebra H_1 is isomorphic to $I^oD^oE^o \cong I^o$ and the algebras H_r and H_l, both trivial, are the second duals of the end-points of I. The resulting algebra is a semilattice sum of these three subalgebras and is isomorphic to I.

The second duals of the two remaining intervals are calculated in a similar fashion.

As a final remark, let us note that the methods we used in this paper do not work in the case of bounded finite-dimensional convex sets, such as open circles, which are not quasipolytopes. To find a duality for the class of bounded convex sets, in general, would require different methods.

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