DISCRETE MATHEMATICS
EXERCISES
PART 4. GENERATING FUNCTIONS. INDUCTION. RECURRENCES.

1. Find generating functions of the following sequences:
   a) $a_n = \alpha^n$, $n = 0, 1, 2, \ldots$, $\alpha \in R$,
   b) $a_n = \begin{cases} 1, & n = 0, 1, \ldots, N, \\ 0, & n > N \end{cases}$,
   c) $a_n = \begin{cases} n+1, & n = 0, 1, \ldots, N, \\ 0, & n > N \end{cases}$,
   d) $a_n = \alpha n$, $n = 0, 1, 2, \ldots$, $\alpha \in R$,
   e) $a_n = n^2$, $n = 0, 1, 2, \ldots$,
   f) $a_n = n\alpha^n$, $n = 0, 1, 2, \ldots$, $\alpha \in R$.
2. Find a generating function $F(x)$ for the sequence A_n if the generating function $f(x)$ for a_n is given and:
   a) $A_n = a_{n+1}$, $n = 0, 1, 2, \ldots$,
   b) $A_n = a_{n+k}$, $n = 0, 1, 2, \ldots$, k is a fixed positive integer,
   c) $A_n = a_{n+1} - a_n$, $n = 0, 1, 2, \ldots$,
   d) $A_n = n \cdot a_n$, $n = 0, 1, 2, \ldots$,
   e) $A_n = \begin{cases} a_{n-1}, & n = 1, \ldots, \\ 0, & n = 0 \end{cases}$
3. Use generating functions to find a_n if:
   a) $a_n = 6n + a_{n-1}$, for $n \geq 1$ and $a_0 = 0$,
   b) $a_{n+2} = 2a_{n+1} + 3a_n$ for $n \geq 0$ and $a_0 = 1$, $a_1 = 2$,
   c) $a_n = -a_{n-1} + 2a_{n-2}$ for $n \geq 2$ and $a_0 = 1$, $a_1 = 2$.
4. Find a coefficient of x^{12} in
   a) $(1 + x^3 + x^6 + x^9 + \ldots)^7$,
   b) $(x + x^2 + x^3 + x^4)^5$,
   c) $x^2(1 - x)^{12}$.
5. Find a coefficient of x^{20} in $(x + x^2 + x^3 + x^4 + x^5)(x^2 + x^3 + x^4 + \ldots)^5$.
6. Use generating functions to find the number of ways to select 10 balls from a large pile of red, white and blue balls if:
   a) the selection has at least 2 balls of each color,
   b) the selection has at least 2 red balls.
7. Find the number of ways to select $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ fruits from a pile of 3 apples, 5 oranges and 2 bananas. (Hint: Use generating functions.)
8. How many ways are there to divide 2 blue, 5 red and 9 white balls into equal unordered piles?
9. Show that:
   a) $F_{n+m} = F_n F_m + F_{n-1} F_{m-1}$,
   b) $(F_n)^2 - F_{n+1} F_{n-1} = (-1)^n$,
   where F_n are the Fibonacci numbers.