Properties of quadratic forms
associated with positive definite Hankel matrices

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(joint project with Christian Berg)

Every strictly positive definite Hankel matrix $H = \{m_{k+l}\}_{k,l=0}^{\infty}$ gives rise to the positive definite quadratic form on $\mathcal{F} \subset \ell^2$

$$Q(f,g) = \sum_{k,l=0}^{\infty} m_{k+l} f_k \overline{g_l},$$

where $\mathcal{F}$ denotes the sequences with finitely many nonzero terms. By the Hamburger theorem, there exists a finite measure μ, with infinite support on the real line, such that

$$m_k = \int_{-\infty}^{\infty} x^k \, d\mu(x) \qquad (*)$$

There are two entirely different cases, when the form Q is closable:

1) $\operatorname{supp}\mu \in (-1,1)$ or $m_n \to 0$, the result obtained by Yafaev

(2) The sequence $\{m_n\}$ is indeterminate, i.e. the measure μ in (1) is not uniquely determined. In particular $\sum m_n^{-1} < \infty$, joint result with Berg .

For a measure satisfying $(*)$, we study the operator A_μ, with $D(A_\mu) = \mathcal{F}$, given by

$$\mathcal{F} \ni g \stackrel{A_\mu}{\longmapsto} \sum_{k=0}^{\infty} g_k x^k \in L^2(\mu).$$

As $Q(f,g) = (A_\mu f, A_\mu g)$, the form Q is closable if and only if the operator A_μ is closable.

We are going to study the properties of $\overline{A}_\mu$, the closure of A_μ. In case (2) the operator $\overline{A}_\mu$ is a bijection from its domain onto $L^2(\mu)$, for any N-extremal measure μ, i.e. a measure μ for which the polynomials are dense in $L^2(\mu)$.

In case (1) the operator $\overline{A}_\mu$ may be surjective only when the set $\operatorname{supp}\mu$ is discrete in $(-1,1)$ and concentrated on a sequence of points x_n satisfying

$$\sum (1 - |x_n|) < \infty$$

and

$$\mu(\{x_n\}) \geq c(1 - |x_n|)$$

for a positive constant c.

The problem of surjectivity in case (1) is closely related to the Carleson theorem on interpolation in $H^2(\mathbb{D})$ space.